SIMONS' CONE AND EQUIVARIANT MAXIMIZATION OF THE FIRST p-LAPLACE EIGENVALUE

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ABSTRACT. We consider an optimization problem for the first Dirichlet eigenvalue of the p-Laplacian on a hypersurface in $\mathbb{R}^{2n}$, with $n \geq 2$. If $p \geq 2n-1$, then among hypersurfaces in $\mathbb{R}^{2n}$ which are $O(n) \times O(n)$-invariant and have one fixed boundary component, there is a surface which maximizes the first Dirichlet eigenvalue of the p-Laplacian. This surface is either Simons' cone or a C^1 hypersurface, depending on p and n. If n is fixed and p is large, then the maximizing surface is not Simons' cone. If $p = 2$ and $n \leq 5$, then Simons' cone does not maximize the first eigenvalue.

1. Introduction

In this article we consider an optimization problem for the first Dirichlet eigenvalue of the p-Laplacian. This problem is motivated by Simons' cone and by the Faber-Krahn inequality. Simons' cone was the first example of a singular area minimizing cone. Almgren [2] showed that the only area minimizing hypercones in $\mathbb{R}^4$ are hyperplanes. Simons [22] extended this to higher dimensions up to $\mathbb{R}^7$ and established the existence of a singular stable minimal hypercone in $\mathbb{R}^8$, given by

$$\left\{(x_1, \ldots, x_4, y_1, \ldots, y_4) \in \mathbb{R}^8 : x_1^2 + \ldots + x_4^2 = y_1^2 + \ldots + y_4^2 \leq 1\right\} \tag{1.1}$$

Bombieri, De Giorgi, and Giusti [5] showed that Simons' cone is area minimizing. That is, Simons' cone has less volume than any other hypersurface in $\mathbb{R}^8$ with the same boundary. Lawson [13] and Simoes [21] gave more examples of area minimizing hypercones.

The Faber-Krahn inequality states that among domains in $\mathbb{R}^n$ with fixed volume, the ball minimizes the first Dirichlet eigenvalue of the p-Laplacian for every $1 < p < \infty$. The p-Laplacian Δ_p is defined by

$$\Delta_p f = \operatorname{div}\left(|\nabla f|^{p-2} \nabla f\right) \tag{1.2}$$

The Dirichlet eigenvalues of the p-Laplacian on a smoothly bounded domain Ω in $\mathbb{R}^n$ are the numbers λ such that the equation $-\Delta_p \varphi = \lambda |\varphi|^{p-2} \varphi$ admits a weak solution in $W_0^{1,p}(\Omega)$. The p-Laplacian admits a smallest eigenvalue, denoted $\lambda_{1,p}(\Omega)$. Lindqvist [15] showed this eigenvalue is simple on a connected domain, meaning the corresponding eigenfunction is unique up to normalization. If $\mathrm{Lip}_0(\Omega)$ is the set of Lipschitz functions $f : \Omega \to \mathbb{R}$ which vanish on the boundary of Ω, then $\lambda_{1,p}(\Omega)$ can be characterized variationally by

$$\lambda_{1,p}(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla f|^p}{\int_\Omega |f|^p} : f \in \mathrm{Lip}_0(\Omega)\right\} \tag{1.3}$$

This characterization and the Pólya-Szegö inequality [20] imply the Faber-Krahn inequality. Moreover, Brothers and Ziemer [6] proved a uniqueness result. In particular, the ball is the only minimizer with smooth boundary.

We consider a similar optimization problem for the first Dirichlet eigenvalue of the p-Laplacian on a hypersurface in $\mathbb{R}^{2n}$, for $n \geq 2$. Let $G = O(n) \times O(n)$ and consider the usual action of G on $\mathbb{R}^{2n}$. Fix an orbit $\mathcal{O}$ of dimension $2n-2$. Let $\mathcal{S}$ be the set of all C^1 immersed G-invariant hypersurfaces in $\mathbb{R}^{2n}$ with one boundary component, given by $\mathcal{O}$. For a hypersurface Σ in $\mathcal{S}$, the immersion of Σ into $\mathbb{R}^{2n}$ induces a continuous Riemannian metric on Σ. Let $\mathrm{Lip}_0(\Sigma)$ denote the set of Lipschitz functions $f : \Sigma \to \mathbb{R}$ which vanish on $\mathcal{O}$, and let dV be the Riemannian measure on Σ. Let $\lambda_{1,p}(\Sigma)$ denote the first Dirichlet eigenvalue of the p-Laplacian, which is given by

$$\lambda_{1,p}(\Sigma) = \inf\left\{ \frac{\int_\Sigma |\nabla f|^p \, dV}{\int_\Sigma |f|^p \, dV} : f \in \mathrm{Lip}_0(\Sigma) \right\} \tag{1.4}$$

If $\mathcal{O}$ is the product of two spheres of the same radius R, then let Γ be Simons' cone, defined by

$$\Gamma = \left\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n} : x_1^2 + \ldots + x_n^2 = y_1^2 + \ldots + y_n^2 \leq R^2 \right\} \tag{1.5}$$

Let $\mathrm{Lip}_0(\Gamma)$ denote the set of Lipschitz functions $f : \Gamma \to \mathbb{R}$ which vanish on $\mathcal{O}$. Note that $\Gamma \setminus \{0\}$ is a smooth immersed hypersurface. This immersion induces a Riemannian metric on $\Gamma \setminus \{0\}$. Let dV be the Riemannian measure. Then define

$$\lambda_{1,p}(\Gamma) = \inf\left\{ \frac{\int_{\Gamma\setminus\{0\}} |\nabla f|^p \, dV}{\int_{\Gamma\setminus\{0\}} |f|^p \, dV} : f \in \mathrm{Lip}_0(\Gamma) \right\} \tag{1.6}$$

The following theorem states that if $p \geq 2n-1$, then there is a hypersurface which maximizes the eigenvalue $\lambda_{1,p}$. This hypersurface is either Simons' cone or a C^1 hypersurface in $\mathcal{S}$.

Theorem 1.1. *Fix $n \geq 2$ and $p \geq 2n-1$. If $\mathcal{O}$ is the product of two spheres of different radii, then there is a C^1 embedded surface Σ_p^* in $\mathcal{S}$ such that*

$$\lambda_{1,p}(\Sigma_p^*) = \sup\left\{ \lambda_{1,p}(\Sigma) : \Sigma \in \mathcal{S} \right\} \tag{1.7}$$

If $\mathcal{O}$ is the product of two spheres of the same radius and if

$$\lambda_{1,p}(\Gamma) < \sup\left\{ \lambda_{1,p}(\Sigma) : \Sigma \in \mathcal{S} \right\} \tag{1.8}$$

then there is a C^1 embedded surface Σ_p^* in $\mathcal{S}$ such that

$$\lambda_{1,p}(\Sigma_p^*) = \sup\left\{ \lambda_{1,p}(\Sigma) : \Sigma \in \mathcal{S} \right\} \tag{1.9}$$

We note that in the definition of the set $\mathcal{S}$, the assumption that the surfaces are G-invariant is essential for these results. In fact the Nash-Kuiper theorem [18, 12] implies that there are C^1 hypersurfaces in $\mathbb{R}^{2n}/G$ with boundary given by $\mathcal{O}$ which have arbitrarily large first eigenvalue $\lambda_{1,p}$.

A natural problem motivated by Theorem 1.1 is to determine if (1.8) holds, i.e. if Simons' cone maximizes $\lambda_{1,p}$. For fixed n and large p, we show that Simons' cone does not maximize $\lambda_{1,p}$. For the case $p = 2$ and $n \leq 5$, we also show that Simons' cone does not maximize $\lambda_{1,2}$.

Theorem 1.2. *Assume $\mathcal{O}$ is the product of two spheres of the same radius. For each n, there is a value p_n such that if $p \geq p_n$, then*

$$\lambda_{1,p}(\Gamma) < \sup\Big\{\lambda_{1,p}(\Sigma) : \Sigma \in \mathcal{S}\Big\} \tag{1.10}$$

If $n \leq 5$ and $p = 2$, then

$$\lambda_{1,2}(\Gamma) < \sup\Big\{\lambda_{1,2}(\Sigma) : \Sigma \in \mathcal{S}\Big\} \tag{1.11}$$

In particular, for the cases $n = 4$ and $n = 5$, Simons' cone is area minimizing, but does not maximize the eigenvalue $\lambda_{1,2}$. This is in contrast to the inverse relationship that the eigenvalue and the volume of a domain often exhibit. More accurately, the eigenvalues of a domain Ω in $\mathbb{R}^n$, are inversely related to the Cheeger constant $h(\Omega)$, which is defined by

$$h(\Omega) = \inf\left\{\frac{|\partial U|}{|U|} : U \subset \Omega\right\} \tag{1.12}$$

Here U is a smoothly bounded open subset of Ω, and $|\partial U|$ is the $(n-1)$-dimensional volume of ∂U, while $|U|$ is the n-dimensional volume of U. Cheeger's inequality states that

$$\lambda_{1,p}(\Omega) \geq \left(\frac{h(\Omega)}{p}\right)^p \tag{1.13}$$

Cheeger [7] first proved this inequality for the case $p = 2$. Lefton and Wei [14], Matei [16], and Takeuchi [23] extended this inequality to the case $1 < p < \infty$. Moreover, Kawohl and Fridman [11] showed that

$$\lim_{p\to 1} \lambda_{1,p}(\Omega) = h(\Omega) \tag{1.14}$$

We observe that the relationship between the eigenvalue $\lambda_{1,p}$ and the Cheeger constant is strongest for small p. For large p, the eigenvalue $\lambda_{1,p}$ is more strongly related to the inradius of Ω, denoted $\mathrm{inrad}(\Omega)$. Juutinen, Lindqvist, and Manfredi [10] proved that

$$\lim_{p\to\infty} \Big(\lambda_{1,p}(\Omega)\Big)^{1/p} = \frac{1}{\mathrm{inrad}(\Omega)} \tag{1.15}$$

Moreover, Poliquin [19] showed that for each $p > n$, there is a constant $C_{n,p}$, independent of Ω, such that

$$\Big(\lambda_{1,p}(\Omega)\Big)^{1/p} \geq \frac{C_{n,p}}{\mathrm{inrad}(\Omega)} \tag{1.16}$$

In light of (1.15) and (1.16), it is not surprising that Simons' cone does not maximize $\lambda_{1,p}$ for large p. We remark that Grosjean [9] established a result similar to (1.15) on a compact Riemannian manifold, with the inradius replaced by half the diameter of the manifold. Valtorta [25] and Naber and Valtorta [17] obtained lower bounds for the first eigenvalue of the p-Laplacian in terms of the diameter on a compact Riemannian manifold.

A similar problem to the one described in Theorem 1.1 is to maximize the first Dirichlet eigenvalue among surfaces of revolution in $\mathbb{R}^3$ with one fixed boundary component. This problem has been studied for the case $p = 2$. It follows from a result of Abreu and Freitas [1] that the disc maximizes the first Dirichlet eigenvalue. In fact the disc maximizes all of the Dirichlet eigenvalues [3]. Moreover, it follows from a result of Colbois, Dryden, and El Soufi [8] that a flat n-dimensional ball in $\mathbb{R}^{n+1}$ maximizes the first Dirichlet eigenvalue among $O(n)$-invariant hypersurfaces in $\mathbb{R}^{n+1}$ with the same boundary. Among surfaces of revolution in $\mathbb{R}^3$ with two fixed boundary components, there is a smooth surface which maximizes the first Dirichlet eigenvalue [4].

The argument we use to prove Theorem 1.1 is a development of the argument used in [4] to maximize Laplace eigenvalues on surfaces of revolution in $\mathbb{R}^3$. For the case where p is large, the proof of Theorem 1.2 is a simple application of (1.15). For the case where $p = 2$, we use a variational argument. In the next section, we reformulate Theorem 1.1 and Theorem 1.2 as statements about curves in the orbit space $\mathbb{R}^{2n}/G$. In the third section, we prove a low regularity version of Theorem 1.1. In the fourth section, we complete the proof of Theorem 1.1. In the fifth section, we prove Theorem 1.2.

2. Reformulation

In this section, we reformulate Theorem 1.1 and Theorem 1.2 as statements about curves in the orbit space $\mathbb{R}^{2n}/G$. Identify $\mathbb{R}^{2n}/G$ with a quarter plane

$$\mathbb{R}^{2n}/G = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\} \tag{2.1}$$

A point (x, y) is identified with the orbit

$$\left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} : x_1^2 + \ldots + x_n^2 = x^2, y_1^2 + \ldots + y_n^2 = y^2\right\} \tag{2.2}$$

Let g be the orbital distance metric on $\mathbb{R}^{2n}/G$, i.e. $g = dx^2 + dy^2$. Define a function $F : \mathbb{R}^{2n}/G \to \mathbb{R}$ which maps an orbit to its $(2n-2)$-dimensional volume in $\mathbb{R}^{2n}$. There is a constant c_n such that

$$F(x, y) = c_n \cdot x^{n-1} y^{n-1} \tag{2.3}$$

Let (x_0, y_0) be the coordinates of the orbit $\mathcal{O}$. By symmetry, we may assume that

$$x_0 \geq y_0 > 0 \tag{2.4}$$

For a C^1 curve $\alpha : [0, 1] \to \mathbb{R}^{2n}/G$, let $L_g(\alpha)$ be the length of α with respect to g. Let $\mathcal{C}$ be the set of C^1 curves $\alpha : [0, 1] \to \mathbb{R}^{2n}/G$ which satisfy the following properties. First $\alpha(0) = (x_0, y_0)$ and $\alpha(1)$ is in the boundary of $\mathbb{R}^{2n}/G$. Second $\alpha(t)$ is in the interior of $\mathbb{R}^{2n}/G$ for every t in $[0, 1)$. Third $|\alpha'(t)|_g = L_g(\alpha)$ for every t in $[0, 1]$. Fourth α intersects the boundary of $\mathbb{R}^{2n}/G$ away from the origin, and the intersection is orthogonal. If α is a curve in $\mathcal{C}$, let $F_\alpha = F \circ \alpha$. Let $\mathrm{Lip}_0([0, 1])$ be the set of Lipschitz functions $w : [0, 1] \to \mathbb{R}$ which vanish at zero. Then define

$$\lambda_{1,p}(\alpha) = \inf\left\{\frac{\int_0^1 \frac{|w'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w|^p F_\alpha |\alpha'|_g\, dt} : w \in \mathrm{Lip}_0([0, 1])\right\} \tag{2.5}$$

For a function w in $\mathrm{Lip}_0([0, 1])$, the Rayleigh quotient of w is

$$\frac{\int_0^1 \frac{|w'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w|^p F_\alpha |\alpha'|_g\, dt} \tag{2.6}$$

Note that if α is in $\mathcal{C}$, then there is a corresponding surface Σ in $\mathcal{S}$ such that α parametrizes the projection of Σ in $\mathbb{R}^{2n}/G$. Moreover $\lambda_{1,p}(\Sigma) = \lambda_{1,p}(\alpha)$, because the first eigenfunction on Σ is G-invariant. Furthermore, if Σ is a surface in $\mathcal{S}$ and

$\lambda_{1,p}(\Sigma)$ is non-zero, then Σ is connected and there is a curve α in $\mathcal{C}$ corresponding to Σ. In particular,

$$\sup\Big\{\lambda_{1,p}(\Sigma) : \Sigma \in \mathcal{S}\Big\} = \sup\Big\{\lambda_{1,p}(\alpha) : \alpha \in \mathcal{C}\Big\} \tag{2.7}$$

If $x_0 = y_0 = R$, then define a curve $\sigma : [0,1] \to \mathbb{R}^{2n}/G$ by

$$\sigma(t) = (1-t)\cdot(R,R) \tag{2.8}$$

Let $F_\sigma = F \circ \sigma$ and define

$$\lambda_{1,p}(\sigma) = \inf\left\{\frac{\int_0^1 \frac{|w'|^p F_\sigma}{|\sigma'|_g^{p-1}}\,dt}{\int_0^1 |w|^p F_\sigma |\sigma'|_g\,dt} : w \in \mathrm{Lip}_0([0,1])\right\} \tag{2.9}$$

This curve corresponds to Simons' cone Γ in $\mathbb{R}^{2n}/G$, and $\lambda_{1,p}(\Gamma) = \lambda_{1,p}(\sigma)$.

Lemma 2.1. *Fix $n \geq 2$ and $p \geq 2n-1$. If $x_0 \neq y_0$ then there is a simple curve α in $\mathcal{C}$ such that*

$$\lambda_{1,p}(\alpha) = \sup\Big\{\lambda_{1,p}(\beta) : \beta \in \mathcal{C}\Big\} \tag{2.10}$$

If $x_0 = y_0$ and if

$$\lambda_{1,p}(\sigma) < \sup\Big\{\lambda_{1,p}(\beta) : \beta \in \mathcal{C}\Big\} \tag{2.11}$$

then there is a simple curve α in $\mathcal{C}$ such that

$$\lambda_{1,p}(\alpha) = \sup\Big\{\lambda_{1,p}(\beta) : \beta \in \mathcal{C}\Big\} \tag{2.12}$$

Lemma 2.1 immediately yields Theorem 1.1. In the third section of the article, we prove a low regularity version of Lemma 2.1. In the fourth section, we complete the proof of Lemma 2.1.

Lemma 2.2. *Assume $x_0 = y_0$. For each n, there is a value p_n such that if $p \geq p_n$, then*

$$\lambda_{1,p}(\sigma) < \sup\Big\{\lambda_{1,p}(\alpha) : \alpha \in \mathcal{C}\Big\} \tag{2.13}$$

If $n \leq 5$ and $p = 2$, then

$$\lambda_{1,2}(\sigma) < \sup\Big\{\lambda_{1,2}(\alpha) : \alpha \in \mathcal{C}\Big\} \tag{2.14}$$

Lemma 2.2 immediately yields Theorem 1.2. We prove Lemma 2.2 in the fifth section of the article. For the case where p is large, the proof is a simple application of (1.15). For the case where $p = 2$, we use a variational argument.

3. Existence

In this section we prove a low regularity version of Lemma 2.1. We first extend the defintion of $\lambda_{1,p}$ to low regularity curves. Define a Riemannian metric h on the interior of $\mathbb{R}^{2n}/G$ by

$$h = F^2 \cdot g = c_n^2 \cdot x^{2n-2} y^{2n-2}\Big(dx^2 + dy^2\Big) \tag{3.1}$$

The length of a curve in $\mathbb{R}^{2n}/G$ with respect to h is the $(2n-1)$-dimensional volume of the corresponding G-invariant hypersurface in $\mathbb{R}^{2n}$. Define an equivalence relation on $\mathbb{R}^{2n}/G$ such that each point in the interior is only equivalent to itself, and any two points on the boundary are equivalent. Let Q be the quotient space of

$\mathbb{R}^{2n}/G$ with respect to this equivalence relation. Let Q_0 be the image of the interior of $\mathbb{R}^{2n}/G$ under the quotient map. Let Q_B denote the remaining point in Q which is the image of the boundary of $\mathbb{R}^{2n}/G$. Then $Q = Q_0 \cup \{Q_B\}$. We view Q as a metric space, with distance function induced by h. The function $F : \mathbb{R}^{2n}/G \to \mathbb{R}$ induces a function on Q, which we also denote by F. Let $\alpha : [c,d] \to Q$ be a Lipschitz curve such that $\alpha(t) \neq Q_B$ for all t in $[c,d)$ and $\alpha(d) = Q_B$. Let $\mathrm{Lip}_0([c,d])$ be the set of Lipschitz functions $w : [c,d] \to \mathbb{R}$ which vanish at c. Let $F_\alpha = F \circ \alpha$ and define

$$\lambda_{1,p}(\alpha) = \inf \left\{ \frac{\int_c^d \frac{|w'|^p F_\alpha^p}{|\alpha'|_h^{p-1}} \, dt}{\int_c^d |w|^p |\alpha'|_h \, dt} : w \in \mathrm{Lip}_0([c,d]) \right\} \tag{3.2}$$

If the integrand in the numerator takes the form $0/0$ at some point in $[c,d]$, then we interpret the integrand as being equal to zero at this point. If the Rayleigh quotient takes the form $0/0$, then we interpret the Rayleigh quotient as being infinite. Let $\mathcal{R}_h$ be the set of Lipschitz curves $\alpha : [0,1] \to Q$ such that $\alpha(0) = (x_0, y_0)$ and $\alpha(1) = Q_B$ and $\alpha(t)$ is in Q_0 for every t in $[0,1)$. Note that a curve in $\mathcal{C}$ can be identified with a curve in $\mathcal{R}_h$, by composing with the quotient map $\mathbb{R}^{2n}/G \to Q$. Making this identification, the definitions (2.5) and (3.2) are the same.

For a Lipschitz curve $\gamma : [c,d] \to Q$, let $L_h(\gamma)$ denote the length of γ. In the following lemma, we prove that reparametrizing a curve by arc length with respect to h does not decrease the eigenvalue.

Lemma 3.1. *Let $\gamma : [c,d] \to Q$ be a Lipschitz curve such that $\gamma(c) = (x_0, y_0)$ and $\gamma(d) = Q_B$. Assume that $\gamma(t) \neq Q_B$ for all t in $[c,d)$. Define $\ell_h : [c,d] \to [0,1]$ by*

$$\ell_h(t) = \frac{1}{L_h(\gamma)} \int_c^t |\gamma'(u)|_h \, du \tag{3.3}$$

There is a curve β in $\mathcal{R}_h$ such that $\beta(\ell_h(t)) = \gamma(t)$ for all t in $[c,d]$. Moreover $|\beta'(t)|_h = L_h(\beta)$ for almost every t in $[0,1]$, and $L_h(\beta) = L_h(\gamma)$. Furthermore $\lambda_{1,p}(\beta) \geq \lambda_{1,p}(\gamma)$ for every $p \geq 2n-1$.

Proof. Define $\eta : [0,1] \to [c,d]$ by

$$\eta(s) = \min \Big\{ t \in [c,d] : \ell_h(t) = s \Big\} \tag{3.4}$$

Note that η may not be continuous, but $\beta = \gamma \circ \eta$ is in $\mathcal{R}_h$, and $\beta(\ell_h(t)) = \gamma(t)$ for all t in $[c,d]$. Also $|\beta'(t)|_h = L_h(\gamma)$ for almost every t in $[0,1]$, so $L_h(\beta) = L_h(\gamma)$. Let $F_\gamma = F \circ \gamma$ and $F_\beta = F \circ \beta$. Let w be a function in $\mathrm{Lip}_0([0,1])$. Define $v = w \circ \ell_h$. Then v is in $\mathrm{Lip}_0([c,d])$, and changing variables yields

$$\lambda_{1,p}(\gamma) \leq \frac{\int_c^d \frac{|v'|^p F_\gamma^p}{|\gamma'|_h^{p-1}} \, dt}{\int_c^d |v|^p |\gamma'|_h \, dt} = \frac{\int_0^1 \frac{|w'|^p F_\beta^p}{|\beta'|_h^{p-1}} \, dt}{\int_0^1 |w|^p |\beta'|_h \, dt} \tag{3.5}$$

Since w is arbitrary, this implies that $\lambda_{1,p}(\gamma) \leq \lambda_{1,p}(\beta)$. ∎

In the following lemma, we bound the length $L_h(\gamma)$ of a curve γ in $\mathcal{R}_h$ in terms of the eigenvalue $\lambda_{1,p}(\gamma)$.

Lemma 3.2. *Fix $p \geq 2n-1$. There is a constant C_p such that for any γ in $\mathcal{R}_h$,*

$$L_h(\gamma) \leq \frac{C_p}{\lambda_{1,p}(\gamma)} \tag{3.6}$$

Proof. Let β be the reparametrization given by Lemma 3.1 so that $\lambda_{1,p}(\beta) \geq \lambda_{1,p}(\gamma)$ and $|\beta'(t)|_h = L_h(\gamma)$ for almost every t in $[0,1]$. Let $r > 0$ be a small number. Define $w : [0,1] \to \mathbb{R}$ by

$$w(t) = \begin{cases} \frac{L_h(\gamma)}{r} \cdot t & 0 \leq t \leq \frac{r}{L_h(\gamma)} \\ 1 & \frac{r}{L_h(\gamma)} \leq t \leq 1 \end{cases} \tag{3.7}$$

Let $F_\beta = F \circ \beta$. Then there is a constant C_p, which is independent of γ and β, such that

$$\lambda_{1,p}(\gamma) \leq \lambda_{1,p}(\beta) \leq \frac{\int_0^{\frac{r}{L_h(\gamma)}} \frac{|w'|^p F_\beta^p}{|\beta'|_h^{p-1}}\, dt}{\int_{\frac{r}{L_h(\gamma)}}^1 |w|^p |\beta'|_h \, dt} \leq \frac{C_p}{L_h(\gamma)} \tag{3.8}$$

□

The purpose of the next lemma is to show that there is an eigenvalue maximizing sequence of curves in $\mathcal{R}_h$ whose images are contained in a fixed compact subset of Q. Let $\rho_0 = \sqrt{x_0^2 + y_0^2}$ and define

$$K = \Big\{(x,y) \in \mathbb{R}^{2n}/G : x^2 + y^2 \leq \rho_0^2\Big\} \tag{3.9}$$

Let Q_K be the image of K under the quotient map $\mathbb{R}^{2n}/G \to Q$.

Lemma 3.3. *Let α be a curve in $\mathcal{R}_h$. There is a curve β in $\mathcal{R}_h$ such that $\beta(t)$ is in Q_K for all t in $[0,1]$ and $\lambda_{1,p}(\beta) \geq \lambda_{1,p}(\alpha)$ for all $p \geq 2n-1$.*

Proof. There are functions $r_\alpha : [0,1) \to \mathbb{R}$ and $\theta_\alpha : [0,1) \to [0,\pi/2]$ such that for all t in $[0,1)$,

$$\alpha(t) = \Big(r_\alpha(t)\cos\theta_\alpha(t), r_\alpha(t)\sin\theta_\alpha(t)\Big) \tag{3.10}$$

Define a function $r_\beta : [0,1) \to [0,\rho_0]$ by

$$r_\beta(t) = \min\Big(\frac{\rho_0^2}{r_\alpha(t)}, r_\alpha(t)\Big) \tag{3.11}$$

Then define a curve β in $\mathcal{R}_h$ so that $\beta(1) = Q_B$ and for all t in $[0,1)$,

$$\beta(t) = \Big(r_\beta(t)\cos\theta_\alpha(t), r_\beta(t)\sin\theta_\alpha(t)\Big) \tag{3.12}$$

Then β is in $\mathcal{R}_h$ and $\beta(t)$ is in Q_K for all t in $[0,1]$. Let $F_\alpha = F \circ \alpha$ and $F_\beta = F \circ \beta$. For all $p \geq 2n-1$ and for almost every t in $[0,1]$,

$$\frac{(F_\alpha(t))^p}{|\alpha'(t)|_h^{p-1}} \leq \frac{(F_\beta(t))^p}{|\beta'(t)|_h^{p-1}} \tag{3.13}$$

Also $|\alpha'(t)|_h \geq |\beta'(t)|_h$ for almost every t in $[0,1]$. Therefore $\lambda_{1,p}(\alpha) \leq \lambda_{1,p}(\beta)$ for all $p \geq 2n-1$. □

We can now establish the existence of an eigenvalue maximizing curve in $\mathcal{R}_h$. For $p \geq 2n-1$, define

$$\Lambda_p = \sup\Big\{\lambda_{1,p}(\alpha) : \alpha \in \mathcal{R}_h\Big\} \tag{3.14}$$

Lemma 3.4. *Fix $p \geq 2n-1$. There is a curve α in $\mathcal{R}_h$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$ and $\alpha(t)$ is in Q_K for all t in $[0,1]$.*

Proof. Let $\{\gamma_j\}$ be a sequence in $\mathcal{R}_h$ such that

$$\lim_{j\to\infty} \lambda_{1,p}(\gamma_j) = \Lambda_p \tag{3.15}$$

By Lemma 3.3, we may assume that $\gamma_j(t)$ is in Q_K for every j and every t in $[0,1]$. Using Lemma 3.1, we may assume that $|\gamma_j'(t)|_h = L_h(\gamma_j)$ for every j and almost every t in $[0,1]$. By Lemma 3.2, the lengths $L_h(\gamma_j)$ are uniformly bounded. By passing to a subsequence, we may assume that the lengths $L_h(\gamma_j)$ converge to some positive number ℓ. The curves γ_j are uniformly Lipschitz. Therefore, by applying the Arzela-Ascoli theorem and passing to a subsequence, we may assume that the curves γ_j converge uniformly to a Lipschitz curve $\gamma : [0,1] \to Q_K$. Moreover $|\gamma'(t)|_h \leq \ell$ for almost every t in $[0,b]$. For each j, define $F_j = F \circ \gamma_j$. Also define $F_\gamma = F \circ \gamma$. Define b in $(0,1]$ by

$$b = \min\Big\{t \in [0,1] : \gamma(t) = Q_B\Big\} \tag{3.16}$$

Let w be in $\mathrm{Lip}_0([0,b])$. Define v to be a function in $\mathrm{Lip}_0([0,1])$ which agrees with w over $[0,b]$ and is constant over $[0,1]$. Then

$$\Lambda_p = \lim_{j\to\infty} \lambda_{1,p}(\gamma_j) \leq \liminf_{j\to\infty} \frac{\int_0^1 \frac{|v'|^p F_j^p}{L_h(\gamma_j)^{p-1}}\, dt}{\int_0^1 |v|^p L_h(\gamma_j)\, dt} \leq \liminf_{j\to\infty} \frac{\int_0^b \frac{|w'|^p F_j^p}{L_h(\gamma_j)^{p-1}}\, dt}{\int_0^b |w|^p L_h(\gamma_j)\, dt} \tag{3.17}$$

Moreover F_j converges to F_γ uniformly over $[0,b]$, because F is continuous on Q_K. Also $L_h(\gamma_j)$ converges to ℓ, so

$$\lim_{j\to\infty} \frac{\int_0^b \frac{|w'|^p F_j^p}{L_h(\gamma_j)^{p-1}}\, dt}{\int_0^b |w|^p L_h(\gamma_j)\, dt} = \frac{\int_0^b \frac{|w'|^p F_\gamma^p}{\ell^{p-1}}\, dt}{\int_0^b |w|^p \ell\, dt} \leq \frac{\int_0^b \frac{|w'|^p F_\gamma^p}{|\gamma'|_h^{p-1}}\, dt}{\int_0^b |w|^p |\gamma'|_h\, dt} \tag{3.18}$$

Therefore

$$\Lambda_p \leq \frac{\int_0^b \frac{|w'|^p F_\gamma^p}{|\gamma'|_h^{p-1}}\, dt}{\int_0^b |w|^p |\gamma'|_h\, dt} \tag{3.19}$$

Since w is arbitrary,

$$\Lambda_p \leq \lambda_{1,p}\Big(\gamma\big|_{[0,b]}\Big) \tag{3.20}$$

Let α be the reparametrization of $\gamma|_{[0,b]}$ given by Lemma 3.1. Then α is in $\mathcal{R}_h$ and $\alpha(t)$ is in Q_K for all t in $[0,1]$. Moreover

$$\lambda_{1,p}(\alpha) \geq \lambda_{1,p}(\gamma|_{[0,b]}) \geq \Lambda_p \tag{3.21}$$

Therfore $\lambda_{1,p}(\alpha) = \Lambda_p$. □

Let $\mathcal{R}_h^+$ be the set of continuous curves $\alpha : [0,1] \to \mathbb{R}^{2n}/G$ such that composition with the quotient map $\mathbb{R}^{2n}/G \to Q$ yields a curve in $\mathcal{R}_h$. We use (3.2) to define $\lambda_{1,p}(\alpha)$ for α in $\mathcal{R}_h^+$. In the next lemma we establish existence of an eigenvalue maximizing curve in $\mathcal{R}_h^+$. We first introduce new coordinates functions on $\mathbb{R}^{2n}/G$. Define $u : \mathbb{R}^{2n}/G \to \mathbb{R}$ and $v : \mathbb{R}^{2n}/G \to [0,\infty)$ by

$$u(x,y) = \frac{1}{2}(x^2 - y^2) \tag{3.22}$$

and

$$v(x,y) = xy \tag{3.23}$$

These coordinates can be used to identify $\mathbb{R}^{2n}/G$ with a half-plane. A key feature of these coordinates is that the function F can be expressed as $F = c_n \cdot v^{n-1}$. Define a function $r = \sqrt{u^2 + v^2}$. Then the metric h can be expressed as

$$h = \frac{c_n^2 \cdot v^{2n-2}}{2r}\Big(du^2 + dv^2\Big) \tag{3.24}$$

By (2.4), we have

$$u(x_0, y_0) \geq 0 \tag{3.25}$$

Lemma 3.5. *Fix $p \geq 2n - 1$. There is a curve α in $\mathcal{R}_h^+$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Moreover $u \circ \alpha$ is monotonically increasing over $[0,1]$.*

Proof. By Lemma 3.4, there is a curve β in $\mathcal{R}_h$ such that $\lambda_{1,p}(\beta) = \Lambda_p$ and $\beta(t)$ is in Q_K for all t in $[0,1]$. Define functions $u_\beta : [0,1) \to \mathbb{R}$ and $v_\beta : [0,1) \to [0,\infty)$ by $u_\beta = u \circ \beta$ and $v_\beta = v \circ \beta$. Note that u_β is bounded over $[0,1)$. Also $v(t) > 0$ for all t in $[0,1)$ and

$$\lim_{t \to 1} v_\beta(t) = 0 \tag{3.26}$$

In particular v_β has a continuous extension to $[0,1]$. Let $v_\alpha : [0,1] \to [0,\infty)$ be this extension. Note that $u_\beta(0) \geq 0$ by (3.25), and define $u_\alpha : [0,1] \to \mathbb{R}$ by

$$u_\alpha(t) = \sup\Big\{|u_\beta(s)| : s \in [0,t)\Big\} \tag{3.27}$$

Then u_α is monotonically increasing, continuous, and bounded. Define a continuous curve $\alpha : [0,1] \to \mathbb{R}^{2n}/G$ so that $u \circ \alpha = u_\alpha$ and $v \circ \alpha = v_\alpha$. Then α is in $\mathcal{R}_h^+$, and $u \circ \alpha$ is monotonically increasing over $[0,1]$. Let $F_\alpha = F \circ \alpha$ and $F_\beta = F \circ \beta$. Note that $F_\alpha = F_\beta$ over $[0,1]$, and $|\alpha'(t)|_h \leq |\beta'(t)|_h$ for almost every t in $[0,1]$. Therefore $\lambda_{1,p}(\alpha) \geq \lambda_{1,p}(\beta)$, hence $\lambda_{1,p}(\alpha) = \Lambda_p$. $\square$

In the following lemma, we establish existence of a maximizing curve α in $\mathcal{R}_h^+$ such that $u \circ \alpha$ is monotonically increasing and $r \circ \alpha$ is monotonically decreasing.

Lemma 3.6. *Fix $p \geq 2n - 1$. There is a curve α in $\mathcal{R}_h^+$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Moreover $u \circ \alpha$ is monotonically increasing over $[0,1]$ and $r \circ \alpha$ is monotonically decreasing over $[0,1]$.*

Proof. By Lemma 3.5, there is a curve γ in $\mathcal{R}_h^+$ such that $\lambda_{1,p}(\gamma) = \Lambda_p$. Moreover $u \circ \gamma$ is monotonically increasing over $[0,1]$. Define $r_\gamma = r \circ \gamma$. Note that r_γ is non-vanishing over $[0,1)$. There is a function $\theta_\gamma : [0,1] \to [0,\pi/2]$ such that

$$\Big(u \circ \gamma, v \circ \gamma\Big) = \Big(r_\gamma \cos\theta_\gamma, r_\gamma \sin\theta_\gamma\Big) \tag{3.28}$$

Define a function $r_\beta : [0,1] \to (0,\infty)$ by

$$r_\beta(t) = \min\Big\{r_\gamma(s) : s \in [0,t]\Big\} \tag{3.29}$$

Define a curve $\beta : [0,1] \to \mathbb{R}^{2n}/G$ such that

$$\Big(u \circ \beta, v \circ \beta\Big) = \Big(r_\beta \cos\theta_\gamma, r_\beta \sin\theta_\gamma\Big) \tag{3.30}$$

Note that β is in $\mathcal{R}_h^+$ and r_β is monotonically decreasing over $[0,1]$. Define a set W by

$$W = \Big\{t \in [0,1] : \beta(t) = \gamma(t)\Big\} \tag{3.31}$$

The isolated points of W are countable, so $\beta'(t) = \gamma'(t)$ for almost every t in W. Note that there are countably many disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots$ such that

$$[0,1) \setminus W = \bigcup_j (a_j, b_j) \tag{3.32}$$

Moreover r_β is constant on each interval (a_j, b_j). For all $p \geq 2n-1$ and for almost every t in $[0,1]$,

$$\frac{(F_\beta(t))^p}{|\beta'(t)|_h^{p-1}} \geq \frac{(F_\gamma(t))^p}{|\gamma'(t)|_h^{p-1}} \tag{3.33}$$

Also $|\beta'(t)|_h \leq |\gamma'(t)|_h$ for almost every t in $[0,1]$. Therefore $\lambda_{1,p}(\beta) \geq \lambda_{1,p}(\gamma)$, so $\lambda_{1,p}(\beta) = \Lambda_p$. Define $\theta_\alpha : [0,1] \to [0, \pi/2]$ by

$$\theta_\alpha(t) = \begin{cases} \theta_\gamma(t) & t \in W \\ \min\Big\{\theta_\gamma(s) : s \in [a_j, t]\Big\} & t \in (a_j, b_j) \end{cases} \tag{3.34}$$

Define a curve α in $\mathcal{R}_h^+$ such that

$$\Big(u \circ \alpha, v \circ \alpha\Big) = \Big(r_\beta \cos\theta_\alpha, r_\beta \sin\theta_\alpha\Big) \tag{3.35}$$

Note that $r \circ \alpha = r_\beta$ is monotonically decreasing over $[0,1]$. Additionally $u \circ \alpha$ is monotonically increasing over W, because $\alpha = \gamma$ over W. Also $u \circ \alpha$ is monotonically increasing over each interval (a_j, b_j), because r_β is constant and θ_α is monotonically decreasing over each of these intervals. Therefore $u \circ \alpha$ is monotonically increasing over $[0,1]$. In order to show that $\lambda_{1,p}(\alpha) \geq \lambda_{1,p}(\beta)$, define a set Z by

$$Z = \Big\{t \in [0,1] : \alpha(t) = \beta(t)\Big\} \tag{3.36}$$

Note that W is contained in Z, and there are countably many disjoint intervals $(c_1, d_1), (c_2, d_2), \ldots$ such that

$$[0,1) \setminus Z = \bigcup_j (c_j, d_j) \tag{3.37}$$

Moreover θ_α and r_β are constant on each interval (c_j, d_j). That is, α is constant on each interval (c_j, d_j). Let w be a function in $\mathrm{Lip}_0([0,1])$ such that

$$\frac{\int_0^1 \frac{|w'|^p F_\alpha^p}{|\alpha'|_h^{p-1}}\,dt}{\int_0^1 |w|^p |\alpha'|_h\,dt} < \infty \tag{3.38}$$

In particular w is constant on each interval (c_j, d_j). Additionally, the isolated points of Z are countable, so $\alpha'(t) = \beta'(t)$ for almost every t in Z. Therefore

$$\lambda_{1,p}(\beta) \leq \frac{\int_0^1 \frac{|w'|^p F_\beta^p}{|\beta'|_h^{p-1}}\,dt}{\int_0^1 |w|^p |\beta'|_h\,dt} \leq \frac{\int_Z \frac{|w'|^p F_\beta^p}{|\beta'|_h^{p-1}}\,dt}{\int_Z |w|^p |\beta'|_h\,dt} = \frac{\int_0^1 \frac{|w'|^p F_\alpha^p}{|\alpha'|_h^{p-1}}\,dt}{\int_0^1 |w|^p |\alpha'|_h\,dt} \tag{3.39}$$

Since w is arbitrary, this shows that $\lambda_{1,p}(\alpha) \geq \lambda_{1,p}(\beta)$. Therefore $\lambda_{1,p}(\alpha) = \Lambda_p$. □

Recall g is the orbital distance metric on $\mathbb{R}^{2n}/G$, i.e. $g = dx^2 + dy^2$. The metric g can also be expressed as

$$g = \frac{1}{2r}\Big(du^2 + dv^2\Big) \tag{3.40}$$

We view $\mathbb{R}^{2n}/G$ as a metric space, with distance function induced by g. Let $\mathcal{R}_g$ be the set of Lipschitz curves $\alpha : [0,1] \to \mathbb{R}^{2n}/G$ such that $\alpha(0) = (x_0, y_0)$ and $\alpha(1)$ is in the boundary of $\mathbb{R}^{2n}/G$ and $\alpha(t)$ in the interior of $\mathbb{R}^{2n}/G$ for every t in $[0,1)$. Note that $\mathcal{R}_g$ is a subset of $\mathcal{R}_h^+$. If α is in $\mathcal{R}_g$ and $F_\alpha = F \circ \alpha$, then

$$\lambda_{1,p}(\alpha) = \inf \left\{ \frac{\int_0^1 \frac{|w'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w|^p F_\alpha |\alpha'|_g \, dt} : w \in \mathrm{Lip}_0([0,1]) \right\} \tag{3.41}$$

The previous lemma can be used to establish existence of an eigenvalue maximizing curve in $\mathcal{R}_h^+$ which has finite length with respect to g. The following lemma shows that reparametrization then yields a curve in $\mathcal{R}_g$. The statement and proof are very similar to Lemma 3.1, with the metric g in place of the metric h. For a curve α in $\mathcal{R}_h^+$, let $L_g(\alpha)$ denote the length of α with respect to g.

Lemma 3.7. *Let β be a curve in $\mathcal{R}_h^+$ and assume that $L_g(\beta)$ is finite. Define $\ell_g : [0,1] \to [0,1]$ by*

$$\ell_g(t) = \frac{1}{L_g(\beta)} \int_0^t |\beta'(u)|_g \, du \tag{3.42}$$

There is a curve α in $\mathcal{R}_g$ such that $\alpha(\ell_g(t)) = \beta(t)$ for all t in $[0,1]$, and $|\alpha'(t)|_g = L_g(\alpha)$ for almost every t in $[0,1]$. Also $\lambda_{1,p}(\alpha) \geq \lambda_{1,p}(\beta)$ for all $p \geq 2$.

Proof. First note that β is locally Lipschitz over $[0,1)$ and continuous over $[0,1]$. Define $\eta : [0,1] \to \mathbb{R}$ by

$$\eta(s) = \min \Big\{ t \in [0,1] : \ell_g(t) = s \Big\} \tag{3.43}$$

Define $\alpha : [0,1] \to \mathbb{R}^{2n}/G$ by $\alpha = \beta \circ \eta$. Note that η may not be continuous, but α is locally Lipschitz over $[0,1)$ and continuous over $[0,1]$. Moreover $\alpha(\ell_g(t)) = \beta(t)$ for all t in $[0,1]$. Therefore $|\alpha'(t)|_g = L_g(\alpha)$ for almost every t in $[0,1]$. In particular α is in $\mathcal{R}_g$. Let $F_\beta = F \circ \beta$ and $F_\alpha = F \circ \alpha$. Let w be in $\mathrm{Lip}_0([0,1])$. Define $v = w \circ \ell_g$. Then v is in $\mathrm{Lip}_0([0,1])$, and changing variables yields

$$\lambda_{1,p}(\beta) \leq \frac{\int_0^1 \frac{|v'|^p F_\beta}{|\beta'|_g^{p-1}}\, dt}{\int_0^1 |v|^p F_\beta |\beta'|_g \, dt} = \frac{\int_0^1 \frac{|w'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w|^p F_\alpha |\alpha'|_g \, dt} \tag{3.44}$$

Since w is arbitrary, this implies that $\lambda_{1,p}(\beta) \leq \lambda_{1,p}(\alpha)$. □

Let $\mathcal{R}_g^*$ be the set of curves α in $\mathcal{R}_g$ which satisfy the following properties. First α is simple and $|\alpha'(t)|_g = L_g(\alpha)$ for almost every t in $[0,1]$. Second $u \circ \alpha(1) > 0$. Third there is a constant $c > 0$ such that for all t in $[0,1]$,

$$v \circ \alpha(t) \geq c(1-t) \tag{3.45}$$

We can now establish existence of an eigenvalue maximizing curve in $\mathcal{R}_g^*$.

Lemma 3.8. *Fix $p \geq 2n-1$. Assume either $x_0 \neq y_0$ or $\Lambda_p > \lambda_{1,p}(\sigma)$. Then there is a curve α in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$.*

Proof. By Lemma 3.6, there is a curve β in $\mathcal{R}_h^+$ such that $\lambda_{1,p}(\beta) = \Lambda_p$. Moreover $u \circ \beta$ is monotonically increasing and $r \circ \beta$ is monotonically decreasing. We claim that $r \circ \beta(1) > 0$. To prove this, suppose that $r \circ \beta(1) = 0$. Then $u \circ \beta$ is identically zero and the reparametrization of β given by Lemma 3.7 is σ. In particular $x_0 = y_0$ and Lemma 3.7 implies that $\lambda_{1,p}(\sigma) = \Lambda_p$. By this contradiction $r \circ \beta(1) > 0$, so

$u \circ \beta(1) > 0$. Since $u \circ \beta(0) \geq 0$, the monotonicity of $r \circ \beta$ and $u \circ \beta$ together imply that $v \circ \beta$ is monotonically decreasing over $[0, 1]$. The monotonicity of $u \circ \beta$ and $v \circ \beta$ together imply that β has finite length with respect to the metric $du^2 + dv^2$. Since $r \circ \beta$ is positive over $[0, 1]$, this implies that β has finite length with respect to g. Let α be the reparametrization of β given by Lemma 3.7. Then α is in $\mathcal{R}_g$ and $|\alpha'(t)|_g = L_g(\alpha)$ for almost every t in $[0, 1]$. Also α is simple, because $u \circ \alpha$ and $v \circ \alpha$ are monotonic. Furthermore $\lambda_{1,p}(\alpha) \geq \lambda_{1,p}(\beta)$, so $\lambda_{1,p}(\alpha) = \Lambda_p$. Moreover $r \circ \alpha$ is monotonically decreasing over $[0, 1]$ and $u \circ \alpha$ is monotonically increasing over $[0, 1]$. Therefore there is a constant $c > 0$ such that if t close to 1 and α is differentiable at t, then

$$(v \circ \alpha)'(t) < -c \tag{3.46}$$

This implies (3.45), so α is in $\mathcal{R}_g^*$. □

4. Regularity

In this section we complete the proof of Lemma 2.1 by establishing regularity of a maximizing curve in $\mathcal{R}_g^*$. The following lemma gives sufficient conditions for a curve in $\mathcal{R}_g$ to admit an eigenfunction. Let $\text{Lip}_0([0, 1))$ be the set of locally Lipschitz functions $w : [0, 1) \to \mathbb{R}$ such that $w(0) = 0$.

Lemma 4.1. *Let α be a curve in $\mathcal{R}_g$. Assume that $u \circ \alpha(1) > 0$. Let $c > 0$ and assume that $|\alpha'(t)|_g \geq c$ for almost every t in $[0, 1]$. Assume that for all t in $[0, 1]$,*

$$v \circ \alpha(t) \geq c(1 - t) \tag{4.1}$$

Fix $p \geq 2$. Then there is a function φ in $\text{Lip}_0([0, 1))$ such that

$$\frac{\int_0^1 \frac{|\varphi'|^p F_\alpha}{|\alpha'|_g^{p-1}} \, dt}{\int_0^1 |\varphi|^p F_\alpha |\alpha'|_g \, dt} = \lambda_{1,p}(\alpha) \tag{4.2}$$

Moreover $\varphi(t) > 0$ for every t in $(0, 1)$.

Proof. Let $F_\alpha = F \circ \alpha$. Let $L^p(\alpha)$ be the set of measurable functions $f : [0, 1] \to \mathbb{R}$ such that

$$\|f\|_{L^p(\alpha)} = \left(\int_0^1 |f|^p F_\alpha |\alpha'(t)|_g \, dt \right)^{1/p} < \infty \tag{4.3}$$

Let $C_0^1([0, 1])$ be the set of continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ such that $f(0) = 0$. Let $W_0^{1,p}(\alpha)$ be the completion of $C_0^1([0, 1])$ with respect to the norm

$$\|f\|_{W_0^{1,p}(\alpha)} = \left(\int_0^1 \frac{|f'|^p F_\alpha}{|\alpha'(t)|_g^{p-1}} \, dt \right)^{1/p} < \infty \tag{4.4}$$

There are positive constants $C_1 < C_2$ such that for all t in $[0, 1]$,

$$C_1 (1 - t)^{n-1} \leq F_\alpha(t) \leq C_2 (1 - t)^{n-1} \tag{4.5}$$

Let B_n be a unit ball in $\mathbb{R}^n$. Identify a function f in $L^p(\alpha)$ or $W_0^{1,p}(\alpha)$ with a radial function $w : B_n \to \mathbb{R}$ defined by

$$w(x) = f(1 - |x|) \tag{4.6}$$

The space $L^p(\alpha)$ is a Banach space, equivalent to the subspace of $L^p(B_n)$ consisting of radial functions. Similarly, the space $W_0^{1,p}(\alpha)$ is a Banach space, equivalent to the subspace of $W_0^{1,p}(B_n)$ consisting of radial functions. In particular $W_0^{1,p}(\alpha)$ is reflexive. Note that a function in $W_0^{1,p}(\alpha)$ is necessarily continuous over $[0,1)$. By the Rellich-Kondrachov theorem, the space $W_0^{1,p}(B_n)$ is compactly embedded in $L^p(B_n)$. Therefore $W_0^{1,p}(\alpha)$ is compactly embedded in $L^p(\alpha)$. Now the direct method in the calculus of variations establishes the existence of a function φ in $W_0^{1,p}(\alpha)$ such that

$$\frac{\int_0^1 \frac{|\varphi'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |\varphi|^p F_\alpha |\alpha'|_g \, dt} = \lambda_{1,p}(\alpha) \tag{4.7}$$

We may assume $\varphi(t) \geq 0$ for all t in $[0,1]$, by possibly replacing φ with $|\varphi|$. Moreover φ weakly satisfies the corresponding Euler-Lagrange equation, i.e.

$$-\left(\frac{|\varphi'|^{p-2}\varphi' F_\alpha}{|\alpha'|_g^{p-1}}\right)' = \lambda_{1,p}(\alpha) F_\alpha |\alpha'|_g (\varphi)^{p-1} \tag{4.8}$$

This equation implies that φ is in $\mathrm{Lip}_0([0,1))$. Furthermore a Harnack inequality of Trudinger [24, Theorem 1.1] implies that φ does not vanish in $(0,1)$. □

In the next lemma we show that for any function w in $\mathrm{Lip}_0([0,1))$, the Rayleigh quotient of w is greater than or equal to $\lambda_{1,p}$.

Lemma 4.2. *Let α be a curve in $\mathcal{R}_g$ and let $F_\alpha = F \circ \alpha$. Let w be in* $\mathrm{Lip}_0([0,1))$. *Fix $p \geq 2$ and assume that*

$$\int_0^1 |w|^p F_\alpha |\alpha'|_g \, dt < \infty \tag{4.9}$$

Then

$$\lambda_{1,p}(\alpha) \leq \frac{\int_0^1 \frac{|w'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w|^p F_\alpha |\alpha'|_g \, dt} \tag{4.10}$$

Proof. For each s in $(0,1)$, define a function w_s in $\mathrm{Lip}_0([0,1])$ by

$$w_s(t) = \begin{cases} w(t) & t \in [0,s] \\ w(s) & t \in [s,1] \end{cases} \tag{4.11}$$

For each s,

$$\lambda_{1,p}(\alpha) \leq \frac{\int_0^1 \frac{|w_s'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w_s|^p F_\alpha |\alpha'|_g \, dt} \tag{4.12}$$

Applying the monotone convergence theorem and Fatou's lemma,

$$\lambda_{1,p}(\alpha) \leq \limsup_{s \nearrow 1} \frac{\int_0^1 \frac{|w_s'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w_s|^p F_\alpha |\alpha'|_g \, dt} \leq \frac{\int_0^1 \frac{|w'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |w|^p F_\alpha |\alpha'|_g \, dt} \tag{4.13}$$

□

In the following lemma we show that if a maximizing curve intersects a small circle at two points, then it must stay inside the circle between those points.

Lemma 4.3. *Fix $p \geq 2n-1$. Let α be a curve in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Let C be a large positive constant. Let (x_1, y_1) be in the interior of $\mathbb{R}^{2n}/G$. Let r_1 be a positive number such that*

$$Cr_1 \leq \min(x_1, y_1) \tag{4.14}$$

Define

$$D = \left\{(x,y) \in \mathbb{R}^{2n}/G : (x-x_1)^2 + (y-y_1)^2 \leq r_1^2\right\} \tag{4.15}$$

Let $0 \leq t_1 < t_2 \leq 1$ and assume $\alpha(t_1)$ and $\alpha(t_2)$ lie on the boundary ∂D. Assume that, for all t in $[t_1, t_2]$,

$$|\alpha(t) - (x_1, y_1)| < 2r_1 \tag{4.16}$$

If C is sufficiently large, independent of x_1, y_1, and r_1, then it follows that $\alpha(t)$ is in D for all t in $[t_1, t_2]$.

Proof. Suppose not. It suffices to consider the case where $\alpha(t)$ lies outside of D for every t in (t_1, t_2). There are Lipschitz functions $r : [t_1, t_2] \to (0, \infty)$ and $\theta : [t_1, t_2] \to \mathbb{R}$ such that for all t in $[t_1, t_2]$,

$$\alpha(t) = \Big(x_1 + r(t)\cos\theta(t), y_1 + r(t)\sin\theta(t)\Big) \tag{4.17}$$

Note that $r_1 < r(t) < 2r_1$ for all t in (t_1, t_2). Define a curve β in $\mathcal{R}_g^*$ by

$$\beta(t) = \begin{cases} \alpha(t) & t \in [0, t_1) \cup (t_2, 1] \\ \Big(x_1 + \frac{r_1^2}{r(t)}\cos\theta(t), y_1 + \frac{r_1^2}{r(t)}\sin\theta(t)\Big) & t \in [t_1, t_2] \end{cases} \tag{4.18}$$

Let $F_\alpha = F \circ \alpha$ and $F_\beta = F \circ \beta$. For all $p \geq 2n-1$ and all t in (t_1, t_2),

$$\frac{F_\beta(t)}{|\beta'(t)|_g^{p-1}} > \frac{F_\alpha(t)}{|\alpha'(t)|_g^{p-1}} \tag{4.19}$$

Also, for all t in (t_1, t_2),

$$F_\beta(t)|\beta'(t)|_g < F_\alpha(t)|\alpha'(t)|_g \tag{4.20}$$

By Lemma 4.1, there is a function φ in $\mathrm{Lip}_0([0,1))$ which is non-vanishing over $(0,1)$ and satisfies

$$\lambda_{1,p}(\beta) = \frac{\int_0^1 \frac{|\varphi'|^p F_\beta}{|\beta'|_g^{p-1}}\,dt}{\int_0^1 |\varphi|^p |\beta'|_g F_\beta\,dt} \tag{4.21}$$

Then by Lemma 4.2,

$$\lambda_{1,p}(\alpha) \leq \frac{\int_0^1 \frac{|\varphi'|^p F_\alpha}{|\alpha'|_g^{p-1}}\,dt}{\int_0^1 |\varphi|^p |\alpha'|_g F_\alpha\,dt} < \frac{\int_0^1 \frac{|\varphi'|^p F_\beta}{|\beta'|_g^{p-1}}\,dt}{\int_0^1 |\varphi|^p |\beta'|_g F_\beta\,dt} = \lambda_{1,p}(\beta) \tag{4.22}$$

This is a contradiction, because $\lambda_{1,p}(\alpha) = \Lambda$. □

The next lemma is a variation of the previous lemma for circles centered on the boundary of $\mathbb{R}^{2n}/G$.

Lemma 4.4. *Fix $p \geq 2n-1$. Let α be a curve in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Let C be a large positive constant. Let x_1 be a positive number. Let r_1 be a positive number such that $Cr_1 \leq x_1$. Define*

$$D = \left\{ (x,y) \in \mathbb{R}^{2n}/G : (x-x_1)^2 + y^2 \leq r_1^2 \right\} \tag{4.23}$$

Let $0 < t_1 < 1$ and assume $\alpha(t_1)$ lies on the boundary ∂D. Assume that, for all t in $[t_1, 1]$,

$$|\alpha(t) - (x_1, 0)| < 2r_1 \tag{4.24}$$

If C is sufficiently large, independent of x_1 and r_1, then it follows that $\alpha(t)$ is in D for all t in $[t_1, 1]$.

Proof. Suppose not. It suffices to consider two cases. In the first case we assume that there is a number t_2 in $(t_1, 1]$ such that $\alpha(t)$ lies outside of D for every t in (t_1, t_2) and $\alpha(t_2)$ lies on the boundary ∂D. In the second case, we assume that $\alpha(t)$ lies outside of D for every t in $(t_1, 1]$. In the second case, define $t_2 = 1$. In either case, define $y_1 = 0$. Then repeating the argument used to prove Lemma 4.3 yields a contradiction. □

In the next lemma, we show that an eigenvalue maximizing curve in $\mathcal{R}_g^*$ is absolutely differentiable.

Lemma 4.5. *Fix $p \geq 2n-1$. Let α be a curve in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Let t_0 be a point in $[0,1)$. Let $\{p_k\}$ be a sequence in $[0, t_0]$ converging to t_0 and let $\{q_k\}$ be a sequence in $[t_0, 1)$ converging to t_0 Assume that $p_k \neq q_k$ for all k. Then*

$$\lim_{k\to\infty} \frac{|\alpha(q_k) - \alpha(p_k)|}{|q_k - p_k|} = L_g(\alpha) \tag{4.25}$$

In particular,

$$\lim_{t\to t_0} \frac{|\alpha(t) - \alpha(t_0)|}{|t - t_0|} = L_g(\alpha) \tag{4.26}$$

Proof. Suppose not. Since α is Lipschitz with constant $L_g(\alpha)$, there is a constant c such that

$$\liminf_{k\to\infty} \frac{|\alpha(q_k) - \alpha(p_k)|}{|q_k - p_k|} < c < L_g(\alpha) \tag{4.27}$$

By passing to subsequences, we may assume that for all k,

$$\frac{|\alpha(q_k) - \alpha(p_k)|}{|q_k - p_k|} < c \tag{4.28}$$

Fix k large and define a curve β in $\mathcal{R}_g^*$ by

$$\beta(t) = \begin{cases} \alpha(t) & 0 \leq t \leq p_k \\ \alpha(p_k) + (t - p_k)\frac{\alpha(q_k)-\alpha(p_k)}{q_k - p_k} & p_k \leq t \leq q_k \\ \alpha(t) & q_k \leq t \leq 1 \end{cases} \tag{4.29}$$

Let $F_\beta = F \circ \beta$. Since α is simple, Lemma 4.1 shows that there is a function φ in $\mathrm{Lip}_0([0,1))$ which is non-vanishing over $(0,1)$ and satisfies

$$\lambda_{1,p}(\beta) = \frac{\int_0^1 \frac{|\varphi'|^p F_\beta}{|\beta'|_g^{p-1}}\, dt}{\int_0^1 |\varphi|^p |\beta'|_g F_\beta \, dt} \tag{4.30}$$

Let $F_\alpha = F \circ \alpha$. If k is sufficiently large, then for all t in (p_k, q_k),

$$\frac{F_\beta(t)}{|\beta'(t)|_g^{p-1}} > \frac{F_\alpha(t)}{|\alpha'(t)|_g^{p-1}} \tag{4.31}$$

Also for all t in (p_k, q_k),

$$F_\beta(t)|\beta'(t)|_g < F_\alpha(t)|\alpha'(t)|_g \tag{4.32}$$

Therefore if k is sufficiently large, then by Lemma 4.2,

$$\lambda_{1,p}(\alpha) \leq \frac{\int_0^1 \frac{|\varphi'|^p F_\alpha}{|\alpha'|_g^{p-1}}\, dt}{\int_0^1 |\varphi|^p |\alpha'|_g F_\alpha \, dt} < \frac{\int_0^1 \frac{|\varphi'|^p F_\beta}{|\beta'|_g^{p-1}}\, dt}{\int_0^1 |\varphi|^p |\beta'|_g F_\beta \, dt} = \lambda_{1,p}(\beta) \tag{4.33}$$

This is a contradiction, because $\lambda_{1,p}(\alpha) = \Lambda_p$. □

Now we can show that an eigenvalue maximizing curve in $\mathcal{R}_g^*$ is differentiable.

Lemma 4.6. *Fix $p \geq 2n-1$. Let α be a curve in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Then α is differentiable over $[0,1)$. Moreover $|\alpha'(t)|_g = L_g(\alpha)$ for every t in $[0,1)$.*

Proof. We first prove that α is right-differentiable over $[0,1)$. Let t_0 be in $[0,1)$ and suppose that α is not right-differentiable at t_0. It follows from Lemma 4.5 that there is a positive constant c and sequences $\{y_k\}$ and $\{z_k\}$ in $(t_0, 1)$ converging to t_0 such that for all k, the points $\alpha(t_0), \alpha(y_k), \alpha(z_k)$ are distinct, and the interior angle at $\alpha(t_0)$ of the triangle with vertices at these points is at least c. By passing to a subsequence we may assume that $y_k < z_k$ for all k. Let x_α and y_α be the component functions of α. Let $C > 0$ be a large constant. Fix a positive constant r_1 with

$$Cr_1 < \min\Big(x_\alpha(t_0), y_\alpha(t_0)\Big) \tag{4.34}$$

For large k,

$$0 < |\alpha(z_k) - \alpha(t_0)| < r_1 \tag{4.35}$$

Then there are two closed discs of radius r_1 which contain $\alpha(z_k)$ and $\alpha(t_0)$ on their boundaries. If C and k are large, then by Lemma 4.3, the point $\alpha(y_k)$ must be in the intersection of these discs. But this implies that the interior angle at $\alpha(t_0)$ of the triangle with vertices at $\alpha(t_0), \alpha(y_k), \alpha(z_k)$ converges to zero as k tends to infinity. By this contradiction α is right-differentiable over $[0,1)$.

A symmetric argument shows that α is left-differentiable over $(0,1)$. Then Lemma 4.5 implies that the left and right derivatives must agree over $(0,1)$ and $|\alpha'(t)|_g = L_g(\alpha)$ for every t in $[0,1)$. □

The following lemma shows that an eigenvalue maximizing curve in $\mathcal{R}_g^*$ is in $\mathcal{C}$.

Lemma 4.7. *Fix $p \geq 2n-1$. Let α be a curve in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. Then α is in $\mathcal{C}$.*

Proof. Note that α is differentiable over $[0,1)$ and $|\alpha'(t)|_g = L_g(\alpha)$ for every t in $[0,1)$ by Lemma 4.6. In order to show that α is continuously differentiable over $[0,1)$, fix t_0 in $[0,1)$ and let $\{s_k\}$ be a sequence in $[0,1)$ converging to t_0. Let x_α and y_α be the component functions of α. Let $C > 0$ be a large constant. Let $r_1 > 0$ be such that

$$Cr_1 < \min\Big(x_\alpha(t_0), y_\alpha(t_0)\Big) \tag{4.36}$$

For large k, there are exactly two closed discs in $\mathbb{R}^{2n}/G$ of radius r_1 which contain $\alpha(s_k)$ and $\alpha(t_0)$ on their boundaries. If k is large, then Lemma 4.3 implies that $\alpha(t)$ must lie in the intersection of these discs for all t between t_0 and s_k. Since α is differentiable over $[0,1)$, and $|\alpha'(t)|_g = L_g(\alpha)$ for all t in $[0,1)$, it follows that

$$\lim_{k\to\infty} |\alpha'(s_k) - \alpha'(t_0)| = 0 \tag{4.37}$$

Therefore α' is continuous at t_0. This proves that α is continuously differentiable over $[0,1)$.

Fix t_1 in $[0,1)$. Let $C > 0$ be a large constant. Let $r_2 > 0$ be such that $Cr_2 < x_\alpha(t_1)$. If t_1 is close to 1, then there are exactly two closed half-discs in $\mathbb{R}^{2n}/G$ of radius r_1 which are centered on the boundary of $\mathbb{R}^{2n}/G$ and contain $\alpha(t_1)$ on their boundaries. Lemma 4.4 implies that $\alpha(t)$ must lie in the intersection of these discs for all t between t_1 and 1. Since α is differentiable over $[0,1]$, and $|\alpha'(t)|_g = L_g(\alpha)$ for all t in $[0,1]$, it follows that

$$\lim_{t\to 1} \alpha'(t) = \Big(0, -L_g(\alpha)\Big) \tag{4.38}$$

This implies that α is continuously differentiable over $[0,1]$ and $\alpha'(1) = (0, -L_g(\alpha))$. Therefore α is in $\mathcal{C}$. □

We can now prove Lemma 2.1.

Proof of Lemma 2.1. If $x_0 \neq y_0$ or $\Lambda_p > \lambda_{1,p}(\sigma)$, then by Lemma 3.8, there is a curve α in $\mathcal{R}_g^*$ such that $\lambda_{1,p}(\alpha) = \Lambda_p$. In particular α is simple. Moreover α is in $\mathcal{C}$ by Lemma 4.7. □

5. Simons' cone

In this section we conclude the article by proving Lemma 2.2. By a scaling argument, it suffices to consider the case $x_0 = y_0 = 1$. We assume that $x_0 = y_0 = 1$ throughout this section. Define a function $\sigma_0 : [0,1] \to \mathbb{R}^{2n}/G$ by

$$\sigma_0(t) = (1-t)^{1/2} \cdot (1,1) \tag{5.1}$$

Note that

$$\lambda_{1,p}(\sigma_0) = \inf \left\{ \frac{2^{p/2} \int_0^1 |w'|^p (1-t)^{n+\frac{p}{2}-\frac{3}{2}}\, dt}{\int_0^1 |w|^p (1-t)^{n-\frac{3}{2}}\, dt} : w \in \mathrm{Lip}_0([0,1]) \right\} \tag{5.2}$$

Furthermore, changing variables shows that $\lambda_{1,p}(\sigma) = \lambda_{1,p}(\sigma_0)$. Recall the coordinate functions u and v, defined in (3.22) and (3.23). For all t in $[0,1]$,

$$\Big(u \circ \sigma_0(t), v \circ \sigma_0(t)\Big) = (0, 1-t) \tag{5.3}$$

For ν in $\mathbb{R}$, let J_ν denote the Bessel function of the first kind of order ν. Let $j_{\nu,1}$ denote the first positive root of J_ν. Define a function φ_σ in $\mathrm{Lip}_0([0,1])$ by

$$\varphi_\sigma(t) = (1-t)^{\frac{3-2n}{4}} J_{n-\frac{3}{2}}(j_{n-\frac{3}{2},1}\sqrt{1-t}) \tag{5.4}$$

In the following lemma, we express the eigenvalues $\lambda_{1,p}(\sigma)$ in terms of the eigenvalues of a unit ball in $\mathbb{R}^{2n-1}$.

Lemma 5.1. *Let B_{2n-1} be the unit ball in $\mathbb{R}^{2n-1}$. For all p,*

$$\lambda_{1,p}(\sigma) = 2^{-p/2}\lambda_{1,p}(B_{2n-1}) \tag{5.5}$$

For the case $p = 2$,

$$\lambda_{1,2}(\sigma) = \frac{2\int_0^1 |\varphi_\sigma'|^2(1-t)^{n-\frac{1}{2}}\,dt}{\int_0^1 |\varphi_\sigma|^2(1-t)^{n-\frac{3}{2}}\,dt} = \frac{1}{2}\cdot j_{n-\frac{3}{2},1}^2 \tag{5.6}$$

Proof. Fix w in $\mathrm{Lip}_0([0,1])$. Let B_{2n-1} be the unit ball in $\mathbb{R}^{2n-1}$. Define a function v in $\mathrm{Lip}_0(B_{2n-1})$ by

$$v(x) = w(1-|x|^2) \tag{5.7}$$

For all p,

$$\frac{2^{p/2}\int_0^1 |w'|^p(1-t)^{n+\frac{p}{2}-\frac{3}{2}}\,dt}{\int_0^1 |w|^p(1-t)^{n-\frac{3}{2}}\,dt} = 2^{-\frac{p}{2}}\cdot\frac{\int_B |\nabla v|^p}{\int_B |v|^p} \tag{5.8}$$

Moreover, any radial function v in $\mathrm{Lip}_0(B_{2n-1})$ can be realized in this way, so (5.5) follows. For the case $p = 2$, it is a classical fact that $\lambda_{1,2}(B_{2n-1}) = j_{n-\frac{3}{2},1}^2$ and that the Rayleigh quotients in (5.8) are minimized when

$$v(x) = |x|^{\frac{3}{2}-n}J_{n-\frac{3}{2}}(j_{n-\frac{3}{2},1}|x|) \tag{5.9}$$

That is, the quotient is minimized when $w = \varphi_\sigma$. Therefore (5.6) holds. □

In the next lemma, we show that if n is fixed and p is large, then $\lambda_{1,p}(\sigma) > \Lambda_p$. The proof is a simple application of a result of Juutinen, Lindqvist, and Manfredi [10, Lemma 1.5]. They showed that if Ω is a smoothly bounded domain in $\mathbb{R}^d$, and if $\mathrm{inrad}(\Omega)$ is the inradius of Ω, then

$$\lim_{p\to\infty}\Big(\lambda_{1,p}(\Omega)\Big)^{1/p} = \frac{1}{\mathrm{inrad}(\Omega)} \tag{5.10}$$

In particular, if $d \geq 1$ is an integer and B_d is a unit ball in $\mathbb{R}^d$, then

$$\lim_{p\to\infty}\Big(\lambda_{1,p}(B_d)\Big)^{1/p} = 1 \tag{5.11}$$

Lemma 5.2. *Fix $n \geq 2$. If p is large, then there is a curve α in $\mathcal{C}$ such that $\lambda_{1,p}(\alpha) > \lambda_{1,p}(\sigma)$.*

Proof. By (5.11) and Lemma 5.1,

$$\lim_{p\to\infty}\Big(\lambda_{1,p}(\sigma)\Big)^{1/p} = 2^{-1/2} \tag{5.12}$$

Define a curve α in $\mathcal{C}$ by $\alpha(t) = (1, 1-t)$. Let B_n be a unit ball in $\mathbb{R}^n$. The hypersurface in $\mathbb{R}^{2n}$ corresponding to α is isometric to $B_n \times \mathbb{S}^{n-1}$. In particular, $\lambda_{1,p}(\alpha) = \lambda_{1,p}(B_n)$. Therefore, by (5.11),

$$\lim_{p\to\infty}\Big(\lambda_{1,p}(\alpha)\Big)^{1/p} = 1 \tag{5.13}$$

Now (5.12) and (5.13) imply that $\lambda_{1,p}(\sigma) < \lambda_{1,p}(\alpha)$, for large p. □

We use a variational argument for the case $p = 2$. For each $s > 0$, define a curve σ_s in $\mathcal{C}$ such that for all t in $[0, 1]$,

$$\Big(u \circ \sigma_s(t), v \circ \sigma_s(t)\Big) = (st, 1 - t) \tag{5.14}$$

For each $s \geq 0$ define functions $P_s : [0, 1) \to \mathbb{R}$ and $Q_s : [0, 1) \to \mathbb{R}$ by

$$P_s(t) = \frac{2(1 - t)^{n-1}((1 - t)^2 + s^2t^2)^{1/4}}{(1 + s^2)^{1/2}} \tag{5.15}$$

and

$$Q_s(t) = \frac{(1 - t)^{n-1}(1 + s^2)^{1/2}}{((1 - t)^2 + s^2t^2)^{1/4}} \tag{5.16}$$

For each $s \geq 0$,

$$\lambda_{1,2}(\sigma_s) = \min\left\{\frac{\int_0^1 |w'(t)|^2 P_s(t)\, dt}{\int_0^1 |w(t)|^2 Q_s(t)\, dt} : w \in \mathrm{Lip}_0([0, 1])\right\} \tag{5.17}$$

For each $s > 0$, let φ_s be the eigenfunction in $\mathrm{Lip}_0([0, 1))$ corresponding to $\lambda_{1,2}(\sigma_s)$, given by Lemma 4.1. Let φ_0 be a scalar multiple of φ_σ. Then for each $s \geq 0$,

$$\lambda_{1,2}(\sigma_s) = \frac{\int_0^1 P_s|\varphi_s'|^2\, dt}{\int_0^1 Q_s|\varphi_s|^2\, dt} \tag{5.18}$$

The eigenfunction φ_s satisfies the associated Euler-Lagrange equation, i.e.

$$-\,(P_s\varphi_s')' = \lambda_{1,2}(\sigma_s)Q_s\varphi_s \tag{5.19}$$

This equation implies that φ_s is twice continuously differentiable over $[0, 1)$. Moreover $\varphi_s'(0)$ is non-zero. For each $s \geq 0$, normalize φ_s so that $\varphi_s'(0) = 1$.

In the next lemma, we show that $\lambda_{1,2}(\sigma_s)$ depends continuously on s. Note that if $s_1 \geq 0$ and $s_2 > 0$, then for all t in $[0, 1)$,

$$\frac{P_{s_1}(t)}{P_{s_2}(t)} = \frac{Q_{s_2}(t)}{Q_{s_1}(t)} \leq \left(\frac{1 + s_2^2}{1 + s_1^2}\right)^{1/2} \cdot \max\left(1, \frac{s_1^{1/2}}{s_2^{1/2}}\right) \tag{5.20}$$

Lemma 5.3. *The function $s \mapsto \lambda_{1,2}(\sigma_s)$ is continuous over $[0, \infty)$.*

Proof. Fix $s_0 \geq 0$. The bounds in (5.20) imply that, for all $s > 0$,

$$\lambda_{1,2}(\sigma_{s_0}) \leq \lambda_{1,2}(\sigma_s) \cdot \frac{1 + s^2}{1 + s_0^2} \cdot \max\left(1, \frac{s_0}{s}\right) \tag{5.21}$$

In particular,

$$\lambda_{1,2}(\sigma_{s_0}) \leq \liminf_{s \to s_0} \lambda_{1,2}(\sigma_s) \tag{5.22}$$

Note that for all $s \geq 0$,

$$\lambda_{1,2}(\sigma_s) \leq \frac{\int_0^1 |\varphi_{s_0}'|^2 P_s(t)\, dt}{\int_0^1 |\varphi_{s_0}|^2 Q_s(t)\, dt} \tag{5.23}$$

Additionally,

$$\lim_{s \to s_0} \frac{\int_0^1 |\varphi_{s_0}'|^2 P_s(t)\, dt}{\int_0^1 |\varphi_{s_0}|^2 Q_s(t)\, dt} = \frac{\int_0^1 |\varphi_{s_0}'|^2 P_{s_0}(t)\, dt}{\int_0^1 |\varphi_{s_0}|^2 Q_{s_0}(t)\, dt} = \lambda_{1,2}(\sigma_{s_0}) \tag{5.24}$$

If $s_0 > 0$, then (5.24) follows from (5.20). For the case $s_0 = 0$, note that φ_0 and φ_0' are bounded and P_s and Q_s are uniformly bounded for small s, because $n \geq 2$. Then (5.24) follows from Lebesgue's dominated convergence theorem. Now

$$\limsup_{s\to s_0} \lambda_{1,2}(\sigma_s) \leq \lambda_{1,2}(\sigma_{s_0}) \tag{5.25}$$

By (5.22) and (5.25), the function $s \mapsto \lambda_{1,2}(\sigma_s)$ is continuous at $s = s_0$. □

Next we show that the eigenfunctions φ_s converge to φ_σ.

Lemma 5.4. *For all t in $[0, 1)$,*

$$\lim_{s\to 0} \varphi_s(t) = \varphi_\sigma(t) \tag{5.26}$$

and

$$\lim_{s\to 0} \varphi_s'(t) = \varphi_\sigma'(t) \tag{5.27}$$

For any $\delta > 0$, the convergence in both limits is uniform over $[0, 1-\delta]$.

Proof. Note that the eigenfunctions φ_s satisfy (5.19). Moreover $\varphi_s(0) = 0$, and the functions φ_s are normalized so that $\varphi_s'(0) = 1$. The convergence now follows from Lemma 5.3 and continuous dependence on parameters. □

Define $D_- : (0, \infty) \to \mathbb{R}$ to be the lower left Dini derivative of the function $s \mapsto \lambda_{1,2}(\sigma_s)$. That is, for each s_0 in $(0, \infty)$,

$$D_-(s_0) = \liminf_{s\nearrow s_0} \frac{\lambda_{1,2}(\sigma_s) - \lambda_{1,2}(\sigma_{s_0})}{s - s_0} \tag{5.28}$$

For each $s > 0$, define functions $\dot{P}_s : [0, 1] \to \mathbb{R}$ and $\dot{Q}_s : [0, 1] \to \mathbb{R}$ by

$$\dot{P}_s(t) = \frac{st^2(1-t)^{n-1}}{((1-t)^2 + s^2t^2)^{3/4}(1+s^2)^{1/2}} - \frac{2s(1-t)^{n-1}((1-t)^2 + s^2t^2)^{1/4}}{(1+s^2)^{3/2}} \tag{5.29}$$

and

$$\dot{Q}_s(t) = \frac{s(1-t)^{n-1}}{((1-t)^2 + s^2t^2)^{1/4}(1+s^2)^{1/2}} - \frac{st^2(1-t)^{n-1}(1+s^2)^{1/2}}{2((1-t)^2 + s^2t^2)^{5/4}} \tag{5.30}$$

The following lemma establishes a lower bound for $D_-(s)$.

Lemma 5.5. *If $s > 0$ is small, then*

$$D_-(s) \geq \frac{\int_0^1 |\varphi_s'|^2 \dot{P}_s - \lambda_{1,2}(\sigma_s)|\varphi_s|^2 \dot{Q}_s \, dt}{\int_0^1 |\varphi_s|^2 Q_s \, dt} \tag{5.31}$$

Proof. Let $s_0 > 0$ be small. Define a function $h : (0, \infty) \to \mathbb{R}$ by

$$h(s) = \frac{\int_0^1 |\varphi_{s_0}'|^2 P_s(t) \, dt}{\int_0^1 |\varphi_{s_0}|^2 Q_s(t) \, dt} \tag{5.32}$$

Note that $\lambda_{1,2}(\sigma_s) \leq h(s)$ for every $s > 0$ by Lemma 4.2, and $\lambda_{1,2}(\sigma_{s_0}) = h(s_0)$. Therefore

$$D_-(s_0) = \liminf_{s\nearrow s_0} \frac{\lambda_{1,2}(\sigma_s) - \lambda_{1,2}(\sigma_{s_0})}{s - s_0} \geq \liminf_{s\nearrow s_0} \frac{h(s) - h(s_0)}{s - s_0} \tag{5.33}$$

It suffices to show that

$$\liminf_{s\nearrow s_0}\frac{h(s)-h(s_0)}{s-s_0}\geq\frac{\int_0^1|\varphi_{s_0}'|^2\dot P_{s_0}-\lambda_{1,2}(\sigma_{s_0})|\varphi_{s_0}|^2\dot Q_{s_0}\,dt}{\int_0^1|\varphi_{s_0}|^2Q_{s_0}\,dt}\tag{5.34}$$

Note that

$$\frac{h(s)-h(s_0)}{s-s_0}=\frac{\int_0^1|\varphi_{s_0}'|^2\frac{P_s-P_{s_0}}{s-s_0}-\lambda_{1,2}(\sigma_{s_0})|\varphi_{s_0}|^2\frac{Q_s-Q_{s_0}}{s-s_0}\,dt}{\int_0^1|\varphi_{s_0}|^2Q_s\,dt}\tag{5.35}$$

By (5.20),

$$\limsup_{s\nearrow s_0}\int_0^1|\varphi_{s_0}|^2Q_s\,dt\leq\int_0^1|\varphi_{s_0}|^2Q_{s_0}\,dt\tag{5.36}$$

Since s_0 is small, there is a $\delta>0$ such that if $0<s<s_0$, then $P_s<P_{s_0}$ and $Q_s>Q_{s_0}$ over $[1-\delta,1]$. Therefore

$$\begin{aligned}\liminf_{s\nearrow s_0}\int_0^1|\varphi_{s_0}'|^2\frac{P_s-P_{s_0}}{s-s_0}-\lambda_{1,2}(\sigma_{s_0})|\varphi_{s_0}|^2\frac{Q_s-Q_{s_0}}{s-s_0}\,dt\\ \geq\int_0^1|\varphi_{s_0}'|^2\dot P_{s_0}-\lambda_{1,2}(\sigma_{s_0})|\varphi_{s_0}|^2\dot Q_{s_0}\,dt\end{aligned}\tag{5.37}$$

To prove this inequality, use the uniform convergence of the integrands over $[0,1-\delta]$ and use Fatou's lemma over $[1-\delta,1]$. Now (5.35), (5.36), and (5.37) imply (5.34), completing the proof. □

Define functions $\ddot P_0:[0,1]\to\mathbb{R}$ and $\ddot Q_0:[0,1]\to\mathbb{R}$ by

$$\ddot P_0(t)=t^2(1-t)^{n-\frac52}-2(1-t)^{n-\frac12}\tag{5.38}$$

and

$$\ddot Q_0(t)=(1-t)^{n-\frac32}-\frac{t^2(1-t)^{n-\frac72}}{2}\tag{5.39}$$

The following lemma gives a sufficient condition to verify that $\lambda_{1,2}(\sigma)<\lambda_{1,2}(\sigma_s)$ for small $s>0$.

Lemma 5.6. *Fix n and assume that*

$$\int_0^1|\varphi_\sigma'|^2\ddot P_0(t)-\lambda_{1,2}(\sigma)|\varphi_\sigma|^2\ddot Q_0(t)\,dt>0\tag{5.40}$$

If s is small and positive, then $\lambda_{1,2}(\sigma_s)>\lambda_{1,2}(\sigma)$.

Proof. By Lemma 5.3, the function $s\mapsto\lambda_{1,2}(\sigma_s)$ is continuous. Therefore it suffices to show that the Dini derivative $D_-(s)$ is positive for small positive s. In particular, it suffices to show that

$$\liminf_{s\searrow 0}s^{-1}D_-(s)\int_0^1|\varphi_s|^2Q_s\,dt>0\tag{5.41}$$

By Lemma 5.5, it suffices to show that

$$\liminf_{s\searrow 0}\int_0^1|\varphi_s'|^2s^{-1}\dot P_s-\lambda_{1,2}(\sigma_s)|\varphi_s|^2s^{-1}\dot Q_s\,dt>0\tag{5.42}$$

Fix a small $\delta > 0$. If s is small and positive, then $\dot{P}_s \geq 0$ over $[1-\delta, 1]$ and $\dot{Q}_s \leq 0$ over $[1-\delta, 1]$. Therefore

$$\begin{aligned}\liminf_{s \searrow 0} \int_0^1 |\varphi_s'|^2 s^{-1} \dot{P}_s - \lambda_{1,2}(\sigma_s)|\varphi_s|^2 s^{-1} \dot{Q}_s \, dt \\ \geq \int_0^1 |\varphi_\sigma'|^2 \ddot{P}_0(t) - \lambda_{1,2}(\sigma)|\varphi_\sigma|^2 \ddot{Q}_0(t)\, dt\end{aligned} \tag{5.43}$$

To prove this inequality, use Lemma 5.3 and Lemma 5.4 to obtain uniform convergence of the integrands over $[0, 1-\delta]$ and use Fatou's lemma over $[1-\delta, 1]$. Now (5.40) and (5.43) imply (5.42), completing the proof. □

Now we verify the condition (5.40) for $n \leq 5$.

Lemma 5.7. *If $n \leq 5$, then*

$$\int_0^1 |\varphi_\sigma'|^2 \ddot{P}_0(t) - \lambda_{1,2}(\sigma)|\varphi_\sigma|^2 \ddot{Q}_0(t)\, dt > 0 \tag{5.44}$$

Proof. Fix $n \geq 2$, and let $\alpha = n - \frac{3}{2}$. Using the identity $\frac{\alpha}{x} J_\alpha(x) - J_\alpha'(x) = J_{\alpha+1}(x)$, the derivative φ_σ' can be expressed as

$$\varphi_\sigma'(t) = \frac{j_{\alpha,1}}{2}(1-t)^{-\frac{\alpha+1}{2}} J_{\alpha+1}(j_{\alpha,1}\sqrt{1-t}) \tag{5.45}$$

Using Lemma 5.1 and changing variables, we have

$$\begin{aligned}\int_0^1 |\varphi_\sigma'|^2 \ddot{P}_0(t) - \lambda_{1,2}(\sigma)|\varphi_\sigma|^2 \ddot{Q}_0(t)\, dt \\ = \int_0^{j_{\alpha,1}} t\Big(|J_{\alpha+1}(t)|^2 + |J_\alpha(t)|^2\Big)\Big(\frac{(j_{\alpha,1}^2 - t^2)^2}{2t^4} - 1\Big)\, dt\end{aligned} \tag{5.46}$$

This integral can be approximated precisely. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \frac{t^2}{2}\Big(J_{\alpha+1}(t)^2 - J_{\alpha+2}(t)J_\alpha(t) + J_\alpha(t)^2 - J_{\alpha+1}(t)J_{\alpha-1}(t)\Big) \tag{5.47}$$

By Lommel's integral,

$$f'(t) = t\Big(|J_{\alpha+1}(t)|^2 + |J_\alpha(t)|^2\Big) \tag{5.48}$$

If $n \leq 5$, it follows that

$$\int_0^{j_{\alpha,1}} t\Big(|J_{\alpha+1}(t)|^2 + |J_\alpha(t)|^2\Big)\, dt < 4 \tag{5.49}$$

Define a function $g : (0, \infty) \to \mathbb{R}$ by

$$g(t) = \frac{(j_{\alpha,1}^2 - t^2)^2}{2t^4} \tag{5.50}$$

For any partition $0 = p_0 < p_1 < \ldots < p_m = j_{\alpha,1}$, the monotonicity of g implies

$$\int_0^{j_{\alpha,1}} t\Big(|J_{\alpha+1}(t)|^2 + |J_\alpha(t)|^2\Big)\frac{(j_{\alpha,1}^2 - t^2)^2}{2t^4}\, dt > \sum_{i=1}^m \Big(f(p_i) - f(p_{i-1})\Big)g(p_i) \tag{5.51}$$

If $n \leq 5$, then choosing a suitable partition shows that

$$\int_0^{j_{\alpha,1}} t\Big(|J_{\alpha+1}(t)|^2 + |J_\alpha(t)|^2\Big)\frac{(j_{\alpha,1}^2 - t^2)^2}{2t^4}\, dt > 4 \tag{5.52}$$

Now (5.46), (5.49), and (5.52) imply (5.44), completing the proof. □

Next we round off a curve σ_s with $s > 0$ to obtain a curve in $\mathcal{C}$.

Lemma 5.8. *If $n \leq 5$, then there is a curve α in $\mathcal{C}$ such that $\lambda_{1,2}(\alpha) > \lambda_{1,2}(\sigma)$.*

Proof. Fix $s > 0$ small. Then $\lambda_{1,2}(\sigma_s) > \lambda_{1,2}(\sigma)$ by Lemma 5.6 and Lemma 5.7. Let $u_s = u \circ \sigma_s$ and $v_s = v \circ \sigma_s$. Note that $u_s(1) = s$. Let L be the line segment in $\mathbb{R}^2$ given by

$$L = \Big\{ \Big(u_s(t), v_s(t) \Big) : t \in [0, 1] \Big\} \tag{5.53}$$

Let δ be a small constant satisfying $0 < \delta < s$. There is a disc D in $\mathbb{R}^2$ which is centered about $(s - \delta, 0)$ such that $L \cap D$ consists of exactly one point. Fix t_0 so that $\sigma(t_0)$ is the point in $L \cap D$. There is a unique continuous function $u_\delta : [0, 1] \to [0, \infty)$ which agrees with u_s over $[0, t_0]$ such that $(u_\delta(t), v_s(t))$ is in the boundary ∂D for all t in $[t_0, 1]$. Let β be the curve in $\mathcal{R}_g$ such that

$$(u \circ \beta, v \circ \beta) = (u_\delta, v_s) \tag{5.54}$$

Note that $F \circ \beta = F \circ \sigma_s$. Let $\varepsilon > 0$ be small. If δ is small, then for all t in $[0, 1]$,

$$|\beta'(t)|_g \leq (1 + \varepsilon) |\sigma_s'(t)|_g \tag{5.55}$$

This yields $\lambda_{1,2}(\beta) \geq (1 + \varepsilon)^{-2} \lambda_{1,2}(\sigma_s)$. If ε is small, then $\lambda_{1,2}(\beta) > \lambda_{1,2}(\sigma)$. Let α be the reparametrization of β given by Lemma 3.7. Then α is in $\mathcal{C}$ and $\lambda_{1,2}(\alpha) > \lambda_{1,2}(\sigma)$. □

We can now prove Lemma 2.2.

Proof of Lemma 2.2. It suffices to consider the case $x_0 = y_0 = 1$, by a scaling argument. The case where n is fixed and p is large is established by Lemma 5.2. The case where $p = 2$ and $n \leq 5$ is established by Lemma 5.8. □

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