A pseudo-Newtonian limit for test motion in arbitrary space-times

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We present a particular low-energy limit of the Hamiltonian of free test particle motion in arbitrary relativistic space-times. As it turns out, this limit gives insight into the general Newtonian limit by providing an intermediate, "pseudo-Newtonian" step, which encompasses some pseudo-Newtonian formulas already present in the literature. If the metric is expressed so as to be diagonal in the coordinate-time components, we are able to derive a description exactly reproducing the spatial shapes of geodesics. In fully general space-times where dragging ("time non-diagonal") terms appear in the metric, the limit at least yields a previously unknown Hamiltonian reproducing exact shapes of null geodesics. Furthermore, if the space-time is stationary, the exact shapes of null geodesics can be also correctly parametrized by coordinate-time; the limit thus provides an alternative Hamiltonian for computations in gravitational lensing.

Relevant astrophysical superpositions of gravitating sources, the addition of electromagnetic fields, and fluid dynamics in the pseudo-Newtonian limit are discussed. Additionally, the method is demonstrated in the case of the Kerr space-time and massive-particle circular orbits, and analogies with respect to the recent pseudo-Kerr Lagrangian by Ghosh *et al.* [1] is commented upon.

I. INTRODUCTION

Pseudo-Newtonian potentials are a common tool in astrophysics used to avoid unnecessarily complicated relativistic formulas while at the same time salvaging at least some of the features of a strong-field gravitational situation in a Newtonian framework (see introduction of Tejeda and Rosswog [2] or Artemova *et al.* [3] for a review). Typically, the pseudo-Newtonian description is used for astrophysical simulations in which most of the dynamics happens in regions where a Newtonian description is fully appropriate but where the dynamics marginally pass into a strongly relativistic mode near a compact object such as a Schwarzschild black hole. Splitting the description into a 3D Newtonian and 4D relativistic part while preserving accuracy is often difficult to conceive, so the researcher has to choose between using a fully Newtonian or a fully relativistic code. Thus, if there is a modified "pseudo-Newtonian" dynamics mimicking relativity which seamlessly coincides with Newtonian dynamics in an appropriate limit, then it can be very useful for the purposes of such astrophysical models.

Even though pseudo-Newtonian potentials have been proposed for over 35 years [3–6], until recently the potentials were not able to accurately reproduce properties of general orbits or to accurately describe the field of a rapidly spinning black hole. However, Tejeda and Rosswog [2, 7] proposed a class of generalized (velocity-dependent) pseudo-Newtonian potentials accurately describing the motion of quite general test-particles in the Schwarzschild and generally any spherically symmetric space-time (the same result on spherically symmetric space-times was almost simultaneously given by Sarkar *et al.* [8]). By a similar but slightly more complicated argument, Ghosh *et al.* [1] gave a generalized pseudo-Newtonian potential for test-particles in the equatorial plane of a slowly spinning Kerr black hole.

Many questions arise in connection with the recent results: Are the potentials also applicable for null-geodesics, i.e. gravitational lensing? Is there a deeper pattern in the way the potentials are formulated? How do the potentials superpose with additional gravitating matter and forces such as electromagnetism? Is it correct to use these relativistic-like potentials along with non-modified Newtonian fluid dynamics? Can we extend the pseudo-Newtonian description to higher spins of the black hole and off-equatorial particles? This paper partially clarifies these questions.

In Section II the notion of reparametrization of phase-space trajectories is introduced to be applied in Sections III and IV to relativistic geodesics in general and so-called "time-diagonal" space-times respectively. The issues such as superposition of gravitating sources, charged particle motion, and fluid dynamics are discussed in Section V; the effectivity of the framework in the Kerr space-time is then investigated in Section VI.

II. REPARAMETRIZATION OF PHASE-SPACE TRAJECTORIES

It is a well known fact often utilized in the theory of classical and celestial mechanics that it is possible to reparametrize a trajectory by a given coordinate using its conjugate momentum as a new Hamiltonian [9]. I.e., if we have a variable q with a conjugate p_q in the phase space of an autonomous dynamical system with a (time-parameter independent) Hamiltonian H and a coordinate

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λ with a conjugate momentum p_λ, it holds that

$$\frac{\mathrm{d}q}{\mathrm{d}\lambda} = \frac{\dot{q}}{\dot{\lambda}} = \frac{\frac{\partial H}{\partial p_q}}{\frac{\partial H}{\partial p_\lambda}}|_{H=const.} = \frac{\partial(-p_\lambda)}{\partial p_q}|_{H=const.}, \quad (1)$$

where the first equality holds only under the assumption of $\dot{\lambda} \neq 0$ and the last equality follows from the implicit function theorem.

Even though the resulting trajectory $q(\lambda)$ might be moving at a different pace with respect to the parameter λ, it will draw the same shape in the *full phase space* of coordinates and canonically conjugate momenta (q, p_q). Thus, for a given initial condition specified in terms of the phase-space variables, the shape of the original trajectory is reproduced exactly.

Nevertheless, we will use this reparametrization technique to obtain pseudo-Newtonian Hamiltonians and Lagrangians in a new pseudo-time with an interpretation of $\mathrm{d}q/\mathrm{d}\lambda$ as true physical velocities. Hence, the velocities will be rescaled by $\mathrm{d}\lambda/\mathrm{d}\tau$ in comparison with the original proper velocities $\mathrm{d}q/\mathrm{d}\tau$. The motion in terms of coordinates q and canonical momenta p_q will be the same, but the conversion from p_q to the "new velocities" $\mathrm{d}q/\mathrm{d}\lambda$ will be often very different.

Once giving the initial condition in terms of coordinates and velocities, the trajectories will be different accordingly because before the reparametrization the initial velocity is plugged into initial $\mathrm{d}q/\mathrm{d}\tau$ whereas in the second case the initial velocity is inserted as the initial $\mathrm{d}q/\mathrm{d}\lambda$. I.e., each trajectory generated by these models will correspond to *some* exact trajectory in the original space-time, but with a rescaled velocity.

On the other hand, integrals of motion are always functions of the phase space which means that if we have a complete set of first integrals of motion, then by giving their values we obtain the exact relativistic orbit even under reparametrization. For instance, in many space-times there is a family of circular orbits uniquely characterized by their position (coordinate shape) and a set of angular momenta. As follows from the previous discussion, such a family of circular orbits will be exactly preserved under reparametrization in the sense of the same position and canonical angular momenta.

III. GEODESICS IN GENERAL SPACE-TIMES

In the following, we mostly use the $G = c = 1$ units and $-+++$ sign convention of the metric $g_{\mu\nu}$. In certain instants we will switch to SI units and indicate so. The Lagrangian of a free test particle in any purely metric theory (such as general relativity) reads

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu,\; \dot{x}^\mu \equiv u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \quad (2)$$

where s is the proper time τ for massive particles and affine parameter λ for massless particles. Under a Legendre transform we obtain the Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} u_\mu u_\nu\,, \quad (3)$$

where u_μ is canonically conjugate with respect to x^μ. It will prove useful that both the Hamiltonian and Lagrangian are an integral of motion with a fixed value for all particles of a given kind; $H = L = -\kappa/2$ where $\kappa = 1$ corresponds to massive particles and $\kappa = 0$ to massless particles.

Say we now have a convenient time coordinate t corresponding to the 0-component of the metric and spatial coordinates labelled by $i, j = 1, 2, 3$ (the presented results are in fact applicable in any dimension). In the following, it will be important that t is a "good" time coordinate in the sense of e.g. being the time of some class of observers at asymptotically flat infinity. Then we can invert the Hamiltonian to obtain

$$\tilde{H} = -u_0 \equiv \mathcal{E} = \omega^i u_i - \sqrt{(\omega^i u_i)^2 - (g^{ij} u_i u_j + \kappa)/g^{00}}, \quad (4)$$

where we have chosen the root which corresponds to particles travelling forward in time under the assumption $g^{00} < 0$, and where $\omega^i \equiv g^{0i}/g^{00}$ is the gravitomagnetic part of the metric. The expression (4) could be used as a "Newtonian" Hamiltonian as it will exactly reproduce the shapes of trajectories in the 3-dimensional coordinate space parametrized by the coordinate time t as long as

$$\frac{\partial H}{\partial u_0} = g^{0\mu} u_\mu \equiv u^0 \neq 0. \quad (5)$$

Nonetheless, there is nothing Newtonian about the motion, which is fully relativistic.

To obtain an expression for the respective generalized velocity-dependent potential one has to pass back to Lagrangian formalism via a Legendre transform. Nevertheless, it should be kept in mind that

$$u_i \neq \frac{\mathrm{d}x^i}{\mathrm{d}t}, \quad (6)$$

and that Legendre transforming back to Lagrangian formalism requires expressing u_j from

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = u^i u^0 = g^{i\nu} u_\nu g^{0\mu} u_\mu \quad (7)$$

with the substitution of equation (4) for u_0. However, the complicated form of equation (4) seems not to allow such an inversion in general so one can only resort to approximations. The approximation used henceforth is that $\mathcal{E} = 1 + \delta$ where δ is a small dimensionless quantity.

It should be noted that in nearly flat space-time and for massive particles this limit yields exactly the dynamics of a non-relativistic particle in the Newtonian gravitational fields. However, such an assertion is not true in strongly curved space-times. Hence, we will call this limit the "pseudo-Newtonian" limit. Taking the four-velocity normalization we obtain up to $\mathcal{O}(\delta^2)$

$$2g^{00}\delta - 2g^{0i}\delta u_i + g^{00} + g^{0i}u_i + g^{ij}u_i u_j \approx -\kappa \tag{8}$$

which can be rearranged as

$$1+\delta \approx H_{\rm PN}\,. \tag{9}$$

where

$$H_{\rm PN} \equiv -\frac{1}{2(1-\omega^i u_i)}\frac{g^{ij}}{g^{00}}u_i u_j - \frac{1}{2(1-\omega^i u_i)}\left(\frac{\kappa}{g^{00}} - 1\right). \tag{10}$$

Hence, we could try to use $H_{\rm PN}$ a new Hamiltonian to reproduce $\mathcal{O}((\mathcal{E}-1)^2)$ error in the local particle evolution. To obtain a better estimate of the error the trajectory is introduced to when taking expression (10) as a new Hamiltonian we recover $H_{\rm PN}$ *exactly* in terms of δ

$$H_{\rm PN} = 1+\delta+\frac{\delta^2}{2(1-\omega^i u_i)} = \mathcal{E} + \frac{(\mathcal{E}-1)^2}{2(1-\omega^i u_i)}. \tag{11}$$

If we take a coordinate q and its canonically conjugate momentum p_q, Hamilton's equations yield

$$\begin{aligned} -\frac{\partial H_{\rm PN}}{\partial q} &= -\frac{\mathcal{E}-\omega^i u_i}{1-\omega^i u_i}\frac{\partial \mathcal{E}}{\partial q} + \frac{(\mathcal{E}-1)^2}{2(1-\omega^i u_i)^2}\frac{\partial}{\partial q}(\omega^i u_i) \\ &= \frac{\mathcal{E}-\omega^i u_i}{1-\omega^i u_i}\frac{{\rm d}p_q}{{\rm d}t} + \frac{(\mathcal{E}-1)^2}{2(1-\omega^i u_i)^2}\frac{\partial}{\partial q}(\omega^i u_i). \end{aligned} \tag{12}$$

Under the interchange $q \leftrightarrow -p_q$ we get a similar result for the other equation of motion. Thus, the pseudo-Newtonian Hamiltonian H_{PN} yields an approximate shape of the geodesic with an $\mathcal{O}(\delta)$-rescaled parametrization with respect to coordinate time and with an $\mathcal{O}(\delta^2)$ deformation term.

It is obvious that the description will fail for $\omega^i u_i \to 1$, so we would like to at least intuitively understand the meaning of the term $\omega^i u_i$ and the importance of $\omega^i u_i \to 1$. The formal vector ω^i represents the local dragging velocity with respect to the t-static observers (see the application to Kerr space-time in Section **??**), and u_i is the specific momentum of the particle. The whole term $\omega^i u_i$ then roughly represents the specific energy a t-static particle would have to receive to acquire specific momentum u_i. It is then intuitive that the low-specific-energy limit $\mathcal{E}-1 \ll 1$ will work well only if the "specific dragging energy" $\omega^i u_i \ll 1$.

The last remark is that in the case of massless particles the trajectory depends only on the direction of the four-velocity and we are thus free to normalize the initial velocity u^μ so that $-u_0 = \mathcal{E} = 1$. As a result, the equations of motion (12) correspond, at least initially, to exact evolution of the light-ray as parametrized by coordinate time t. In the case of a space-time stationary with respect to t, $\mathcal{E}$ is an integral of motion and the t-parametrization can thus be made exact along the *whole* curve by the mentioned normalization. I.e., if we are given an initial direction of the light-ray in terms of $\tilde{u}_i$, we obtain the correct parametrization by using the initial momenta $u_i = \tilde{u}_i/\alpha$ where

$$\alpha = \omega^i \tilde{u}_i - \sqrt{(\omega^i\tilde{u}_i)^2 - g^{ij}\tilde{u}_i\tilde{u}_j/g^{00}}, \tag{13}$$

This Section in fact ends in a different tone than its outset; the Hamiltonian (10) still does not allow for a Legendre transform and the most "solid" result of this Section is the fact that one can use it the $\kappa = 0$ case as an alternative 3D description of exact null geodesics parametrized by coordinate time. In the next Section, more satisfactory results are presented in a restricted class of space-times.

IV. TIME-DIAGONAL SPACE-TIMES

We now investigate the class of metrics for which $g^{0i} = 0$. Such metrics correspond to static space-times, but also to e.g. cosmological or gravitational-wave metrics. For these equation (10) gives (up to a constant)

$$H_{\rm PN} = -\frac{1}{2}g_{00}g^{ij}u_i u_j - \frac{\kappa}{2}(g_{00}+1). \tag{14}$$

The resulting equations of motion can be related to the exact geodesics similarly as in equation (1)

$$\frac{\partial H_{PN}}{\partial p_q} = \mathcal{E}\frac{{\rm d}q}{{\rm d}t} = -g_{00}\frac{{\rm d}q}{{\rm d}s},\; s=\tau,\lambda, \tag{15}$$

and analogously for the equations of motion for momenta. Equation (15) implies that the shape of the orbit generated by this Hamiltonian will be an exact geodesic up to a rescaling of velocities and time. One can understand the rescaling either as a local deformation of proper time (affine parametrization) by $-1/g^{00}$ or a global rescaling of coordinate time by $\mathcal{E}$ (in the case $\mathcal{E}$ is a constant of motion).

Additionally, under the assumption $g_{00} \to -1$ at infinity the Hamiltonian will also generate exact scattering with respect to initial conditions given in terms of proper-time velocities or in terms of canonically conjugate momenta. (Recall that the phase-space trajectory is reproduced exactly by the reparametrizing Hamiltonian.)

The corresponding Lagrangian reads

$$L_{\rm PN} = -\frac{1}{2}\frac{g_{ij}}{g_{00}}\dot{x}^i\dot{x}^j + \frac{\kappa}{2}(g_{00}+1), \tag{16}$$

where in this case $\dot{x}^i \equiv {\rm d}x^i/{\rm d}\tilde{t}$ are the velocities with respect to the rescaled time ${\rm d}\tilde{t} = {\rm d}t/\mathcal{E}$. If we would like to identify a velocity-dependent pseudo-Newtonian potential $\Phi_{\rm PN}$, we must first identify a natural "flat" metric in the coordinate space d_{ij}. The Lagrangian is then simply rewritten into the form

$$L_{\rm PN} = \frac{1}{2} d_{ij}\dot{x}^i\dot{x}^j 1 + \Phi_{\rm PN}(x^i,\dot{x}^i), \tag{17}$$

$$\Phi_{\rm PN} = \frac{\kappa}{2}(g_{00}+1) - \frac{1}{2}(\frac{g_{ij}}{g_{00}} + d_{ij})\dot{x}^i\dot{x}^j. \tag{18}$$

For example, in the case of an asymptotically flat space-time the pseudo-gravitational potential $\Phi_{\rm PN}$ goes asymptotically to zero if $g_{00} \to -1$ and $-g_{ij}/g_{00} \to d_{ij}$.

Another nice feature is the fact that for a gravitational source of mass M the first term of the pseudo-gravitational potential in SI units is a GM/r effect whereas the second one is a GMc^{-2}/r effect. Thus once neglecting $\sim 1/c^2$ terms, we gain exactly the well known non-relativistic weak-field relation $g_{00} = -1 - 2\Phi$.

Additionally, the specific form of the potential (18) allows for an intriguing intuitive interpretation. The first term in the pseudo-Newtonian potential is a purely potential term corresponding to the Newtonian gravitational potential, whereas the second term is a "geometrical correction" of the kinetic energy. For instance, near a Schwarzschild black hole of mass M the centrifugal term $\ell^2/2r^2$ of a particle bearing specific angular momentum ℓ at Schwarzschild radius r gets "geometrically corrected" as

$$\frac{\ell^2}{2r^2} \to \frac{\ell^2}{2r^2}\left(1 - \frac{2M}{r}\right). \tag{19}$$

One can then intuitively explain the in-fall into the black hole merely as a consequence of the geometry "turning off" the centrifugal barrier. The second point to be made on the intuitive interpretation of the structure of the potential (18) is that a photon, as a massless particle, is not affected by the purely potential term and only by the second "geometric" term. (Note also the obvious invariance with respect to conformal rescalings of the metric.)

It is only from the discussion in this Section where the plausibility of the usage of expression (9) as a new Hamiltonian and its Newtonian interpretation is imminent. The presented limit can also be used as a conceptually different understanding of the Newtonian limit; Newtonian-like expressions such as in equation (18) emerge once we force a unified time parametrization, and that is even in the case of fully exact relativistic geodesics.

A. Spherically symmetric space-times

The first example which comes to mind is the Schwarzschild space-time for which the formula (16) gives

$$L_{\rm TR} = \frac{1}{2}\left(\frac{\dot{r}^2}{(1-2M/r)^2} + \frac{r^2(\sin^2\vartheta\,\dot{\phi}^2 + \dot{\vartheta}^2)}{1-2M/r}\right) + \kappa\frac{M}{r}, \tag{20}$$

which for $\kappa = 1$ coincides with the Lagrangian derived from the equations of motion in the Schwarzschild space-time by Tejeda and Rosswog [2]. (The $\kappa = 0$ case giving exact light-rays is proposed only here.) Similarly, one obtains the same formula as in Tejeda and Rosswog [7], Sarkar *et al.* [8] once applying formula (16), $\kappa = 1$ to spherically symmetric space-times.

The authors have found that these pseudo-Newtonian descriptions reproduce exactly the relation between the radii of circular orbits and the particle energy and angular momentum. This is to be anticipated in light of the discussion in Section II. Furthermore, neither the Keplerian or epicyclic frequencies of circular orbits given by the pseudo-Newtonian Lagrangian have been found to differ by more than 10% from the values in the relativistic space-times considered by the authors. Once again, this concordance is easily explained by the fact that the reparametrization introduces a relative time-lag

$$\eta_t = 1 - \mathcal{E}, \tag{21}$$

with the frequencies of close oscillations scaling accordingly. For instance, in the case of the Schwarzschild space-time, the tightest-bound circular orbit has $\mathcal{E} = \sqrt{8/9}$ yielding the highest relative time-lag $\eta_t \approx 0.06$. Similarly in any given space-time, the highest time lag will be derived from the specific binding energy of the tightest bound orbit.

V. APPLICATIONS AND EXTERNAL FORCES

A. Superposition with axi-symmetric sources

A class of astrophysical situations of interest can be described by a superposition of a central black hole and an additional axisymmetric structure such as a slowly rotating gravitating disk or torus. Exact relativistic superpositions of a static black hole with a disc were studied e.g. by Semerák *et al.* [10, 11] and in this Subsection, the pseudo-Newtonian counterpart is briefly investigated.

Static, axially symmetric vacuum space-times are described by the Weyl metric [12]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda-2\nu}(d\rho^2 + dz^2) + e^{-2\nu}\rho^2 d\phi^2, \tag{22}$$

where the cosmological constant is necessarily set to zero, the metric function $\nu(\rho,z)$ satisfies the Poisson equation in cylindrical coordinates with respect to the gravitating matter-density distribution, and $\lambda(\rho,z)$ is obtained through a line integral of a quadratic function of derivatives of ν

$$\lambda = \int_{\rm axis}^{\rho,z} \rho\left\{\left[(\nu_{,\rho})^2 - (\nu_{,z})^2\right]d\rho + 2\nu_{,\rho}\nu_{,z}dz\right\}. \tag{23}$$

Hence, the functions ν superpose linearly for various gravitating sources and λ will always contain nonlinear cross-terms. For Weyl metrics formula (16) yields

$$L_{\rm W} = \frac{1}{2}\left[e^{2\lambda-4\nu}(\dot{\rho}^2 + \dot{z}^2) + e^{-4\nu}\rho^2\dot{\phi}^2\right] + \frac{\kappa}{2}(1 - e^{2\nu}). \tag{24}$$

We would now like to see how do additional axially symmetric matter sources superpose with the Tejeda-Rosswog potential, i.e. the pseudo-Newtonian potential of a Schwarzschild black hole. In that case it is useful to switch to the pseudo-Schwarzschild coordinates r, ϑ (ϕ stays the same)

$$\rho = \sqrt{r(r-2M)}\sin\vartheta, z = (r-M)\cos\vartheta \tag{25}$$

where M is the black hole mass. In terms of these coordinates, a Schwarzschild black hole is represented by the metric function

$$\nu_{\rm schw} = \ln\left(1 - \frac{2M}{r}\right). \tag{26}$$

Once we add the $\nu_{\rm E}$ of an external gravitating source we obtain the Lagrangian of the superposition

$$L_{\rm sup} = \frac{1}{2}\Big[\frac{e^{2\lambda_{\rm E}-4\nu_{\rm E}}}{(1-2M/r)^2}\dot{r}^2 + \frac{e^{-4\nu_{\rm E}}r^2}{1-2M/r}(e^{2\lambda_{\rm E}}\dot{\vartheta}^2 + \sin^2\vartheta\dot{\phi}^2)\Big] - \frac{1}{2}\Big[1 - e^{2\nu_{\rm E}}(1-\frac{2M}{r})\Big], \tag{27}$$

where $\lambda_{\rm E}$ will once again be a function acquired by a line integral involving both $\nu_{\rm E}$ and Schwarzschild terms (see eq. (23)). These "cross terms" in $\lambda_{\rm E}$ do not allow for a simple linear superposition of sources in the pseudo-Newtonian limit.

The only case when it is possible to simply add the Newtonian gravitational potential $\nu_{\rm E}$ of the additional source into the Lagrangian (20) as a simple additional term is when

1. $\nu_{\rm E}\nu_{\rm schw} \ll 1\,,\ \nu_{{\rm E},\alpha}\nu_{{\rm schw},\beta} \ll 1\,, \alpha,\beta = \rho, z$ everywhere, and
2. $\nu_{\rm E}/c^2 \ll 1$ (in SI units).

Otherwise additional corrections would arise. For instance, the Newtonian gravitational potential of perturbing faraway axi-symmetric halos surely fulfil these conditions and can be safely added to the pseudo-Schwarzschild Lagrangian without further complications.

B. Electromagnetic forces

The Hamiltonian of a charged particle with specific charge q in an electromagnetic field A^μ reads

$$H_{\rm EM} = \frac{1}{2}g^{\mu\nu}(\pi_\mu - qA_\mu)(\pi_\nu - qA_\nu)\,, \tag{28}$$

where $\pi_\mu = u_\mu + qA_\mu$ is canonically conjugate to x^μ. Analogously to Section III we invert the expression for the constant value of the Hamiltonian $H_{\rm EM} = -\kappa/2$ to get a Hamiltonian of coordinate-time parametrized electro-geodesics

$$\tilde{H} = -\pi_0 \equiv \mathcal{E}_{\rm EM} = \mathcal{E} + qA_0 = \sqrt{-g_{00}(\kappa + g^{ij}(\pi_i - qA_i)(\pi_j - qA_j))} + qA_0. \tag{29}$$

However, there seems not to be a satisfactory pseudo-Newtonian limit when $A_0 \neq 0$ i.e. when $\pi_0 \neq u_0$. In most astrophysical situations this will not be an issue because electrostatic fields play a negligible role in the dynamics. Alternatively, one can use the gauge freedom $A'_\mu = A_\mu + \partial_\mu\alpha$ to eliminate A_0. Once $A_0 = 0$, the pseudo-Newtonian limit gives trivially

$$H_{\rm PN,EM} = -\frac{1}{2(1-\omega^i u_i)}\frac{g^{ij}}{g^{00}}u_i u_j - \frac{1}{2(1-\omega^i u_i)}\left(\frac{\kappa}{g^{00}} - 1\right), \tag{30}$$

with the substitution $u_i = \pi_i - qA_i$. The reparametrization and deformation of the geodesic is exactly analogous to equation (12). Hence, the addition of electromagnetism up to the gauge fix $A_0 = 0$ is exactly in the lines of the usual minimal coupling.

We are only able to express the Hamiltonian in a gauge-independent manner in time-diagonal metrics

$$H_{\rm EM,stat} = -\frac{1}{2}g_{00}g^{ij}u_i u_j - \frac{\kappa}{2}(g_{00}+1) + qA_0\,, \tag{31}$$

where the $u_i = \pi_i - qA_i$ substitution is again understood. The corresponding Lagrangian reads

$$L_{\rm EM,stat} = -\frac{1}{2}\frac{g_{ij}}{g_{00}}\dot{x}^i\dot{x}^j + \frac{\kappa}{2}(g_{00}+1) - q(A_0 + \dot{x}^j A_j)\,. \tag{32}$$

The expression for the gauge-independent Hamiltonian (31) was obtained by a qualified guess and can be easily verified to yield gauge-independent equations of motion.

C. Fluid dynamics

Consider the Boltzmann equation (BE) for a phase-space distribution $f(q^i, p_i)$

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} = \frac{\delta f}{\delta t}. \tag{33}$$

For simplicity, we assume no collision term (it is trivial to add it along the following analysis). When integrating the zeroth moment $\int \bullet {\rm d}^3p$ of the BE, under the assumptions of vanishing boundary terms, we obtain the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial q^i}\left(n\langle\dot{q}^i\rangle\right) = 0, \tag{34}$$

where $n = \int f{\rm d}^3p$, $\langle\bullet\rangle = \int \bullet f{\rm d}^3p/n$, and we have used $\partial H/\partial p_i = \dot{q}^i$. Under similar assumptions we can integrate the first moment $\int \bullet p_j {\rm d}^3p$ of the BE to obtain

$$\frac{\partial}{\partial t}\left(n\langle p_j\rangle\right) + \frac{\partial}{\partial q^i}\left(n\langle\dot{q}^i p_j\rangle\right) + \delta^i_j n\langle\mathcal{A}_i\rangle = 0, \tag{35}$$

where $\mathcal{A}_i = \partial H/\partial q^i$.

So far the discussion only required that $f, f\partial H/\partial p^i, ... \to 0$ as $p_j \to \pm\infty$ to get rid of boundary terms in the integration. One of the already mentioned problems of these equations is that for the general pseudo-Newtonian Hamiltonian (10) we cannot express $p = p(\dot{q}, ...)$ in an elegant form to obtain a physical description of the fluid in terms of its macroscopic velocity.

Additionally, the issue of definition of pressure and viscous stresses $\sim \langle \dot{q}^i p_j \rangle$, and the assumptions on the fluid (such as a thermal equilibrium) to obtain the effective acceleration $\langle \mathcal{A}_i \rangle$ must be discussed. However, such a discussion is out of the scope of the current paper and should be presented once the pseudo-Newtonian fluid description is applied in a specific physical setting.

VI. THE KERR SPACE-TIME

In Boyer-Lindquist coordinates t, r, θ, ϕ we have the non-zero inverse metric components

$$\begin{aligned} g^{tt} &= -\frac{\mathcal{A}}{\Delta\Sigma}, \\ g^{rr} &= \frac{\Delta}{\Sigma}, g^{\theta\theta} = \frac{1}{\Sigma}, \\ g^{\phi\phi} &= \frac{\Delta - a^2 \sin^2\theta}{\Delta\Sigma \sin^2\theta}, \\ g^{t\phi} &= -\frac{2Mra}{\Delta\Sigma}, \end{aligned} \tag{36}$$

where $\Sigma = r^2 + a^2 \cos^2\vartheta$, $\Delta = r^2 - 2Mr + a^2$ and $\mathcal{A} = (r^2 + a^2)^2 - a^2 \Delta \sin^2\vartheta$. The corresponding pseudo-Newtonian Hamiltonian (9) then reads

$$\begin{aligned} H_{\rm PNK} = &\frac{1}{2\mathcal{A}(1 - \omega u_\phi)} \left(\Delta^2 u_r^2 + \Delta u_\theta^2 + \frac{\Delta - a^2 \sin^2\theta}{\sin^2\theta} u_\phi^2 \right) \\ &+ \frac{1}{2(1 - \omega u_\phi)} \left(\frac{\Delta\Sigma}{\mathcal{A}} + 1 \right), \end{aligned} \tag{37}$$

where $\omega \equiv g^{t\phi}/g^{tt} = 2Mra/\mathcal{A}$. Here the Hamiltonian is equal to 1 for a particle at rest at infinity, and thus represents the total particle specific energy including rest mass. The Hamiltonian is exactly equivalent to the Tejeda-Rosswog potential (20) at $a = 0$.

To obtain a Lagrangian, we would have to invert for u_ϕ from the non-trivial

$$\dot{\phi} = \frac{\partial H_{\rm PNK}}{\partial u_\phi} \tag{38}$$

However, the resulting relations seem to be too complicated and pathological to be of any use.

A. Circular orbits in the equatorial plane

When considering the case $\omega u_\phi < 1$ and regions above the horizon $\Delta > 0$, the Hamiltonian is positive-definite in u_θ and u_r and we may use the following effective potential for the analysis of turning points of orbits in the equatorial plane $\theta = \pi/2$

$$\mathcal{V}_{\rm eff} = \frac{2a^2(M + r) - 2M\left(u_\phi^2 + r^2\right) + r\left(u_\phi^2 + 2r^2\right)}{2\left(a^2 r + 2aM(a - u_\phi) + r^3\right)}. \tag{39}$$

The circular orbits are given by the roots of $\mathcal{V}'_{\rm eff} = 0$, i.e. the roots of

$$\begin{aligned} &M a u_{\phi c}^3 \\ &+(r^3 - 3Mr^2 - 2Ma^2) u_{\phi c}^2 \\ &+(6Mar^2 - 4M^2 ar + 2Ma^3) u_{\phi c} \\ &+(-Mr^4 - 2Mar^2 + 4M^2 a^2 r - Ma^4) u_{\phi c} \\ &= 0 \end{aligned} \tag{40}$$

The discriminant of the polynomial (40) understood as a polynomial in $u_{\phi c}$ is positive for most r and a which means that there are three real u_ϕ corresponding to three distinct circular orbits at almost every r. Two of the roots correspond to the usual families of corotating and counter-rotating orbits and the "third root" corresponds to a particular family of counter-rotating unstable circular orbits (thus yielding them less physical). The said "third root" $u_{\phi(3)}$ goes to $-\infty$ as $a \to 0$ and stays at highly negative values for $a < 1$ (see Subsection VI B for further discussion).

The physicality of a given root must also be verified by checking that the circular orbit with $r_c, u_{\phi c}, a$ is above the singularity of the effective potential (39), i.e. it must hold that

$$a^2 r_c + 2aM(a - u_{\phi c}) + r_c^3 > 0\,. \tag{41}$$

B. Angular momentum, frequency and energy of circular orbits

In Figure 1 the radial distribution of angular momenta of the usual corotating and counter-rotating circular orbits is compared with the exact Kerr distribution [13]

$$u_{\phi({\rm K})} = \frac{M^{1/2} r^{1/2} - 2Mar^{-1} + M^{1/2} a^2 r^{-3/2}}{\sqrt{1 - 3Mr^{-1} + 2M^{1/2} a r^{-3/2}}}, \tag{42}$$

where the counter-rotating case is obtained by $a \to -a$. Both the stable corotating and counter-rotating orbits exhibit very satisfactory agreement with the exact Kerr relation and up to $a \to M$ and $r \approx 4M$ the third $u_{\phi(3)}$ root is too large in magnitude to be of any physical influence.

For $a = M$ there is a minimal magnitude of the third root $u_{\phi(3)} \approx -13M$ at $r \approx 4.3M$ but for smaller a and larger r the value of $u_{\phi(3)}$ rapidly grows in magnitude; for $a = M$, $r = 5M$ it is around $-45M$ and for $a = 0.8M, r = 4.3M$ it is $\approx -25M$. It should be again stressed that the physical significance of the root is also negligible due to the fact that the respective circular orbit is unstable and counter-rotating with respect to the centre.

Nevertheless, Figure 1 does not show the fact that the behaviour of *unstable* circular orbits (i.e. the sector on the left from the local minimum of the angular momentum distribution) is not very satisfactory. For all values of $a/M \in (0,1)$ the corotating unstable circular orbits extend all the way to the horizon and take a finite value of $u_{\phi c}$ there (this horizon-value of $u_{\phi c}$ diverges as $a \to 0^+$).

Furthermore, the horizon is pathological also because a radially in-falling particle gets frozen there, very much like in the case of the Tejeda-Rosswog potential [2], or like in the case of the exact-relativistic infall as observed from infinity. Hence, the dynamics should be cut off somewhere between the marginally stable orbit and the horizon $\Delta = 0$.

The angular frequency of the orbits is given by $\Omega = \dot{\phi} = \partial H_{\rm PNK}/\partial u_\phi$ for the pseudo-Kerr case, and for the exact Kerr the angular frequency of corotating circular orbits is given as

$$\Omega = \frac{1}{a + r^{3/2} M^{-1/2}}, \tag{43}$$

where only values above the photon circular orbit $r_{\rm ph} = 2M(1 + \cos[2\arccos(-a/M)/3])$ are physical. We only plot a comparison of the corotating case in Fig. 2 and note that the tendencies of the counter-rotating case are quite similar.

The specific energy of circular orbits in the Kerr space-time reads

$$\mathcal{E}_{\rm (K)} = \frac{r^2 - 2Mr + a\sqrt{Mr}}{r\sqrt{r^2 - 3Mr + 2a\sqrt{Mr}}}. \tag{44}$$

The energy of the pseudo-Kerr circular orbits is given simply by substituting $u_{\phi c}$ into $H_{\rm PNK}$. The relations for the corotating case are compared in Figure 2. It should be stressed that the plot ranges do not show the growing energies of unstable circular orbits extending to the horizon.

C. Small perturbations of circular orbits

Let $\delta r, \delta u_r, \delta\theta, \delta u_\theta$ be small deviations from the stable circular orbits. At the point of reflectional symmetry $\theta = \pi/2$ all the first $\partial/\partial\theta$ derivatives of the Hamiltonian vanish and for $u_\theta = 0$ first $\partial/\partial u_\theta$ derivatives are also zero. Hence, the linearised equations decouple into two sectors corresponding to the purely radial (epicyclic) and purely vertical oscillations. The equations for the purely radial oscillations read

$$\begin{pmatrix} 0 & -\frac{\partial^2 H}{\partial^2 r} \\ \frac{\partial^2 H}{\partial^2 u_r} & 0 \end{pmatrix} \begin{pmatrix} \delta u_r \\ \delta r \end{pmatrix} = \begin{pmatrix} \delta \dot{u}_r \\ \delta \dot{r} \end{pmatrix}. \tag{45}$$

Because we are considering $u_r = 0$, the diagonal terms corresponding to first $\partial/\partial u_r$ derivatives of the Hamiltonian are also zero. Assuming $\sim e^{i\kappa t}$ oscillating solutions we obtain the epicyclic frequency

$$\kappa = \left(\frac{\partial^2 H}{\partial^2 u_r} \frac{\partial^2 H}{\partial^2 r} \right)^{1/2}, \tag{46}$$

where the expression is evaluated at $\theta = \pi/2$, $u_r = u_\theta = 0$, $u_\phi = u_{\phi(C)}$. Similarly for the purely vertical oscillations

$$\Omega_{\rm v} = \left(\frac{\partial^2 H}{\partial^2 u_\theta} \frac{\partial^2 H}{\partial^2 \theta} \right)^{1/2}. \tag{47}$$

Expressions (46) and (47) along with the substitution of the appropriate $u_{\phi c}$ give the oscillation frequencies analytically. However, the explicit form is rather involved and can be easily obtained via symbolic software so we only compare their values with the oscillation frequencies for the stable corotating circular orbits in the Kerr space-time in Fig. 3.

The conclusion drawn from the examination of Figures 1-3 is clear: The correspondence between pseudo-Kerr and exact-Kerr properties of stable circular orbits is excellent for $a/M \in (0, 0.8)$ and borderline-satisfactory for $a/M \in (0.8, 0.9)$. Furthermore, in the approximate region $a/M \in (0.9, 1)$, both in the exact Kerr space-time and in the pseudo-Kerr dynamics, a qualitative transition is taking place. In this region, the precise rate of the transition and the various critical points such as vanishing and appearance of extrema of the various distributions are not faithfully reproduced in the pseudo-Kerr dynamics. On the other hand, even though the quantitative differences are large, the qualitative correspondence is recovered for $a = M$.

D. Special radii

As already noted, the main issue of the presented pseudo-Kerr Hamiltonian is the non-existence of a photon circular orbit ($u_{\phi c} \to \infty$ singularity in the radial distribution of angular momentum). However, all the other features such as the marginally stable and the marginally bound orbit are recovered.

The marginally stable orbit is given by the zero of the epicyclic frequency $\kappa = 0$ and the marginally bound orbit is given by the specific energy equal to one $H = E_{\rm tot} = 1$. But since the marginally stable orbit has $\delta = 0$ or $\mathcal{E} = 1$, the discussion in Section III precludes that both the position and frequency of the marginally bound orbits will be reproduced exactly in the pseudo-Kerr dynamics. Indeed, we have verified by numerical root-finding that the radius of the marginally bound is exactly equal to the Kerr value [14]

$$r_{\rm mb} = 2M - a + 2\sqrt{M^2 - a^2}. \tag{48}$$

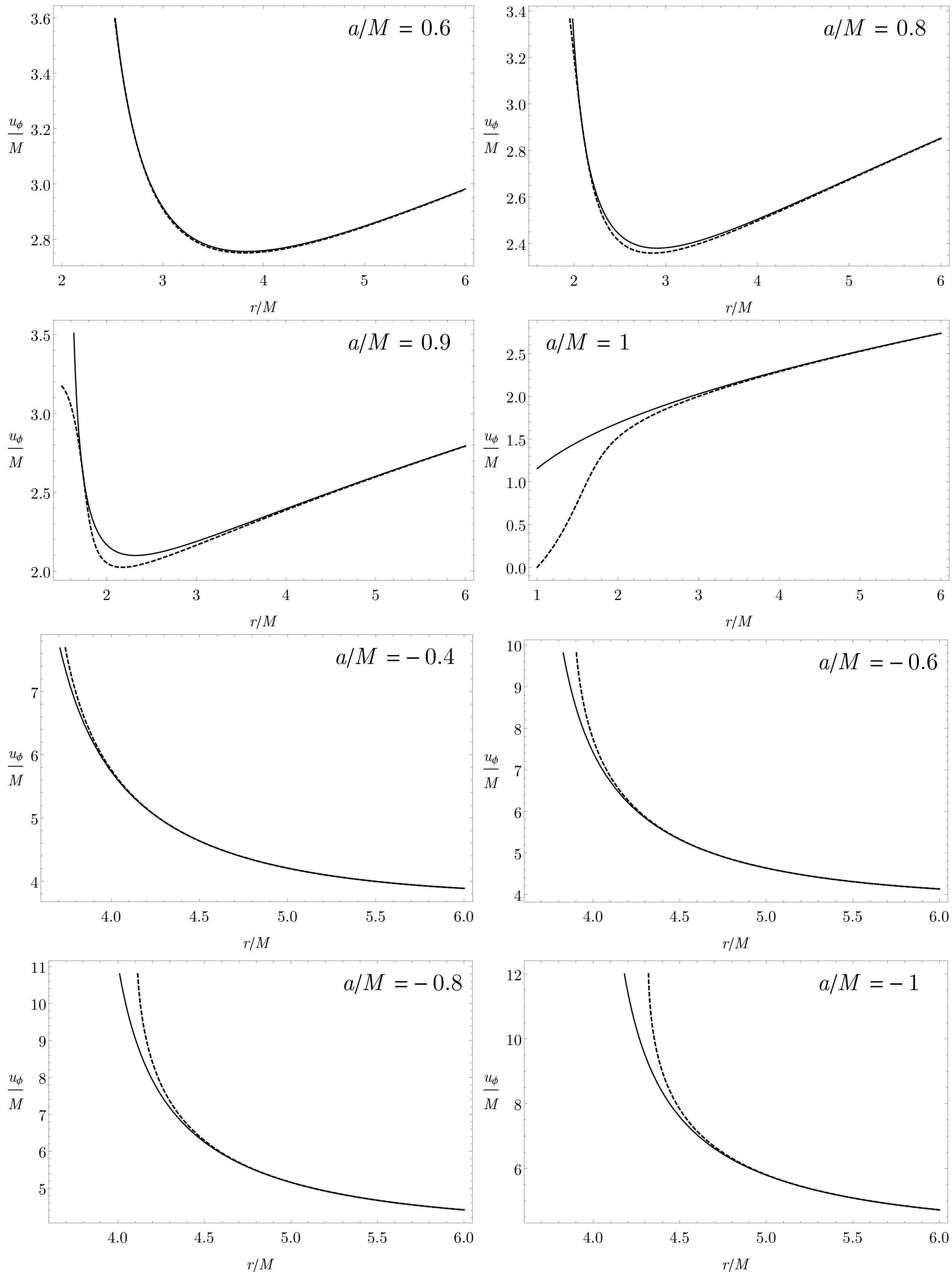

FIG. 1. Angular momenta u_ϕ of corotating (top) and counter-rotating (bottom) circular orbits of radius r in the Kerr space-time (solid line) compared with the distribution as given by the pseudo-Kerr Hamiltonian (dashed). For $r > 6M$ or $a < 0.6M$ the differences are virtually zero. The ranges for the corotating case are chosen to clearly document the vicinity of the marginally stable orbit.

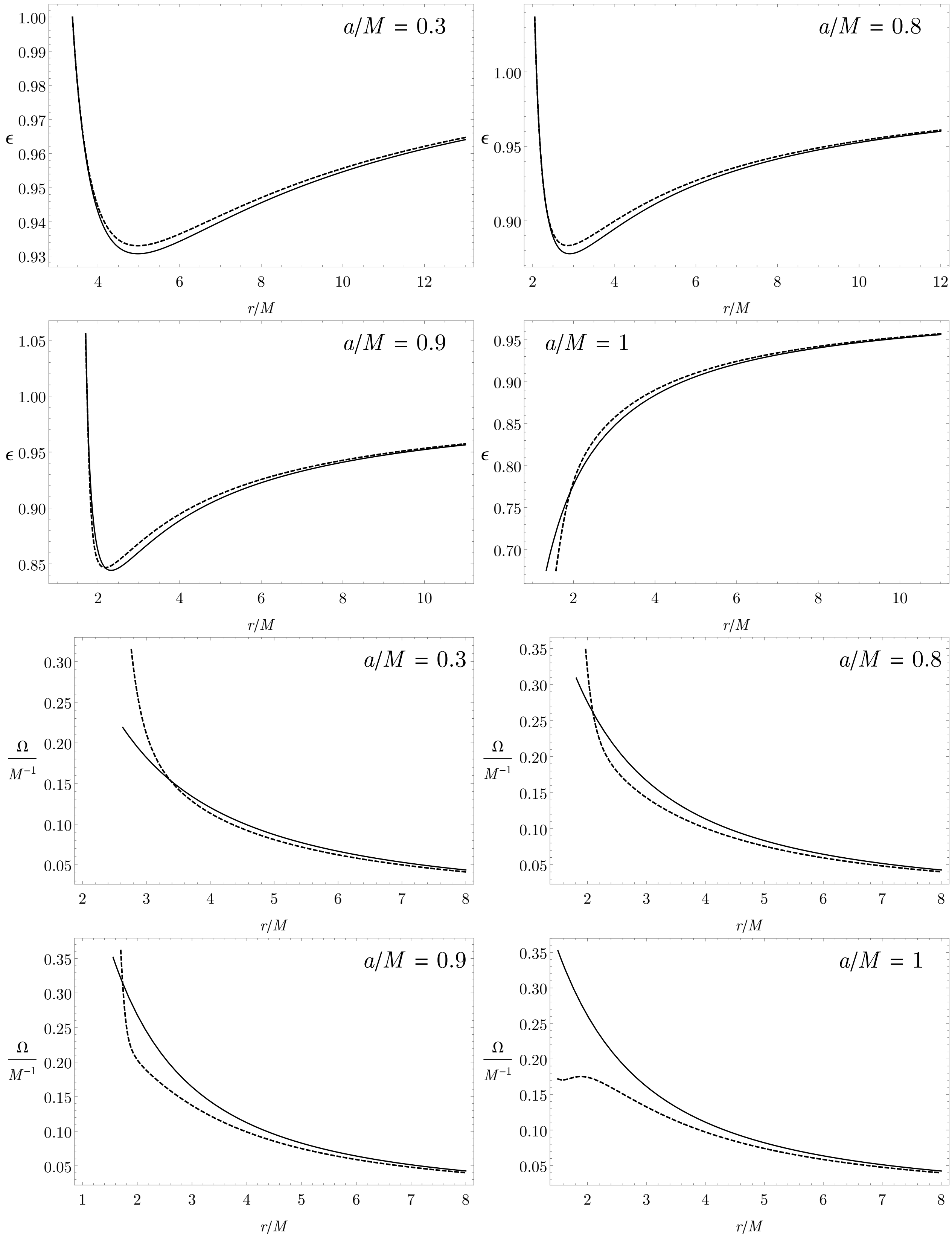

FIG. 2. Specific energy (top) and angular frequency (bottom) of corotating circular orbits of radius r in the Kerr space-time (full line) compared with the distribution as given by the pseudo-Kerr Hamiltonian (dashed). The plots of the angular frequencies always show the endpoint of the exact-Kerr relation whereas the pseudo-Kerr relations continue to grow up to the horizon. In the specific energy case, the exact-Kerr relation diverges at the endpoint for $a < M$ and thus cannot be depicted.

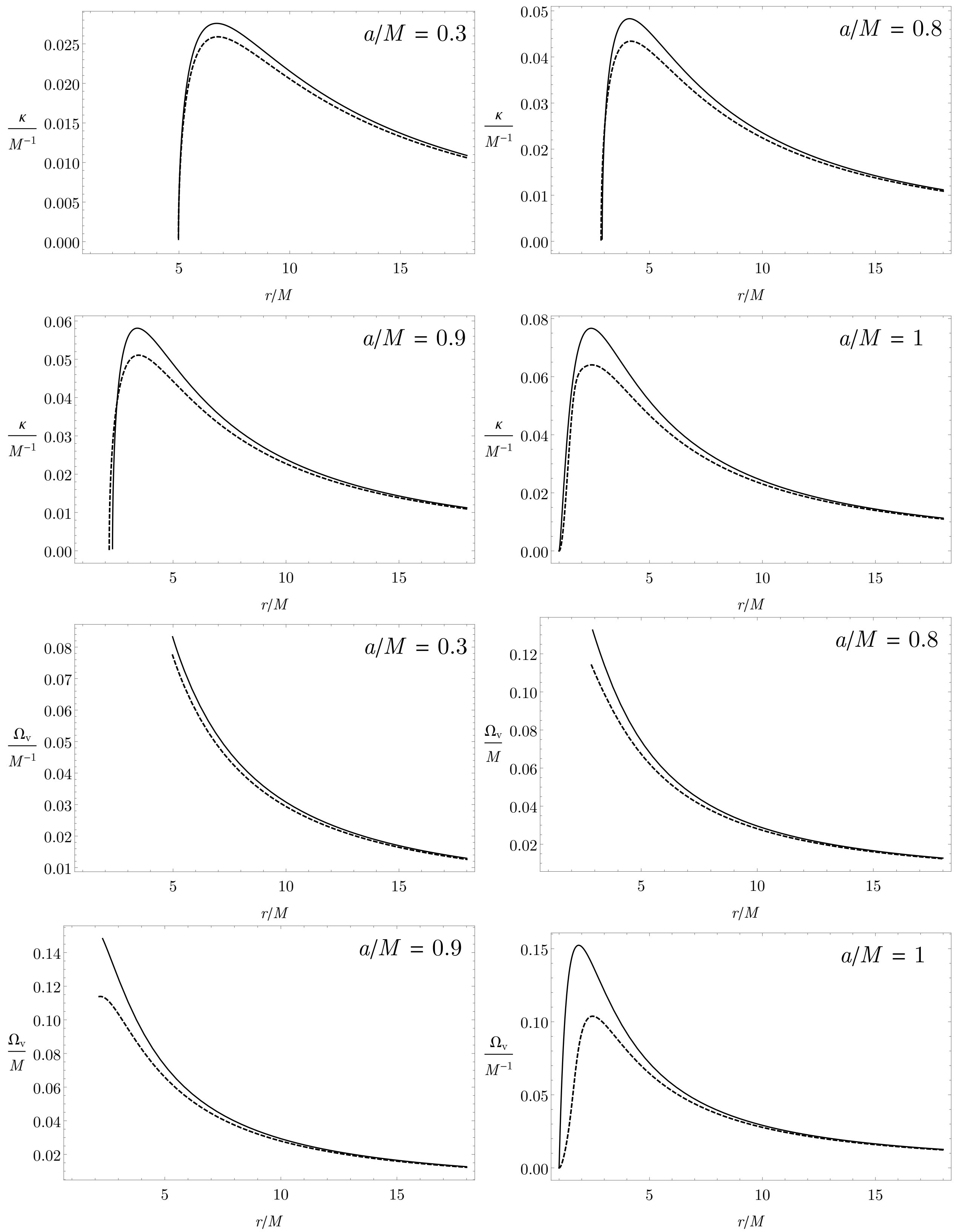

FIG. 3. Epicyclic (top) and vertical (bottom) oscillation frequency of stable corotating circular orbits of radius r in the Kerr space-time (solid line) compared with the distribution as given by the pseudo-Kerr Hamiltonian (dashed). Even though the relative error is larger than in the case of angular momentum u_ϕ, the qualitative correspondence is satisfied for every value of $a \lesssim 0.8M$. Around $a/M \in (0.9, 1)$ a qualitative transition starts taking place during which the correspondence is broken to be recovered for $a = M$.

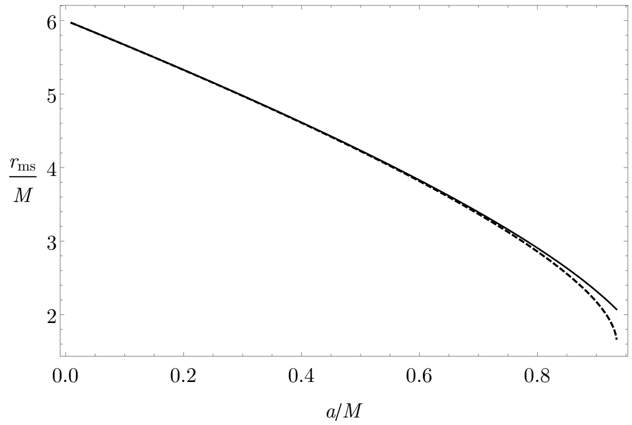

FIG. 4. Positions of marginally stable orbits r_{ms} in the Kerr space-time (full line) compared with the positions of marginally stable orbits as given by the pseudo-Kerr Hamiltonian (dashed). The exact-Kerr relation continues up to $a = M$ whereas the pseudo-Kerr one ends at $a = a_{crit} \approx 0.934$.

On the other hand, the marginally stable orbit vanishes "too soon" in the pseudo-Kerr dynamics, at $a_{crit} \approx 0.934M$. This critical spin value a_{crit} can be understood as the ultimate limit of applicability of the presented Hamiltonian. The exact-Kerr marginally stable orbit reads

$$r_{ms(K)} = 3M + Z_2 - \sqrt{(3M - Z_1)(3M + Z_1 + 2Z_2)},$$
$$Z_1 = M + (M^2 - a^2)^{1/3} \left[(M + a)^{1/3} + (M - a)^{1/3}\right],$$
$$Z_2 = \sqrt{3a^2 + Z_1^2}. \tag{49}$$

The value of r_{ms} for $a/M \in (0, 0.934)$, as obtained by numerical root finding in the pseudo-Kerr dynamics, is compared with the exact-Kerr values in Fig. 4. Once again we see that the correspondence is very strong for $a \lesssim 0.8$ and reasonable up to $a \approx 0.9M$.

E. Note on practical simulations

The Hamiltonian (37) might seem sufficiently effective for practical purposes, but it is also rather complicated and might not be worth the extra computational cost for a large range of situations. Nonetheless, since the presented Lagrangians and Hamiltonians have a seamless Newtonian limit, it is possible to naturally "switch off" parts of the dynamics for different regions of space-time and do not spend computational time on them.

The deviation of the pseudo-Kerr Hamiltonian (37) from the Hamiltonian of a test particle in the field of a Newtonian monopole are of two kinds, the static field corrections $\sim M/r$ and $\sim a^2/r^2$, and the dragging term $\sim Mau_\phi/r^3$ (all to leading order in $1/r$). To obtain an estimate independent of u_ϕ and a, we use the leading-order value for a circular orbit $u_\phi \approx \sqrt{Mr}$ and the maximum spin $a = M$ to get $Mau_\phi/r^3 \sim M^{5/2}/r^{5/2}$ and $a^2/r^2 \sim M^2/r^2$. Since we are only interested in $r > M$, we can conclude that the dragging term will always be less significant than the spin-static term $\sim M^2/r^2$.

For the sake of computation-time saving it is then convenient to choose a small dimensionless inaccuracy tolerance ε and switch between the near-blackhole dynamics in the following way (in the following discussion $M \leftrightarrow GM/c^2$). If $M/r < \varepsilon$ use purely Newtonian dynamics; if $\varepsilon < M/r < \sqrt{\varepsilon}$, use the Tejeda-Rosswog dynamics; if $M/r > \sqrt{\varepsilon}$, use the pseudo-Kerr Hamiltonian (37).

Since the intrinsic error of the Tejeda-Rosswog and the presented pseudo-Kerr dynamics is at least in orders of units of percent, a reasonable tolerance is $\varepsilon = 0.01$ because then the switch introduces about the same error as the approximate dynamics themselves. Hence, the pseudo-Kerr Hamiltonian should only be used from the near-horizon cut-off up to $r \approx 10M$ and the Tejeda-Rosswog Hamiltonian from $r \approx 10M$ up to $r \approx 100M$. Beyond $r \approx 100M$ it is pointless to use other than Newtonian dynamics, unless, of course, describing extremely fast objects for which none of the mentioned approximations are suited.

F. Remarks on the Ghosh-Sarkar-Bhadra Lagrangian

The dynamics presented by Ghosh *et al.* [1] are concordant with the presented approach in the idea of a low-energy limit, not, however, in the idea of phase-space reparametrization. Instead of covariant velocity components u_i, the dynamics are constructed by a series of ansatzes using the contravariant (canonically *non*-conjugate) components u^i. As a consequence, the dynamics are restricted only to the equatorial plane and it seems that there is no simple characterization of the Ghosh-Sarkar-Bhadra Lagrangian in terms of reparametrized geodesics.

Nonetheless, the latter approach seems to be plagued by analogous problems as the one presented in the current paper. Namely, the Ghosh-Sarkar-Bhadra Lagrangian

$$L_{GSB} = \frac{1}{2(r-2M)^2(1+\gamma\dot{\phi})}\left(\frac{r^3(r-2M)}{\Delta}\dot{r}^2 + \Delta r^2 \dot{\phi}^2\right) + \frac{M}{r}(1-\gamma\dot{\phi}), \tag{50}$$

where $\gamma = 2Ma/(r-2M)$, has a Hamiltonian form complicated beyond usefulness. For the angular momenta of circular orbits $\lambda_{\mathrm{GSB(C)}}$ it holds that

$$\lambda_{\mathrm{GSB(C)}} = \frac{-Q \pm \sqrt{Q^2 - 4R}}{2},$$
$$Q = \frac{4a^3 rM - 6Mar(r^2 + a^2)}{a^2 r(r - 2M) - r(r - 3M)(r^2 + a^2)}, \quad (51)$$
$$R = \frac{M(r^2 + a^2)[r(r^3 + 3a^2) - 2a^2 r]}{a^2 r(r - 2M) - r(r - 3M)(r^2 + a^2)}.$$

What was not clearly stated or shown in the original paper is the fact that this angular momentum distribution has a singularity at

$$a^2(r_{\mathrm{s}} - 2M) - (r_{\mathrm{s}} - 3M)(r_{\mathrm{s}}^2 + a^2) = 0, \quad (52)$$

for which the solution varies quite uniformly from $r_{\mathrm{s}} = 3M$ for $a = 0$ to $r_{\mathrm{s}} \approx 3.1M$ for $a = M$.

Even though the authors state that the marginally bound circular orbit exists up to $a \approx 0.7M$ and that the potential is thus useful up to such values, there is a possible issue with the marginally bound orbit; the angular momentum distribution (51) crosses the singularity (52) before reaching the radius of the marginally bound orbit already for $a \gtrsim 0.45M$. Amongst other things, this means that the Keplerian circular orbits have a "singular pause" before reaching the marginally bound orbit and the matter density of a stationary accretion disc would quite probably exhibit non-physical behaviour at the singular $r = r_{\mathrm{s}}$.

Hence, the Ghosh-Sarkar-Bhadra Lagrangian should be considered as useful only for $r \gtrsim 3.1M$, and if the marginally bound orbit is important in the given model, only $a \lesssim 0.45M$ should be considered. The point where even the marginally stable orbit collides with this singularity is $a \approx 0.7M$ (which is probably also the reason why the authors were not able to find the marginally stable orbit beyond that spin). Amongst other things, this means that for a near-Keplerian accretion disk near a center with spin $a \approx 0.7M$ the singularity (52) is very near its edge and exotic effects might ensue. Thus, it seems commendable to use the Lagrangian (50) only for spins well below $a \approx 0.7M$.

VII. CONCLUSION

We have shown that it is possible to formulate a general pseudo-Newtonian limit for geodesics in arbitrary space-times. The Hamiltonian (10) as given for a *general* space-time has a pathological inversion $\dot{x}^i(u_i, ...) \to u_i(\dot{x}^i, ...)$, and thus provides only a less direct description of particle motion (and no Lagrangian formalism). Nevertheless, under the initial-condition normalization (13), the Hamiltonian (10) yields exact null geodesics parametrized by coordinate time, thus possibly presenting a useful tool for gravitational lensing.

Generally, the pseudo-Newtonian limit behaves very well in "time-diagonal" space-times, i.e. where $g^{0i} = 0$. In these space-times, the description through physical velocities and accelerations is possible, and even the produced trajectories of massive particles correspond, at least in shape, to exact geodesics. A quick investigation into the possible applications of the pseudo-Newtonian limit given in Section V shows that the fluid dynamics, electromagnetic forces, or external gravitating sources must be handled in a less-than-naive approach.

An obvious benefit of the presented results is a unified framework for the estimate and understanding of errors in the particle dynamics of Tejeda and Rosswog [2, 7]. Even though the original ambition was to extend the results of Tejeda and Rosswog to the Kerr space-time, the usefulness of the presented limit seems to be, in fact, in the extension to light geodesics and to general time-diagonal space-times. The "time-non-diagonal" space-times ($g^{0i} \neq 0$) yield only a moderately satisfactory pseudo-Newtonian limit, as is demonstrated also in Section VI in the case of the Kerr space-time.

The current outlook is to finish the fluid formulation as sketched in Subsection V C, and apply it in the physical context of accretion onto a spinning compact center. The factor of the type $\sim 1/(1 - \omega u_\phi)$, as seen e.g. in the Hamiltonian (37), complicates the Legendre transform to Lagrangian formalism and to a direct description of the fluid in terms of its velocity. In the current moment it seems that the only workaround towards a relatively straight-forward Legendre transform is to expand the dragging factor as $1/(1-\omega u_\phi) \approx 1+\omega u_\phi - \omega^2 u_\phi^2 + ...$.

Nevertheless, the mentioned expansion in ωu_ϕ is essentially an expansion in the powers of a/M and would thus be inaccurate for the case of supermassive black holes, which are generally assumed to have spins $a \sim M$. On the other hand, even the fastest spinning known pulsars are estimated to have spins $a \lesssim 0.4M$ (e.g. [15]), so the $\mathcal{O}(a^2)$ expansion should yield reasonably accurate results for that case. The development of the pseudo-Newtonian framework for accretion onto neutron stars (utilizing an appropriate *non-Kerr* metric such as the one by Manko and Novikov [16]) or slowly spinning black holes will be the subject of future papers.

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