SPECTRAL ESTIMATES FOR THE HEISENBERG LAPLACIAN ON CYLINDERS

HYNEK KOVAŘÍK, BARTOSCH RUSZKOWSKI, AND TIMO WEIDL

ABSTRACT. We study Riesz means of eigenvalues of the Heisenberg Laplacian with Dirichlet boundary conditions on a cylinder in dimension three. We obtain an inequality with a sharp leading term and an additional lower order term.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain. We consider the Heisenberg Laplacian on Ω with Dirichlet boundary condition, formally given by

$$\mathrm{A}(\Omega) \ := \ -X_1^2 - X_2^2 \quad \text{with} \quad X_1 \ := \ \partial_{x_1} + \frac{x_2}{2}\partial_{x_3}, \quad X_2 \ := \ \partial_{x_2} - \frac{x_1}{2}\partial_{x_3}\,.$$

This operator is associated with the closure of the quadratic form

$$\mathrm{a}[u] \ := \ \int_\Omega |X_1 u(x)|^2 + |X_2 u(x)|^2 \ \mathrm{d}x, \tag{1.1}$$

initially defined on $C_0^\infty(\Omega)$. It is known, see e.g. [8], [3], [10], that $\mathrm{A}(\Omega)$ has purely discrete spectrum. We denote by $(\lambda_k(\Omega))_{k\in\mathbb{N}}$ the non-decreasing unbounded sequence of the eigenvalues of $\mathrm{A}(\Omega)$, where we repeat entries according to their finite multiplicities. We are interested in uniform upper bounds on the quantity

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- = \sum_{k=1}^\infty (\lambda_k(\Omega)-\lambda)_-\,.$$

In [3] Hansson and Laptev proved the following Berezin-type inequality for $\mathrm{A}(\Omega)$:

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \ \leq \ \frac{|\Omega|}{96}\,\lambda^3 \qquad \forall\, \lambda > 0. \tag{1.2}$$

It is also shown in [3] that

$$\sum_{k=1}^\infty (\lambda - \lambda_k(\Omega))_+ \ = \ \frac{|\Omega|}{96}\,\lambda^3 + o(\lambda^3) \qquad \text{as} \quad \lambda \to +\infty, \tag{1.3}$$

which implies that the constant $\frac{1}{96}$ on the right hand side of (1.2) is sharp.

Nevertheless, the authors of the present paper proved, see [10], that inequality (1.2) can be improved in the following sense; for a any bounded domain $\Omega \subset \mathbb{R}^3$, there exists a constant $C(\Omega) > 0$ such that for any $\lambda \geq 0$ it holds

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \ \leq \ \max\left\{0,\ \frac{|\Omega|}{96}\,\lambda^3 - C(\Omega)\,\lambda^2\right\}. \tag{1.4}$$

In other words, a negative remainder term of a lower order can be added to the right hand side of (1.2) without violating the inequality.

In this paper we will prove that the order of the remainder term in (1.4) can be further improved if we consider cylindrical domains of the type $\Omega = \omega \times (a, b)$, where $\omega \subset \mathbb{R}^2$ is open and bounded, and $a, b \in \mathbb{R}$ are such that $a < b$. In particular for cylinders wit convex cross-section ω our main result, Theorem 2.3, implies that

$$\mathrm{Tr}(\mathrm{A}(\Omega) - \lambda)_- \ \leq \ \max\left\{0, \frac{|\Omega|}{96}\,\lambda^3 - \frac{\lambda^{2+\frac{1}{4}}}{2^7 \cdot 3^{5/2}}\, \frac{|\Omega|}{\mathrm{R}(\omega)^{3/2}}\right\}, \tag{1.5}$$

where $\mathrm{R}(\omega)$ is the in-radius of ω, see Corollary 2.7. The proof of (1.5) is based on the unitary equivalence of $\mathrm{A}(\Omega)$ to the two-dimensional Laplacian with constant magnetic field. To estimate the remainder term we use a boundary estimate for the magnetic Laplacian based on an application of a Hardy inequality in the spirit of [4], see Proposition 3.1.

2. **Notation and main results**

As for the cross-section ω, throughout the paper we will suppose that the following condition is satisfied.

Assumption 2.1. *The open domain $\omega \subset \mathbb{R}$ is bounded and simply connected with Lipschitz boundary.*

In the sequel we will decompose the vector $x = (x', x_3) \in \mathbb{R}^3$. Let us denote by

$$\delta(x') \ := \ \mathrm{dist}\left(x', \partial\omega\right), \tag{2.1}$$

the distance function between a given $x' \in \omega$ and $\partial\omega$. The in-radius of ω is then given by

$$\mathrm{R}(\omega) := \sup_{x' \in \omega} \delta(x')\,.$$

Hardy inequality. Let $c = c(\omega)$ be defined by

$$c^{-2} := \inf_{u \in C_0^\infty(\omega)} \frac{\int_\omega |\nabla_{x'} u(x')|^2 \, \mathrm{d}x'}{\int_\omega |u(x')/\delta(x')|^2 \, \mathrm{d}x'}\,, \tag{2.2}$$

where $\nabla_x' := (\partial_{x_1}, \partial_{x_2})$. Clearly, c is the best constant in Hardy's inequality

$$\int_\omega \frac{u(x')^2}{\delta(x')^2}\, \mathrm{d}x' \ \leq \ c^2 \int_\omega |\nabla_{x'} u(x')|^2 \, \mathrm{d}x', \qquad u \in C_0^\infty(\omega). \tag{2.3}$$

Remark 2.2. Under assumption 2.1 it follows from [1] that

$$2 \ \leq \ c \ \leq \ 4. \tag{2.4}$$

The best possible value of c is $c = 2$. For a survey on Hardy inequalities we refer to [12], [6].

To continue we define for any $\beta > 0$ the set ω^β by

$$\omega^\beta \ := \ \left\{x' \in \omega \,|\, \delta(x') < \beta\right\}\,.$$

Finally, we introduce the quantity

$$l(\omega) := (b - a) \inf_{0 < \beta \leq \mathrm{R}(\omega)} \frac{|\omega^\beta|}{\beta}\,.$$

Now can state the main result of this paper.

Theorem 2.3. *Let $\Omega := \omega \times (a,b)$ and let c is given by (2.2). Then*

$$\operatorname{Tr}(\mathrm{A}(\Omega) - \lambda)_- \le \max\left\{0, \frac{|\Omega|}{96}\lambda^3 - \lambda^{\frac{2c+5}{c+2}} \frac{(1+\frac{2}{c})}{96} l(\omega)^{\frac{2c+2}{c+2}} |\Omega|^{-\frac{c}{c+2}} (4c+4)^{-\frac{2c+2}{c+2}}\right\} \tag{2.5}$$

holds for all $\lambda \ge 0$.

Remark 2.4. Note that the order of the remainder term is larger than λ^2 for any $c > 0$. So far the order of the second term in the asymptotic expansion (1.3) it is not known.

Remark 2.5. For analogous improvements of the classical spectral estimates for Dirichlet Laplacian on bounded domains we refer to [2], [11], [13], [14] and references therein.

Remark 2.6. Following [10] it can be shown that $l(\omega)$ is strictly positive. In particular, it holds $l(\omega) \ge (b-a)\mathrm{R}(\omega)\pi$.

Corollary 2.7. *Let $\Omega := \omega \times (a,b)$. If ω is convex, then*

$$\operatorname{Tr}(\mathrm{A}(\Omega) - \lambda)_- \le \max\left\{0, \frac{1}{96}|\Omega|\lambda^3 - \lambda^{2+\frac{1}{4}} \frac{1}{2^7 \cdot 3^2\sqrt{3}} \frac{|\Omega|}{\mathrm{R}(\omega)^{3/2}}\right\}$$

holds for all $\lambda \ge 0$.

Proof. In case that ω is convex we have $c = 2$ in (2.2) , see e.g. [6]. In addition, $\frac{|\omega^\beta|}{\beta}$ is a decreasing function of β on $(0, \mathrm{R}(\omega)]$, see [9, Lemma 4.2]. Hence we compute

$$l(\omega) = \frac{|\Omega|}{\mathrm{R}(\omega)}$$

and simplify the constant in Theorem 2.3. □

3. Preliminaries

3.1. **Magnetic Dirichlet Laplacian.** Let $\mathrm{P_{k,B}}$ be the orthogonal projection onto the kth Landau level $\mathrm{B(2k-1)}$ of the Landau Hamiltonian with constant magnetic field for $\mathrm{B} > 0$ in $L^2(\mathbb{R}^2)$ and $\mathrm{k} \in \mathbb{N}$. Denote by $\mathrm{P_{k,B}}(x,y)$ the integral kernel of $\mathrm{P_{k,B}}$. We will need these well-known characteristics

$$\begin{aligned}
\mathrm{P_{k,B}}(y,y) &= \frac{1}{2\pi}\mathrm{B}, \qquad \text{where } y \in \mathbb{R}^2,\\
\int_{\mathbb{R}^2}\left(\int_\Omega |\mathrm{P_{k,B}}(x,y)|^2 \ \mathrm{d}y\right) \mathrm{d}x &= \int_\Omega \left(\int_{\mathbb{R}^2} \mathrm{P_{k,B}}(x,y)\mathrm{P_{k,B}}(y,x) \ \mathrm{d}x\right) \mathrm{d}y \qquad (3.1)\\
&= \int_\Omega \mathrm{P_{k,B}}(y,y) \ \mathrm{d}y = \frac{\mathrm{B}}{2\pi}|\Omega|.
\end{aligned}$$

3.2. **A boundary estimate for the Heisenberg Laplacian.** In this subsection we will derive a boundary estimate for the operator $\mathrm{A}(\Omega)$ which will be crucial in estimating the size of the remainder term in Theorem 2.3.

Proposition 3.1. *Let $\Omega := \omega \times (a,b) \subset \mathbb{R}^3$ and let c be given by (2.2). Then*

$$\int_a^b \int_{\omega^\beta} |u(x',x_3)|^2 \ \mathrm{d}x' \ \mathrm{d}x_3 \ \le \ c^{2+\frac{2}{c}}\beta^{2+\frac{2}{c}} \left\|\mathrm{A}(\Omega)\, u\right\|_{L^2(\Omega)} \left\|\mathrm{A}(\Omega)^{1/c}\, u\right\|_{L^2(\Omega)} \tag{3.2}$$

holds for all $u \in \operatorname{Dom}(\mathrm{A}(\Omega))$ and any $\beta > 0$

For the proof we need the following Lemma.

Lemma 3.2. *Let* $\Omega := \omega \times (a,b) \subset \mathbb{R}^3$. *Then for all* $u \in d[a]$, *the form domain of the closure of* (1.1), *and any* $\beta > 0$ *we have*

$$\int_a^b \int_\omega \frac{|u(x',x_3)|^2}{\delta(x')^2} \, \mathrm{d}x' \, \mathrm{d}x_3 \le c^2 \, a[u].$$

Proof. Let u be in $C_0^\infty(\Omega)$. In addition let us denote by $\mathcal{F}_3$ the Fourier transform in x_3-direction, which is a unitary map in $L^2(\mathbb{R})$. Because Ω is a cylinder, the function $|\mathcal{F}_3 u(x',\xi_3)|$ lies in $\mathrm{H}_0^1(\omega)$ for fixed $\xi_3 \in \mathbb{R}$. Therefore we can apply inequality (2.3) to get

$$\begin{aligned}\int_a^b \int_\omega \frac{|u(x',x_3)|^2}{\delta(x')^2} \, \mathrm{d}x' \, \mathrm{d}x_3 &= \int_{\mathbb{R}} \int_\omega \left(\frac{|\mathcal{F}_3 u(x',\xi_3)|}{\delta(x')} \right)^2 \, \mathrm{d}x' \, \mathrm{d}\xi_3 \\ &\le c^2 \int_{\mathbb{R}} \int_\omega \big(\nabla_{x'} |\mathcal{F}_3 u(x',\xi_3)| \big)^2 \, \mathrm{d}x' \, \mathrm{d}\xi_3.\end{aligned}$$

Now we set

$$\mathbf{A}(x') := \frac{1}{2}(-x_2, x_1), \tag{3.3}$$

and apply the diamagnetic inequality which states that

$$|\nabla |\psi|| \ \le \ |(\mathrm{i}\nabla + \mathbf{A})\psi| \qquad \text{a. e.} \tag{3.4}$$

holds for all $\psi \in H^1(\omega)$, see e.g. [7]. This gives

$$\int_{\mathbb{R}} \int_\omega \big(\nabla_{x'} |\mathcal{F}_3 u(x',\xi_3)| \big)^2 \, \mathrm{d}x' \, \mathrm{d}\xi_3 \le \ \int_{\mathbb{R}} \int_\omega \big| \big(\mathrm{i}\nabla_{x'} + \xi_3 \mathbf{A}(x') \big) \mathcal{F}_3 u(x',\xi_3) \big|^2 \, \mathrm{d}x' \, \mathrm{d}\xi_3.$$

Integration by parts in the x_3-direction yields the inequality for $u \in C_0^\infty(\Omega)$. A density argument completes the proof. □

Proof of Proposition 3.1. We follow the proof of [4, Thm 4]. Let us fix $u \in \mathrm{Dom}(\mathrm{A}(\Omega))$ and set

$$\varphi(x) := (\max\{\delta(x'), \beta\})^{-1/c}.$$

for $x := (x', x_3) \in \Omega$ and $\beta > 0$. In what follows we will use the notation

$$\nabla_{\mathbb{H}} = (X_1, X_2)$$

to denote the Heisenberg gradient. First we check that $\varphi u \in d[a]$. Since $u \in \mathrm{Dom}(\mathrm{A}(\Omega)) \subseteq d[a]$, $\varphi \in H_0^1(\omega)$ and get

$$\int_\Omega |\nabla_{\mathbb{H}}(\varphi(x) u(x))|^2 \, \mathrm{d}x \le 2 \int_\Omega |\varphi(x) \nabla_{\mathbb{H}} u(x)|^2 \, \mathrm{d}x + 2 \int_\Omega |\nabla_{x'} \varphi(x)|^2 |u(x)|^2 \, \mathrm{d}x. \tag{3.5}$$

Note that we used here $\nabla_{\mathbb{H}} \varphi(x) = \nabla_{x'} \varphi(x)$ for all $x \in \Omega$. The Eikonal equation

$$|\nabla_{x'} \varphi(x)|^2 = 1 \quad \text{a.e. } x \in \Omega, \tag{3.6}$$

and the boundedness of Ω yield the finiteness of (3.5). Hence $\varphi u \in d[a]$ and we may use Lemma 3.2 to get

$$\begin{aligned} c^{-2} \int_\Omega \frac{|\varphi(x) u(x)|^2}{\delta(x')^2} \, \mathrm{d}x &\le \int_\Omega |\varphi(x) \nabla_{\mathbb{H}} u(x) + u(x) \nabla_{\mathbb{H}} \varphi(x)|^2 \, \mathrm{d}x \\ &= \big\langle \varphi^2 \nabla_{\mathbb{H}} u, \, \nabla_{\mathbb{H}} u \big\rangle + \big\langle u, \, |\nabla_{\mathbb{H}} \varphi|^2 u \big\rangle \\ &\quad + \frac{1}{2} \big\langle \nabla_{\mathbb{H}} u, \, u \nabla_{\mathbb{H}}(\varphi^2) \big\rangle + \frac{1}{2} \big\langle u \nabla_{\mathbb{H}}(\varphi^2), \, \nabla_{\mathbb{H}} u \big\rangle, \end{aligned}$$

where we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega)$. An integration by parts in the last two terms yields

$$c^{-2} \int_\Omega \frac{|\varphi(x)u(x)|^2}{\delta(x')^2}\, \mathrm{d}x \,\le\, \mathrm{Re}\big\langle\, \varphi^2 u,\, \mathrm{A}(\Omega) u \big\rangle + \big\langle\, u,\, |\nabla_{\mathbb{H}}\, \varphi|^2 u \big\rangle.$$

Next we will estimate the first term on the right hand side. To this end we use Lemma 3.2, which gives

$$\delta^{-2} \le c^2 \mathrm{A}(\Omega)$$

in the operator sense. Then, by the Heinz inequality, see [5, Lemma 4.20],

$$\varphi^4 \le (\delta^{-2})^{2/c} \le \big(c^2 \mathrm{A}(\Omega)\big)^{2/c}.$$

Since $\mathrm{A}(\Omega)^{-1/c}$ is bounded in $L^2(\Omega)$ we obtain

$$\big\|\varphi^2 \mathrm{A}(\Omega)^{-1/c}\big\| \le c^{2/c},$$

where $\|\cdot\|$ stands for the operator norm in $L^2(\Omega)$. Hence

$$|\big\langle\, \mathrm{A}(\Omega)u,\, \varphi^2 u\big\rangle| = |\big\langle\, \mathrm{A}(\Omega)u,\, \varphi^2 \mathrm{A}(\Omega)^{-1/c}\mathrm{A}(\Omega)^{1/c} u\big\rangle| \le \|\mathrm{A}(\Omega)u\|_{L^2(\Omega)} c^{2/c} \big\|\mathrm{A}(\Omega)^{1/c}u\big\|_{L^2(\Omega)}.$$

So we arrive at

$$c^{-2} \int_\Omega \frac{|\varphi(x)u(x)|^2}{\delta(x')^2}\, \mathrm{d}x \le \|\mathrm{A}(\Omega)u\|_{L^2(\Omega)} c^{2/c} \big\|\mathrm{A}(\Omega)^{1/c}u\big\|_{L^2(\Omega)} + \big\langle\, u,\, |\nabla_{\mathbb{H}}\varphi|^2 u\big\rangle. \tag{3.7}$$

On the other hand, the Eikonal equation (3.6) implies that

$$|\nabla_{\mathbb{H}}\varphi(x)|^2 = |\nabla_{x'}\varphi(x)|^2 = c^{-2}\delta(x')^{-2/c-2}\chi_{\{\delta(x')\ge\beta\}}(x'),$$

where $\chi_{\{\delta(x')>\beta\}}$ is the characteristic function of the set $\{x \in \Omega \,|\, \delta(x') \ge \beta\}$. Inserting the above identity into (3.7) thus yields

$$\int_{\{x\in\Omega|\delta(x')<\beta\}} \frac{u(x)|^2}{\delta(x')^2}\, \mathrm{d}x \le \beta^{2/c}\|\mathrm{A}(\Omega)u\|_{L^2(\Omega)} c^{2+2/c} \big\|\mathrm{A}(\Omega)^{1/c}u\big\|_{L^2(\Omega)}. \tag{3.8}$$

The result now follows from the estimate

$$\int_{\{x\in\Omega|\delta(x')<\beta\}} |u(x)|^2\, \mathrm{d}x \le \beta^2 \int_{\{x\in\Omega|\delta(x')<\beta\}} \frac{u(x)|^2}{\delta(x')^2}\, \mathrm{d}x.$$

□

4. **Proof of Theorem 2.3**

Here and below we write a vector $x \in \mathbb{R}^3$ as $x = (x', x_3)$. Let v_j the orthonormal basis of the eigenfunctions of $\mathrm{A}(\Omega)$ for $j \in \mathbb{N}$;

$$\mathrm{A}(\Omega)v_j \;=\; \lambda_j v_j, \quad \|v_j\|_{L^2(\Omega)} = 1.$$

Let $\mathcal{F}_3$ be the partial Fourier transform in the x_3 variable. Then

$$\mathcal{F}_3 \mathrm{A}(\mathbb{R}^3)\mathcal{F}_3^* \;=\; \Big(\mathrm{i}\partial_{x_1} - \frac{1}{2}x_2\xi_3\Big)^2 + \Big(\mathrm{i}\partial_{x_2} + \frac{1}{2}x_1\xi_3\Big)^2 \;=\; \big(\mathrm{i}\nabla_{x'} + \xi_3 \mathbf{A}(x')\big)^2,$$

where $\mathbf{A}(x')$ is given by (3.3). At this point we use the properties of the magnetic Laplacian, see section 3.1, to obtain

$$\mathcal{F}_3\,\mathrm{A}(\mathbb{R}^3)\,u(x',\xi_3) = \sum_{\mathrm{k}=1}^{\infty} |\xi_3|(2\mathrm{k}-1) \int_{\mathbb{R}^2} \mathrm{P}_{\mathrm{k},\xi_3}(x',y')\mathcal{F}_3 u(y',\xi_3)\;\mathrm{d}y' \tag{4.1}$$

for $\mathcal{F}_3\,u(\cdot,\xi_3)$ in the domain of the magnetic Laplacian.

4.1. **The sharp leading term.** First of all we extend for every $j\in\mathbb{N}$ the eigenfunctions by $v_j(x) := 0$ for all $x\in\Omega^c$. Now we consider

$$\begin{aligned}\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;&=\; \sum_{j:\lambda_j<\lambda} \lambda\|v_j\|^2_{L^2(\mathbb{R}^3)} - \|X_1 v_j\|^2_{L^2(\mathbb{R}^3)} - \|X_2 v_j\|^2_{L^2(\mathbb{R}^3)}\\ &=\; \int_{\mathbb{R}} \sum_{j:\lambda_j<\lambda} \lambda\|\mathcal{F}_3 v_j(\cdot,\xi_3)\|^2_{L^2(\mathbb{R}^2)} - \left\|\left(\mathrm{i}\partial_{x_1} - \tfrac{1}{2}x_2\xi_3\right)\mathcal{F}_3 v_j(\cdot,\xi_3)\right\|^2_{L^2(\mathbb{R}^2)}\;\mathrm{d}\xi_3\\ &\quad-\; \int_{\mathbb{R}} \sum_{j:\lambda_j<\lambda} \left\|\left(\mathrm{i}\partial_{x_2} + \tfrac{1}{2}x_1\xi_3\right)\mathcal{F}_3 v_j(\cdot,\xi_3)\right\|^2_{L^2(\mathbb{R}^2)}\;\mathrm{d}\xi_3.\end{aligned}$$

At this point we apply the spectral decomposition (4.1) of the free Heisenberg Laplacian. An application of Fatou's Lemma then yields

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;\le\; \int_{\mathbb{R}} \sum_{j:\lambda_j<\lambda} \sum_{\mathrm{k}=1}^{\infty} (\lambda - |\xi_3|(2\mathrm{k}-1))_+ \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\;\mathrm{d}\xi_3,$$

where

$$\begin{aligned}f_{j,\mathrm{k},\xi_3}(x') &:= \int_{\mathbb{R}^2} \mathrm{P}_{\mathrm{k},\xi_3}(x',y')\mathcal{F}_3 v_j(y',\xi_3)\;\mathrm{d}y' = \frac{1}{\sqrt{2\pi}}\int_{\Omega} \mathrm{P}_{\mathrm{k},\xi_3}(x',y')\mathrm{e}^{-\mathrm{i}y_3\xi_3} v_j(y',y_3)\;\mathrm{d}y'\;\mathrm{d}y_3\\ &= \frac{1}{\sqrt{2\pi}}\left\langle\, \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\; v_j(\cdot)\right\rangle_{L^2(\Omega)}.\end{aligned}$$

We split the sum as follows;

$$\begin{aligned}\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_- \;\le\; &\int_{\mathbb{R}} \sum_{\mathrm{k}:\lambda>|\xi_3|(2\mathrm{k}-1)} (\lambda - |\xi_3|(2\mathrm{k}-1)) \sum_{j=1}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\;\mathrm{d}\xi_3\\ &- \int_{\mathbb{R}} \sum_{\mathrm{k}:\lambda>|\xi_3|(2k-1)} (\lambda - |\xi_3|(2\mathrm{k}-1)) \sum_{j:\lambda_j\ge\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\;\mathrm{d}\xi_3,\end{aligned} \tag{4.2}$$

noting that the first term on the right hand side is positive and the other one is negative. The completeness of v_j and the traces of $\mathrm{P}_{\mathrm{k},\xi_3}$, see (3.1), yield

$$\sum_{j=1}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \sum_{j=1}^{\infty} \left|\left\langle\, \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\; v_j(\cdot)\right\rangle_{L^2(\Omega)}\right|^2\;\mathrm{d}x' = \frac{|\xi_3|}{4\pi^2}|\Omega|. \tag{4.3}$$

To obtain the sharp leading term in (2.5) we apply this identity in the first integral on the right hand side of (4.2). Using the fact that

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8},$$

we thus get

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_{\mathrm{k}:\lambda>|\xi_3|(2\mathrm{k}-1)} (\lambda-|\xi_3|(2\mathrm{k}-1)) \sum_{j=1}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}\xi_3 \\
&= \frac{|\Omega|}{4\pi^2} \int_{\mathbb{R}} \sum_{\mathrm{k}:\lambda>|\xi_3|(2\mathrm{k}-1)} (\lambda-|\xi_3|(2\mathrm{k}-1))|\xi_3|\, \mathrm{d}\xi_3 \\
&= \frac{|\Omega|}{2\pi^2} \sum_{\mathrm{k}=1}^{\infty} \frac{1}{(2\mathrm{k}-1)^2} \int_0^\infty s(\lambda-s)_+\, \mathrm{d}s \;=\; \frac{|\Omega|}{96}\,\lambda^3.
\end{aligned}$$

Inserting this back into (4.2) gives

$$\operatorname{Tr}(\mathrm{A}(\Omega)-\lambda)_- \le \frac{|\Omega|}{96}\,\lambda^3 - \int_{\mathbb{R}} \sum_{\mathrm{k}:\lambda>|\xi_3|(2k-1)} (\lambda-|\xi_3|(2\mathrm{k}-1)) \sum_{j:\lambda_j\ge\lambda}^{\infty} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}\, \mathrm{d}\xi_3. \quad (4.4)$$

4.2. **The lower order term.** To obtain a suitable lower bound on the second term in (4.4) we use the same technique as in [9]. The key point of this approach is to estimate the quantity

$$\mathcal{R}_\lambda := \sum_{j:\lambda_j\ge\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)}$$

from below by a power function of λ. Note that

$$\begin{aligned}
&\mathcal{R}_\lambda \;=\; \frac{|\xi_3|}{4\pi^2}|\Omega| - \sum_{j:\lambda_j<\lambda} \|f_{j,\mathrm{k},\xi_3}\|^2_{L^2(\mathbb{R}^2)} = \\
&\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\Omega} \left| \mathrm{P}_{\mathrm{k},\xi_3}(x',y')\mathrm{e}^{-\mathrm{i}y_3\xi_3} - \sum_{j:\lambda_j<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y',y_3) \right|^2 \mathrm{d}y'\, \mathrm{d}y_3\, \mathrm{d}x'.
\end{aligned}$$

The inclusion $\omega \supseteq \omega^\beta$ and an application of $|z-w|^2 \ge \frac{1}{2}|z|^2 - |w|^2$, with $z,w\in\mathbb{C}$, imply that

$$\begin{aligned}
\mathcal{R}_\lambda \;\ge\; &\frac{|\xi_3|}{8\pi^2}(b-a)\,|\omega^\beta| \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_a^b\int_{\omega^\beta} \left| \sum_{j:\lambda_j<\lambda} \big\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot)\big\rangle_{L^2(\Omega)} v_j(y',y_3) \right|^2 \mathrm{d}y'\, \mathrm{d}y_3\, \mathrm{d}x'.
\end{aligned}$$

Next we estimate the negative integral. Note that the linear combinations of v_j lie in $\mathrm{Dom}(\mathrm{A}(\Omega))$. Therefore we may apply Proposition 3.1 and obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_a^b\int_{\omega^\beta}\left|\sum_{j:\lambda_j<\lambda}\left\langle \mathrm{P}_{\mathrm{k},\xi_3}(x',\cdot)\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot)\right\rangle_{L^2(\Omega)} v_j(y',y_3)\right|^2 \,\mathrm{d}y'\,\mathrm{d}y_3\right)\,\mathrm{d}x' \\
&\le\ c^{2+2/c}\beta^{2+2/c}\lambda^{1+1/c}\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\sum_{j:\lambda_j<\lambda}\left|\left\langle P_{k,\xi_3}(x',\cdot)\mathrm{e}^{-\mathrm{i}\cdot\xi_3},\, v_j(\cdot)\right\rangle_{L^2(\Omega)}\right|^2\right)\,\mathrm{d}x' \\
&\le\ c^{2+2/c}\beta^{2+2/c}\lambda^{1+1/c}\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_\Omega \left|P_{k,\xi_3}(x',y')\right|^2\,\mathrm{d}y'\,\mathrm{d}x_3\right)\,\mathrm{d}x' \\
&=\ c^{2+2/c}\beta^{2+2/c}\lambda^{1+1/c}\frac{|\Omega|}{4\pi^2}|\xi_3|,
\end{aligned}$$

which yields the following lower bound on $\mathcal{R}_\lambda$:

$$\begin{aligned}
\mathcal{R}_\lambda\ &\ge \frac{|\xi_3|}{8\pi^2}(b-a)\left|\omega^\beta\right| - c^{2+2/c}\beta^{2+2/c}\lambda^{1+1/c}\frac{|\Omega|}{4\pi^2}|\xi_3| \\
&\ge \frac{|\xi_3|}{8\pi^2}\beta\left(l(\omega)-2c^{2+2/c}\beta^{1+2/c}\lambda^{1+1/c}|\Omega|\right).
\end{aligned}$$

Now we set

$$\beta^{1+2/c}=\frac{l(\omega)}{c^{2+2/c}\lambda^{1+1/c}(4+4/c)|\Omega|},$$

which is possible for $\lambda\ge\lambda_1(\Omega)$, because of

$$\beta^{1+2/c}\ \le\ \frac{1}{c^{2+2/c}\lambda_1(\Omega)^{1+1/c}(4+4/c)\mathrm{R}(\omega)}\ \le\ \frac{\mathrm{R}(\omega)^{1+2/c}}{4}\,.$$

The last inequality was obtained by applying Proposition 3.1 to $u=v_1$ and $\beta=\mathrm{R}(\omega)$. Summing up we thus arrive at

$$\mathcal{R}_\lambda\ge\ \frac{|\xi_3|}{8\pi^2}\lambda^{-\frac{c+1}{c+2}}l(\omega)^{\frac{2c+2}{c+2}}|\Omega|^{-\frac{c}{c+2}}(2+4/c)(4c+4)^{-\frac{2c+2}{c+2}}\ =\lambda^{-\frac{c+1}{c+2}}\,G(\Omega)\,|\xi_3|,$$

where

$$G(\Omega):=\frac{l(\omega)^{\frac{2c+2}{c+2}}}{8\pi^2}\,|\Omega|^{-\frac{c}{c+2}}(2+4/c)(4c+4)^{-\frac{2c+2}{c+2}}\,.$$

This in combination with (4.4) gives

$$\mathrm{Tr}(\mathrm{A}(\Omega)-\lambda)_-\le\frac{|\Omega|}{96}\,\lambda^3-G(\Omega)\lambda^{-\frac{c+1}{c+2}}\int_{\mathbb{R}}\sum_{\mathrm{k}:\lambda>|\xi_3|(2k-1)}(\lambda-|\xi_3|(2\mathrm{k}-1))|\xi_3|\,\mathrm{d}\xi_3.$$

To finish the proof we calculate in the same way as in the beginning of the proof:

$$\sum_{\mathrm{k}=1}^\infty\int_0^\infty(\lambda-\xi_3(2\mathrm{k}-1))_+\xi_3\,\mathrm{d}\xi_3\ =\sum_{\mathrm{k}=1}^\infty\frac{1}{(2\mathrm{k}-1)^2}\int_0^\infty s(\lambda-s)_+\,\mathrm{d}s\ =\ \frac{\pi^2\,\lambda^3}{48}\,.$$

This gives the estimate stated in Theorem 2.3.

Aknowledgements

H. K. was supported by the Gruppo Nazionale per Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The support of MIUR-PRIN2010-11 grant for the project "Calcolo delle variazioni" (H. K.) is also gratefully acknowledged.

B. R. was supported by the German Science Foundation through the Research Training Group 1838: *Spectral Theory and Dynamics of Quantum Systems.*

References

[1] A. Ancona. On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$. *J.London Math. Soc.*, 34:274–290, 1986.

[2] A.D. Melas. A lower bound for sums of eigenvalues of the Laplacian. *Proc.Amer.Math.Soc.*, 131:631–636, 2003.

[3] A.M. Hansson and A. Laptev. Sharp spectral inequalities for the Heisenberg Laplacian. *London Math. Soc. Lecture Note Ser., 354*, pages 100–115, 2008.

[4] E.B. Davies. Sharp boundary estimates for elliptic operators. *Math. Proc. Cambridge Philos. Soc.*, 129(1):165–178, 2000.

[5] Edward Brian Davies. *One-parameter semigroups*, volume 15 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1980.

[6] E.B. Davies. A review of Hardy inequalities. In *The Maz'ya anniversary Collection. Vol 2.Oper. Theory Adv. Appl.*, 110, pages 55–67. 1999.

[7] E.H. Lieb, M. Loss. *Analysis*. Amer. Math. Soc., second edition, pages 195-196. 1997.

[8] G.B. Folland. A fundamental solution for a subelliptic Operator. *Bull. Amer. Math. Soc.*, 79:373–376, 1973.

[9] H. Kovařík and T. Weidl. Improved Berezin-Li-Yau inequalities with magnetic field. *Proc. Royal Soc. Edinburgh, Sect. A, 145*, pages 145–160, 2015.

[10] H. Kovařík, B. Ruszkowski and T. Weidl. Melas-type bounds for the Heisenberg Laplacian on bounded domains. *arXiv:1511.04223*, 2015.

[11] H. Kovařík, S. Vugalter and T. Weidl. Two dimensional Berezin-Li-Yau inequalities with a correction term. *Comm. Math. Phys., 287*, pages 959–981, 2009.

[12] B. Opic and A. Kufner. *Hardy-type inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.

[13] S.Y. Yolcu. An improvement to a berezin-li-yau type inequality. *Proc. Amer. Math. Soc., 138*, pages 4059–4066, 2010.

[14] S.Y. Yolcu and T. Yolcu. Estimates for the sums of eigenvalues of the fractional laplacian on a bounded domain. *Commun. Cont. Math., 15*, page 1250048, 2013.

Hynek Kovařík, DICATAM, Sezione di Matematica, Università degli studi di Brescia, Via Branze, 38 - 25123 Brescia, Italy

E-mail address: hynek.kovarik@ing.unibs.it

Bartosch Ruszkowski, Institute of Analysis, Dynamics and Modeling, Universität Stuttgart, PF 80 11 40, D-70569 Stuttgart, Germany

E-mail address: Bartosch.Ruszkowski@mathematik.uni-stuttgart.de

Timo Weidl, Institute of Analysis, Dynamics and Modeling, Universität Stuttgart, PF 80 11 40, D-70569 Stuttgart, Germany

E-mail address: weidl@mathematik.uni-stuttgart.de