Einstein-Podolsky-Rosen correlations and Bell correlations in the simplest scenario

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Einstein-Podolsky-Rosen (EPR) steering is an intermediate type of quantum nonlocality which lies between entanglement and Bell nonlocality. It is interesting to know what kind of states can generate EPR nonlocal correlations in the simplest nontrivial scenario, that is, two projective measurements for each party that share a two-qubit state. In this paper we settle this problem by deriving a necessary and sufficient analytical criterion. It turns out that a two-qubit state can demonstrate EPR nonlocal correlations in this scenario iff it can demonstrate Bell nonlocal correlations. This phenomenon is in sharp contrast with the strict hierarchy expected between the two forms of nonlocality. However, the strict hierarchy can emerge if both parties are restricted to performing mutually unbiased measurements. In addition, we clarify the relationships between entanglement, steering, and Bell nonlocality. In our study, we introduce the concept of restricted local hidden state models, which is useful in connecting different steering scenarios and is of interest in a wider context.

Local measurements on entangled states can generate nonlocal correlations that cannot be reproduced by any classical mechanism [1–4]. This counterintuitive phenomenon is a subject of continuous debates and inspirations for various new ideas [5–7]. These nonlocal correlations also play a key role in many quantum information processing protocols, especially quantum key distribution (QKD) and secure quantum communication [8–10]. Traditionally, nonlocal correlations can be detected by violation of Bell inequalities, which indicates one's inability to construct a local hidden variable (LHV) model that reproduces the correlations [3, 7]. Such correlations are known as Bell nonlocal.

Recently, Wiseman et al. [4, 11] formalized the concept of Einstein-Podolsky-Rosen (EPR) steering [1, 2] and showed that it is an intermediate type of nonlocality that sits between entanglement and Bell nonlocality. EPR steering can be detected by violation of steering inequalities (analog of Bell inequalities) [12–15], which indicates one's inability to construct a hybrid local hidden variable-local hidden state (LHV-LHS) model [4, 11, 16]. The asymmetry in the model reflects the asymmetry in the roles played by the two parties in a steering test [4, 17, 18]. Correlations that do not admit an LHV-LHS model are referred to as EPR nonlocal. EPR-nonlocal correlations are weaker than Bell-nonlocal correlations and are usually easier to generate in experiments [4]. Such correlations have received increasing attentions recently because of their intriguing connections with Bell-nonlocal correlations [19–22] and their potential applications in many quantum information processing protocols [6, 10, 23].

Which states can generate nonlocal correlations under local measurements? This question is of paramount importance not only to foundational studies but also to practical applications. However, it is usually very difficult, if not impossible, to address such a question in general. In view of this situation, it is instructive to first decode simple but nontrivial scenarios. The simplest Bell scenario consists of two parties, Alice and Bob, and two dichotomic measurements for each party [3, 24, 25]. In this case, the set of correlations is Bell nonlocal iff it violates the celebrated Clauser-Horne-Shimony-Holt (CHSH) inequality [24]; in addition, there is a simple criterion on determining which two-qubit states can generate such correlations [26, 27]. The simplest steering scenario also consists of two dichotomic measurements for each party, assuming that Bob (the steered party) can only perform projective measurements. Recently, Cavalcanti et al. introduced an analog CHSH inequality and showed that the set of correlations is EPR nonlocal iff this inequality is violated, assuming Bob performs mutually unbiased measurements [28]. Roy et al. then determined the optimal quantum violation of this inequality [29]. However, little is known beyond this point.

In this paper we show that a two-qubit state can generate EPR-nonlocal correlations in the simplest scenario iff it can violate the analog CHSH inequality introduced in Ref. [28]. Unlike the analysis in Ref. [28], it is not necessary to assume that Bob's measurements are mutually unbiased. We also derive the maximal violation of the analog CHSH inequality by any two-qubit state analytically, thereby determining all two-qubit states that can generate EPR-nonlocal correlations in the simplest steering scenario. It turns out that a two-qubit state can generate EPR-nonlocal correlations in this scenario iff it can generate Bell-nonlocal correlations, in sharp contrast with the strict hierarchy expected between the two forms of nonlocality. As comparison, we also determine the maximal violations of the CHSH and analog CHSH

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inequalities when both parties can only perform mutually unbiased measurements and show that this restriction leads to a strict hierarchy between steering and Bell nonlocality. The relations between concurrence and these maximal violations are clarified as well. Our study settles a fundamental question on steering in the simplest nontrivial scenario, which may serve as a starting point for exploring steering in more complicated scenarios. It also provides valuable insight on the relations between entanglement, steering, and Bell nonlocality. In the course of our study, we introduce the concept of restricted LHS models, which is very useful to connecting different steering scenarios and studying steering in more complicated scenarios.

Suppose Alice and Bob share a bipartite state ρ and they can perform measurements in the sets $M_{\rm A}$ and $M_{\rm B}$, respectively, which are referred to as their *measurement assemblages* henceforth. Here we shall mainly focus on projective measurements as represented by Hermitian operators. Let $p(a,b|A,B)$ be the probability of obtaining the outcomes a and b when Alice and Bob perform measurements $A\in M_{\rm A}$ and $B\in M_{\rm B}$, respectively. Then the state ρ is *steerable* in this scenario if the set of probability distributions $p(a,b|A,B)$ does not admit an LHV-LHS model [4, 11] as

$$p(a,b|A,B)=\sum_\lambda p_\lambda p(a|A,\lambda)p(b|B,\rho_\lambda) \qquad (1)$$

for all $A\in M_{\rm A}$ and $B\in M_{\rm B}$. Here $p(a|A,\lambda)$ for each A and λ denotes an arbitrary probability distribution, while $p(b|B,\rho_\lambda)=\operatorname{tr}(\rho_\lambda B_b)$ denotes the quantum probability of outcome b when Bob performs the measurement B on the state ρ_λ, where B_b is the operator corresponding to the outcome b. The difference between the two conditional probability distributions reflects different roles Alice and Bob play in the steering test.

What is the simplest steering scenario? This question has attracted a lot of attention recently [25, 28, 30]. To demonstrate steering, Alice and Bob need to share an entangled state, and the simplest candidate is a two-qubit state. Alice also needs a choice over her measurements, and the simplest measurement assemblage would consist of two projective measurements. It is not necessary for Bob to have a choice over different measurements, but the outcomes in his measurement assemblage cannot commute with each other pairwise, so the span of these outcomes has dimension at least 3; the simplest measurement assemblage satisfying this property consists of either two projective measurements or one trine measurement [25]. To summarize, in the simplest steering scenario, Alice and Bob share a two-qubit state, and there are two choices for the measurement setting:

1. Two projective measurements for Alice and Bob, respectively.

2. Two projective measurements for Alice and one trine measurement for Bob.

As we shall see shortly, a state is steerable in scenario 1 iff it is steerable in scenario 2; in other words, the two scenarios are equivalent as far as demonstrating steering is concerned. In practice, it is usually easier to perform two projective measurements than one trine measurement, so we shall focus on scenario 1 in the following discussions, but many of our results are also applicable to scenario 2 thanks to the above observation.

Now the steering scenario looks pretty simple, but it is still highly nontrivial to determine the steerability of a generic two-qubit state. To make further progress, it is instructive to note that the scenario we are considering is analogous to the simplest Bell scenario, in which the CHSH inequality is the only nontrivial Bell inequality up to permutations [24, 27]. In addition, the CHSH inequality is a (pure) correlation inequality. This observation suggests that it might be fruitful to look into those correlations between Alice and Bob's measurements in the study of steering.

Suppose Alice and Bob share the two-qubit state

$$\rho=\frac{1}{4}(I\otimes I+\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\otimes I+I\otimes\boldsymbol{\beta}\cdot\boldsymbol{\sigma}+\sum_{i,j=1}^{3}t_{ij}\sigma_i\otimes\sigma_j), \quad (2)$$

where σ_j for $j=1,2,3$ are three Pauli matrices, $\boldsymbol{\sigma}$ is the vector composed of these Pauli matrices, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the Bloch vectors for Alice and Bob, respectively, and $T=(t_{ij})$ is the correlation matrix. The projective measurements of Alice and Bob are specified by $\{A,A'\}=\{\boldsymbol{a}\cdot\boldsymbol{\sigma},\boldsymbol{a}'\cdot\boldsymbol{\sigma}\}$ and $\{B,B'\}=\{\boldsymbol{b}\cdot\boldsymbol{\sigma},\boldsymbol{b}'\cdot\boldsymbol{\sigma}\}$, respectively, where $\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}'$ are unit vectors in dimension 3. The two outcomes of each measurement are denoted by $\pm$. The correlation function between A and B is given by

$$\langle AB\rangle=\operatorname{tr}[\rho(A\otimes B)]=\boldsymbol{a}^{\rm T}T\boldsymbol{b}; \qquad (3)$$

the other three correlation functions $\langle AB'\rangle$, $\langle A'B\rangle$, $\langle A'B'\rangle$ take on a similar form. The set of correlations is EPR nonlocal if it does not admit an LHV-LHS model as discussed in Ref. [28],

$$\langle AB\rangle=\sum_\lambda p_\lambda {\rm E}(A,\lambda){\rm E}(B,\rho_\lambda) \qquad (4)$$

for all $\{A,A'\}$ and $\{B,B'\}$. Here ${\rm E}(A,\lambda)=p(+|A,\lambda)-p(-|A,\lambda)$, ${\rm E}(B,\rho_\lambda)=p(+|B,\rho_\lambda)-p(-|B,\rho_\lambda)$, with p_λ, $p(\pm|A,\lambda)$, $p(\pm|B,\rho_\lambda)$ being special instances of the probability distributions in (1).

Which two-qubit states can generate EPR nonlocal correlations under the simplest steering scenario? This question is a starting point for decoding steering in more complicated scenarios. It is also of intrinsic interest to understanding the connection and distinction between steering and Bell nonlocality. However, no complete solution has been found in the literature. Here we shall resolve this problem completely by deriving a simple analytical criterion that is both necessary and sufficient.

The first step towards our goal is the simple but crucial observation that under the simplest steering scenario, Bob's measurements can always be replaced by two mutually unbiased measurements without affecting the steerability. To demonstrate this point, we need to introduce several additional concepts, which are also applicable to general steering scenarios.

If Alice performs the measurement $A \in M_{\mathrm{A}}$ on the bipartite state ρ and obtains the outcome a, then Bob's subnormalized reduced state is given by $\rho_{a|A} = \mathrm{tr}_{\mathrm{A}}[(A_a \otimes I)\rho]$, with $\sum_a \rho_{a|A} = \rho_{\mathrm{B}} := \mathrm{tr}_{\mathrm{A}}(\rho)$. The set of subnormalized states $\{\rho_{a|A}\}_a$ for a given measurement A is an *ensemble* for ρ_{B}, and the whole collection of ensembles $\{\rho_{a|A}\}_{a,A}$ is a *state assemblage* [31]. The state assemblage $\{\rho_{a|A}\}_{a,A}$ is *unsteerable* if there exists an LHS model [4, 11, 15, 21, 32, 33]:

$$\rho_{a|A} = \sum_\lambda p_\lambda p(a|A,\lambda)\rho_\lambda, \tag{5}$$

where $p(a|A,\lambda) \geq 0$, $\sum_a p(a|A,\lambda) = 1$, and $p_\lambda\rho_\lambda$ are a collection of subnormalized states that sum up to ρ_{B} and thus form an ensemble for ρ_{B}.

To see the connection between the LHS model in (5) and the LHV-LHS model in (1), we need to introduce the concept of restricted LHS models. Let $\mathcal{B}(\mathcal{H})$ be the space of all operators acting on Bob's Hilbert space $\mathcal{H}$ and $\mathcal{V} \leq \mathcal{B}(\mathcal{H})$ a subspace. The assemblage $\{\rho_{a|A}\}_{a,A}$ admits a *$\mathcal{V}$-restricted LHS model* if

$$\mathrm{tr}(\Pi\rho_{a|A}) = \sum_\lambda p_\lambda p(a|A,\lambda)\,\mathrm{tr}(\Pi\rho_\lambda), \quad \forall \Pi \in \mathcal{V}, \tag{6}$$

where $p(a|A,\lambda)$ and ρ_λ satisfy the same constraints as in (5). In that case, the state assemblage $\{\rho_{a|A}\}_{a,A}$ is also called $\mathcal{V}$-unsteerable. The assemblage is unsteerable iff it is $\mathcal{B}(\mathcal{H})$-unsteerable. Any $\mathcal{V}$-unsteerable assemblage is also $\mathcal{W}$-unsteerable if $\mathcal{W} \leq \mathcal{V}$. In particular, an unsteerable assemblage is $\mathcal{V}$-unsteerable for any $\mathcal{V} \leq \mathcal{B}(\mathcal{H})$.

Let $\mathcal{R}$ be the space spanned by all the outcomes B_b in Bob's measurement assemblage. Then the set of probability distributions $p(a,b|A,B)$ admits an LHV-LHS model iff the state assemblage $\{\rho_{a|A}\}_{a,A}$ is $\mathcal{R}$-unsteerable. This conclusion follows from the following equation

$$p(a,b|A,B) = \mathrm{tr}[\rho(A_a \otimes B_b)] = \mathrm{tr}(\rho_{a|A}B_b). \tag{7}$$

Note that the existence of such a model does not depend on any detail of Bob's measurement assemblage except for the span $\mathcal{R}$. When $\{\rho_{a|A}\}_{a,A}$ is unsteerable, the set of $p(a,b|A,B)$ admits an LHV-LHS model irrespective of Bob's measurement assemblage. The converse holds if Bob's measurement assemblage is informationally complete, that is, $\mathcal{R} = \mathcal{B}(\mathcal{H})$, but may fail otherwise. For example, if Bob can perform only one projective measurement, then the set of $p(a,b|A,B)$ always admits an LHV-LHS model, irrespective of the state ρ and the measurements performed by Alice [25].

Now we can explain why steering scenarios 1 and 2 mentioned previously are equivalent and why it is sufficient to consider mutually unbiased measurements for Bob. The former is simply because the span of outcomes of any trine measurement on a qubit can be realized by two projective measurements, and vice versa. By the same token, in scenario 1, the steerability of the state does not change when the orientations of Bob's measurement vectors vary as long as the span of these vectors remains the same. Similarly, the existence of an LHV-LHS model of the set of correlations between Alice and Bob's measurements does not change, although the correlations themselves and the existence criterion on the model may depend on Bob's measurements. In view of this observation, we may assume that the two projective measurements of Bob are mutually unbiased without loss of generality.

Recently, Cavalcanti et al. [28] introduced an analog CHSH inequality,

$$\begin{aligned}&\sqrt{\langle (A+A')B\rangle^2 + \langle (A+A')B'\rangle^2} \\ &+ \sqrt{\langle (A-A')B\rangle^2 + \langle (A-A')B'\rangle^2} \leq 2,\end{aligned} \tag{8}$$

and showed that the set of correlations $\langle AB\rangle$, $\langle AB'\rangle$, $\langle A'B\rangle$, $\langle A'B'\rangle$ between Alice and Bob in the simplest steering scenario is EPR nonlocal iff the inequality is violated, assuming Bob's measurements are mutually unbiased. The violation of the inequality implies that the set of probability distributions $p(a,b|A,B)$ is EPR nonlocal. They also claimed that the converse is true, but there is a gap in their reasoning. Our discussion does not rely on this claim.

It is instructive to compare the analog CHSH inequality with the conventional CHSH inequality [24]

$$\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle \leq 2. \tag{9}$$

Note that the left hand side in (8) is never smaller and usually larger than the counterpart in (9). This observation may suggest that it is easier to generate EPR nonlocal correlations than Bell-nonlocal correlations. Here comes the surprise.

Theorem 1. *A two-qubit state can generate Bell-nonlocal correlations in the simplest nontrivial scenario iff it can generate EPR nonlocal correlations.*

This theorem is a consequence of the following theorem and the above conclusion that it is sufficient to consider mutually unbiased measurements for Bob.

Theorem 2. *The maximal violation S of the analog CHSH inequality by any two-qubit state with correlation matrix T is equal to the maximal violation of the CHSH inequality, namely, $S = 2\sqrt{\lambda_1+\lambda_2}$, where λ_1, λ_2 are the two larger eigenvalues of TT^{T}. Both inequalities are violated iff $\lambda_1+\lambda_2 > 1$.*

Remark 1. *The eigenvalues of TT^{T} coincide with that of $T^{\mathrm{T}}T$ and happen to be the squares of the singular values of T or T^{T}. Strictly speaking, $S = 2\sqrt{\lambda_1+\lambda_2}$ represents the maximal violation only when it is not smaller* *than 2. To avoid verbosity, we leave the reader to make a necessary modification when needed.*

Here the maximal violation S may be taken as a steering measure in the simplest steering scenario. Theorem 2 implies that any two-qubit state with $S > 2$ is steerable by two projective measurements. For Bell-diagonal states, the converse is also true according to our earlier study [30]. Surprisingly, in this case the steering measure S introduced here coincides with that introduced in Ref. [30] based on a quite different approach. Now the set of probability distributions $p(a,b|A,B)$ admits an LHV-LHS model iff the set of correlations $\langle AB\rangle$, $\langle AB'\rangle$, $\langle A'B\rangle$, $\langle A'B'\rangle$ does. In other words, the analog CHSH inequality is the only nontrivial inequality that characterizes the set of EPR-local probability distributions in this scenario.

Proof. The maximal violation of the CHSH inequality was derived by the Horodecki family [26]; cf. Refs. [27, 30]. To understand its connection with the violation of the analog CHSH inequality, it is instructive to reproduce the derivation here.

$$\begin{aligned}&\langle AB\rangle+\langle A'B\rangle+\langle AB'\rangle-\langle A'B'\rangle\\&=(\boldsymbol{a}^{\mathrm{T}}+\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}+(\boldsymbol{a}^{\mathrm{T}}-\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}'\\&\leq|T^{\mathrm{T}}(\boldsymbol{a}+\boldsymbol{a}')|+|T^{\mathrm{T}}(\boldsymbol{a}-\boldsymbol{a}')|\leq 2\sqrt{\lambda_1+\lambda_2},\end{aligned}\tag{10}$$

where the first inequality is saturated when $\boldsymbol{b}$ and $\boldsymbol{b}'$ align with $T^{\mathrm{T}}(\boldsymbol{a}+\boldsymbol{a}')$ and $T^{\mathrm{T}}(\boldsymbol{a}-\boldsymbol{a}')$, respectively, and the second one is saturated when $\boldsymbol{a}$ and $\boldsymbol{a}'$ are eigenvectors corresponding to the two larger eigenvalues of TT^{T}.

The maximal violation of the analog CHSH inequality can be derived as follows,

$$\begin{aligned}&\sqrt{\langle (A+A')B\rangle^2+\langle (A+A')B'\rangle^2}\\&\quad+\sqrt{\langle (A-A')B\rangle^2+\langle (A-A')B'\rangle^2}\\&=\sqrt{[(\boldsymbol{a}^{\mathrm{T}}+\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}]^2+[(\boldsymbol{a}^{\mathrm{T}}+\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}']^2}\\&\quad+\sqrt{[(\boldsymbol{a}^{\mathrm{T}}-\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}]^2+[(\boldsymbol{a}^{\mathrm{T}}-\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}']^2}\\&\leq|T^{\mathrm{T}}(\boldsymbol{a}+\boldsymbol{a}')|+|T^{\mathrm{T}}(\boldsymbol{a}-\boldsymbol{a}')|\leq 2\sqrt{\lambda_1+\lambda_2}.\end{aligned}\tag{11}$$

Note that $\boldsymbol{b}$ and $\boldsymbol{b}'$ are orthogonal by assumption. Here the first inequality is saturated when they form an orthonormal basis in the span of $T^{\mathrm{T}}(\boldsymbol{a}+\boldsymbol{a}')$ and $T^{\mathrm{T}}(\boldsymbol{a}-\boldsymbol{a}')$ or, equivalently, the span of $T^{\mathrm{T}}\boldsymbol{a}$ and $T^{\mathrm{T}}\boldsymbol{a}'$. The second one is saturated under the same condition as in (10). This observation confirms the first statement of the theorem. As an implication, the CHSH and analog CHSH inequalities can be violated iff $\lambda_1+\lambda_2>1$. □

What we have demonstrated in the above proof is actually stronger than stated in Theorem 2: the maximal violation of the analog CHSH inequality for fixed measurements of Alice is also equal to that of the CHSH inequality, though the maximum may depend on the measurements of Alice. The optimal measurements of Alice can always be chosen to be mutually unbiased. However, it is usually not necessary to do so. For example, when $T=\mathrm{diag}(t_1,t_2,t_3)$ with $1\geq t_1\geq t_2\geq |t_3|$, the maximal violations of the CHSH and analog CHSH inequalities are both equal to $2\sqrt{t_1^2+t_2^2}$. The optimal measurement directions can take on the form $\boldsymbol{a}=(t_1,t_2,0)/\sqrt{t_1^2+t_2^2}$ and $\boldsymbol{a}'=(t_1,-t_2,0)/\sqrt{t_1^2+t_2^2}$. Quite surprisingly, $\boldsymbol{a}$ and $\boldsymbol{a}'$ are almost parallel to each other when t_2/t_1 approaches to zero, which is the case for rank-2 Bell-diagonal states with almost equal nonzero eigenvalues [30].

Although the maximal violation of the analog CHSH inequality is the same as that of the CHSH inequality, there is a crucial difference in Bob's measurements required to saturate these inequalities. In the former scenario, it is not necessary to align Bob's measurements as long as the span of his measurement vectors is the same as that of $T^{\mathrm{T}}\boldsymbol{a}$ and $T^{\mathrm{T}}\boldsymbol{a}'$. In addition, it suffices to consider mutually unbiased measurements for both Alice and Bob to attain the maximal violation. In the latter scenario, by contrast, Bob's measurements need to be aligned according to Alice's measurements, and it is usually impossible to attain the maximal violation if their measurements are both mutually unbiased. So there is a strict hierarchy between steering and Bell nonlocality under the restriction to mutually unbiased measurements.

Theorem 3. *Suppose both Alice and Bob can only perform mutually unbiased measurements. The maximal violation of the analog CHSH inequality by any two-qubit state with correlation matrix T is still equal to $S=2\sqrt{\lambda_1+\lambda_2}$, where λ_1,λ_2 are the two larger eigenvalues of TT^{T}. The maximal violation of the CHSH inequality is equal to $S_{\mathrm{M}}=\sqrt{2}(\sqrt{\lambda_1}+\sqrt{\lambda_2})$.*

Remark 2. *Note that $\sqrt{2}(\sqrt{\lambda_1}+\sqrt{\lambda_2})\leq 2\sqrt{\lambda_1+\lambda_2}$, and the inequality is saturated iff $\lambda_1=\lambda_2$.*

Proof. The conclusion on the analog CHSH inequality is clear from the proof of Theorem 2, so it remains to consider the CHSH inequality. Since A and A' are mutually unbiased, we have $\boldsymbol{a}\perp\boldsymbol{a}'$ and similarly $\boldsymbol{b}\perp\boldsymbol{b}'$.

$$\begin{aligned}&\langle AB\rangle+\langle A'B\rangle+\langle AB'\rangle-\langle A'B'\rangle\\&=(\boldsymbol{a}^{\mathrm{T}}+\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}+(\boldsymbol{a}^{\mathrm{T}}-\boldsymbol{a}'^{\mathrm{T}})T\boldsymbol{b}'\\&=\sqrt{2}(\boldsymbol{c}^{\mathrm{T}}T\boldsymbol{b}+\boldsymbol{c}'^{\mathrm{T}}T\boldsymbol{b}')\leq\sqrt{2}(\sqrt{\lambda_1}+\sqrt{\lambda_2}),\end{aligned}\tag{12}$$

where $\boldsymbol{c}=(\boldsymbol{a}+\boldsymbol{a}')/\sqrt{2}$ and $\boldsymbol{c}'=(\boldsymbol{a}-\boldsymbol{a}')/\sqrt{2}$ are orthonormal. The inequality follows from the variational characterization of singular values [34]; it is saturated when $\boldsymbol{c},\boldsymbol{c}'$ are the left singular vectors corresponding to the two larger singular values of T, and $\boldsymbol{b},\boldsymbol{b}'$ are the corresponding right singular vectors. □

In general, entanglement is necessary but not sufficient to guarantee steering or EPR-nonlocal correlations. For entangled Bell-diagonal states, according to our early study [30], the steering measure S and the concurrence C [35] satisfy the relation

$$\frac{2\sqrt{2}}{3}(1+2C)\leq S\leq 2\sqrt{1+C^2}.\tag{13}$$

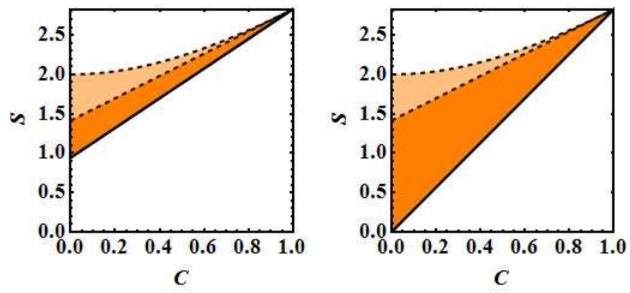

FIG. 1. Range of values of the steering measure S (the maximal violation of the CHSH and analog CHSH inequality) for given concurrence. Left plot: entangled Bell-diagonal states; right plot: general two-qubit states. For comparison, the range of values of $S_{\rm M}$ is rendered in dark orange, where $S_{\rm M}$ is the maximal violation of the CHSH inequality when both parties are restricted to performing mutually unbiased measurements. The maximal violation of the analog CHSH inequality does not change under this restriction.

The relation between $S_{\rm M}$ and C can be derived using the same method in Ref. [30] with the result

$$\frac{2\sqrt{2}}{3}(1+2C) \leq S_{\rm M} \leq \sqrt{2}(1+C); \tag{14}$$

see Fig. 1. Both lower bounds are saturated by Werner states, and upper bounds by rank-2 Bell-diagonal states.

In view of Theorem 2, the relation between S and C for general two-qubit states is the same as that between the maximal violation of the CHSH inequality and the concurrence as determined by Verstraete et al. [36], that is,

$$2\sqrt{2}C \leq S \leq 2\sqrt{1+C^2}. \tag{15}$$

Here the upper bound is saturated by pure states and rank-2 Bell-diagonal states, while the lower bound is saturated by quasi-distillable states [36, 37]. Recall that a quasi-distillable state is a mixture of a maximally entangled state and an orthogonal pure product state. These states are the worst in generating EPR and Bell-nonlocal correlations among states with the same concurrence. As an implication of (15), any state with $C > 1/\sqrt{2}$ can violate the analog CHSH inequality and is thus steerable.

The relation between $S_{\rm M}$ and C for general two-qubit states can be derived using a similar method for deriving (15), with the result

$$2\sqrt{2}C \leq S_{\rm M} \leq \sqrt{2}(1+C). \tag{16}$$

Again, the upper bound is saturated by pure states and rank-2 Bell-diagonal states, while the lower bound is saturated by quasi-distillable states. On the other hand, the upper bound in (16) is usually much smaller than that in (15), especially when C is small; see Fig. 1. As a consequence, any two-qubit state that can violate the CHSH inequality under mutually unbiased measurements for both Alice and Bob has concurrence at least $C > \sqrt{2}-1$. In particular, not all entangled pure states can violate the CHSH inequality in this scenario although they can under optimal measurements. Therefore, the restriction to mutually unbiased measurements severely limits the capability of two-qubit states in generating Bell-nonlocal correlations, in sharp contrast with the steering scenario, in which this is not a limitation. This observation is instructive to clarifying the distinction between the two forms of nonlocality. It is also of intrinsic interest to understanding the relation between nonlocality and incompatibility of observables.

In summary, we studied the generation of EPR-nonlocal correlations in the simplest nontrivial scenario and determined which two-qubit states are useful for this purpose. It turns out that a two-qubit state can generate EPR-nonlocal correlations in this scenario iff it can generate Bell-nonlocal correlations, despite the strict hierarchy between the two forms of nonlocality in general. However, the hierarchy emerges if both parties can only perform mutually unbiased measurements. The relations between entanglement, steering, and Bell nonlocality were then clarified. In the course of our study, we introduced the concept of restricted LHS models, which is useful beyond the focus of this work.

Note added: upon completion of this paper, we noticed a highly-related work of Costa and Angelo [38].

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