ON THE ARITHMETIC OF LANDAU-GINZBURG MODEL OF A CERTAIN CLASS OF THREEFOLDS

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Abstract. We prove that the Apéry constants for a certain class of Fano threefolds can be obtained as a special value of a higher normal function.

1. Introduction

The application of normal functions in areas peripheral to Hodge theory has emerged as a topic of research over the last decade [3],[4],[9],[15],[16],[18]; areas related to physics have accounted for much of this growth. The goal of this work is to use normal functions to give a 'motivic' meaning to constants arising in quantum differential equations associated to a certain class of Landau-Ginzburg models.

In [3], there is a explicit computation of a higher normal function associated with the Landau-Ginzburg mirror of a rank 4 Fano threefold, which turns out to be the value of a Feynman Integral. We want to present a similar approach, but instead of a Feynman integral, we will express some Apéry constants ([2],[14],[10],[11]) in terms of a special values of the associated higher normal functions.

Landau-Ginzburg models are the natural object for 'mirrors' of Fano manifolds; more precisely, mirror symmetry relates a Fano variety with a dual object, which is a variety equipped with a non-constant complex valued function. For example, a LG model for $\mathbb{P}^2$ is a family of elliptic curves and more generally, the LG model of a Fano n-fold is a family of Calabi-Yau $(n-1)$-folds. In general, mirror symmetry relates symplectic properties of a Fano variety with algebraic ones of the mirror and vice versa.

In this work we will be mainly concerned with the Landau-Ginzburg models for a special class of threefolds, namely the ones whose associated local system is of rank three, with a single nontrivial involution exchanging two maximally unipotent monodromy points. Looking at the classification in [5], one finds the short list V_{12}, V_{16}, V_{18} and "R_1", where the first three are rank 1 Fanos appearing in [14] and the latter is a rank 4 threefold with $-K^3 = 24$ (K the canonical divisor). The involutions for these LG models have essentially been described in [14] and [3]. In the presence of an involution, it is possible to move the degeneracy locus of a higher cycle from the fiber over 0 to its involute, a property which we use for the construction of the desired normal function.

Let $\mathbb{P}_{\Delta^\circ}$ be a toric degeneration of any of the varieties considered above; then each one of these will have a mirror Landau-Ginzburg model, which is a family of $K3$ surfaces in $\mathbb{P}_\Delta$, that can be constructed as follows. Let ϕ be a Minkowski

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polynomial for Δ, then the family of $K3$ is:

$$X_t := \overline{\{1 - t\phi(\mathbf{x}) = 0\}} \subset \mathbb{P}_\Delta \tag{1.1}$$

Let

$$\omega_t = \frac{1}{(2\pi i)^2} Res_{X_t}\left(\frac{\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}}{1 - t\phi}\right) \tag{1.2}$$

and γ_t the invariant vanishing cycle about $t = 0$. We define the period of ϕ by

$$\Pi_\phi(t) = \int_{\gamma_t} \omega_t = \sum a_n t^n \tag{1.3}$$

where a_n is the constant term of ϕ^n. We say that a_n is the period sequence of ϕ. Consider a polynomial differential operator $L = \sum F_k(t)P_k(D_t)$ where $P_k(D_t)$ is a polynomial in $D_t = t\frac{d}{dt}$, then $L \cdot \Pi_\phi(t) = 0$ is equivalent to a linear recursion relation. In practice, to compute L one uses knowledge of the first few terms of the period sequence and linear algebra to guess the recursion relation. The operator L is called a *Picard Fuchs operator*.

Example 1.1. The Picard-Fuchs operator for the threefold V_{12} is:

$$D^3 - t(1 + 2D)(17D^2 + 17D + 5) + t^2(D + 1)^3 \tag{1.4}$$

More generally, one also gets the same linear recursion on the power-series coefficients b_n of solutions of inhomogeneous equations $L(\,\cdot\,) = G$, G a polynomial in t, for $n \geq \deg(G)$.

Definition 1.2 ([14])**.** Given a linear homogeneous recurrence R and two solutions $a_n, b_n \in \mathbb{Q}$ with $a_0 = 1, b_0 = 0, b_1 = 1$, if there is a L-function $L(x)$ and $c \in \mathbb{Q}^*$ such that:

$$\lim \frac{b_n}{a_n} = cL(x_0) \tag{1.5}$$

We say that 1.5 is the Apéry constant of R.

When we have a family of Calabi-Yau manifolds, a common way to look for Apéry constants is by considering the Picard-Fuchs equation. As described above, the coefficients of the power series expansion of the solutions of this equation satisfy a recurrence and in some cases the Apéry constant exists, see [2] for a wide class of examples. Beyond this "classical" case, we can also talk about quantum recurrences, which are recurrences arising from solutions of the Quantum differential equations satisfied by the quantum periods, which are defined using quantum products, see [13].

In [14], Golyshev uses quantum recurrences of the threefolds $V_{10}, V_{12}, V_{14}, V_{16}, V_{18}$ to find Apéry constants; his method is basically to use a result of Beukers [[14], Proposition 3.3] for the rational cases and apply a different approach for the non-rational ones. In the course of the proof of his results, he also describes the involution we mentioned above, but only for V_{12}, V_{16} and V_{18}. The main theorem of this manuscript is:

Theorem 1.3. *Let X be a Fano threefold, in the special class described above. Then there is a higher normal function $\mathcal{N}$, arising from a family of motivic cohomology classes on the fibers of the LG model, such that the Apéry constant is equal to $\mathcal{N}(0)$.*

As an immediate corollary of this result and Borel's theorem, the Apéry constant for these cases must be a $\mathbb{Q}$-linear combination of $\zeta(3)$ and $(2\pi i)^3$, which provides a uniform conceptual explanation of this feature of the results in [14] and [3].

Remark 1.4. We note that throughout this paper, the cycle groups are taken modulo torsion ($\otimes\mathbb{Q}$).

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2. Construction of the "toric" motivic classes

We assume the reader is familiar with the basic notions of Toric geometry, see [7] for a brief review or [8] for a more comprehensive treatment. Let

$$\phi = \sum a_{\mathbf{m}}\mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}] \tag{2.1}$$

be a Laurent polynomial with coefficients in $\mathbb{C}$ and Δ be the Newton polytope associated with ϕ, which we will assume to be reflexive. (A list of all 3-dimensional reflexive polytopes is available at [5].) We briefly review the construction of the anti-canonical bundle and the facet divisors on the toric variety $\mathbb{P}_\Delta$. Let x, y, z be the toric coordinates on $\mathbb{P}_\Delta$ and for each codimension 1 face $\sigma \in \Delta(1)$, choose a point $\mathbf{o}_\sigma$ with integral coordinates, and write $\mathbb{R}_\sigma$ for the 2-plane through σ . Then take a basis $\mathbf{m}_1, \mathbf{m}_2$ for the translate $(\mathbb{R}^3_\sigma \cap \mathbb{Z}^3) - \mathbf{o}_\sigma$ and complete it to a basis $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ for $\mathbb{Z}^3$ such that

$$\mathbb{R}_{\geq 0}\langle \pm\mathbf{m}_1, \pm\mathbf{m}_2, \mathbf{m}_3\rangle \supset \Delta - \mathbf{o}_\sigma \tag{2.2}$$

Change coordinates, by setting $x_j^\sigma = \mathbf{x}^{\mathbf{m}_j}, j = 1, 2, 3$. Consider the subset

$$\mathbb{D}^*_\sigma = \{x_1^\sigma, x_2^\sigma \in \mathbb{C}^*\} \cap \{x_3^\sigma = 0\} \tag{2.3}$$

of $\mathbb{P}_\Delta$; let $\mathbb{D}_\sigma$ be the Zariski closure of $\mathbb{D}^*_\sigma$, and set

$$\mathbb{D} := \sum_{\sigma\in\Delta(1)} [\mathbb{D}_\sigma] = \mathbb{P}_\Delta \backslash (\mathbb{C}^*)^3. \tag{2.4}$$

Henceforth we shall write x, y, z for x_1, x_2, x_3.

A standard result in toric geometry is that the sheaf $\mathcal{O}(\mathbb{D})$ is ample and in case Δ is reflexive; it is also the anti-canonical sheaf for $\mathbb{P}_\Delta$, and hence $\mathbb{P}_\Delta$ is Fano in this case.

Given non vanishing holomorphic functions $f_1, \dots, f_n$ on a quasi-projective variety Y, we denote the higher Chow cycle given by the graph of the f_j in $Y \times (\mathbb{P}^1)^n$ by $\langle f_1, \dots, f_n\rangle \in CH^n(Y, n)$.

Definition 2.1. A 3 dimensional Laurent polynomial ϕ is tempered if the symbol $\langle x^\sigma, y^\sigma\rangle_{D^*_\sigma} \in CH^2(D^*_\sigma, 2)$ is trivial, for all facets σ, where $D^*_\sigma \subset \mathbb{D}^*_\sigma$ is the zero locus of the facet polynomial ϕ_σ.

Remark 2.2. The definition above can be restated as follows: For X_t a general $K3$ surface of the family induced by ϕ, let $X_t^* = X_t \cap (\mathbb{C}^*)^3$; then ϕ is tempered if the image of the higher Chow cycle $\xi_t := \langle x, y, z\rangle_{X_t^*} \in CH^3(X_t^*, 3)$ under all residue maps vanishes. (Equivalently, viewed as an element of Milnor K-theory $K_3^M(\mathbb{C}(X_t))$, ξ_t belongs to the kernel of the Tame symbol, cf. [17].)

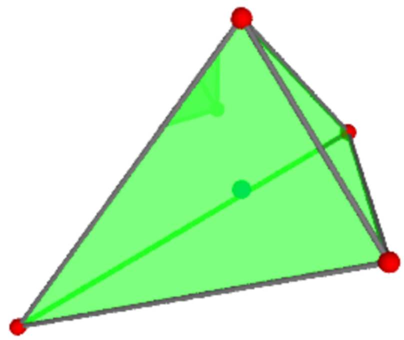

FIGURE 1. Newton polytope for the Laurent polynomial $\phi = x + y + z + (xyz)^{-1}$. Taken from [5].

In this work, we will focus on a special class of Laurent polynomials, namely Minkowski polynomials. See [1] for the basic definitions and properties of Minkowski polynomials.

Example 2.3. Consider the Minkowski polynomial $\phi = x + y + z + (xyz)^{-1}$ with Newton polytope Δ with vertices $(1,0,0),(0,1,0),(0,0,1)$ and $(-1,-1,-1)$, see figure 2. Let σ be the facet with vertices $(1,0,0),(0,1,0),(-1,-1,-1)$ and fix $(-1,-1,-1)$ as the 'origin' of the facet. Then clearly one possible choice of the new toric coordinates is:

$$\begin{aligned} x^\sigma &= x^2yz \\ y^\sigma &= xy^2z \\ z^\sigma &= x^{-1} \end{aligned} \tag{2.5}$$

Moreover $\mathbb{D}_\sigma^* = \{z^\sigma = 0\}$, so that D_σ^* is given by the zero locus of the facet polynomial $\phi_\sigma = 1 + x^\sigma + y^\sigma$. Therefore $Res_{D_\sigma^*}\langle x,y,z\rangle_{X_t^*} = Res_{z^\sigma=0}\langle x^\sigma, y^\sigma, z^\sigma\rangle_{X_t^*} = \langle x^\sigma, y^\sigma\rangle_{D_\sigma^*} = \langle x^\sigma, -1-x^\sigma\rangle = 0$. Similarly, any other facet σ of this polytope has the property that $\langle x^\sigma, y^\sigma\rangle_{D_\sigma^*} = 0$.

The fact that the symbol $\langle x^\sigma, y^\sigma\rangle_{D_\sigma^*}$ is trivial for all facets is not a coincidence; in fact, this is always the case for three-dimensional Minkowski polynomials. More precisely, we have:

Proposition 2.4. Every three-dimensional Minkowski polynomial is tempered.

Proof. In general, it is not true that every Laurent polynomial is tempered; one of the features of Minkowski polynomials is that they give rise to a decomposition in terms of rational irreducible subvarieties, a fact that will be strongly used below. We use the equivalent definition of tempered as presented in remark 2.2.

Noting that $D_\sigma := \mathbb{D}_\sigma \cap X_t$ and $D = \mathbb{D} \cap X_t = \cup D_\sigma$ are independent of $t \neq 0$, and $X_t^* = X_t \setminus D$, let $\imath : D \to X_t$ and $\jmath : X_t^* \to X_t$ be the natural inclusions. The localization exact sequence for higher Chow groups reads:

$$\cdots \to CH^2(D,3) \xrightarrow{\imath_*} CH^3(X_t,3) \xrightarrow{\jmath^*} CH^3(X_t^*,3) \xrightarrow{Res_D} CH^2(D,2)\cdots \tag{2.6}$$

Now in general, D_σ is reducible, with components determined by the Minkowski decomposition of σ. Write $D = \cup D_i$ as the resulting union of irreducible curves, and $D_i^* = D_i \setminus \cup_j (D_i \cap D_j)$. By the localization sequence (for D_i), we have

$$CH^2(D_i,2) = \ker\left\{ CH^2(D_i^*,2) \overset{Res_{ij}}{\to} \oplus_j CH^1(D_i \cap D_j, 1)\right\}. \tag{2.7}$$

Since the edge polynomials of a Minkowski polynomial are cyclotomic,[1] for every i,j the composition

$$CH^3(X_t^*,3) \overset{Res_i}{\to} CH^2(D_i^*,2) \overset{Res_{ij}}{\to} \oplus_j CH^1(D_i\cap D_j,1) \tag{2.8}$$

sends ξ_t to zero. By (2.7), we therefore have $Res_i\xi \in CH^2(D_i,2)$ for every i. Since in dimension 3 the irreducible pieces of a lattice Minkowski decomposition are either segments or triangles with no interior points, all the D_i are rational and smooth. Moreover, since both the Minkowski polynomial and the decomposition of the facet polynomials are defined over $\bar{\mathbb{Q}}$, the D_i are rational over $\bar{\mathbb{Q}}$. Now the $Res_i\xi$ are clearly defined over $\bar{\mathbb{Q}}$ (as the $Res_\sigma \xi_t = \langle x^\sigma, y^\sigma\rangle$ are), and so belong to $CH^2(\mathbb{P}^1,2) \cong K_2(\bar{\mathbb{Q}}) = \{0\}$,

Therefore $Res_i\xi_t$ is trivial, and ϕ is tempered by Remark 2.2. □

Remark 2.5. The notion of Minkowski polynomial for dimension greater than 3 is not yet well understood. However, if we assume the lattice polytopes in the Minkowski decompositions of facets have no interior points, then the proof above will extend to dimension 4, since we would still have rationality of the D_i (as above), and no significant problems appear in the local-global spectral sequence for higher Chow groups.

3. The Higher normal function $\mathcal{N}$

Recall that if S is a smooth projective variety, then

$$H^n_{\mathcal{M}}(S,\mathbb{Q}(n)) \cong CH^n(S,n) \cong Gr^n_\gamma K_n(S). \tag{3.1}$$

Not every member of our family X_t is smooth, but we can still have an element in the motivic cohomology. Such elements can be explicitly represented via higher Chow (double) complexes, so that we can still use standard formulas for Abel-Jacobi maps [20, §8]:

$$AJ^{m,n} : H^n_{\mathcal{M}}(X_t,\mathbb{Q}(n)) \to H^{n-1}(X_t, \mathbb{C}/\mathbb{Q}(n)). \tag{3.2}$$

The Landau-Ginzburg models for the threefolds V_{12}, V_{16}, V_{18}, and R_1, may be defined by (the Zariski closure of) the families $\{1-t\phi=0\}$, with ϕ given by:

$$\begin{aligned}
V_{12} : \phi &= \frac{(1+x+z)(1+x+y+z)(1+z)(y+z)}{xyz}\\
V_{16} : \phi &= \frac{(1+x+y+z)(1+z)(1+y)(1+x)}{xyz}\\
V_{18} : \phi &= \frac{(x+y+z)(x+y+z+xy+xz+yz+xyz)}{xyz}\\
R_1 : \phi &= \frac{(1+x+y+z)(xyz+xy+xz+yz)}{xyz}
\end{aligned} \tag{3.3}$$

[1] in fact the roots are ± 1

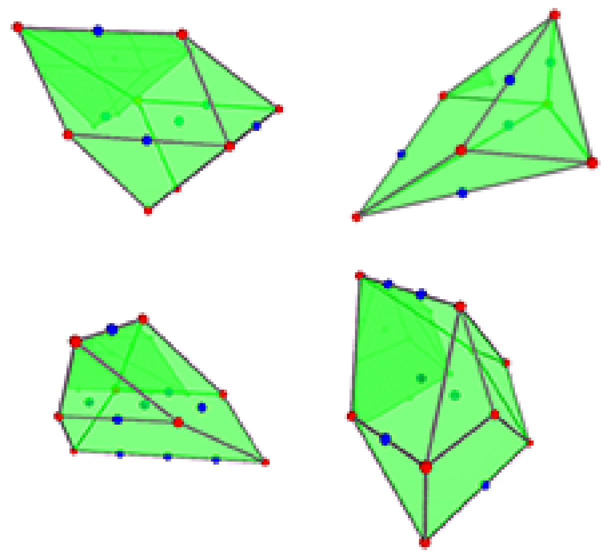

FIGURE 2. Newton polytopes for (top) V_{18}, R_1 and (bottom) V_{12}, V_{16} respectively. Taken from [5].

As these families of K3s all have Picard rank 19, their Picard-Fuchs operators take the form $D_{PF} = \sum_{i=0}^{3} F_k(t)(D_t)^k$, with $F_i(t)$ relatively prime polynomials. We call $F_3(t) =: \sigma(D_{PF})$, which is taken to be monic, the *symbol* of D_{PF}. In the four cases the symbols are

$$1 - 34t + t^2,\ 1 - 24t + 16t^2,\ 1 - 18t - 27t^2, \text{ and } 1 - \tfrac{5}{16}t + \tfrac{1}{64}t^2, \tag{3.4}$$

respectively.

We shall adopt the notation $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$ for the total space of each family, $\mathcal{X}^\circ = \mathcal{X} \setminus X_0 \xrightarrow{\pi^\circ} \mathbb{A}^1_{\frac{1}{t}}$ and $\mathcal{X}_\circ = \mathcal{X} \setminus X_\infty \xrightarrow{\pi_\circ} \mathbb{A}^1_t$, for restrictions. Henceforward, X will denote any threefold in the list $V_{12}, V_{16}, V_{18}, R_1$.

Proof of theorem 1.3. Associated to X is a Newton polytope Δ, and to the latter we associate a Minkowski polynomial ϕ. Since by the proposition above, ϕ is tempered, the family of higher Chow cycles lifts to a class $[\Xi] \in CH^3(\mathcal{X}^\circ, 3)$ [[9],theorem 3.8], yielding by restriction a family of motivic cohomology classes $[\Xi_t] \in H^3_{\mathcal{M}}(X_t, \mathbb{Q}(3))$ on the Landau-Ginzburg model. (On the smooth fibers these are just higher Chow cycles.)

The local system $\mathbb{V} = R^2_{tr}\pi_*\mathbb{Z}$ associated to the Landau-Ginzburg model of X has the following singular points:

- $V_{12} : t = 0, 17 \pm 12\sqrt{2}, \infty$
- $V_{16} : t = 0, 12 \pm 8\sqrt{2}, \infty$
- $V_{18} : t = 0, 9 \pm 6\sqrt{3}, \infty$
- $R_1 : t = 0, 4, 16, \infty$

(Besides 0 and ∞, these are just the roots of $\sigma(D_{PF})$.)

In each case, we have an involution $\iota(t) = \frac{M}{t}$, ($M = 1, \frac{1}{16}, \frac{-1}{27}, 64$), exchanging say t_1 and t_2 with $0 < |t_1| < |t_2| < \infty$. The involution ι gives then a correspondence

$I \in Z^2(\mathcal{X} \times \iota^*\mathcal{X})$ which gives a rational isomorphism between $\mathbb{V}$ and $\iota^*\mathbb{V}$. Since I induces an isomorphism, the vanishing cycle γ_t at $t = 0$ is sent to a rational multiple of the vanishing cycle μ_t at $t = \infty$. Hence in a neighborhood of $t = 0$, we have:

$$\int_{\gamma_t} I^*\omega_{\iota(t)} = \int_{I_*\gamma_t} \omega_{\iota(t)} = n\int_{\mu_{\iota(t)}} \omega_{\iota(t)}, n \in \mathbb{Q}^* \tag{3.5}$$

Moreover, as a section of the Hodge bundle, ω_t has a simple zero at $t = \infty$ and no zero or poles anywhere else. So $I^*\omega_{\iota(t)} = Ct\omega_t$, for some $C \in \mathbb{C}^*$. If we set $A(t) = \int_{\gamma_t} \omega_t$, then $A(0) = 1$, and it follows that

$$\begin{aligned} C = \lim_{t\to 0} \frac{n}{(2\pi i)^2 A(t)} \int_{\mu_{\iota(t)}} Res_{X_{\iota(t)}} \left(\frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}}{t - M\phi} \right) \\ = -\frac{n}{M} Res^3_p \left(\frac{dx \wedge dy \wedge dz}{xyz \cdot \phi(x, y, z)} \right), \end{aligned} \tag{3.6}$$

where $p \in \text{sing}(X_\infty)$ is the point to which $\mu_{\iota(t)}$ contracts. Hence C is rational and $\tilde{\omega} := I^*\omega$ is a rational multiple of $t\omega$.

Now let $\tilde{\Xi} := I^*\Xi \in H^3_{\mathcal{M}}(\mathcal{X}_\circ, \mathbb{Q}(3))$ be the pullback of the cycle, with fiberwise slices $\tilde{\Xi}_t$. If AJ is the Abel-Jacobi map[2] as above, then

$$AJ^{3,3}([\tilde{\Xi}_t]) \in H^2(X_t, \mathbb{C}/\mathbb{Q}(3)). \tag{3.7}$$

Taking $\mathcal{R}_t$ to be any lift of this class to $H^2(X_t, \mathbb{C})$, we may define a normal function by:

$$\mathcal{N}(t) := \langle \mathcal{R}_t, \omega_t \rangle \tag{3.8}$$

By [9, Prop. 4.1], $\mathcal{N}(t)$ has a power series of radius of convergence $|t_2| > |t_1|$. Moreover, by [9, p. 474], we have

$$D_{PF}(\mathcal{N}(t)) = \sigma(D_{PF})\mathcal{Y}(t), \tag{3.9}$$

where $\mathcal{Y}(t) = (2\pi i)^2 \langle \tilde{\omega_t}, \nabla^2_{D_t}\omega_t \rangle$ is the Yukawa coupling.

Applying [9, Rem. 4.4], the right-hand side of (3.9) takes the form kt, where (in view of (3.4)) $k = \lim_{t\to 0} \frac{\mathcal{Y}(t)}{t}$. By writing ω_t in terms of a basis of $e^{\frac{log(t)}{(2\pi i)}N}\mathbb{V}$ about $t = 0$, we find that $k = C\kappa$ where κ is the (rational) nonzero entry of N^2. We conclude that

$$D_{PF}(\mathcal{N}(t)) = kt, k \in \mathbb{Q}^*. \tag{3.10}$$

Finally, if $A(t) = \sum a_n t^n$ is the period sequence, then $B(t) = \sum b_n t^n = -\mathcal{N}(t) + A(t)\mathcal{N}(0)$ is another solution for the Picard-Fuchs equation, so that

$$\mathcal{N}(t) = \sum (a_n \mathcal{N}(0) - b_n)t^n.$$

Since the radii of convergence for the generating series of a_n and b_n are both $|t_1| < |t_2|$, while that of $a_n\mathcal{N}(0) - b_n$ is $|t_2|$, it follows that $\frac{b_n}{a_n} \to \mathcal{N}(0)$. □

Corollary 3.1. $\mathcal{N}(0)$ is (up to $\mathbb{Q}(3)$) a multiple of $\zeta(3)$.

[2]In smooth fibers, AJ takes a rather simple form in terms of currents, see [19]

Proof. The proof is a direct consequence of the following commutative diagram (See [20]):

$$\begin{array}{ccc} H^3_{\mathcal{M}}(X_0,\mathbb{Q}(3)) & \xrightarrow{\cong} & K_5^{ind}(\mathbb{Q}) \\ \downarrow{\scriptstyle AJ^{3,3}} & & \downarrow{\scriptstyle r_b} \\ J^{3,3}(X_0) & \xrightarrow[\cong]{} & \frac{\mathbb{C}}{\mathbb{Q}(3)} \end{array} \tag{3.11}$$

Where the lower isomorphism is the pairing with ω_0 and r_b is the Borel regulator. The Abel-Jacobi map then reduces to the Borel regulator and by Borel's theorem it has to be multiple of $\zeta(3)$. □

Remark 3.2. An explicit computation of $\mathcal{N}(0)$ for R_1 has been written in [3]; the computation for V_{12} was done by M. Kerr and will be available in a forthcoming paper. Below we present the explicit computation of $\mathcal{N}(0)$ in the case V_{16}:

Example 3.3. Consider V_{16} which has a Minkowski polynomial given by $\phi = (x+1)(y+1)(z+1)(1+x+y+z)$; We change the coordinates to simplify the computations and use the same idea as [3]. The normal function $\mathcal{N}$ at 0 takes the following form:

$$\mathcal{N}(0) = \int_{\nabla} R\{x, y, (1-x-y)\} \tag{3.12}$$

Where ∇ is the "membrane" $\nabla = \big\{(\text{x,y}) : -1 \le y \le 1 \ , \ -y \le x \le 1 \ \big\}$. We have:

$$\begin{aligned} \mathcal{N}(0) &= \int_{\nabla} log(y) dlog\big(1-x-y\big) \wedge dlog(x) \\ &= \int_{-1}^{1} log(y)\Big(\int_{-y}^{1} \frac{dx}{x(1-x-y)}\Big)dy \\ &= \int_{-1}^{1} log(y)\Big(\int_{-y}^{1} \frac{dx}{x(1-y)} + \int_{-y}^{1} \frac{dx}{(1-y)(1-x-y)}\Big)dy \\ &= 2\int_{-1}^{1} log(y)\frac{log(-y)}{(1-y)}dy \\ &\equiv 4\int_{-1}^{1} log(1-y)\frac{log(y)}{y}dy \qquad \text{mod} \qquad \mathbb{Q}(3) \\ &\equiv -4\sum_{k\ge 1}\frac{1}{k}\int_{-1}^{1} log(y)y^{k-1}dy \qquad \text{mod} \qquad \mathbb{Q}(3) \\ &\equiv 8\sum_{k\ odd}\frac{1}{k^3} \qquad \text{mod} \qquad \mathbb{Q}(3) \\ &\equiv 7\zeta(3) \qquad \text{mod} \qquad \mathbb{Q}(3) \end{aligned} \tag{3.13}$$

where the $\mathbb{Q}(3)$ reflects the local ambiguity of $\mathcal{N}$ by a $\mathbb{Q}(3)$-period of $\tilde{\omega}$ (owing to the choice of lift $\mathcal{R}$). Since the Apéry constant is a real number, we normalize $\mathcal{N}$ locally by adding such a period to obtain $\mathcal{N}(0) = 7\zeta(3)$.

4. Concluding Remarks

The proof of Theorem 1.3 makes use of an involution of the family over $t \mapsto \pm\frac{M}{t}$ to produce a cycle with no residues on the $t = 0$ fiber, but with nontorsion associated normal function. That is, we use the involution to transport the residues of the cycle we *do* know how to construct (via temperedness) to over $t = \infty$.

What is absolutely certain is that without a second maximally unipotent monodromy fiber (at $t = \infty$ in our four examples), such a normal function cannot exist. This follows from injectivity of the topological invariant into

$$Hom_{\mathrm{MHS}}(\mathbb{Q}(0), H^3(\mathcal{X}^*, \mathbb{Q}(3))) \subset \oplus_{\lambda\in\Sigma} Hom_{\mathrm{MHS}}(\mathbb{Q}(0), H_2(X_\lambda, \mathbb{Q})),$$

where $\Sigma \subset \mathbb{P}^1$ denotes the discriminant locus. As an immediate consequence, nothing like Theorem 1.3 can possibly hold for Golyshev's V_{10} and V_{14} examples.

While we could broaden the search to all local systems with more than one maximally unipotent monodromy point, those having an involution (or some other automorphism) represent our best chance for constructing cycles. Though it is required to apply a couple of the tools of[9] as written, the $h^2_{tr}(X_t) = 3$ assumption is perhaps less essential; if we drop this, there are many other LG local systems with "potential involutivity". Inspecting data from [5], we see that the period sequences $35, 49, 52, 53, 55, 59, 60, 62, 97$ and 151 have monodromies that suggest the presence of an involution. This is something we will investigate in future works.

Finally, we omitted one case with $h^2_{tr}(X_t) = 3$ ad an involution, namely B_4 (cf. [5]). This is because there is a second involution, namely $t \mapsto -t$, wich probably rules out a meaninful Apéry constant (as $|t_1| = |t_2|$).

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