THE ANGEHRN–SIU TYPE EFFECTIVE FREENESS FOR QUASI-LOG CANONICAL PAIRS

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Abstract. We prove the Angehrn–Siu type effective freeness and effective point separation for quasi-log canonical pairs. As a natural consequence, we obtain that these two results hold for semi-log canonical pairs. One of the main ingredients of our proof is the inversion of adjunction for quasi-log canonical pairs, which is established in this paper.

1. Introduction

The theory of mixed Hodge structures plays an important role in the recent developments of the minimal model program (see, for example, [KS11]). Now we have various powerful vanishing theorems based on the theory of mixed Hodge structures on cohomology with compact support (see [Fuj09], [Fuj14a], and so on). They are much sharper than the Kawamata–Viehweg vanishing theorem and the (algebraic version of) Nadel vanishing theorem. By these new vanishing theorems, the fundamental theorems of the minimal model program were established for quasi-log canonical (qlc, for short) pairs (see [Fuj09], [Fuj14a], and so on). Note that the notion of quasi-log structures was first introduced by Ambro in [Amb03]. The category of qlc pairs is very large and contains kawamata log terminal pairs, log canonical pairs, quasi-projective semi-log canonical pairs (see [Fuj14c, Theorem 1.1]), and so on. The notion of qlc pairs seems to be indispensable for the cohomological study of semi-log canonical pairs (see [Fuj14c]). In this paper, we formulate the Angehrn–Siu type effective freeness and effective point separation for qlc pairs and prove them in the framework of quasi-log structures. Of course, our results generalize [AS95], [Kol97, 5.8, 5.9], and [Fuj10, Theorems 1.1 and 1.2]. The effective freeness for qlc pairs is as follows.

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Theorem 1.1 (Effective freeness)**.** *Let $[X, \omega]$ be a projective qlc pair such that ω is an $\mathbb{R}$-Cartier divisor and let M be a Cartier divisor on X such that $N = M - \omega$ is ample. Let $x \in X$ be a closed point. We assume that there are positive numbers $c(k)$ with the following properties.*

(1) *If $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety, then*
$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$
(2) *The numbers $c(k)$ satisfy the inequality:*
$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \leq 1.$$

Then $\mathscr{O}_X(M)$ has a global section not vanishing at x.

A key ingredient of the proof of Theorem 1.1 is the inversion of adjunction for qlc pairs (see Theorem 2.10). We will formulate and prove it in Section 2.

Remark 1.2. In Theorem 1.1, we have $H^1(X, \mathscr{I}_W \otimes \mathscr{O}_X(M)) = 0$, where W is the minimal qlc stratum of $[X, \omega]$ passing through x and $\mathscr{I}_W$ is the defining ideal sheaf of W on X (see Theorem 2.7). Therefore, the natural restriction map
$$H^0(X, \mathscr{O}_X(M)) \to H^0(W, \mathscr{O}_W(M))$$
is surjective. Thus, by replacing X with W, we can assume that X is irreducible in Theorem 1.1.

By suitably modifying the proof of Theorem 1.1, we can prove the following effective point separation for qlc pairs without any difficulties.

Theorem 1.3 (Effective point separation)**.** *Let $[X, \omega]$ be a projective qlc pair such that ω is an $\mathbb{R}$-Cartier divisor and let M be a Cartier divisor on X such that $N = M - \omega$ is ample. Let $x_1, x_2 \in X$ be two closed points. We assume that there are positive numbers $c(k)$ with the following properties.*

(1) *If $Z \subset X$ is an irreducible (positive dimensional) subvariety that contains x_1 or x_2, then*
$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$
(2) *The numbers $c(k)$ satisfy the inequality:*
$$\sum_{k=1}^{\dim X} 2^{1/k} \frac{k}{c(k)} \leq 1.$$

Then $\mathscr{O}_X(M)$ has a global section separating x_1 and x_2.

Remark 1.4. In Theorem 1.3, let W_1, W_2 be the minimal qlc stratum of $[X, \omega]$ passing through x_1, x_2 respectively. Possibly after switching x_1 and x_2, we can assume that $\dim W_1 \leq \dim W_2$. We put $W = W_1 \cup W_2$ with the reduced structure. Then, by adjunction, $[W, \omega|_W]$ has a natural quasi-log structure with only qlc singularities induced by $[X, \omega]$ (see Theorem 2.7). Moreover, we have $H^1(X, \mathscr{I}_W \otimes \mathscr{O}_X(M)) = 0$, where $\mathscr{I}_W$ is the defining ideal sheaf of W on X (see Theorem 2.7). Therefore, the natural restriction map

$$H^0(X, \mathscr{O}_X(M)) \to H^0(W, \mathscr{O}_W(M))$$

is surjective. Thus, as in Remark 1.2, we can replace X with W in Theorem 1.3.

By [Fuj14c, Theorem 1.1], we know that any quasi-projective semi-log canonical pair has a natural quasi-log structure with only qlc singularities, which is compatible with the original semi-log canonical structure. Therefore, Theorem 1.1 and Theorem 1.3 also hold for semi-log canonical pairs. For the precise statements, see Corollary 3.5 and Corollary 4.6. Our proof of Theorem 1.1 and Theorem 1.3 works very well in the category of quasi-log schemes. On the other hand, it does not seem to work well in the category of semi-log canonical pairs. This is one of the key points of formulating and proving the effective freeness and effective point separation for qlc pairs.

The paper is organized as follows. In Section 2, we recall some basic definitions and properties of quasi-log schemes. Then we formulate and prove the inversion of adjunction for qlc pairs, which will play a crucial role in this paper. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.3. The proof of Theorem 1.3 is essentially the same as the proof of Theorem 1.1.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. We note that a *scheme* means a separated scheme of finite type over $\mathbb{C}$. For the details of the theory of quasi-log schemes, see [Fuj14a, Chapter 6] and [Fuj14b]. We also note that [Fuj11a] is a gentle introduction to the theory of quasi-log schemes.

2. Inversion of adjunction

Let us recall the definition of quasi-log schemes, which was first introduced by Ambro in [Amb03]. For the details of the theory of quasi-log schemes, see [Fuj14a, Chapter 6].

Definition 2.1 ([Fuj14a, Chapter 6] and [Amb03, Definition 4.1])**.** A quasi-log scheme is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C_i\}$ of reduced and irreducible subschemes of X such that there exists a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$.
(2) The natural map $\mathscr{O}_X \to f_*\mathscr{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism
$$\mathscr{I}_{X_{-\infty}} \cong f_*\mathscr{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$
where $\mathscr{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C_i\}$ coincides with the images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

The morphism $f : (Y, B_Y) \to X$ is usually called a quasi-log resolution of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$. If $X_{-\infty} = \emptyset$, then we usually say that $[X, \omega]$ is a quasi-log canonical pair (a qlc pair, for short) or $[X, \omega]$ is a quasi-log scheme with only qlc singularities.

We note that X may be reducible and is not necessarily equidimensional in Definition 2.1 (see Example 2.5 below). We give some remarks on Definition 2.1.

Remark 2.2. Definition 2.1 may look slightly different from the definition in [Amb03]. In [Amb03], Ambro only assumes that (Y, B_Y) is an *embedded normal crossing pair* (see [Amb03, Section 2] and [Fuj14a, Chapter 5] for the definitions and examples of *embedded normal crossing pair* and see [Fuj14b, Appendix] for the difference between *embedded normal crossing pair* and *globally embedded simple normal crossing pair*). By [Fuj14a] and [Fuj14b], we see that Definition 2.1 is equivalent to the original definition in [Amb03].

Remark 2.3. By [Amb03, Remark 4.2], $[X, \omega]$ is a qlc pair if and only if the coefficients of B_Y are ≤ 1, that is, B_Y is a subboundary $\mathbb{R}$-divisor. In this case, we have $\mathscr{O}_X \cong f_*\mathscr{O}_Y(\lceil -(B_Y^{<1}) \rceil)$. In particular, we see that f is surjective and $\mathscr{O}_X \cong f_*\mathscr{O}_Y$. Therefore, f has connected fibers and X is seminormal. In particular, X is reduced.

Remark 2.4. The subvariety C_i in Definition 2.1 is called a *qlc stratum* of $[X, \omega]$. A *qlc center* of $[X, \omega]$ means a qlc stratum of $[X, \omega]$ which is not an irreducible component of X.

We give some examples of qlc pairs to see why the notion of qlc pairs is very important.

Example 2.5. Every log canonical pair (X, Δ) defines a natural quasi-log structure on $[X, K_X + \Delta]$ to make $[X, K_X + \Delta]$ a qlc pair. This idea played a crucial role in [KK10] to prove that log canonical pairs have only Du Bois singularities. Let $\{C_i\}_{i \in I}$ be the set of log canonical centers of (X, Δ). We put $W = \bigcup_{i \in J} C_i$ for any $\emptyset \neq J \subset I$ with the reduced structure. Then, by adjunction (see Theorem 2.7), $[W, (K_X + \Delta)|_W]$ has a natural quasi-log structure with only qlc singularities induced by $[X, K_X + \Delta]$.

Example 2.6. A quasi-projective semi-log canonical pair (X, Δ) has a natural quasi-log structure on $[X, K_X + \Delta]$ to make $[X, K_X + \Delta]$ a qlc pair. For the details, see [Fuj14c, Theorem 1.1].

The above examples show that we need the theory of quasi-log schemes to understand log canonial pairs and semi-log canonical pairs deeply.

The following theorem is one of the key results of the theory of quasi-log schemes, which heavily depends on the theory of mixed Hodge structures on cohomology with compact support.

Theorem 2.7 ([Amb03, Theorems 4.4 and 7.3] and [Fuj14a, Chapter 6])**.** *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

(1) (Adjunction). *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*

(2) (Vanishing theorem). *Assume that $\pi : X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathscr{I}_{X'} \otimes \mathscr{O}_X(L)) = 0$ for every $i > 0$, where $\mathscr{I}_{X'}$ is the defining ideal sheaf of X' on X.*

Note that an $\mathbb{R}$-Cartier divisor is called nef and log big over S with respect to $[X, \omega]$ if it is nef and big over S and big over S on every qlc stratum of $[X, \omega]$ when it is restricted to that stratum.

We will use the following two lemmas in the proof of Theorem 1.1 and Theorem 1.3.

Lemma 2.8. *Let $[X, \omega]$ be a qlc pair and B be an effective $\mathbb{R}$-Cartier divisor on X. Assume that* $\mathrm{Supp} B$ *contains no qlc centers of $[X, \omega]$. Then $[X, \omega + B]$ has a natural quasi-log structure induced by $[X, \omega]$.*

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution, where (Y, B_Y) is a globally embedded simple normal crossing pair. By taking further blow-ups, we can assume that $(Y, B_Y + f^*B)$ is a globally embedded simple normal crossing pair. We note that $B_Y^{=1}$ and $\mathrm{Supp} f^*B$ have no common components by the assumption. We have $K_Y + B_Y + f^*B \sim_{\mathbb{R}} f^*(\omega + B)$. We put

$$\begin{aligned}\mathscr{I}_{\mathrm{Nqlc}(X,\omega+B)} &= f_*\mathscr{O}_Y(\lceil -(B_Y + f^*B)^{<1}\rceil - \lfloor (B_Y + f^*B)^{>1}\rfloor) \\ &\subset f_*\mathscr{O}_Y(\lceil -(B_Y^{<1})\rceil) = \mathscr{O}_X.\end{aligned}$$

It gives the so-called ideal sheaf of the non-qlc locus $\mathrm{Nqlc}(X, \omega + B)$. By construction, there is a collection of subschemes $\{\overline{C}_i\}$ coincides with the image of $(Y, B_Y + f^*B)$-strata. Note that $\{\overline{C}_i\} \supset \{C_i\}$ by construction, where $\{C_i\}$ is the set of qlc strata of $[X, \omega]$. The above conditions give a natural quasi-log structure of $[X, \omega + B]$. □

Lemma 2.9 ([Fuj15, Lemma 4.6]). *Let $[X, \omega]$ be a qlc pair such that X is irreducible. Let B be an effective $\mathbb{R}$-Cartier divisor on X. Then $[X, \omega + B]$ has a natural quasi-log structure, which coincides with the original quasi-log structure of $[X, \omega]$ outside* $\mathrm{Supp}B$.

The following theorem was suggested by Fujino, which is one of the main ingredients of the proof of Theorem 1.1 and Theorem 1.3 just as [OT87, Theorem 1] playing a crucial role in [AS95] in the original analytic case and [Kaw07, Theorem] in [Fuj10] for log canonical pairs.

Theorem 2.10 (Inversion of Adjunction, Osamu Fujino). *Let $[X, \omega]$ be a qlc pair and B be an effective $\mathbb{R}$-Cartier divisor on X such that* $\mathrm{Supp}B$ *contains no qlc centers of $[X, \omega]$. Let X' be a union of some qlc strata of $[X, \omega]$, Then $[X, \omega + B]$ is qlc in a neighborhood of X' if and only if $[X', \omega|_{X'} + B|_{X'}]$ is qlc.*

Before we prove Theorem 2.10, we have an important remark.

Remark 2.11. In Theorem 2.10, $[X', \omega|_{X'}]$ is a qlc pair by adjunction (see Theorem 2.7). By assumption, we see that $B|_{X'}$ contains no qlc centers of $[X', \omega|_{X'}]$. Then, by Lemma 2.8, we have a natural quasi-log structure on $[X', \omega|_{X'} + B|_{X'}]$.

Proof of Theorem 2.10. We take a quasi-log resolution $f : (Z, \Delta_Z) \to X$, where (Z, Δ_Z) is a globally embedded simple normal crossing pair. By taking some suitable blow-ups, we may assume that the union of all strata of (Z, Δ_Z) mapped to X', which is denoted by Z', is a union of some irreducible components of Z. We put $(K_Z + \Delta_Z)|_{Z'} = K_{Z'} + \Delta_{Z'}$ and $Z'' = Z - Z'$. Without loss of generality, we may assume that $(Z, \Delta_Z + f^*B)$ is also a globally embedded simple normal crossing pair.

We put $\Theta_Z = \Delta_Z + f^*B$. By adjunction, $f : (Z', \Delta_{Z'}) \to X'$ is a quasi-log resolution of $[X', \omega|_{X'}]$. We put $K_{Z'} + \Theta_{Z'} = (K_Z + \Theta_Z)|_{Z'}$. First, we may assume that $[X, \omega + B]$ is qlc in a neighborhood of X'. Then $\Theta_{Z'}$ is a subboundary $\mathbb{R}$-divisor. By construction, $f : (Z', \Theta_{Z'}) \to X'$ is a quasi-log resolution of $[X', \omega|_{X'} + B|_{X'}]$. Therefore, $[X', \omega|_{X'} + B|_{X'}]$ is also qlc. Next, we assume that $[X, \omega+B]$ is not qlc in a neighborhood of X'. By replacing B with $(1-\varepsilon)B$ for $0 < \varepsilon \ll 1$, we may assume that $\Delta_Z^{=1} = (\Delta_Z + f^*B)^{=1} = \Theta_Z^{=1}$. Note that

$$\mathscr{I}_{\mathrm{Nqlc}(X, \omega+B)} = f_*\mathscr{O}_Z(\lceil -(\Theta_Z^{<1})\rceil - \lfloor \Theta_Z^{>1} \rfloor)$$

by definition. We put

$$\widetilde{X} = X' \cup \mathrm{Nqlc}(X, \omega + B)$$

and consider the quasi-log structure of $[\widetilde{X}, (\omega+B)|_{\widetilde{X}}]$ induced by $[X, \omega + B]$. Then, by adjunction, we obtain

$$\mathscr{I}_{\mathrm{Nqlc}(\widetilde{X}, (\omega+B)|_{\widetilde{X}})} = f_*\mathscr{O}_{Z'}(\lceil -(\Theta_{Z'}^{<1})\rceil - \lfloor \Theta_{Z'}^{>1} \rfloor).$$

We note that $\Delta_Z^{=1} = \Theta_Z^{=1}$ (see Remark 2.12 below). By assumption, $\mathscr{I}_{\mathrm{Nqlc}(\widetilde{X}, (\omega+B)|_{\widetilde{X}})}$ is nontrivial on X' because $\mathrm{Nqlc}(X, \omega + B) = \mathrm{Nqlc}(\widetilde{X}, (\omega + B)|_{\widetilde{X}})$. By construction, we can see that $f : (Z', \Theta_{Z'}) \to X'$ is also a quasi-log resolution of $[X', \omega|_{X'} + B|_{X'}]$. Therefore,

$$\mathscr{I}_{\mathrm{Nqlc}(X', \omega|_{X'} + B|_{X'})} = \mathscr{I}_{\mathrm{Nqlc}(\widetilde{X}, (\omega+B)|_{\widetilde{X}})}$$

is nontrivial. Thus, we obtain that $[X', \omega|_{X'} + B|_{X'}]$ is not qlc. □

Remark 2.12. In the proof of Theorem 2.10, we may assume that $\Delta_Z^{=1} = \Theta_Z^{=1}$ by replacing B with $(1-\varepsilon)B$ for $0 < \varepsilon \ll 1$. By this condition $\Delta_Z^{=1} = \Theta_Z^{=1}$, the union of all strata of (Z, Θ_Z) mapped to $\widetilde{X}$ is Z'. Therefore, $f : (Z', \Theta_{Z'}) \to \widetilde{X}$ is a quasi-log resolution of $[\widetilde{X}, (\omega + B)|_{\widetilde{X}}]$. This is a key point of the proof of Theorem 2.10.

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The main result of this section is as follows.

Proposition 3.1 ([Kol97, Theorem 6.4], [Fuj10, Proposition 2.1])**.** *Let $[X, \omega]$ be a projective qlc pair such that ω is an $\mathbb{R}$-Cartier divisor. Assume that X is irreducible. Let N be an ample $\mathbb{R}$-divisor on X and $x \in X$ be a closed point. Assume that there are positive numbers $c(k)$ for $1 \le k \le \dim X$ with the following properties.*

(1) *If $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety, then*

$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$

(2) *The numbers $c(k)$ satisfy the inequality:*

$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \le 1.$$

Then there is an effective $\mathbb{R}$-Cartier divisor $D \sim_{\mathbb{R}} cN$ with $0 \le c < 1$ and an open neighborhood $x \in X^0 \subset X$ such that

(i) *$[X^0, (\omega + D)|_{X^0}]$ is qlc, and*
(ii) *x is a qlc center of $[X^0, (\omega + D)|_{X^0}]$.*

Note that $[X, \omega + D]$ has a natural quasi-log structure by Lemma 2.9.

To prove this proposition, we need some preparations.

Lemma 3.2. *Let $[X, \omega]$ be an irreducible qlc pair and $x \in X$ be a general smooth point. Let B_x be an effective $\mathbb{R}$-Cartier divisor such that $\operatorname{mult}_x B_x > \dim_x X$. Then $[X, \omega + B_x]$ is not qlc at x.*

Proof. By Lemma 2.9, $[X, \omega + B_x]$ has a natural quasi-log structure. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. Since x is a general smooth point, we may assume that every stratum of $(Y, \operatorname{Supp} B_Y)$ is smooth over a nonempty Zariski open neighborhood U_x of x. We can assume that U_x is smooth. By taking a blow-up along an irreducible component E of $f^{-1}(x)$, we can directly check that $[X, \omega + B_x]$ is not qlc at x by $\operatorname{mult}_x B_x > \dim_x X$. □

Proposition 3.3. *Let X be a projective irreducible variety with $\dim X = n$. Let ω be an $\mathbb{R}$-Cartier divisor on X. Assume that there exists a nonempty Zariski open set $U \subset X$ such that $[U, \omega|_U]$ is a qlc pair. Let H be an ample $\mathbb{R}$-divisor on X such that $H^n > n^n$. Let x be a closed point of U such that no qlc centers of $[U, \omega|_U]$ contain x. Then there is an effective $\mathbb{R}$-Cartier divisor B_x on X such that $B_x \sim_{\mathbb{R}} H$ and that $[U, (\omega + B_x)|_U]$ is not qlc at x.*

Proof. Let us consider $X \times \mathbb{A}^1 \to \mathbb{A}^1$ and take a general irreducible curve C' on $X \times \mathbb{A}^1$ passing through $(x, 0) \in X \times \mathbb{A}^1$. Since C' is a general curve, $C' \to \mathbb{A}^1$ is finite. Let $\nu : C \to C'$ be the normalization. By taking the base change of $X \times \mathbb{A}^1 \to \mathbb{A}^1$ by $C \to C' \to \mathbb{A}^1$, we obtain $p_2 : X \times C \to C$. By construction, there exists a section $s : C \to X \times C$ of p_2 such that $s(C)$ is passing through $(x, 0) \in X \times C$ for some $0 \in C$. By [Fuj11b, Lemma 12.2] (in which it was assumed that a variety should be normal, but the normality is not used in the proof), we can find an effective $\mathbb{R}$-Cartier divisor B on $X \times C$ such that $B \sim_{\mathbb{R}} p_1^*H$, where $p_1 : X \times C \to X$ is the first projection and that $\mathrm{mult}_{s(C)} B > n$. By shrinking C, we can assume that B contains no fibers of p_2. By shrinking U, we can further assume that $[U, \omega|_U]$ contains no qlc centers. We consider the natural quasi-log structure on $[U \times C, p_1^*\omega|_U + p_2^*0 + B|_{U\times C}]$ by Lemma 2.8. Note that $p_2^*0 \cong U$ is a qlc center of this quasi-log structure. We assume that $[U, (\omega + B_x)|_U]$ is qlc at x, where $B_x = B|_{p_2^*0}$. By applying the inversion of adjunction (see Theorem 2.10) to $[U \times C, p_1^*\omega|_U + p_2^*0]$, $B|_{U\times C}$, and $p_2^*0 \cong U$, we see that $[U \times C, p_1^*\omega|_U + p_2^*0 + B|_{U\times C}]$ is qlc in a neighborhood of $(x, 0)$ since $[U, (\omega + B_x)|_U]$ is qlc at x. Then we obtain that $[U \times C, p_1^*\omega|_U + p_2^*0 + B|_{U\times C}]$ is qlc at $(s(t), t)$ if t is sufficiently close to $0 \in C$. Thus, by adjunction, $[U, (\omega + B|_{p_2^*t})|_U]$ is qlc at $s(t)$ if t is sufficiently close to $0 \in C$ and is general in C. On the other hand, $[U, (\omega + B|_{p_2^*t})|_U]$ is not qlc at $s(t)$ for general $t \in C$ by Lemma 3.2. This is a contradiction. Therefore, $[U, (\omega + B_x)|_U]$ is not qlc at x. This means that B_x is a desired effective $\mathbb{R}$-Cartier divisor. □

The following proposition was established for kawamata log terminal pairs in [Kol97, Theorem 6.7.1] and for log canonical pairs in [Fuj10, Proposition 2.7].

Proposition 3.4. *Let X be a projective irreducible variety and ω be an $\mathbb{R}$-Cartier divisor on X. Assume that there exists a nonempty Zariski open set $U \subset X$ such that $[U, \omega|_U]$ is a qlc pair. Let $x \in U$ be a closed point and Z be the minimal qlc stratum of $[U, \omega|_U]$ passing through x with $k = \dim Z > 0$. Let H be an ample $\mathbb{R}$-divisor on X such that $H^k \cdot \overline{Z} > k^k$, where $\overline{Z}$ is the closure of Z in X. Then there are an effective $\mathbb{R}$-Cartier divisor $B \sim_{\mathbb{R}} H$, a real number $0 < c < 1$, and an open neighborhood $x \in X^0 \subset U$ such that:*

1. *$[X^0, (\omega + cB)|_{X^0}]$ is qlc, and*
2. *there is a minimal qlc stratum Z_1 of $[X^0, (\omega + cB)|_{X^0}]$ passing through x with $\dim Z_1 < \dim Z$.*

Proof. Since $[U, \omega|_U]$ is a qlc pair, $[Z, \omega|_Z]$ has a qlc structure by adjunction (see Theorem 2.7). Note that Z is normal at x because Z is minimal. Of course, no qlc centers of $[Z, \omega|_Z]$ contain x. By Proposition 3.3, there is an effective $\mathbb{R}$-Cartier divisor $F_Z \sim_{\mathbb{R}} mH|_{\overline{Z}}$ such that $[Z, (\omega + \frac{1}{m}F_Z)|_Z]$ is not qlc at x. Furthermore, as in [Fuj11b, Lemma 12.2], we can assume that $H = H_1 + a_2H_2 + \cdots + a_tH_t$ where H_1 is an ample $\mathbb{Q}$-divisor such that $H_1^k \cdot \overline{Z} > k^k$, a_i is a positive real number and H_i is an ample Cartier divisor for every $i \geq 2$, and that $F_Z = F_1 + a_2F_2 + \cdots + a_tF_t$ with $F_i \sim_{\mathbb{Q}} mH_i|_{\overline{Z}}$ for every i. By replacing

m and F_Z with mk and kF_Z for some large positive integer k, we can take m as large as we want. Especially, for every i, we can find m such that mH_i is an ample Cartier divisor and that $F_i \sim mH_i|_{\overline{Z}}$ for every i. We may further assume that

$$H^1(X, \mathscr{I}_{\overline{Z}} \otimes \mathscr{O}_X(mH_i)) = 0$$

for every i by Serre's vanishing theorem, where $\mathscr{I}_{\overline{Z}}$ is the ideal sheaf of $\overline{Z}$ on X, and that $\mathscr{I}_{\overline{Z}} \otimes \mathscr{O}(mH_i)$ is globally generated for every i. By the following short exact sequence:

$$0 \to \mathscr{I}_{\overline{Z}} \otimes \mathscr{O}_X(mH_i) \to \mathscr{O}_X(mH_i) \to \mathscr{O}_{\overline{Z}}(mH_i) \to 0,$$

we obtain that the natural restriction map

$$H^0(X, \mathscr{O}_X(mH_i)) \to H^0(\overline{Z}, \mathscr{O}_{\overline{Z}}(mH_i))$$

is surjective. Therefore, we can take $D_i \in |mH_i|$ on X such that $D_i|_{\overline{Z}} = F_i$ for every i. We put $F = D_1 + a_2D_2 + \cdots + a_tD_t$. Then $F|_{\overline{Z}} = F_Z$ and $F \sim_{\mathbb{R}} mH$. Since Z is a minimal qlc stratum of $[U, \omega|_U]$ passing through x, in a neighborhood $x \in X^0 \subset U$, we may assume that Supp$F$ contains no qlc centers of $[U, \omega|_U]$. By the inversion of adjunction (see Theorem 2.10), $[X^0, (\omega + \frac{1}{m}F)|_{X^0}]$ is not qlc at x. Since we assumed that $\mathscr{I}_{\overline{Z}} \otimes \mathscr{O}_X(mH_i)$ is globally generated for every i, by choosing D_i general for every i, we obtain that $[X^0, (\omega + \frac{1}{m}F)|_{X^0}]$ is qlc on $X^0 \setminus Z$. By the above argument, we have constructed an $\mathbb{R}$-Cartier divisor $F \sim_{\mathbb{R}} mH$ on X such that

(1) $F|_{\overline{Z}} = F_Z$;
(2) $[X^0, (\omega + \frac{1}{m}F)|_{X^0}]$ is qlc on $X^0 \setminus Z$;
(3) $[X^0, (\omega + \frac{1}{m}F)|_{X^0}]$ is qlc at the generic point of Z;
(4) $[X^0, (\omega + \frac{1}{m}F)|_{X^0}]$ is not qlc at $x \in Z$.

We put $B = \frac{1}{m}F$. Let c be the maximal real number such that $[X^0, (\omega + cB)|_{X^0}]$ is qlc at x. Then, after shrinking X^0 further, we have a new minimal qlc center Z_1 of $[X^0, (\omega + cB)|_{X^0}]$ passing through x. Note that Z (after restriction) is also a qlc center in this new qlc structure too. Therefore, we have that $x \in Z_1 \subset Z$ and $\dim Z_1 < \dim Z$. □

Now, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let Z_1 be the minimal qlc stratum of $[X, \omega]$ passing through x. If $\dim Z_1 = 0$, then it is done. If $\dim Z_1 > 0$, then, by Proposition 3.4, we can find $x \in D_1 \sim_{\mathbb{R}} \frac{k_1}{c(k_1)}N$, $0 < c_1 < 1$ and an open neighborhood X^0 of x such that $[X^0, (\omega + c_1D_1)|_{X^0}]$ is qlc and $k_2 = \dim Z_2 < k_1$ where Z_2 is the minimal qlc strutum of $[X^0, (\omega + c_1D_1)|_{X^0}]$ passing through x. By Lemma 2.9, we can consider the natural quasi-log structure $[X, \omega + c_1D_1]$, which coincides with the one of $[X^0, (\omega+c_1D_1)|_{X^0}]$ when restricting to X^0. Repeat this argument by Proposition 3.4. Finally, we get a sequence $\dim X \geq k_1 > k_2 > \cdots > k_t > 0$, where $k_i \in \mathbb{Z}$ with the following properties:

(1) there is an effective $\mathbb{R}$-Cartier divisor $x \in D_i \sim_{\mathbb{R}} \frac{k_i}{c(k_i)}N$ for every i;
(2) there is a real number $0 < c_i < 1$ for every i;
(3) $[X, \omega + \sum_{i=1}^{t} c_iD_i]$ is qlc in a neighborhood X^0 of x;
(4) x is a qlc center of $[X, \omega + \sum_{i=1}^{t} c_iD_i]$.

We put $D = \sum_{i=1}^{t} c_iD_i$. Then D has the desired properties. Note that $0 \leq c = \sum_{i=1}^{t} c_i\frac{k_i}{c(k_i)} < 1$ and $D \sim_{\mathbb{R}} cN$. □

Proof of Theorem 1.1. By Remark 1.2, we may assume that X is irreducible. Let D be an $\mathbb{R}$-Cartier divisor constructed in Proposition 3.1. By Lemma 2.9, we still have a natural quasi-log structure on $[X, \omega + D]$. Then, by the construction of X^0 in Proposition 3.1, x is still a qlc center of this quasi-log structure on $[X, \omega + D]$ and is disjoint from $\mathrm{Nqlc}(X, \omega + D)$. We consider $\widetilde{X} = x \cup \mathrm{Nqlc}(X, \omega + D)$. By adjunction (see Theorem 2.7), $\widetilde{X}$ has a quasi-log structure induced by $[X, \omega + D]$. Now we have the following short exact sequence:

$$0 \to \mathscr{I}_{\widetilde{X}} \to \mathscr{O}_X \to \mathscr{O}_{\widetilde{X}} \to 0.$$

Since $M - (\omega + D) \sim_{\mathbb{R}} (1 - c)N$ is ample, $H^1(X, \mathscr{I}_{\widetilde{X}} \otimes \mathscr{O}_X(M)) = 0$ by the vanishing theorem in Theorem 2.7. Therefore, the natural restriction map

$$H^0(X, \mathscr{O}_X(M)) \to H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(M))$$

is surjective. Note that $x \cap \mathrm{Nqlc}(X, \omega + D) = \emptyset$, Thus we obtain that

$$H^0(X, \mathscr{O}_X(M)) \to H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(M)) \to H^0(x, \mathscr{O}_{\widetilde{X}}(M)) = \mathscr{O}_{\widetilde{X}}(M) \otimes \mathbb{C}(x)$$

is surjective. This is what we wanted. □

As a direct consequence of Theorem 1.1, we have:

Corollary 3.5 (Effective freeness for semi-log canonical pairs)**.** *Let (X, Δ) be a projective semi-log canonical pair. Let M be a Cartier divisor on X such that $N = M - (K_X + \Delta)$ is ample. Let $x \in X$ be a closed point. We assume that there are positive numbers $c(k)$ with the following properties.*

(1) *If* $x \in Z \subset X$ *is an irreducible (positive dimensional) subvariety, then*
$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$
(2) *The numbers* $c(k)$ *satisfy the inequality:*
$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \leq 1.$$

Then $\mathscr{O}_X(M)$ *has a global section not vanishing at* x.

Proof. By [Fuj14c, Theorem 1.1], we see that $[X, K_X+\Delta]$ has a natural quasi-log structure with only quasi-log canonical singularities, which is compatible with the original semi-log canonical structure of (X, Δ). For the details, see [Fuj14c]. Then, by Theorem 1.1, we know that $\mathscr{O}_X(M)$ has a global section not vanishing at x. □

4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. By Remark 1.4, we may assume that $X = W_1 \cup W_2$, where W_1 (resp. W_2) is the minimal qlc stratum of $[X, \omega]$ passing through x_1 (resp. x_2) with $\dim W_1 \leq \dim W_2$. We put $V = W_1 \cap W_2$. Then we obtain that either $V \subsetneq W_2$ or $V = W_2$. If $V \subsetneq W_2$, then V is a union of qlc centers of W_2 and $x_2 \notin V$ by [Amb03, Proposition 4.8] (see also [Fuj14a]).

First, we slightly generalize Proposition 3.4 as follows.

Proposition 4.1. *Let* X *be a projective irreducible variety and* ω *be an* $\mathbb{R}$*-Cartier divisor on* X. *Assume that there exists a nonempty Zariski open set* $U \subset X$ *such that* $[U, \omega|_U]$ *is a qlc pair. Let* $x \in U$ *be a closed point and* Z *be the minimal qlc stratum of* $[U, \omega|_U]$ *passing through* x *with* $k = \dim Z > 0$. *Let* V *be a qlc center of* $[U, \omega|_U]$ *disjoint from* x. *Let* H *be an ample* $\mathbb{R}$*-divisor on* X *such that* $H^k \cdot \overline{Z} > k^k$, *where* $\overline{Z}$ *is the closure of* Z *in* X. *Then there are an effective* $\mathbb{R}$*-Cartier divisor* $B \sim_{\mathbb{R}} H$, *a real number* $0 < c < 1$, *and an open neighborhood* $x \in X^0 \subset U$ *such that:*

(1) $[X^0, (\omega + cB)|_{X^0}]$ *is qlc, and*
(2) *there is a minimal qlc stratum* Z_1 *of* $[X^0, (\omega + cB)|_{X^0}]$ *passing through* x *with* $\dim Z_1 < \dim Z$.
(3) V *is still a qlc stratum of* $[X^0, (\omega+cB)|_{X^0}]$ *disjoint from* x *after restrition.*

Proof. In the proof of Proposition 3.4, we can choose F so general that V is not contained in Supp$B$, where $B = \frac{1}{m}F$. Then, by Lemma 2.9, V is still a qlc center of the new quasi-log structure of $[X^0, (\omega + cB)|_{X^0}]$. □

Following the same line as in Section 3, we have a generalization of Proposition 3.1. The proof of Proposition 4.2 is obvious.

Proposition 4.2. *Let $[X, \omega]$ be a projective qlc pair such that ω is an $\mathbb{R}$-Cartier divisor. Assume that X is irreducible. Let N be an ample $\mathbb{R}$-divisor on X and $x \in X$ be a closed point. Let V be a qlc center of $[X, \omega]$ disjoint from x. Assume that there are positive numbers $c(k)$ for $1 \le k \le \dim X$ with the following properties.*

(1) *If $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety, then*
$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$
(2) *The numbers $c(k)$ satisfy the inequality:*
$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \le 1.$$

Then there is an effective $\mathbb{R}$-Cartier divisor $D \sim_{\mathbb{R}} cN$ with $0 \le c < 1$ and an open neighborhood $x \in X^0 \subset X$ such that

(i) *$[X^0, (\omega + D)|_{X^0}]$ is qlc, and*
(ii) *x is a qlc center of $[X^0, (\omega + D)|_{X^0}]$.*
(iii) *V is still a qlc center of $[X^0, (\omega + D)|_{X^0}]$ disjoint from x after restriction.*

Note that $[X, \omega + D]$ has a natural quasi-log structure by Lemma 2.9 and V is a qlc center of $[X, \omega + D]$.

We give a proof of Theorem 1.3 when $W_1 \cap W_2 \subsetneq W_2$.

Proof of Theorem 1.3 when $V = W_1 \cap W_2 \subsetneq W_2$. This proof is essentially the same as the proof of Theorem 1.1. Here, we will prove that $\mathscr{I}_{W_1} \otimes \mathscr{O}_X(M)$ has a global section not vanishing at x_2. Note that such a section obviously separates x_1 and x_2. Since we have a natural isomorphism
$$H^0(X, \mathscr{I}_{W_1} \otimes \mathscr{O}_X(M)) \cong H^0(W_2, \mathscr{I}_V \otimes \mathscr{O}_{W_2}(M))$$
by $X = W_1 \cup W_2$ and $V = W_1 \cap W_2$, we only need to prove that $\mathscr{I}_V \otimes \mathscr{O}_{W_2}(M)$ has a global section not vanishing at x_2. Let us consider the qlc pair $[W_2, \omega|_{W_2}]$ by adjunction. We take an $\mathbb{R}$-Cartier divisor D on W_2 constructed as in Proposition 4.2. Then x_2 is a qlc center of the induced new quasi-log structure on $[W_2, \omega|_{W_2} + D]$ and is disjoint from $\mathrm{Nqlc}(W_2, \omega|_{W_2} + D)$. Note that V is still a qlc center of this new quasi-log structure. We put $\widetilde{W} = x_2 \cup V \cup \mathrm{Nqlc}(W_2, \omega|_{W_2} + D)$. By

adjunction (see Theorem 2.7), $\widetilde{W}$ has a quasi-log structure induced by $[W_2, \omega|_{W_2} + D]$. Now we have the following short exact sequence:

$$0 \to \mathscr{I}_{\widetilde{W}} \to \mathscr{O}_{W_2} \to \mathscr{O}_{\widetilde{W}} \to 0.$$

Since $M - (\omega + D) \sim_{\mathbb{R}} (1-c)N$ is ample, $H^1(W_2, \mathscr{I}_{\widetilde{W}} \otimes \mathscr{O}_{W_2}(M)) = 0$ by the vanishing theorem in Theorem 2.7. Therefore, the natural restriction map

$$H^0(W_2, \mathscr{O}_{W_2}(M)) \to H^0(\widetilde{W}, \mathscr{O}_{\widetilde{W}}(M))$$

is surjective. We put $\widetilde{V} = \mathrm{Nqlc}(X, \omega + D) \cup V$. Then $x_2 \cap \widetilde{V} = \emptyset$, Thus we obtain that

$$H^0(W_2, \mathscr{O}_{W_2}(M)) \to H^0(\widetilde{W}, \mathscr{O}_{\widetilde{W}}(M)) = H^0(\widetilde{V}, \mathscr{O}_{\widetilde{V}}(M)) \oplus \mathbb{C}(x_2)$$

is surjective. By taking a pull-back of $0 \oplus 1$, we get what we wanted. □

Remark 4.3. In the case when $V \subsetneq W_2$, the assumptions in Theorem 1.3 can be replaced as follows.

(1) If $x_2 \in Z \subset X$ is an irreducible (positive dimensional) subvariety, then
$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$
(2) The numbers $c(k)$ satisfy the inequality:
$$\sum_{k=1}^{\dim W_2} \frac{k}{c(k)} \leq 1.$$

This is obvious by the above proof of Theorem 1.3 for the case when $V \subsetneq W_2$.

From now on, we treat the case when $V = W_1 = W_2$.

Proposition 4.4. *Let X be a projective irreducible variety with* $\dim X = n$. *Let ω be an $\mathbb{R}$-Cartier divisor on X. Assume that there exists a nonempty Zariski open set $U \subset X$ such that $[U, \omega|_U]$ is a qlc pair. Let H be an ample $\mathbb{R}$-divisor on X such that $H^n > 2n^n$. Let x_1, x_2 be two closed points of U such that no qlc centers of $[U, \omega|_U]$ contain x_1, x_2. Then there is an effective $\mathbb{R}$-Cartier divisor B_{x_1,x_2} on X such that $B_{x_1,x_2} \sim_{\mathbb{R}} H$ and that $[U, (\omega + B_{x_1,x_2})|_U]$ is not qlc at x_1, x_2.*

Proof. As in the proof of Proposition 3.3, we can take a smooth pointed affine curve $0 \in C$ such that the second projection $p_2 : X \times C \to C$ has two sections s_1 and s_2, $s_1(C)$ (resp. $s_2(C)$) is passing through $(x_1, 0)$ (resp. $(x_2, 0)$) on $X \times C$, and $p_1(s_i(C))$ is a general curve on X for $i = 1, 2$, where p_1 is the first projection $X \times C \to X$. Then we can find an effective $\mathbb{R}$-Cartier divisor B on $X \times C$ such that $B \sim_{\mathbb{R}} p_1^* H$

and $\mathrm{mult}_{s_i(C)} B > n$ for $i = 1, 2$. The same arguments as in the proof of Proposition 3.3 produce a desired effective $\mathbb{R}$-Cartier divisor B_{x_1,x_2} on X. We leave the details as an exercise for the reader. □

Thus we can generalize Proposition 3.4 to the following one without any difficulties.

Proposition 4.5. *Let X be a projective irreducible variety and ω be an $\mathbb{R}$-Cartier divisor on X. Assume that there exists a nonempty Zariski open set $U \subset X$ such that $[U, \omega|_U]$ is a qlc pair. Let $x_1, x_2 \in U$ be two closed points and Z be the common minimal qlc stratum of $[U, \omega|_U]$ passing through x_1, x_2 with $k = \dim Z > 0$. Let H be an ample $\mathbb{R}$-divisor on X such that $H^k \cdot \overline{Z} > 2k^k$, where $\overline{Z}$ is the closure of Z in X. Then there are an effective $\mathbb{R}$-Cartier divisor $B \sim_{\mathbb{R}} H$, a real number $0 < c < 1$, and an open neighborhood $x_1, x_2 \in X^0 \subset U$ such that:*

1. *$[X^0, (\omega + cB)|_{X^0}]$ is qlc at one point of x_1, x_2, say at x_1,*
2. *there is a minimal qlc stratum Z_1 of $[X^0, (\omega + cB)|_{X^0}]$ passing through x_1 with $\dim Z_1 < \dim Z$, and*
3. *$[U, (\omega + B)|_U]$ is not qlc at x_2 or there exists a qlc center of $[U, (\omega + B)|_U]$ passing through x_2.*

Proof of Theorem 1.3 when $V = W_1 = W_2$. We use Proposition 4.5 to cut down $X = W_1 = W_2$. Then we obtain $[X, \omega + cB]$, where $0 < c < 1$ is a real number, such that:

1. B is an effective $\mathbb{R}$-Cartier divisor such that $B \sim_{\mathbb{R}} \frac{2^{1/k}k}{c(k)} N$, where $k = \dim X$.
2. $[X, \omega + cB]$ is qlc at one point of x_1, x_2, say at x_1,
3. there is a minimal qlc stratum $\overline{Z_1}$ of $[X, \omega + cB]$ passing through x_1 with $\dim \overline{Z_1} < \dim Z$, and
4. $[X, \omega + cB]$ is not qlc at x_2 or there is a qlc center of $[X, \omega + cB]$ passing through x_2.

If $[X, \omega + cB]$ is qlc at both x_1 and x_2 and if x_1 and x_2 are still stay on the same minimal qlc center of $[X, \omega + cB]$, then we apply Proposition 4.5 again. By repeating this process finitely many times, we will obtain the situation where there is a suitable effective $\mathbb{R}$-Cartier divisor B', such that $[X, \omega + B']$ is not qlc at one of x_1 and x_2, or x_1 and x_2 are on different minimal qlc centers of $[X, \omega + cB]$. Then we will go to the first case proved by Proposition 4.1 (and Section 3). We leave the details as an exercise for the reader. Thus we get what we want. □

As a corollary of Theorem 1.3, we have:

Corollary 4.6 (Effective point separation for semi-log canonical pairs)**.** *Let (X, Δ) be a projective semi-log canonical pair. Let M be a Cartier* *divisor on X such that $N = M - (K_X + \Delta)$ is ample. Let $x_1, x_2 \in X$ be two closed points. We assume that there are positive numbers $c(k)$ with the following properties.*

(1) *If $Z \subset X$ is an irreducible (positive dimensional) subvariety that contains x_1 or x_2, then*

$$N^{\dim Z} \cdot Z > c(\dim Z)^{\dim Z}.$$

(2) *The numbers $c(k)$ satisfy the inequality:*

$$\sum_{k=1}^{\dim X} 2^{1/k} \frac{k}{c(k)} \leq 1.$$

Then $\mathscr{O}_X(M)$ has a global section separating x_1 and x_2.

Proof. See the proof of Corollary 3.5. □

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