PERIODIC GROUPS FROM MINIMAL ACTIONS OF THE INFINITE DIHEDRAL GROUP

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Abstract. We give an explicit construction transforming an arbitrary minimal non-free action of the infinite dihedral group on the Cantor set into an orbit-equivalent action of a finitely generated amenable periodic group. In particular, we construct first examples of simple infinite finitely generated amenable periodic groups.

1. Introduction

Groups of Burnside type (infinite finitely generated periodic groups) are important examples for the theory of amenable groups. They are never obviously non-amenable, since they do not contain free subgroups. They are also never elementary amenable, see [3, Theorem 2.3]. Note that the fact that the class of groups without free subgroups and the class of elementary amenable groups are distinct is proved in [3] precisely using the existence of infinite finitely generated periodic groups.

Periodic groups were the first examples to show that neither class (groups without free subgroups and elementary amenable groups) does not coincide with the class of amenable groups. They were the first examples of non-amenable groups without free subgroups (free Burnside groups and Tarski monsters, see [17, 1]), and the first example of a non-elementary amenable group (the Grigorchuk group [6, 7]).

New examples of non-elementary amenable groups were discovered recently as groups naturally associated with dynamical systems. It was shown by K. Juschenko and N. Monod [9] that the *topological full group* of a minimal homeomorphism of the Cantor set is amenable. Here the topological full group of a (cyclic in this case) group G acting on a Cantor set $\mathcal{X}$ is the group of all homeomorphisms $h : \mathcal{X} \longrightarrow \mathcal{X}$ such that for every $\zeta \in \mathcal{X}$ there exists a neighborhood U of ζ and an element $g \in G$ such that $g|_U = h|_U$. An action of a group is said to be *minimal* if all its orbits are dense. It was proved earlier by H. Matui [10] that if τ is a minimal homeomorphism of the Cantor set, then the topological full group of the cyclic group generated by τ has simple derived subgroup, and if the homeomorphism is expansive, then the derived subgroup is finitely generated. This provides, by the results of K. Juschenko and N. Monod, the first examples of infinite simple finitely generated amenable groups.

The aim of this paper is to show that minimal dynamical systems can be also used to construct simple amenable periodic groups. In particular, we construct the first examples of simple amenable groups of Burnside type (previous examples of simple periodic groups—Olshanskii-Tarski monsters—are non-amenable).

We show that any minimal non-free action of the infinite dihedral group on a Cantor set can be modified to produce a periodic group.

Namely, let a and b be homeomorphisms of a Cantor set $\mathcal{X}$ such that $a^2 = b^2 = 1$, the group generated by a and b acts minimally on $\mathcal{X}$, and b has a fixed point ξ. Choose a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ of $\mathcal{X} \setminus \{\xi\}$ into a finite number of disjoint b-invariant open subsets such that the closure of each set P_i contains ξ. Consider now the homeomorphisms b_i of $\mathcal{X}$ acting as b on P_i and identically on $\mathcal{X} \setminus P_i$. The homeomorphisms b_i generate a group K isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Consider a subgroup $H < K$ not containing b, but such that for every i there exists $g_i \in H$ acting as b on P_i.

Theorem 1.1. *The group $G_{\mathcal{P},H}$ generated by $H \cup \{a\}$ is periodic. Moreover, the topological full group of $G_{\mathcal{P},H}$ is periodic and amenable.*

We prove amenability of the full group by embedding it into a topological full group of a minimal homeomorphism of a Cantor set.

It is shown in [16] that if the action of a group G on a Cantor set is expansive, then the topological full group contains an infinite simple finitely generated subgroup.

Our procedure of building a periodic group from a dihedral group is very close to the original construction of the Grigorchuk's group. In fact, the Grigorchuk group is defined in the original paper [5] in a way that is very similar to the above description of $G_{\mathcal{P},H}$, and is particular case of it. The original proof of periodicity of the Grigorchuk group, however, used self-similarity of the action on the Cantor set in a very essential way. Our proof of periodicity is very "soft": it uses only the large-scale structure of the orbits of the dihedral group and elementary properties of minimal group actions. This makes it possible to generalize periodicity of the Grigorchuk group to a very wide class of groups of dynamical origin. Orbital graphs of the Grigorchuk groups were studied in great detail by Y. Vorobets in [19]; a large part of our construction is based on his results.

2. Preliminaries on group actions

2.1. **Graphs of actions.** All graphs in this section are oriented, loops and multiple edges are allowed. Their edges (and sometimes vertices) are labeled. Distances between vertices in such graphs are measured ignoring the orientation. Similarly, connectedness and connected components also are defined ignoring the orientation. Isomorphisms must preserve orientation and labeling. A graph is called *rooted* if one vertex, called the *root*, is marked. Every morphism of rooted graphs must map the root to the root.

We denote a ball of radius r with center in a vertex v of a graph Γ by $B_v(r)$. It is considered to be a rooted graph (with root v). Its set of edges is the set of all edges of Γ connecting the vertices of $B_v(r)$. The orientation and labeling are inherited from Γ.

Let G be a group generated by a finite set S and acting by homeomorphisms on a compact metric space $\mathcal{X}$. For $\zeta \in \mathcal{X}$, the *orbital graph* Γ_ζ is the graph with the set of vertices equal to the orbit $G\zeta$ of ζ, in which for every $\eta \in G\zeta$ and $s \in S$ there is an arrow from η to $s(\eta)$ labeled by s.

The graph Γ_ζ is naturally isomorphic to the *Schreier graph* of the group G modulo the stabilizer G_ζ of ζ. The Schreier graph of G modulo a subgroup H is, by definition, the graph with the set of vertices equal to the set of cosets gH, $g \in G$, in which for every coset gH and every generator $s \in S$ there is an arrow from gH to sgH labeled by s.

Denote by $G_{(\zeta)}$ the subgroup of G consisting of all elements $g \in G$ such that ζ is an interior point of the set of fixed points of g. The *graph of germs* $\tilde{\Gamma}_\zeta$ is the Schreier graph of G modulo $G_{(\zeta)}$. Note that $G_{(\zeta)}$ is a normal subgroup of G_ζ, hence the map $hG_{(\zeta)} \mapsto hG_\zeta$ induces a Galois covering of graphs $\tilde{\Gamma}_\zeta \longrightarrow \Gamma_\zeta$ with the group of deck transformations $G_\zeta/G_{(\zeta)}$.

The vertices of $\tilde{\Gamma}_\zeta$ are identified with *germs* of elements of G at ζ. Here a germ is an equivalence class of a pair (g, ζ), where two pairs (g_1, ζ) and (g_2, ζ) are equivalent if there exists a neighborhood U of ζ such that $g_1|_U = g_2|_U$.

Definition 2.1. A point $\zeta \in \mathcal{X}$ is said to be *G-regular*, if $G_{(\zeta)} = G_\zeta$, i.e., if every element $g \in G$ fixing ζ fixes pointwise a neighborhood of ζ.

Note that for every $g \in G$ the set of points $\zeta \in \mathcal{X}$ such that $g(\zeta) = \zeta$ but $g \notin G_{(\zeta)}$ is equal to the boundary of the set of fixed points of g. It follows that this set is closed and nowhere dense. Consequently, if G is countable (in particular, if G is finitely generated), the set of G-regular points is co-meager (residual).

Note also that $gG_\zeta g^{-1} = G_{g(\zeta)}$ and $gG_{(\zeta)}g^{-1} = G_{(g(\zeta))}$ for all $\zeta \in \mathcal{X}$ and $g \in G$, which implies that the set of G-regular points is G-invariant.

Let (Γ_1, v_1), (Γ_2, v_2) be connected labeled graphs with marked vertices (roots). Define the distance $d((\Gamma_1, v_1), (\Gamma_2, v_2))$ between them as $2^{-(R+1)}$, where R is the maximal integer such that the balls $B_{v_1}(R) \subset \Gamma_1$ and $B_{v_2}(R) \subset \Gamma_2$ of radius R with centers in v_1 and v_2 are isomorphic as rooted graphs. Define the metric to be equal to 1 if such R does not exist. This metric defines a natural topology on the space $\mathcal{G}$ of all isomorphism classes of connected oriented rooted labeled graphs. If we fix a finite set of labels, then the space of all such graphs is a compact space.

Definition 2.2. The action of G on $\mathcal{X}$ is said to be *minimal* if all G-orbits are dense in $\mathcal{X}$.

Proposition 2.1. *Suppose that the action of G on $\mathcal{X}$ is minimal. Let $\zeta \in \mathcal{X}$ be a G-regular point. Then for every ball $B_\xi(r)$ of Γ_ζ there exists a number $R > 0$ such that for every $\eta \in \mathcal{X}$ there exists a vertex η' of Γ_η such that $d(\eta, \eta') \le R$ and the rooted balls $B_\xi(r)$ and $B_{\eta'}(r)$ are isomorphic.*

Proof. The ball $B_\xi(r)$ can be described by a finite system of equations and inequalities of the form $g_1(\xi) = g_2(\xi)$ or $g_1(\xi) \ne g_2(\xi)$, for pairs of elements $g_1, g_2 \in G$ of length at most r. Since the set of G-regular points is G-invariant, the point ξ is G-regular, hence each such equality or inequality is valid for all points of some neighborhood of ξ. It follows that there exists a neighborhood N of ξ such that for every $\eta' \in N$ the balls $B_\xi(r)$ and $B_{\eta'}(r)$ of the corresponding orbital graphs are isomorphic as rooted graphs.

For every point $\eta \in \mathcal{X}$ there exists an element $g \in G$ such that $g(\eta) \in N$. The set of sets of the form $g^{-1}(N)$ cover $\mathcal{X}$, and by compactness there exists a finite subcover $g_1^{-1}(N), g_2^{-1}(N), \ldots, g_n^{-1}(N)$. Let R be the maximal length of the elements g_i with respect to the generating set S. Then for every $\eta \in \mathcal{X}$ there exists g_i such that $g_i(\eta) \in N$, and hence the balls $B_\xi(r)$ and $B_{\eta'}(r)$ are isomorphic for $\eta' = g_i(\eta)$. Distance from η to η' is not more than R. □

Suppose that $\mathcal{X}$ is a Cantor set. Fix a partition $\mathcal{V} = \{V_i\}_{i=1,\ldots,s}$ of $\mathcal{X}$ into a disjoint union of a finite number of clopen sets. We can label in the graphs Γ_ζ and $\tilde{\Gamma}_\zeta$ the vertices according to the partition $\mathcal{V}$: a vertex $g(\zeta)$ or (g, ζ) is labeled by

V_i if $g(\zeta) \in V_i$. It is easy to see that Proposition 2.1 remains to be true for such vertex labeled graphs, since labels of vertices are locally constant on $\mathcal{X}$.

2.2. **Topological full groups.** Let G be a group acting on a Cantor set $\mathcal{X}$. The *topological full group* $\mathsf{F}(G, \mathcal{X})$ of the action is the group of all homeomorphisms $h : \mathcal{X} \longrightarrow \mathcal{X}$ such that for every $\zeta \in \mathcal{X}$ there exists a neighborhood U of ζ and an element $g \in G$ such that $h|_U = g|_U$. Topological full groups were introduced in [4]. See the papers [10, 12, 13, 11] for various properties of topological full groups of group actions and étale groupoids.

Let $U \subset \mathcal{X}$ be a clopen set, and let $g_1, g_2, \ldots, g_n \in G$ be such that the sets $U_1 = g_1(U), U_2 = g_2(U), \ldots, U_n = g_n(U)$ are pairwise disjoint. Then for every permutation $\alpha \in \mathsf{S}_n$ we get the corresponding element h_α of the topological full group acting by the rule:

$$h_\alpha(\zeta) = \begin{cases} g_j g_i^{-1}(\zeta) & \text{if } \zeta \in U_i \text{ and } \alpha(i) = j; \\ \zeta & \text{if } \zeta \notin \bigcup_{i=1}^n U_i. \end{cases}$$

The map $\alpha \longrightarrow h_\alpha$ is a monomorphism from S_n to $\mathsf{F}(G)$. Denote by $\mathsf{S}(G, \mathcal{X})$ the subgroup of $\mathsf{F}(G, \mathcal{X})$ generated by images of such monomorphisms for all possible choices of U and g_i. Similarly, denote by $\mathsf{A}(G, \mathcal{X})$ the subgroup generated by images of the alternating subgroups $\mathsf{A}_n < \mathsf{S}_n$ for all such monomorphisms.

The following is proved in [16].

Theorem 2.2. *If the action of G on $\mathcal{X}$ is minimal, then $\mathsf{A}(G, \mathcal{X})$ is simple. If the action of G on $\mathcal{X}$ is expansive and has infinite orbits, then $\mathsf{A}(G, \mathcal{X})$ is finitely generated.*

Here an action of G on $\mathcal{X}$ is said to be expansive if there exists $\delta > 0$ such that $d(g(\zeta_1), g(\zeta_2)) < \delta$ for all $g \in G$ implies $\zeta_1 = \zeta_2$ (where d is a metric on $\mathcal{X}$ compatible with the topology). An action on a Cantor set $(G, \mathcal{X})$ is expansive if and only if there exists a G-equivariant homeomorphism from $\mathcal{X}$ to a closed G-invariant subset of A^G for some finite alphabet A.

3. Perturbations of dihedral groups

3.1. **Construction.** Let a, b be involutive homeomorphisms of a Cantor set $\mathcal{X}$ such that the dihedral group $\langle a, b\rangle$ acts minimally on $\mathcal{X}$, and b has a fixed point $\xi \in \mathcal{X}$.

If a set of generators S of a group G consists of elements of order two, then we will consider the orbital graphs and graphs of germs as non-oriented, so that an edge connecting two vertices v_1 and v_2 labeled by $s \in S$ replaces two arrows labeled by s: one from v_1 to v_2 and one from v_2 to v_1 (if the edge is not a loop).

Lemma 3.1. *The orbital graphs of $\langle a, b\rangle$ are either one-ended or two-ended infinite chains. The graphs of germs are two-ended infinite chains. The orbital graphs of regular points are two-ended infinite chains.*

Proof. Since the action is minimal, its orbits are infinite. The Schreier graphs of the infinite dihedral group D_∞ are either infinite chains (one-ended or two-ended), or finite chains, or finite cycles. The latter two cases are impossible, since then we have a finite orbit.

Suppose that a graph of germs is a one-ended infinite chain. Then the endpoint of the chain is a fixed point of one of the generators. Since this is a graph of germs, it follows that the generator fixes this point together with every point of a neighborhood. But then, by minimality, there are other points of the orbit where the germ of the generator is trivial, which is a contradiction.

Orbital graphs of generic points coincide with their graphs of germs, therefore they are two-ended chains. □

Lemma 3.2. *For every $n \geq 1$ there exists a partition of $\mathcal{X} \setminus \{\xi\}$ into a disjoint union of open b-invariant subsets $P_1, P_2, \ldots, P_n$ such that each set P_i accumulates on ξ.*

Proof. Let $U_k, k \geq 0$, be a descending sequence of clopen neighborhoods of ξ such that $U_0 = \mathcal{X}$ and $\bigcap_{k\geq 0} U_k = \{\xi\}$. For example, one can take U_n to be equal to the ball of radius $1/n$ for some ultrametric compatible with the topology on $\mathcal{X}$.

Then $V_k = U_k \cap b(U_k)$ is a descending sequence of clopen b-invariant neighborhoods of ξ such that $\bigcap_{k\geq 1} V_k = \{\xi\}$. Remove all repetitions, so that $V_k \neq V_{k+1}$ for every k.

Choose an arbitrary partition of the set of non-negative integers into n disjoint infinite subsets $I_1, I_2, \ldots, I_n$, and define $P_i = \bigcup_{k\in I_i} V_k \setminus V_{k+1}$. □

Choose a partition $\mathcal{P}$ of $\mathcal{X} \setminus \{\xi\}$ into n open sets satisfying the conditions of Lemma 3.2. Choose a subgroup $H < (\mathbb{Z}/2\mathbb{Z})^n$ not containing $(1, 1, \ldots, 1)$ and such that for every $i = 1, 2, \ldots, n$ the projection $\pi_i : H \longrightarrow \mathbb{Z}/2\mathbb{Z}$ onto the ith coordinate of the direct product $(\mathbb{Z}/2\mathbb{Z})^n$ is surjective.

For $h \in H$ denote by b_h the homeomorphism:

$$b_h(\zeta) = \begin{cases} \xi & \text{if } \zeta = \xi; \\ b(\zeta) & \text{if } \xi \in P_i \text{ and } \pi_i(h) = 1; \\ \zeta & \text{if } \xi \in P_i \text{ and } \pi_i(h) = 0. \end{cases}$$

Note that since $(1, 1, \ldots, 1)$ is not in H, for every $h \in H$ there exists P_i such that b_h is identical on P_i.

Let $G = G_{\mathcal{P},H}$ be the group generated by the set $\{a\} \cup \{b_h \ : \ h \in H\}$. Note that the map $h \mapsto b_h$ is an isomorphism of the group $\{b_h\}_{h\in H} < G$ with H. It follows that every element of G can be written in the form $a^{\epsilon_1} b_{h_1} a b_{h_2} a \cdots b_{h_n} a^{\epsilon_2}$, where $h_i \in H$ and $\epsilon_i \in \{0, 1\}$.

Example 3.1. Consider the space $\{0, 1\}^\infty$ of right-infinite sequences $x_1x_2\ldots$ over the binary alphabet $\{0, 1\}$. Define the transformations a and b by the rules:

$$a(0x_2x_3\ldots) = 1x_2x_3\ldots, \qquad a(1x_2x_3\ldots) = 0x_2x_3\ldots$$

and

$$b(0x_2x_3\ldots) = 0a(x_2x_3\ldots), \qquad b(1x_2x_3\ldots) = 1b(x_2x_3\ldots).$$

It is easy to show that a and b are of order 2 and that the cyclic group generated by ab has dense orbits. In fact, ab is conjugate to the binary odometer. The homeomorphism b has a unique fixed point $\xi = 111\ldots$. See Figure 1 for a description of the action of a and b.

The sets $W_n = \underbrace{11\ldots1}_{n \text{ times}}0\{0, 1\}^\infty$, for $n \geq 0$, of sequences starting with exactly n ones form a partition of $\{0, 1\}^\infty \setminus \{\xi\}$ into clopen b-invariant subsets.

Consider the partition $P_0 = \bigcup_{k=0}^\infty W_{3k}$, $P_1 = \bigcup_{k=0}^\infty W_{3k+1}$, $P_2 = \bigcup_{k=0}^\infty W_{3k+2}$ of $\{0, 1\}^\infty \setminus \{\xi\}$, and the subgroup $H = \{h_1 = (1, 1, 0), h_2 = (1, 0, 1), h_3 = (0, 1, 1), (0, 0, 0)\}$. The corresponding group $G_{\mathcal{P},H}$ is the *first Grigorchuk group*,

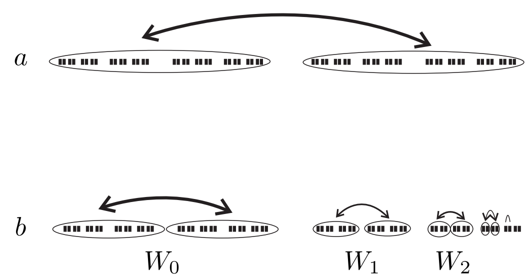

FIGURE 1. The dihedral group

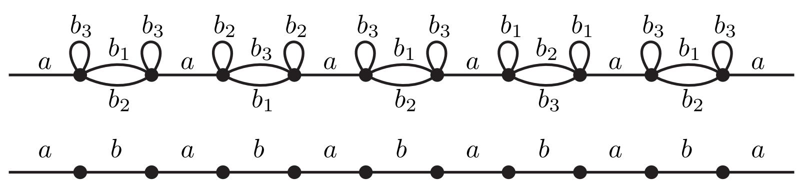

FIGURE 2. The orbital graphs of the Grigorchuk group and the dihedral group

introduced in [5]. Its generators $a, b_{h_1}, b_{h_2}, b_{h_3}$ are usually denoted a, b, c, d. Choosing different partitions P_0, P_1, P_2 equal to unions of the sets W_k, we get all groups from the family of Grigorchuk groups G_w studied in [7].

For every $\zeta \in \mathcal{X}$ and $h \in H$ we have $b_h(\zeta) = b(\zeta)$ or $b_h(\zeta) = \zeta$. For every $\zeta \in \mathcal{X}$ there exists $h \in H$ such that $b_h(\zeta) = b(\zeta)$, since every projection map $\pi_i : H \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is onto. It follows that the orbital graphs of G are just decorated versions of the orbital graphs of $D_\infty = \langle a, b\rangle$. The orbital graphs of D_∞ are obtained from the orbital graphs of G by removing loops labeled by b_h at points that are not fixed points of b and replacing each multiple edge labeled by some elements b_h, $h \in H$, by one edge labeled by b. See Figure 2 where an orbital graph of the Grigorchuk group and the corresponding orbital graph of the dihedral group are shown (where b_{h_i} are denoted b_i).

Let Γ_ξ be the orbital graph of the special point ξ in G, and let Γ'_ξ be obtained from Γ_ξ by removing all loops labeled by b_h at the root ξ.

Consider the graph Ξ with the set of vertices $H \times \Gamma'_\xi$ in which two vertices (h_1, v_1) and (h_2, v_2) are connected by an edge labeled by $s \in \{a\} \cup \{b_h\}_{h\in H}$ either if $h_1 = h_2$ and v_1 and v_2 are vertices of Γ'_ξ connected by an edge labeled by s, or if $v_1 = v_2 = \xi$ and $s = b_{h_1+h_2}$. In other words, we take $|H|$ copies of Γ'_ξ, and then connect their roots ξ by a full graph (the Cayley graph of $H < G$). See Figure 3, where the graph Ξ for the Grigorchuk group is shown. The group H acts on Ξ in the natural way: $h_1(h_2, v) = (h_1 + h_2, v)$.

Proposition 3.3. *The graph of germs $\tilde{\Gamma}_\xi$ is isomorphic to Ξ. The action of the group of deck transformations $G_\xi/G_{(\xi)} \cong H$ of the covering $\tilde{\Gamma}_\xi \longrightarrow \Gamma_\xi$ coincides with the natural action of H on Ξ.*

Proof. If $\zeta \neq \xi$, then the germ (b_h, ζ) is equal either to (id, ζ), or to (b, ζ). It follows that every germ (g, ξ) is equal to a germs of the form $(g' b_h, \xi)$, where $g' \in$

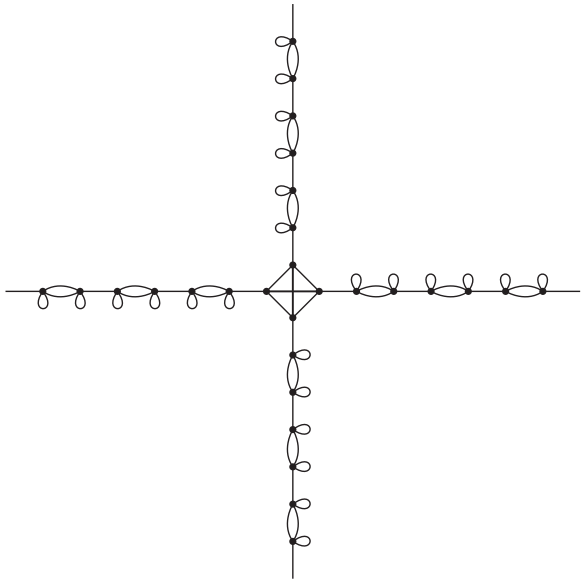

FIGURE 3. The graph of germs Ξ

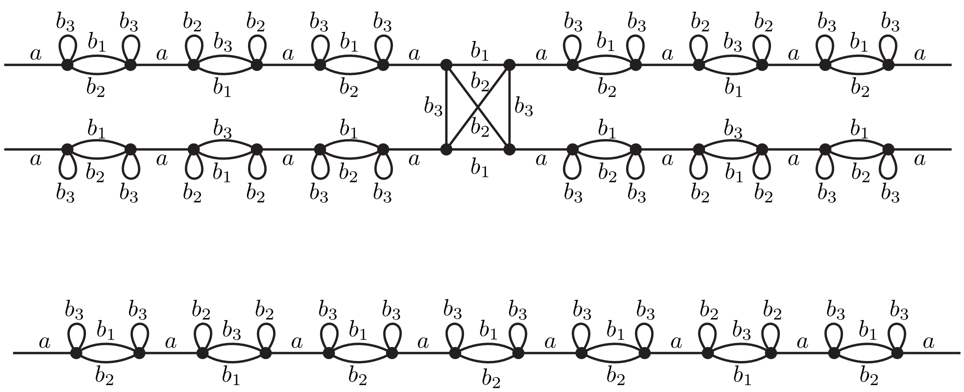

FIGURE 4. A covering λ_i

$\{a, ba, aba, baba, \ldots\}$. Identify the germ $(g'b_h, \xi)$ with the vertex $(h, v) \in \Xi$, where $v = g'(\xi) = g(\xi)$. It is easy to check that this identification is an isomorphism of graphs. The statement about the action by deck transformations also follows directly from the description of the germs (g, ξ). $\square$

For each $i = 1, \ldots, n$, denote by Λ_i the quotient of $\tilde{\Gamma}_\xi$ by the action of $\ker \pi_i$. It is the graph obtained by taking two copies $\{0\} \times \Gamma'_\xi$ and $\{1\} \times \Gamma'_\xi$ of Γ_ξ; connecting the roots $(0, \xi)$ and $(1, \xi)$ by edges labeled by b_h for $h \in H$ such that $\pi_i(H) = 1$; and adding loops at both vertices $(0, \xi)$, $(1, \xi)$ labeled by b_h for $h \in H$ such that $\pi_i(h) = 0$. Denote by $\lambda_i : \tilde{\Gamma}_\xi \longrightarrow \Lambda_i$ the natural covering map. In terms of Ξ, it is given by the rule $\lambda_i(h, v) = (\pi_i(h), v)$. See Figure 4 where a coverings λ_i for the Grigorchuk group is shown.

Proposition 3.4. *If $\zeta_n \in P_i$, $n \geq 1$, is a sequence of regular points converging to ξ, then the rooted orbital graphs Γ_{ζ_n} converge to Λ_i in the space $\mathcal{G}$ of rooted labeled graphs.*

For the definition of the space of rooted graphs, see Subsection 2.1.

Proof. For a given positive integer r consider the ball $B_\xi(r)$ of radius r in the graph of germs $\tilde{\Gamma}_\xi$. It is given by a set of equalities and inequalities of germs of the form $(g_1, \xi) = (g_2, \xi)$ or $(g_1, \xi) \neq (g_2, \xi)$ for elements $g_1, g_2 \in G$ of length at most r. If $(g_1, \xi) = (g_2, \xi)$, then $g_1(\zeta) = g_2(\zeta)$ for all ζ belonging to a neighborhood of ξ. If $g_1(\xi) \neq g_2(\xi)$, then we also have $g_1(\zeta) \neq g_2(\zeta)$ for all ζ in a neighborhood of ξ. Suppose that $(g_1, \xi) \neq (g_2, \xi)$ but $g_1(\xi) = g_2(\xi)$. Then $(g_1, \xi) = (gb_{h_1}, \xi)$ and $(g_2, \xi) = (gb_{h_2}, \xi)$ for some $g \in \langle a, b\rangle$ and $h_1, h_2 \in H$. If $\pi_i(h_1 + h_2) = 0$, then $b_{h_1}|_{P_i} = b_{h_2}|_{P_i}$, hence $g_1(\zeta) = g_2(\zeta)$ for all $\zeta \in N \cap P_i$ for some neighborhood N of ξ. If $\pi_i(h_1 + h_2) = 1$, then $g_1(\zeta) \neq g_2(\zeta)$ for all regular points $\zeta \in N \cap P_i$ for some neighborhood N of ξ, since $b_{h_1+h_2}|_{P_i} = b|_{P_i}$ and the set of fixed points of b is nowhere dense.

We see that for all regular points $\zeta \in N \cap P_i$, where N is a sufficiently small neighborhood of ξ, the ball $B_\zeta(m)$ of the orbital graph Γ_ζ is equal to the quotient of the ball $B_\xi(m) \subset \tilde{\Gamma}_\xi$ by the action of the kernel of the projection π_i. □

3.2. **Periodicity.**

Theorem 3.5. *The group $\mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$ is periodic.*

Proof. Let $g \in \mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$. In the rest of the proof, when we talk about orbital graphs and graphs of germs of $G = G_{\mathcal{P},H}$ we add a labeling of vertices according to a partition of $\mathcal{X}$ into pieces on which g acts as same element of G. Propositions 3.3 and 3.4 obviously remain to be valid for such vertex labeled graphs.

Let m be the maximal length of elements of G describing local action of g. Then for every $\zeta \in \mathcal{X}$ the image $g(\zeta)$ belongs to the ball $B_\zeta(m)$ in the orbital graph Γ_ζ, and is uniquely determined by the labels of the edges of $B_\zeta(m)$.

Below under *segment* of an orbital graph of a regular point we mean a finite connected subgraph of the orbital graph with all labelings of vertices and edges.

Lemma 3.6. *For every segment Σ of an orbital graph of a regular point, a subsegment $\Delta \subset \Sigma$ of edge-length m, and a vertex v of Δ there exists an embedding φ of Σ into an orbital graph of a regular point and an integer $k \geq 1$ such that $g^k(\varphi(v)) \in \varphi(\Delta)$.*

Proof. Suppose that it is not true for some Σ, Δ, and $v \in \Delta$, i.e., that for every orbital graph Γ and an embedding $\varphi : \Sigma \longrightarrow \Gamma$ the sequence $g^k(\varphi(v))$, $k \geq 1$, does not come back to $\varphi(\Delta)$. Since for every vertex u the distance from u to $g(u)$ is not more than m, the sequence $g^k(\varphi(v))$, $k \geq 1$, always stays in one of the two connected components of $\Gamma \setminus \varphi(\Delta)$. It follows that $g^k(\varphi(v))$ converges to one of the two ends of the graph Γ.

By Proposition 2.1, there exists an embedding of Σ into the ray Γ'_ξ. It follows that each Λ_i contains two symmetric copies $\varphi_+, \varphi_- : \Sigma \longrightarrow \Lambda_i$ of Σ, see Figure 5. If $g^k(\varphi_+(v))$, $k \geq 1$, converges to one end of Λ_i, then $g^k(\varphi_-(v))$, $k \geq 1$, converges to the other end, by the symmetry of Λ_i. Let Σ' be a segment of Λ_i containing both copies of Σ.

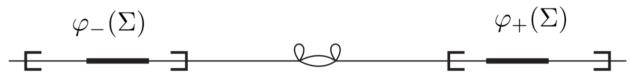

FIGURE 5. Embedding Σ into Λ_i

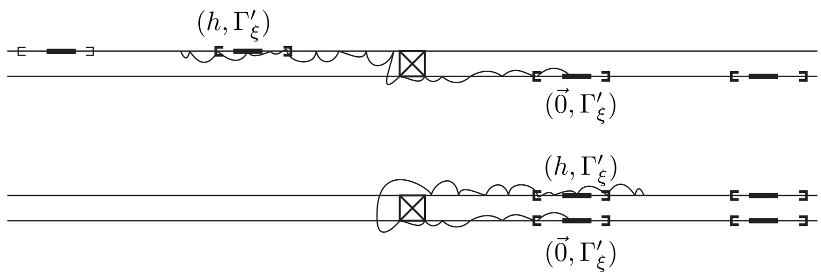

FIGURE 6. Coming back

There exists an embedding $\Sigma' \longrightarrow \Gamma'_\xi$. Consider the corresponding copy $\varphi' : \Sigma' \longrightarrow (\vec{0}, \Gamma'_\xi)$ of Σ' in the ray $(\vec{0}, \Gamma'_\xi)$ of the graph of germs $\tilde{\Gamma}_\xi$.

Consider the image $\lambda_i \circ \varphi'(\Sigma')$ of $\varphi'(\Sigma')$ in any Λ_i. It belongs to the ray $(0, \Gamma_\xi)$ of Λ_i. Then for some $* \in \{+, -\}$ the sequence $g^k(\lambda_i \circ \varphi' \circ \varphi_*(v))$ will converge to the other end $(1, \Gamma'_\xi)$ of Λ_i.

It follows that the sequence $g^k(\varphi' \circ \varphi_*(v))$ will converge in $\tilde{\Gamma}_\xi$ to an end different from $(\vec{0}, \Gamma'_\xi)$. Denote $w = \varphi' \circ \varphi_*(v)$. Suppose that $g^k(w)$ converges to the end (h, Γ'_ξ). Since $(1, 1, \dots, 1) \notin H$, there exists a projection $\lambda_j : \tilde{\Gamma}_\xi \longrightarrow \Lambda_j$ such that $\lambda_j((h, \Gamma'_\xi)) = \lambda_j(\vec{0}, \Gamma'_\xi)$. Then the sequence $\lambda_j(g^k(w))$ will move from one connected component of $\Lambda_j \setminus \lambda_j \circ \varphi_1 \circ \varphi_*(\Delta)$ to another, which is a contradiction. See Figure 6, where projections of $\tilde{\Gamma}_\xi$ onto Λ_i and Λ_j are shown. □

Let Δ be as in the Lemma 3.6, and let $v_0, v_1, \dots, v_m$ be the list of its vertices. According to the lemma, there exists a copy of Δ in an orbital graph Γ of a regular point such that $g^{k_0}(v_0) \in \Delta$ for some $k_0 \geq 1$. Let Σ_0 be a sufficiently big segment of Γ containing Δ such that the sequence $g^k(v_0)$ for $k = 0, 1, \dots, k_0$ is inside Σ_0 and is defined in Σ_0. Then $g^{k_0}(v_0) \in \Delta$ in every copy of Σ_0 in every orbital graph.

Apply now Lemma 3.6 for $\Sigma = \Sigma_0$ and for the vertex v_1 of Δ. We will find an orbital graph with a copy of Σ_0 in which both sequences $g^k(v_0)$ and $g^k(v_1)$ return back to Δ. Therefore there exists a segment Σ_1 containing Δ such that $g^k(v_0)$ and $g^k(v_1)$ return to Δ in every orbital graph containing Σ_1. Continuing in this way we will find a segment Σ_m such that every vertex of Δ returns inside Σ_m back to Δ under some positive power of g. It follows that orbit of every vertex of $\Delta \subset \Sigma_m$ is finite and contained in Σ_m.

Let Γ be an orbital graph of any regular point. By Proposition 2.1, there exists $R > 0$ such that for every vertex u of Γ there exists a copy of Σ_m on both sides of u on distances at most R. Let M be the number of vertices of Σ_m. Then for every vertex u of Γ either the sequence $g^k(u)$ includes a point of one of the neighboring copies of Δ, or always stays between them. In the first case the length of the orbit is not more than M, in the second case it is less than $2R + 2M$. It follows that the lengths of all g-orbits of vertices of Γ are uniformly bounded, hence there exists n

such that g^n acts trivially on the vertices of Γ. But the set of vertices of Γ is dense in $\mathcal{X}$, so $g^n = 1$. □

3.3. **Amenability and simple groups.**

Proposition 3.7. *Every finitely generated subgroup of* $\mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$ *can be embedded into the topological full group of a minimal action of* $\mathbb{Z}$ *on a Cantor set.*

Proof. Let $G_1 \leq \mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$ be a subgroup generated by a finite set S. Let $\mathcal{U}$ be a finite partition of $\mathcal{X}$ into disjoint clopen subsets such that for every element $s \in S$ and every $U \in \mathcal{U}$ the restriction $s|_U$ is equal to $g|_U$ for some $g \in G_{\mathcal{P},H}$.

Let Γ_ζ be the orbital graph (with respect to the original generating set of $G_{\mathcal{P},H}$) of a regular point $\zeta \in \mathcal{X}$ with vertices labeled by the elements of $\mathcal{U}$ to which they belong. It is a bi-infinite chain. Identify the vertices of the chain with integers, so that adjacent vertices of the chain are identified with integers n, m such that $|n - m| = 1$. Let $n \in \mathbb{Z}$, and let v be the corresponding vertex of Γ_ζ. Let v_{-1} and v_1 be the vertices corresponding to $n-1$ and $n+1$, respectively. Let a_n be the tuple $(x_{-1}, x_0, x_1, E_1, E_{-1}, L_{-1}, L_0, L_1)$, where x_{-1}, x_0, x_1 are the labels of the vertices v_{-1}, v, v_1, respectively, E_1, E_{-1} are the sets of labels of the edges connecting v_1 to v and v_{-1} to v, respectively, and L_{-1}, L_0, L_1 are the sets of labels of the loops at v_{-1}, v_0, v_1, respectively.

Then $w_\zeta = (a_n)_{n\in\mathbb{Z}}$ is a sequence over a finite alphabet, and it uniquely determines, up to an isomorphism, the graph Γ_ζ. Note that the action of every generator of G_1 on the orbit of ζ is uniquely determined by the isomorphism class of Γ_ζ, and hence by the sequence w_ζ.

Let $\mathcal{W}$ be the set of all sequences $w = \ldots y_{-1}y_0y_1 \ldots$ such that every finite subword of w is a subword of w_ζ. The set $\mathcal{W}$ is obviously a closed and shift-invariant set. Note that since the point ζ is regular, for every finite subword u of w_ζ there exists $R > 0$ such that for every $i \in \mathbb{Z}$ there exists $j \in \mathbb{Z}$ such that $|i - j| \leq R$ and $x_j x_{j+1} \ldots x_{j+|u|-1} = u$, see Proposition 2.1. This in turn implies that the action of the shift on $\mathcal{W}$ is minimal.

Since the action of every element $s \in S$ on a vertex η of Γ_ζ is uniquely determined by the isomorphism class of a ball of uniformly bounded radius with center in η, the action of s induces a homeomorphism of $\mathcal{W}$ equal to an element of the topological full group of the shift, hence G_1 is isomorphic to a subgroup of the full group of a minimal $\mathbb{Z}$-subshift. □

Theorem 3.8. *The group* $\mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$ *is amenable.*

Proof. By a theorem of K. Juschenko and N. Monod [9], the topological full group of a minimal homeomorphism group of a Cantor set is amenable. Proposition 3.7 implies then that every finitely generated subgroup of $\mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$ is amenable, hence $\mathsf{F}(G_{\mathcal{P},H}, \mathcal{X})$ is amenable. □

Theorem 3.9. *Suppose that the action of* $\langle a, b\rangle$ *on* $\mathcal{X}$ *is expansive. Then the action of* $G_{\mathcal{P},H}$ *on* $\mathcal{X}$ *is also expansive, and the group* $\mathsf{A}(G_{\mathcal{P},H}, \mathcal{X})$ *is simple and finitely generated.*

Proof. Let $\delta > 0$ be such that $d(g(\zeta), g(\eta)) < \delta$ for all $g \in \langle a, b\rangle$ implies $\zeta = \eta$.

Consider an arbitrary pair $1 \leq i, j \leq n$ of indices, and the corresponding homomorphism $\pi_i \oplus \pi_j : H \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The image of this homomorphism can not be zero, and can not be equal to any of the direct summands of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, since the homomorphisms π_i, π_j are surjective. It follows that the element $1 \oplus 1$ belongs to the image, i.e., that there exists $h \in H$ such that $\pi_i(h) = \pi_j(h) = 1$.

It follows that for every two points $\zeta, \eta \in \mathcal{X}$ there exists $h \in H$ such that $b_h(\zeta) = b(\zeta)$ and $b_h(\eta) = b(\eta)$. Consequently, for every $g \in \langle a, b\rangle$ there exists $g' \in G_{\mathcal{P},H}$ such that $g'(\zeta) = g(\zeta)$ and $g'(\eta) = g(\eta)$.

Suppose that $d(g(\zeta), g(\eta)) < \delta$ for all $g \in G_{\mathcal{P},H}$. Then, by the above, we have $d(g(\zeta), g(\eta)) < \delta$ for all $g \in \langle a, b\rangle$, which implies, by expansivity of $(\langle a, b\rangle, \mathcal{X})$, that $\zeta = \eta$. Thus, $(G_{\mathcal{P},H}, \mathcal{X})$ is also expansive. Properties of $\mathsf{A}(G_{\mathcal{P},H}, \mathcal{X})$ follow now from Theorem 2.2. □

4. Examples

4.1. **Irrational rotation.** Consider the circle $\mathbb{R}/\mathbb{Z}$ and the rotation $x \mapsto x + \theta \pmod 1$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. By the classical Kroneker's theorem, the action of $\mathbb{Z}$ generated by the rotation is minimal.

Consider now the interval $[0, 1] \subset \mathbb{R}$ with the natural order on it. Denote by $\mathrm{frac}(x)$ the fractional part of $x \in \mathbb{R}$, i.e., the point of $[0, 1)$ equal to x modulo $\mathbb{Z}$.

Let us replace in the interval $[0, 1]$ each point $\alpha = \mathrm{frac}(n\theta)$, for $n \in \mathbb{Z} \setminus 0$, by two copies α_{+0} and α_{-0}. Let $\mathcal{X}$ be the obtained set with the natural order (where $\alpha_{-0} < \alpha_{+0}$). Consider $\mathcal{X}$ with the order topology, i.e., topology given by the basis of open sets of the form $(\alpha, \beta) = \{x \in \mathcal{X} \ : \alpha < x < \beta\}$. Note that since the set $\{\mathrm{frac}(n\theta) \ : \ n \in \mathbb{Z}\}$ is dense in $[0, 1]$, the set of intervals of the form $[\alpha+0, \beta-0] = (\alpha-0, \beta+0)$ for $\alpha, \beta \in \theta\mathbb{Z} \pmod 1$ is a basis of the topology. These intervals are clopen, hence $\mathcal{X}$ is totally disconnected. It is also easy to see that $\mathcal{X}$ has no isolated points, and is compact, hence it is homeomorphic to the Cantor set. We also denote $0_{-0} = 1$, $0_{+0} = 0$ (according to the natural cyclic order on $\mathbb{R}/\mathbb{Z}$), and $\alpha_{-0} = \alpha_{+0} = \alpha$ if α does not belong to the set $\mathrm{frac}(n\theta) \ : \ n \in \mathbb{Z}\}$.

Let $Q : \mathcal{X} \longrightarrow \mathbb{R}/\mathbb{Z}$ be the natural quotient map identifying α_{+0} with α_{-0} for every $\alpha = \mathrm{frac}(n\theta)$, $n \in \mathbb{Z}$.

The rotation $x \mapsto x + \theta$ on $\mathbb{R}/\mathbb{Z}$ is naturally lifted by Q to a homeomorphism $\rho : \mathcal{X} \longrightarrow \mathcal{X}$ by the rule $\rho(\alpha_{+0}) = \mathrm{frac}(\alpha + \theta)_{+0}$ and $\rho(\alpha_{-0}) = \mathrm{frac}(\alpha - \theta)_{-0}$. The homeomorphism ρ is also minimal.

Note that unlike the rotation of the circle, the homeomorphism $\rho : \mathcal{X} \longrightarrow \mathcal{X}$ is expansive, since for any pair of points $x, y \in \mathcal{X}$ there exists n such that $\rho^n(x)$ and $\rho^n(y)$ belong to different pieces of the partition $\mathcal{X} = [0, \theta_{-0}] \sqcup [\theta_{+0}, 1]$.

Consider the following transformations of $\mathcal{X}$:

$$a(\alpha_{+0}) = \mathrm{frac}(\theta - \alpha)_{-0}, \qquad a(\alpha_{-0}) = \mathrm{frac}(\theta - \alpha)_{+0}$$

and

$$b(\alpha_{+0}) = \mathrm{frac}(-\alpha)_{-0}, \qquad b(\alpha_{-0}) = \mathrm{frac}(-\alpha)_{+0}.$$

In other words, a and b are reflections of the circle $\mathbb{R}/\mathbb{Z}$ with respect to the diameters $\{\theta/2, (\theta + 1)/2\}$ and $\{0, 1/2\}$, naturally lifted to $\mathcal{X}$.

Note that $ab = \rho$, hence the action of the dihedral group $\langle a, b\rangle$ on $\mathcal{X}$ is minimal and expansive. Note that the homeomorphism b has a unique fixed point $1/2 \in \mathcal{X}$ (the point 0 is doubled in $\mathcal{X}$, and its copies $0, 1 \in \mathcal{X}$ are switched by b). Using Lemma 3.2, we can find a partition $\mathcal{P}$ of $\mathcal{X} \setminus \{1/2\}$ into an arbitrary number of open b-invariant sets accumulating on $1/2$. Then the corresponding groups $\mathsf{A}(G_{\mathcal{P},H}, \mathcal{X})$ are finitely generated, simple, amenable, and periodic.

4.2. **Constructing D_∞-shifts from $\mathbb{Z}$-shifts.** Let us show how expansivity (resp. minimality) conditions for D_∞- and $\mathbb{Z}$-actions are related.

Proposition 4.1. *Suppose that involutive homeomorphisms a and b generate an expansive action of D_∞ on a Cantor set $\mathcal{X}$. Then there exists a finite alphabet A, a permutation $\iota : A \longrightarrow A$ such that $\iota^2 = 1$, and a $\mathbb{Z}$-subshift $\mathcal{S} \subset A^{\mathbb{Z}}$ such that there exists a homeomorphism $\mathcal{X} \longrightarrow \mathcal{S}$ conjugating the action of the generators a and b with the homeomeomorphisms of $\mathcal{S}$ given by the formulas:*

$$a(w)(n) = \iota(w(-n)), \qquad b(w)(n) = \iota(w(1-n))$$

for every $w \in \mathcal{S}$ and $n \in \mathbb{Z}$.

Proof. There exists a partition $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of D_∞ into clopen sets such that every point $\zeta \in \mathcal{X}$ is uniquely determined by its *itinerary*, which is defined as the map $I_\zeta : D_\infty \longrightarrow \mathcal{U}$ given by the condition $g(\zeta) \in I_\zeta(g)$. We may assume that $\mathcal{U}$ is a-invariant, i.e., that for every $U \in \mathcal{U}$ the set $a(U)$ belongs to $\mathcal{U}$. Otherwise, we can replace $\mathcal{U}$ by the partition induced by $\mathcal{U}$ and $a(\mathcal{U})$: two points ζ_1, ζ_2 belong to one piece of the induced partition if and only if they belong to one piece of $\mathcal{U}$ and to one piece of $a(\mathcal{U})$.

Then for every $\zeta \in \mathcal{X}$ and $g \in D_\infty$ we have $I_\zeta(g) = a(I_\zeta(ag))$, so that I_ζ, and hence ζ, are uniquely determined by the sequence $I_\zeta((ab)^n)$, $n \in \mathbb{Z}$. Let us denote $J_\zeta(n) = I_\zeta((ab)^n)$.

The set of sequences of the form $J_\zeta(n)$ is obviously a closed shift-invariant subset of the full shift $\mathcal{U}^{\mathbb{Z}}$.

Let us describe the action of a and b on the sequences $J_\zeta(n)$. We have $J_{a(\zeta)}(n) = I_\zeta((ab)^n a) = I_\zeta(a(ba)^n) = a(I_\zeta((ba)^n)) = a(J_\zeta(-n))$ and $J_{b(\zeta)}(n) = I_\zeta((ab)^n b) = I_\zeta(a(ba)^{n-1}) = a(I_\zeta((ba)^{n-1}) = a(J_\zeta(1-n))$. We can set the permutation ι of $\mathcal{U}$ equal to the action of a. □

Proposition 4.2. *Let a and b be involutive homeomorphisms of a Cantor set $\mathcal{X}$. If the action of $\langle ab\rangle$ on $\mathcal{X}$ is minimal, then the action of $\langle a, b\rangle$ is minimal too.*

If the action of $\langle a, b\rangle$ is minimal, then either the action of $\langle ab\rangle$ is minimal, or $\mathcal{X}$ can be split into a disjoint union of two clopen $\langle ab\rangle$-invariant sets S_1, S_2 such that the action of $\langle ab\rangle$ on each of these sets is minimal, and $a(S_1) = b(S_1) = S_2$ and $a(S_2) = b(S_2) = S_1$.

In particular, if a or b have a fixed point, then D_∞-minimality is equivalent to $\mathbb{Z}$-minimality.

Proof. The first statement is obvious. Suppose that the $\langle a, b\rangle$-action is minimal. If $A \subset \mathcal{S}$ is a closed non-empty $\langle ab\rangle$-invariant set, then $a(A)$ is also a closed $\langle ab\rangle$-invariant set (since $(ab)a(A) = a(ba)A = a(ab)^{-1}(A) = a(A)$). It follows that $a(A) \cap A$ and $a(A) \cup A$ are closed and $\langle a, b\rangle$-invariant. Consequently, $a(A) \cup A = \mathcal{S}$, and either $a(A) \cap A = \mathcal{S}$, or $a(A) \cap A = \emptyset$. It follows that either the $\langle ab\rangle$-action is minimal, or $\mathcal{S}$ is split into two disjoint clopen ab-invariant subsets S_1, S_2 such that $a(S_1) = b(S_1) = S_2$ and $a(S_2) = b(S_2) = S_1$, and each $\langle ab\rangle$-orbit is dense either in S_1 or in S_2. □

4.3. **Groups from the Thue-Morse sequence.** As an example of application of Proposition 4.1, consider the Thue-Morse shift. Let τ be the substitution (i.e., an endomorphism of the free monodi $\{0, 1\}^*$) given by:

$$\tau : 0 \mapsto 01, \qquad 1 \mapsto 10.$$

The words $\tau^n(0)$ converge to an infinite sequence $0110100110010110\ldots$. Let $\mathcal{S}$ be the set of all bi-infinite sequences $w = \ldots x_{-1}x_0x_1\ldots$ such that every subword of w is a subword of $\lim_{n\to\infty}\tau^n(0)$. It is known that $\mathcal{S}$ is a minimal subshift (see [2, Example 10.9.3]).

Note that the words $\tau^2(0)$ and $\tau^2(1)$ are symmetric (are palindromes):

$$\tau^2(0) = 0110, \qquad \tau^2(1) = 1001.$$

It follows by induction that $\tau^{2n}(0)$ and $\tau^{2n}(1)$ are palindromes for all $n \geq 1$. Consequently, the shift $\mathcal{S}$ is invariant under the transformation $\ldots x_{-2}x_{-1}x_0x_1x_2\ldots \mapsto \ldots x_2x_1x_0x_{-1}x_{-2}\ldots$.

Let a and b be the homeomorphisms of $\mathcal{S}$ defined by

$$a(w)(n) = w(-n), \qquad b(w)(n) = w(1-n).$$

In other words, a "flips" a sequence around the letter number zero, b "flips" a sequence around the space between the letters number zero and one. Then $\langle a, b\rangle$ is an infinite dihedral group acting minimally and expansively on $\mathcal{S}$. Note that $\tau^{2(n-1)}(0)$ is the middle part of $\tau^{2n}(0)$, so that the limit of the sequence of the words

$$\tau^{2n}(0) = x_{-2^{2n-1}+1}\ldots x_{-1}x_0 \,.\, x_1x_2\ldots x_{2^{2n-1}}$$

is a bi-infinite word $\xi = \ldots 01101001 \,.\, 10010110\ldots$ such that $b(\xi) = \xi$.

Let V_n be the set of words $\zeta = \ldots y_{-2}y_{-1}y_0 \,.\, y_1y_2y_3\ldots \in \mathcal{S}$ such that

$$y_{-2^{2n-1}+1}\ldots y_{-1}y_0 \,.\, y_1y_2\ldots y_{2^{2n-1}} = \tau^{2n}(0).$$

Then $(V_n)_{n\geq 1}$ is a strictly descending sequence of b-invariant clopen neighborhood of ξ, which can be used, as in Lemma 3.2, to construct groups $G_{\mathcal{P},H}$. The corresponding groups $\mathsf{A}(G_{\mathcal{P},H}, \mathcal{S})$ will be simple and finitely generated.

4.4. **Iterated monodromy groups of Chebyshev polynomials.** In some sense the opposite condition to expansiveness is *residual finiteness* of the action. We say that an action of a group G on a Cantor set $\mathcal{X}$ is residually finite if the G-orbit of every clopen subsets of $\mathcal{X}$ is finite. An action is residually finite if and only if there exists a homeomorphism $\Phi : \mathcal{X} \longrightarrow \partial T$ of $\mathcal{X}$ with the boundary of a locally finite rooted tree T and an action of G on T by automorphisms such that Φ is G-equivariant (with respect to the action of G on ∂T induced by the action on T), see [8, Proposition 6.4].

Suppose that a dihedral group $\langle a, b\rangle$ acts by automorphisms on a rooted tree T, so that the action is transitive on every level L_n of T (here L_n is the set of all vertices of T on distance n from the root). The latter condition is equivalent to minimality of the action on ∂T. Suppose that b has a fixed point $\xi \in \partial T$. The point ξ is represented by an infinite path $(v_0, v_1, \ldots)$, where v_0 is the root, and $v_k \in L_k$. Let $V_k = \partial T_{v_k}$ be the subset of ∂T consisting of all simple rooted paths passing through v_k. It is a clopen b-invariant set. The sets V_k can be used as in the proof of Lemma 3.2 to construct a partition $\mathcal{P}$ and a group $G_{\mathcal{P},H}$, which will also act on the rooted tree T.

Examples of residually finite actions of dihedral groups appear naturally as the iterated monodromy groups of the Chebyshev polynomials $T_d(x) = \cos(d \arccos x)$, which are described below.

If g is an automorphism of the rooted tree of finite words over the alphabet $\{1, 2, \ldots, d\}$, then we write its action on the boundary $\{1, 2, \ldots, d\}^\infty$ of the tree

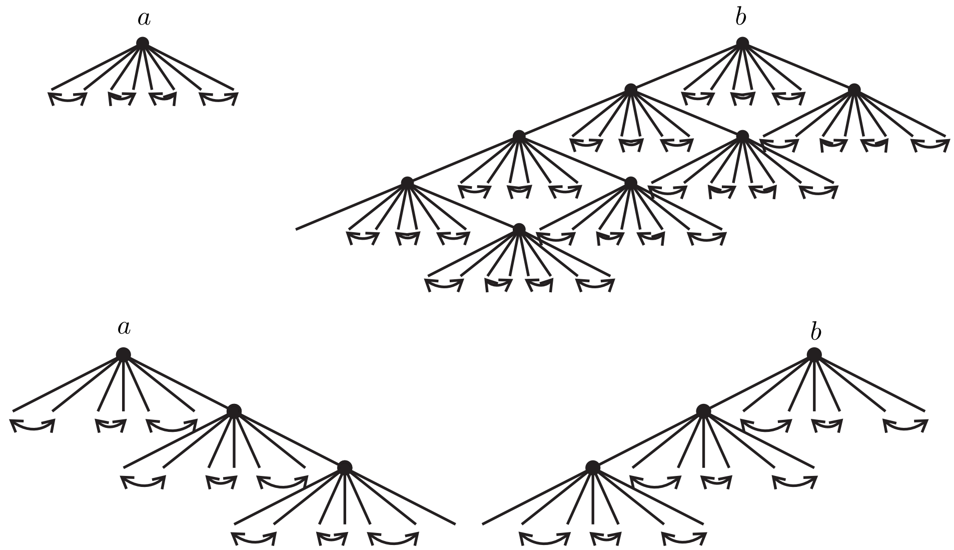

Figure 7. Iterated monodromy groups of Chebyshev polynomials

using recursive formulas of the form $g = \sigma(g_1, g_2, \ldots, g_d)$, where σ is a permutation of the alphabet, and g_i are automorphisms of the tree, such that

$$g(x_1x_2\ldots) = \sigma(x_1)g_{x_1}(x_2x_3\ldots)$$

for all $x_1x_2\ldots \in \{1,2,\ldots,d\}^\infty$. We denote by id the identity transformation.

Then the iterated monodromy group of T_d for even d is generated by

$$a = \sigma_1(id, id, \ldots, id), \qquad b = \sigma_2(b, id, id, \ldots, id, a),$$

where σ_1 and σ_2 are the permutations $(1,2)(3,4)\cdots(d-1,d)$ and $(2,3)(4,5)\cdots(d-2,d-1)$, respectively.

The iterated monodromy group of T_d for odd d is generated by

$$a = \sigma_1(id, id, \ldots, id, a), \qquad b = \sigma_2(b, id, id, \ldots, id),$$

where $\sigma_1 = (1,2)(3,4)\cdots(d-2,d-1)$ and $\sigma_2 = (2,3)(4,5)\cdots(d-1,d)$. See Figure 7 for a schematic description of the action of a and b, and see [14, Proposition 6.12.6] for details.

We see that in both cases the point $\xi = 111\ldots$ is a fixed point of b. Denote by W_k, for $k \ge 0$, the set of sequences $x_1x_2\ldots \in \{1,2,\ldots,d\}^\infty$ such that $x_i = 1$ for all $i \le k$, and $x_{k+1} \ne 1$. Then W_k are disjoint clopen b-invariant sets, and $\bigcup_{k\ge 0} W_k = \{1,2,\ldots,d\}^\infty \setminus \{\xi\}$. Consider a partition $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$ of the set of non-negative integers into n disjoint infinite sets, and define $P_i = \bigcup_{k\in I_i} W_k$. Then the partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ can be used to construct periodic groups of the form $G_{\mathcal{P},H}$.

If the partition $\mathcal{P}$ is invariant under the shift $x_1x_2\ldots \mapsto x_2x_3\ldots$, and H is invariant under the permutation induced by the shift on $\mathcal{P}$, then the group $G_{\mathcal{P},H}$ is the iterated monodromy group of an orbispace uniformization of the action of the Chebyshev polynomial on its Julia set (which is the interval $[-1,1]$). In particular, the group $G_{\mathcal{P},H}$ will be generated by a finite automaton. See more details in [15].

The corresponding groups for the case $d = 2$ were considered before by Z. Šunić, see [18].

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