GENERALIZED HUYGENS TYPES INEQUALITIES FOR BESSEL AND MODIFIED BESSEL FUNCTIONS

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Abstract. In this paper, we present a generalization of the Huygens types inequalities involving Bessel and modified Bessel functions of the first kind.

Keywords: Bessel functions, Modified Bessel functions, Turán type inequalities, Huygens inequalities.

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1. Introduction

This inequality

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>3 \tag{1}$$

which holds for all $x\in(0,\pi/2)$ is known in literature as Huygens's inequality [5]. The hyperbolic counterpart of (1) was established in [8] as follows:

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>3,\ x>0. \tag{2}$$

The inequalities (1) and (2) were respectively refined in [5] as

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>2\frac{x}{\sin x}+\frac{x}{\tan x}>3 \tag{3}$$

for $0<x<\frac{\pi}{2}$ and

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x}>2\frac{x}{\sinh x}+\frac{x}{\tanh x}>3,\ x\neq 0. \tag{4}$$

In [12], Zhu give some new inequalities of the Huygens type for circular functions, hyperbolic functions, and the reciprocals of circular and hyperbolic functions, as follows:

Theorem A The following inequalities

$$(1-p)\frac{\sin x}{x}+p\frac{\tan x}{x}>1>(1-q)\frac{\sin x}{x}+q\frac{\tan x}{x} \tag{5}$$

holds for all $x \in (0, \pi/2)$ if and only if $p \geq 1/3$ and $q \leq 0$.
Theorem B The following inequalities

$$(1-p)\frac{\sinh x}{x} + p\frac{\tanh x}{x} > 1 > (1-q)\frac{\sinh x}{x} + q\frac{\tanh x}{x} \tag{6}$$

holds for all $x \in (0, \infty)$ if and only if $p \leq 1/3$ and $q \geq 1$.

Recently, the author of this paper extend and sharpen inequalities (5) and (6) for the Bessel and modified Bessel functions to the following results in [7].
Theorem C Let $-1 < \nu \leq 0$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the Huygens type inequalities

$$(1-p)\mathcal{J}_{\nu+1}(x) + p\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} > 1 > (1-q)\mathcal{J}_{\nu+1}(x) + q\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \tag{7}$$

holds for all $(x \in (0, j_{\nu,1})$, if and only if, $p \geq \frac{\nu+1}{\nu+2}$ and $q \leq 0$.
Theorem D Let $\nu > -1$, the following inequalities

$$(1-p)\,\mathcal{I}_{\nu+1}(x) + p\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} > 1 > (1-q)\,\mathcal{I}_{\nu+1}(x) + q\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)}, \tag{8}$$

holds for all $x \in (0, \infty)$ if and only if $p \leq \frac{\nu+1}{\nu+2}$ and $q \geq 1$.

For $\nu > -1$ and consider the function $\mathcal{J}_\nu : \mathbb{R} \longrightarrow (-\infty, 1]$, defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n\geq} \frac{\left(\frac{-1}{4}\right)^n}{(\nu+1)_n n!} x^{2n},$$

where Γ is the gamma function, $(\nu+1)_n = \Gamma(\nu+n+1)/\Gamma(\nu+1)$ for each $n \geq 0$, is the well-known Pochhammer (or Appell) symbol, and J_ν defined by

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)},$$

stands for the Bessel function of the first kind of order ν. It is worth mentioning that in particular the function J_ν reduces to some elementary functions, like sine and cosine. More precisely, in particular we have:

$$\mathcal{J}_{-1/2}(x) = \sqrt{\pi/2}.x^{1/2} J_{-1/2}(x) = \cos x, \tag{9}$$

$$\mathcal{J}_{1/2}(x) = \sqrt{\pi/2}.x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \tag{10}$$

For $\nu > -1$, let us consider the function $\mathcal{I}_\nu : \mathbb{R} \longrightarrow [1, \infty)$, defined by

$$\mathcal{I}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} I_\nu(x) = \sum_{n\geq} \frac{\left(\frac{1}{4}\right)^n}{(\nu+1)_n n!} x^{2n},$$

where I_ν is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n\geq 0} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \text{ for all } x \in \mathbb{R}.$$

It is worth mentioning that in particular we have

$$\mathcal{I}_{-1/2}(x) = \sqrt{\pi/2}.x^{1/2} I_{-1/2}(x) = \cosh x, \tag{11}$$

$$\mathcal{I}_{1/2}(x) = \sqrt{\pi/2}.x^{1/2} I_{-1/2}(x) = \frac{\sinh x}{x}. \tag{12}$$

$$\mathcal{I}_{3/2}(x) = 3\sqrt{\pi/2}.x^{-3/2} I_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right). \tag{13}$$

In this note, we present a generalization of the Huygens type inequalities (1) and (2) for Bessel and modified Bessel functions.

2. **Lemmas**

In order to establish our main results, we need several lemmas, which we present in this section.

Lemma 1. [10] *Let a_n and b_n $(n = 0, 1, 2, ...)$ be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$. If $b_n > 0$ for $n = 0, 1, ..,$ and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n = 0, 1, 2...,$ then the function $\frac{A(x)}{B(x)}$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 2. [6, 2, 9] *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If $\frac{f'}{g'}$ is increasing (or decreasing) on (a, b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a, b).*

Lemma 3. *(Turán type inequality for modified Bessel function) The following Turán type inequality*

$$\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x) < \frac{\nu+2}{\nu+1}\,\mathcal{I}_{\nu+1}^2(x), \tag{14}$$

holds for all $\nu > -1$ and $x \in \mathbb{R}$. In particular, the following Turán type inequality

$$\cosh(x)\left(\cosh(x) - x\sinh(x)\right) < \sinh^2(x) \tag{15}$$

is valid for all $x \in \mathbb{R}$.

Proof. By using the Cauchy product

$$\mathcal{I}_\mu(x)\mathcal{I}_\nu(x) = \sum_{n\geq 0} \frac{\Gamma(\nu+1)\Gamma(\mu+1)\Gamma(\nu+\mu+2n+1)x^{2n}}{2^{2n}\Gamma(n+1)\Gamma(\nu+\mu+n+1)\Gamma(\mu+n+1)\Gamma(\nu+n+1)} \tag{16}$$

we have

$$\frac{\nu+2}{\nu+1}\mathcal{I}_{\nu+1}^2(x)-\mathcal{I}_{\nu+2}(x)\mathcal{I}_\nu(x)=\sum_{n\geq 0}\frac{\Gamma(\nu+1)\Gamma(\nu+3)\Gamma(2\nu+2n+3)}{2^{2n}\Gamma(n+1)\Gamma(2\nu+n+3)\Gamma(\nu+n+2)\Gamma(\nu+n+3)}x^{2n}\geq 0$$

for all $x\in\mathbb{R}$ and $\nu>-1$. On the other hand, observe that using (9), (10), and (14) in particular $\nu=-1/2$, the Turán type inequality (14) becomes (15). ■

3. **Main results**

We first obtain the further result concerning the generalized Huygens inequality to the Bessel functions described as Theorem 1.

Theorem 1. *Let $-1<\nu\leq 0$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the Huygens types inequalities*

$$1>\left(1-\frac{\nu+2}{\nu+1}\right)\mathcal{J}_\nu(x)+\frac{\nu+2}{\nu+1}\frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)} \tag{17}$$

holds for all $x\in(0,j_{\nu,1})$

Proof. Let $\nu>-1$, we consider the function

$$F_\nu(x)=\frac{1-\mathcal{J}_\nu(x)}{\frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)}-\mathcal{J}_\nu(x)},\quad 0<x<j_{\nu,1}.$$

For $0<x<j_{\nu,1}$, let

$$f_{\nu,1}(x)=1-\mathcal{J}_\nu(x)\text{ and }f_{\nu,2}(x)=\frac{\mathcal{J}_\nu(x)}{\mathcal{J}_{\nu+1}(x)}-\mathcal{J}_\nu(x).$$

Now, by again using the differentiation formula

$$\mathcal{J}_\nu'(x)=-\frac{x}{2(\nu+1)}\mathcal{J}_{\nu+1}(x) \tag{18}$$

we get

$$f_{\nu,1}'(x)=\frac{x\mathcal{J}_{\nu+1}(x)}{2(\nu+1)},$$

and

$$f_{\nu,2}'(x)=\frac{x\mathcal{J}_{\nu+1}(x)}{2(\nu+1)}+\frac{\frac{x}{2(\nu+2)}\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)-\frac{x}{2(\nu+1)}\mathcal{J}_{\nu+1}^2(x)}{\mathcal{J}_{\nu+1}^2(x)}.$$

Thus

$$\frac{f_{\nu,1}'(x)}{f_{\nu,2}'(x)}=\frac{1}{1+\frac{1}{\mathcal{J}_{\nu+1}(x)}\left(\frac{(\nu+1)\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}{(\nu+2)\mathcal{J}_{\nu+1}^2(x)}-1\right)}. \tag{19}$$

Let

$$h_\nu(x)=\frac{(\nu+1)\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}{(\nu+2)\mathcal{J}_{\nu+1}^2(x)}-1.$$

From the Turán type inequality [3]

$$\mathcal{J}^2_{\nu+1}(x) - \mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x) > 0. \tag{20}$$

where $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$, we conclude that $h_\nu(x) \leq 0$ for all $x \in (0, j_{\nu,1})$. On the other hand, differentiation again and simplifying give

$$h'_\nu(x) = \frac{(\nu+1)x}{(\nu+2)\mathcal{J}^4_{\nu+1}(x)}\left[\mathcal{J}_{\nu+1}(x)\mathcal{J}_{\nu+2}(x)\left(\frac{\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}{\nu+2} - \frac{\mathcal{J}^2_{\nu+1}(x)}{2(\nu+1)}\right) - \frac{\mathcal{J}_\nu(x)\mathcal{J}^3_{\nu+1}(x)\mathcal{J}_{\nu+3}(x)}{2(\nu+3)}\right]. \tag{21}$$

By (20) and (21) we easily get

$$h'_\nu(x) \leq \frac{x\nu\mathcal{J}_{\nu+1}(x)\mathcal{J}_{\nu+2}(x)}{2(\nu+2)^2} - \frac{x\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)\mathcal{J}_{\nu+3}(x)}{2(\nu+2)(\nu+3)\mathcal{J}_{\nu+1}(x)}.$$

In fact, since the function $\nu \mapsto \mathcal{J}_\nu(x)$ is increasing ([3], Theorem 3) on $(-1, \infty)$ for all fixed $x \in (-j_{\nu,1}, j_{\nu,1})$, and $\mathcal{J}_\nu(x) \in (0, 1]$, we conclude that, by (21), the function $h'_\nu(x)$ is decreasing on $(0, j_{\nu,1})$, for all $\nu \in (-1, 0]$. Indeed, the function $x \mapsto \mathcal{J}_\nu(x)$ is decreasing on $[0, j_{\nu,1})$ ([3], Theorem 3) and nonnegative, which implies that the function $x \mapsto \frac{1}{\mathcal{J}_{\nu+1}(x)}\left(\frac{(\nu+1)\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}{(\nu+2)\mathcal{J}^2_{\nu+1}(x)} - 1\right)$ is decreasing on $(0, j_{\nu,1})$, for all $\nu \in (-1, 0]$, as a product of two functions one is increasing and nonnegative and other is decreasing and negative. So, the function $x \mapsto \frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)}$ is increasing $(0, j_{\nu,1})$, for all $\nu \in (-1, 0]$, and consequently the function $x \mapsto F_\nu(x)$ is increasing $(0, j_{\nu,1})$, for all $\nu \in (-1, 0]$, by Lemma 2. Using L'Hospital rule and (19) yields

$$\lim_{x\to 0} F_\nu(x) = \frac{\nu+2}{\nu+1}.$$

Finally, for each $\nu > -1$ and $x \in (0, j_{\nu,1})$ one has $0 \leq \mathcal{J}_\nu(x) \leq 1$, and with this the proof of inequality (17) is complete. ■

Remark 1. *Using the relation (9), (10) and $j_{-1/2,1} = \pi/2$ and from the extended Huygens type inequality (17) for $\nu = -\frac{1}{2}$ we obtain the inequality (1).*

In the next Theorem, we establish the analogue of inequality (17) involving the modified Bessel functions.

Theorem 2. *Let $\nu > -1$. Then the Huygens types inequality*

$$1 > \left(1 - \frac{\nu+2}{\nu+1}\right)\mathcal{I}_\nu(x) + \frac{\nu+2}{\nu+1}\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)} \tag{22}$$

holds for all $x \in (0, \infty)$.

Proof. Let $\nu > -1$, we define the function G_ν on $(0, \infty)$ by

$$G_\nu(x) = \frac{1 - \mathcal{I}_\nu(x)}{\frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)} - \mathcal{I}_\nu(x)} = \frac{g_{\nu,1}(x)}{g_{\nu,2}(x)},$$

where $g_{\nu,1}(x) = 1 - \mathcal{I}_\nu(x)$ and $g_{\nu,2}(x) = \frac{\mathcal{I}_\nu(x)}{\mathcal{I}_{\nu+1}(x)} - \mathcal{I}_\nu(x)$. By using the differentiation formula [[11], p. 79]

$$\mathcal{I}'_\nu(x) = \frac{x}{2(\nu+1)}\mathcal{I}_{\nu+1}(x) \tag{23}$$

can easily show that

$$\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)} = \frac{1}{1 + \frac{1}{\mathcal{I}_{\nu+1}(x)}\left(\frac{(\nu+1)\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)}{(\nu+2)\mathcal{I}^2_{\nu+1}(x)} - 1\right)}. \tag{24}$$

Now, for $\nu > -1$, we define the function k_ν by:

$$k_\nu(x) = \frac{(\nu+1)\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)}{(\nu+2)\mathcal{I}^2_{\nu+1}(x)} - 1.$$

From the Turán type inequality (14) (see Lemma 3), we conclude that $k_\nu(x) \leq 0$ for all $x \in \mathbb{R}$. On the other hand, using the Cauchy product 16, we get

$$\frac{(\nu+1)\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)}{(\nu+2)\mathcal{I}^2_{\nu+1}(x)} = \frac{\sum_{n=0}^\infty a_n x^{2n}}{\sum_{n=0}^\infty b_n x^{2n}},$$

where $a_n(\nu) = \frac{\Gamma^2(\nu+2)\Gamma(2\nu+2n+3)}{2^{2n}\Gamma(n+1)\Gamma(\nu+n+1)\Gamma(\nu+n+3)}$ and $b_n(\nu) = \frac{\Gamma^2(\nu+2)\Gamma(2\nu+2n+3)}{2^{2n}\Gamma(n+1)\Gamma^2(\nu+n+2)}$ for all $n = 0, 1, ...$ So, for all $n = 0, 1, ...$, we have

$$c_n(\nu) = \frac{a_n(\nu)}{b_n(\nu)} = \frac{\Gamma^2(\nu+n+2)}{\Gamma(\nu+n+1)\Gamma(\nu+n+3)} = \frac{\nu+n+1}{\nu+n+2},$$

we conclude that $c_n(\nu)$ is increasing for $n = 0, 1, ...$, and the function $x \mapsto k_\nu(x)$ is increasing on $(0, \infty)$, by Lemma 1. Since the function $x \mapsto \frac{1}{\mathcal{I}_{\nu+1}(x)}$ is decreasing and nonnegative on $(0, \infty)$ and the function $x \mapsto k_\nu(x)$ is increasing and negative on $(0, \infty)$, we conclude that $x \mapsto \frac{g'_{\nu,1}(x)}{g'_{\nu,2}(x)}$ is decreasing on $(0, \infty)$, and consequently the function $x \mapsto G_\nu(x)$ is decreasing on $(0, \infty)$, by Lemma 1. Therefore, from the L'Hospital rule and (24) yields

$$\lim_{x\to 0} G_\nu(x) = \frac{\nu+2}{\nu+1}.$$

Moreover, using the fact $\mathcal{I}(x) \geq 1$, we get the Huygens type inequality (22). So, the proof of Theorem 2 is complete. ■

Remark 2. *1. From the relations (11) and (12) we find that the inequality (22) is the generalization of inequality (2).*
2. Since the function $x \mapsto G_\nu(x)$ is decreasing on $(0, \infty)$, and using the asymptotic formula [[1], p. 377]

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{4\nu^2 - 1}{1!(8x)} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2} - ...\right]$$

which holds for large values of x *and for fixed* $\nu > -1$*, we obtain*

$$\lim_{x \longrightarrow \infty} G_\nu(x) = 1.$$

Then, the following inequality [3]

$$\mathcal{I}_{\nu+1}(x) \leq \mathcal{I}_\nu(x),$$

holds for all $x \in \mathbb{R}$ *and* $\nu > -1$.

3. Using the relation (12) and (13) from the extended Huygens type inequality (22) for $\nu = 1/2$*, we obtain the following inequality*

$$9 > \frac{\sinh x}{x}\left(-6 + \frac{5x^3}{x\cosh x - \sinh x}\right),$$

which holds for all $x \in \mathbb{R}$.

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