The Dynamics of the Forest Graph Operator

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Abstract

In 1966, Cummins introduced the "tree graph": the tree graph $\mathbf{T}(G)$ of a graph G (possibly infinite) has all its spanning trees as vertices, and distinct such trees correspond to adjacent vertices if they differ in just one edge, i.e., two spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The tree graph of a connected graph need not be connected. To obviate this difficulty we define the "forest graph": let G be a labeled graph of order α, finite or infinite, and let $\mathfrak{N}(G)$ be the set of all labeled maximal forests of G. The forest graph of G, denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests F_1, F_2 of G form an edge if and only if they differ exactly by one edge, i.e., $F_2 = F_1 - e + f$ for some edges $e \in F_1$ and $f \notin F_1$.

Using the theory of cardinal numbers, Zorn's lemma, transfinite induction, the axiom of choice and the well-ordering principle, we determine the $\mathbf{F}$-convergence, $\mathbf{F}$-divergence, $\mathbf{F}$-depth and $\mathbf{F}$-stability of any graph G. In particular it is shown that a graph G (finite or infinite) is $\mathbf{F}$-convergent if and only if G has at most one cycle of length 3. The $\mathbf{F}$-stable graphs are precisely K_3 and K_1. The $\mathbf{F}$-depth of any graph G different from K_3 and K_1 is finite. We also determine various parameters of $\mathbf{F}(G)$ for an infinite graph G, including the number, order, size, and degree of its components.

Keywords: Forest graph operator; Graph dynamics.

Mathematics Subject Clasification 2010: Primary 05C76; Secondary 05C05, 05C63

1 Introduction

A *graph dynamical system* is a set X of graphs together with a mapping $\phi : X \to X$ (see Prisner [12]). We investigate the graph dynamical system on finite and infinite graphs defined by the forest graph operator $\mathbf{F}$, which transforms G to its graph of maximal forests.

Let G be a labeled graph of order α, finite or infinite. (All our graphs are labeled.) A *spanning tree* of G is a connected, acyclic, spanning subgraph of G; it exists if and only if G is connected. Any acyclic subgraph of G, connected or not, is called a *forest* of G. A forest F of G is said to be *maximal* if there is no forest F' of G such that F is a proper subgraph of F'. The tree graph $\mathbf{T}(G)$ of G has all the spanning trees of G as vertices, and distinct such trees are adjacent vertices if they differ in just one edge [12, 15]; i.e., two spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The *iterated tree graphs* of G are defined by $\mathbf{T}^0(G) = G$ and $\mathbf{T}^n(G) = \mathbf{T}(\mathbf{T}^{n-1}(G))$ for $n > 0$. There are several results on tree graphs. See [1, 18, 11] for connectivity of the tree graph, [8, 13, 16, 19, 4, 7, 10, 3, 6] for bounds on the order of $\mathbf{T}(G)$ (that is, on the number of spanning trees of G), [2, 14] for Hamilton circuits in a tree graph.

There is one difficulty with iterating the tree graph operator. The tree graph of an infinite connected graph need not be connected [2, 14], so $\mathbf{T}^2(G)$ may be undefined. For example, $\mathbf{T}(K_{\aleph_0})$ is disconnected (see Corollary 2.5 in this paper; $\aleph_0$ denotes the cardinality of the set $\mathbb{N}$ of natural numbers); therefore $\mathbf{T}^2(K_{\aleph_0})$ is not defined. To obviate this difficulty with iterated tree graphs, and inspired by the tree graph operator $\mathbf{T}$, we define a forest graph operator. Let $\mathfrak{M}(G)$ be the set of all maximal forests of G. The *forest graph* of G, denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{M}(G)$ in which two maximal forests F_1, F_2 form an edge if and only if they differ by exactly one edge. The *forest graph operator* (or *maximal forest operator*) on graphs, $G \mapsto \mathbf{F}(G)$, is denoted by $\mathbf{F}$. Zorn's lemma implies that every connected graph contains a spanning tree (see [5]); similarly, every graph has a maximal forest. Hence, the forest graph always exists. Since when G is connected, maximal forests are the same as spanning trees, then $\mathbf{F}(G) = \mathbf{T}(G)$; that is, the tree graph is a special case of the forest graph. We write $\mathbf{F}^2(G)$ to denote $\mathbf{F}(\mathbf{F}(G))$, and in general $\mathbf{F}^n(G) = \mathbf{F}(\mathbf{F}^{n-1}(G))$ for $n \geq 1$, with $\mathbf{F}^0(G) = G$.

Definition 1.1. A graph G is said to be $\mathbf{F}$-*convergent* if $\{\mathbf{F}^n(G) : n \in \mathbb{N}\}$ is finite; otherwise it is $\mathbf{F}$-*divergent.*

A graph H is said to be an $\mathbf{F}$-*root* of G if $\mathbf{F}(H)$ is isomorphic to G, $\mathbf{F}(H) \cong G$. The $\mathbf{F}$-*depth* of G is

$$\sup\{n \in \mathbb{N} : G \cong \mathbf{F}^n(H) \text{ for some graph } H\}.$$

The $\mathbf{F}$-depth of a graph G that has no $\mathbf{F}$-root is said to be zero.

The graph G is said to be $\mathbf{F}$-*periodic* if there exists a positive integer n such that $\mathbf{F}^n(G) = G$. The least such integer is called the $\mathbf{F}$-*periodicity* of G. If $n = 1$, G is called $\mathbf{F}$-*stable.*

This paper is organized as follows. In Section 2 we give some basic results. In later sections, using Zorn's lemma, transfinite induction, the well ordering principle and the theory of cardinal numbers, we study the number of $\mathbf{F}$-roots and determine the $\mathbf{F}$-convergence, $\mathbf{F}$-divergence, $\mathbf{F}$-depth and $\mathbf{F}$-stability of any graph G. In particular we show that: i) A graph G is $\mathbf{F}$-convergent if and only if G has at most one cycle of length 3. ii) The $\mathbf{F}$-depth of any graph G different from K_3 and K_1 is finite. iii) The $\mathbf{F}$-stable graphs are precisely K_3 and K_1. iv) A graph that has one $\mathbf{F}$-root has innumerably many, but only some $\mathbf{F}$-roots are important.

2 Preliminaries

For standard notation and terminology in graph theory we follow Diestel [5] and Prisner [12].

Some elementary properties of infinite cardinal numbers that we use are (see, e.g., Kamke [9]):

(1) $\alpha + \beta = \alpha.\beta = \max(\alpha, \beta)$ if α, β are cardinal numbers and β is infinite. In particular, $2.\beta = \aleph_0.\beta = \beta$.

(2) $\beta^n = \beta$ if β is an infinite cardinal and n is a positive integer.

(3) $\beta < 2^\beta$ for every cardinal number.

(4) The number of finite subsets of an infinite set of cardinality β is equal to β.

We consider finite and infinite labeled graphs *without multiple edges or loops*. An *isthmus* of a graph G is an edge e such that deleting e divides one component of G into two of $G - e$. Equivalently, an isthmus is an edge that belongs to no cycle. Each isthmus is in every maximal forest, but no non-isthmus is.

Let $\mathfrak{C}(G)$ and $\mathfrak{N}(G)$ denote the set of all possible cycles and the set of all maximal forests of a graph G, respectively. Note that a maximal forest of G consists of a spanning tree in each component of G. A fundamental fact, whose proof is similar to that of the existence of a maximal forest, is the following forest extension lemma:

Lemma 2.1. *In any graph G, every forest is contained in a maximal forest.*

Lemma 2.2. *If G is a complete graph of infinite order α, then $|\mathfrak{N}(G)| = 2^\alpha$.*

Proof: Let $G = (V, E)$ be a complete graph of order α (α infinite), i.e., $G = K_\alpha$. Let v_1, v_2 be two vertices of G and $V' = V \setminus \{v_1, v_2\}$. Then for every $A \subseteq V'$ there is a spanning tree T_A such that every vertex of A is adjacent only to v_1 and every vertex of $V' \setminus A$ is adjacent only to v_2. It is easy to see that $T_A \neq T_B$ whenever $A \neq B$. As the cardinality of the power set of V' is 2^α, there are at least 2^α spanning trees of G. Since G is connected, the maximal forests are the spanning trees; therefore $|\mathfrak{N}(G)| \geq 2^\alpha$. Since the degree of each vertex is α and G contains α vertices, the total number of edges in G is $\alpha.\alpha = \alpha$. The edge set of a maximal forest of G is a subset of E and the number of all possible subsets of E is 2^α. Therefore, G has at most 2^α maximal forests, i.e., $|\mathfrak{N}(G)| \leq 2^\alpha$. Hence $|\mathfrak{N}(G)| = 2^\alpha$. ■

For two maximal forests of G, F_1 and F_2, let $d(F_1, F_2)$ denote the distance between them in $\mathbf{F}(G)$. We connect this distance to the number of edges by which F_1, F_2 differ; the result is elementary but we could not find it anywhere in the literature. We say F_1, F_2 *differ by l edges* if $|E(F_1) \setminus E(F_2)| = |E(F_2) \setminus E(F_1)| = l$.

Lemma 2.3. *Let l be a natural number. For two maximal forests F_1, F_2 of a graph G, if $|E(F_1) \setminus E(F_2)| = l$, then $|E(F_2) \setminus E(F_1)| = l$. Furthermore, F_1 and F_2 differ by exactly l edges if and only if $d(F_1, F_2) = l$.*

We cannot apply to an infinite graph the simple proof for finite graphs, in which the number of edges in a maximal forest is given by a formula. Therefore, we prove the lemma by edge exchange.

Proof: We prove the first part by induction on l. Let F_1, F_2 be maximal forests of G and let $E(F_1) \setminus E(F_2) = \{e'_1, e'_2, \ldots, e'_k\}$, $E(F_2) \setminus E(F_1) = \{e_1, e_2, \ldots, e_l\}$. If $l = 0$ then $k = 0 = l$ because $F_2 = F_1$. Suppose $l > 0$; then $k > 0$ also. Deleting e_l from F_2 divides a tree of F_2 into two trees. Since these trees are in the same component of G, there is an edge of F_1 that connects them; this edge is not e_1 so it is not in F_2; therefore, it is an e'_i, say e'_k. Let $F'_2 = F_2 - e_l + e'_k$. Then $E(F_1) \setminus E(F'_2) = \{e'_1, e'_2, \ldots, e'_{k-1}\}$, $E(F_2) \setminus E(F_1) = \{e_1, e_2, \ldots, e_{l-1}\}$. By induction, $k - 1 = l - 1$.

We also prove the second part by induction on l. Assume F_1, F_2 differ by exactly l edges and define F'_2 as above. If $l = 0, 1$, clearly $d(F_1, F_2) = l$. Suppose $l > 1$. In a shortest path from F_1 to F_2, whose length is $d(F_1, F_2)$, each successive edge of the path can increase the number of edges not in F_1 by at most 1. Therefore, F_1 and F_2 differ by at most $d(F_1, F_2)$ edges. That is, $l \leq d(F_1, F_2)$. Conversely, $d(F_1, F'_2) = l - 1$ by induction and there is a path in $\mathbf{F}(G)$ from F_1 to F'_2 of length $l-1$, then continuing to F_2 and having total length l. Thus, $d(F_1, F_2) \leq l$. ■

From the above lemma we have two corollaries.

Corollary 2.4. *For any graph G, $\mathbf{F}(G)$ is connected if and only if any two maximal forests of G differ by at most a finite number of edges.*

Corollary 2.5. *If $G = K_\alpha$, α infinite, then $\mathbf{F}(G)$ is disconnected.*

Lemma 2.6. *Let G be a graph with α vertices and β edges and with no isolated vertices. If either α or β is infinite, then $\alpha = \beta$.*

Proof: We know that $|E(G)| \leq |V(G)|^2$, i.e., $\beta \leq \alpha^2$ so if β is infinite, α must also be infinite. We also know, since each edge has two endpoints, that $|V(G)| \leq 2|E(G)|$, i.e., $\alpha \leq 2.\beta$ so if α is infinite, then β must be infinite. Now assuming both are infinite, $\alpha^2 = \alpha$ and $2.\beta = \beta$, hence $\alpha = \beta$. ■

The following lemmas are needed in connection with $\mathbf{F}$-convergence and $\mathbf{F}$-divergence in Section 5 and $\mathbf{F}$-depth in Section 6.

Lemma 2.7. *Let G be a graph. If K_n (for finite $n \geq 2$) is a subgraph of G, then $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $\mathbf{F}(G)$.*

Proof: Let G be a graph such that K_n ($n \geq 2$, finite) is a subgraph of G with vertex labels $v_1, v_2, \ldots, v_n$. Then there is a path $L = v_1, v_2, \ldots, v_n$ of order n in G. Let F be a maximal forest of G such that F contains the path L. In F if we replace the edge $v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1}$ by any other edge $v_i v_j$ where $i = 1, \ldots, \lfloor n/2 \rfloor$ and $j = \lfloor n/2 \rfloor + 1, \ldots, n$, we get a maximal forest F_{ij}. Since there are $\lfloor n^2/4 \rfloor$ such edges $v_i v_j$, there are $\lfloor n^2/4 \rfloor$ maximal forests F_{ij} (of which one is F). Any two forests F_{ij} differ by one edge. It follows that they form a complete subgraph in $\mathbf{F}(G)$. Therefore $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $\mathbf{F}(G)$. ■

Lemma 2.8. *If G has a cycle of (finite) length n with $n \geq 3$, then $\mathbf{F}(G)$ contains K_n.*

Proof: Suppose that G has a cycle C_n of length n with edge set $\{e_1, e_2, \ldots, e_n\}$. Let $P_i = C_n - e_i$ for $i = 1, 2, \ldots, n$ and let F_1 be a maximal forest of G containing the path P_1. Define $F_i = F_1 \setminus P_1 \cup P_i$ for $i = 2, 3, \ldots, n$. These F_i's are maximal forests of G and any two of them differ by exactly one edge, so they form a complete graph K_n in $\mathbf{F}(G)$. ∎

In particular, $\mathbf{F}(C_n) = K_n$.

Lemma 2.9. *Suppose that G contains K_n, where $n \geq 3$. Then $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$.*

Proof: Cayley's formula states that K_n has n^{n-2} spanning trees. Cummins [2] proved that the tree graph of a finite connected graph is Hamiltonian. Therefore, $\mathbf{F}(K_n)$ contains $C_{n^{n-2}}$. Let F_{T_0} be a spanning tree of G that extends one of the spanning trees T_0 of the K_n subgraph. Replacing the edges of T_0 in F_{T_0} by the edges of any other spanning tree T of K_n, we have a spanning tree F_T that contains T. The F_T's for all spanning trees T of K_n are n^{n-2} spanning trees of G that differ only within K_n; thus, the graph of the F_T's is the same as the graph of the T's, which is Hamiltonian. That is, $\mathbf{F}(G)$ contains $C_{n^{n-2}}$. By Lemma 2.8, $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$. ∎

We do not know exactly what graphs $\mathbf{F}(K_n)$ and $\mathbf{F}^2(K_n)$ are.

Lemma 2.10. *If G has two edge disjoint triangles, then $\mathbf{F}^2(G)$ contains K_9.*

Proof: Suppose that G has two edge disjoint triangles whose edges are e_1, e_2, e_3 and f_1, f_2, f_3, respectively. The union of the triangles has exactly 9 maximal forests F'_{ij}, obtained by deleting one e_i and one f_j from the triangles. Extend F'_{11} to a maximal forest F_{11} and let F_{ij} be the maximal forest $F_{11} \setminus E(F'_{11}) \cup F_{ij}$, for each $i, j = 1, 2, 3$. The nine maximal forests F'_{ij}, and consequently the maximal forests F_{ij} in $\mathbf{F}(G)$, form a Cartesian product graph $C_3 \times C_3$, which contains a cycle of length 9. By Lemma 2.8, $\mathbf{F}^2(G)$ contains K_9. ∎

We now show that repeated application of the forest graph operator to many graphs creates larger and larger complete subgraphs.

Lemma 2.11. *If G has a cycle of (finite) length n with $n \geq 4$ or it has two edge disjoint triangles, then for any finite $m \geq 1$, $\mathbf{F}^m(G)$ contains K_{m^2}.*

Proof: We prove this lemma by induction on m.

Case 1: Suppose that G has a cycle C_n of length n ($n \geq 4$, n finite). By Lemma 2.8, $\mathbf{F}(G)$ contains K_n as a subgraph, which implies that $\mathbf{F}(G)$ contains K_4. By Lemma 2.9, $\mathbf{F}^3(G)$ contains K_{16} and in particular it contains K_{3^2}.

Case 2: Suppose that G has two edge disjoint triangles. By Lemma 2.10 $\mathbf{F}^2(G)$ contains K_9 as a subgraph. It follows by Lemma 2.7 that $\mathbf{F}^3(G)$ contains $K_{\lfloor 9^2/4 \rfloor} = K_{20}$ as a subgraph. This implies that $\mathbf{F}^3(G)$ contains K_{3^2} as a subgraph.

By Cases 1 and 2 it follows that the result is true for $m = 1, 2, 3$. Let us assume that the result is true for $m = l \geq 3$, i.e., that $\mathbf{F}^l(G)$ contains K_{l^2} as a subgraph. By Lemma 2.7 it follows that $\mathbf{F}(\mathbf{F}^l(G))$ has a subgraph $K_{\lfloor l^4/4 \rfloor}$. Since $\lfloor l^4/4 \rfloor > (l+1)^2$, it follows that $\mathbf{F}^{l+1}(G)$ contains $K_{(l+1)^2}$. By the induction hypothesis $\mathbf{F}^m(G)$ contains K_{m^2} for any finite $m \geq 1$. ∎

With Lemma 2.9 it is clearly possible to prove a much stronger lower bound on complete subgraphs of iterated forest graphs, but Lemma 2.11 is good enough for our purposes.

Lemma 2.12. *A forest graph that is not K_1 has no isolated vertices and no isthmi.*

Proof: Let $G = \mathbf{F}(H)$ for some graph H. Consider a vertex F of G, that is, a maximal forest in H. Let e be an edge of F that belongs to a cycle C in H. Then there is an edge f in C that is not in F and $F' = F - e + f$ is a second maximal forest that is adjacent to F in G. Since C has length at least 3, it has a third edge g. If g is not in F, let $F'' = F - e + g$. If g is in F, let $F'' = F - g + f$. In both cases F'' is a maximal forest that is adjacent to F and F'. Thus, F is not isolated and the edge FF' in G is not an isthmus.

Suppose $F, F' \in \mathfrak{N}(H)$ are adjacent in G. That means there are edges $e \in E(F)$ and $e' \in E(F')$ such that $F' = F - e + e'$. Thus, e belongs to the unique cycle in $F + e'$. As shown above, there is an $F'' \in \mathfrak{N}(H)$ that forms a cycle with F and F'. Therefore the edge FF' of G is not an isthmus.

Let $F \in \mathfrak{N}(H)$ be an isolated vertex in G. If H has an edge e not in F, then $F + e$ contains a cycle so F has a neighboring vertex in G, as shown above. Therefore, no such e can exist; in other words, $H = F$ and G is K_1. ∎

3 Basic Properties of an Infinite Forest Graph

We now present a crucial foundation for the proof of the main theorem in Section 5. The *cyclomatic number* $\beta_1(G)$ of a graph G can be defined as the cardinality $|E(G) \setminus E(F)|$ where F is a maximal forest of G.

Proposition 3.1. *Let G be a graph such that $|\mathfrak{C}(G)| = \beta$, an infinite cardinal number. Then:*

i) *$\beta_1(G) = \beta$ and $\beta_1(\mathbf{F}(G)) = 2^\beta$.*

ii) *Both the order of $\mathbf{F}(G)$ and its number of edges equal 2^β. Both the order and the number of edges of G equal β, provided that G has no isolated vertices and no isthmi.*

iii) *$\mathbf{F}(G)$ is β-regular.*

iv) *The order of any connected component of $\mathbf{F}(G)$ is β, and it has exactly β edges.*

v) *$\mathbf{F}(G)$ has exactly 2^β components.*

vi) *Every component of $\mathbf{F}(G)$ has exactly β cycles.*

vii) *$|\mathfrak{C}(\mathbf{F}(G))| = 2^\beta$.*

Proof: Let G be a graph with $|\mathfrak{C}(G)| = \beta$ (β infinite).

i) Let F be a maximal forest of G. The number of cycles in G is not more than the number of finite subsets of $E(G) \setminus E(F)$. This number is finite if $E(G) \setminus E(F)$ is finite, but it cannot be finite because $|\mathfrak{C}(G)|$ is infinite. Therefore $E(G) \setminus E(F)$ is infinite and the number of its finite subsets equals $|E(G) \setminus E(F)| = \beta_1(G)$. Thus, $\beta_1(G) \geq |\mathfrak{C}(G)|$. The number of cycles is at least as large as the number of edges not in F, because every such edge makes a different cycle with F. Thus, $|\mathfrak{C}(G)| \geq \beta_1(G)$. It follows that $\beta_1(G) = |\mathfrak{C}(G)| = \beta$. Note that this proves $\beta_1(G)$ does not depend on the choice of F.

The value of $\beta_1(\mathbf{F}(G))$ follows from this and part (vii).

ii) For the first part, let F be a maximal forest of G and let F_0 be a maximal forest of $G \setminus E(F)$. As $G \setminus E(F)$ has $\beta_1(G) = \beta$ edges by part (i), it has β non-isolated vertices by Lemma 2.6. F_0 has the same non-isolated vertices, so it too has β edges.

Any edge set $A \subseteq F_0$ extends to a maximal forest F_A in $F \cup A$. Since $F_A \setminus F = A$, the F_A's are distinct. Therefore, there are at least 2^β maximal forests in $F_0 \cup F$. The maximal forest F consists of a spanning tree in each component of G; therefore, the vertex sets of components of F are the same as those of G, and so are those of $F_0 \cup F$. Therefore, a maximal forest in $F_0 \cup F$, which consists of a spanning tree in each component of $F_0 \cup F$, contains a spanning tree of each component of G.

We conclude that a maximal forest in $F_0 \cup F$ is a maximal forest of G and hence that there are at least 2^β maximal forests in G, i.e., $|\mathfrak{N}(G)| \geq 2^\beta$. Since G is a subgraph of K_β, and since $|\mathfrak{N}(K_\beta)| = 2^\beta$ by Lemma 2.2, we have $|\mathfrak{N}(G)| \leq 2^\beta$. Therefore $|\mathfrak{N}(G)| = 2^\beta$. That is, the order of $\mathbf{F}(G)$ is 2^β. By Lemmas 2.12 and 2.6, that is also the number of edges of $\mathbf{F}(G)$.

For the second part, note that G has infinite order or else $\beta_1(G)$ would be finite. If G has no isolated vertices and no isthmi, then $|V(G)| = |E(G)|$ by Lemma 2.6. By part (i) there are β edges of G outside a maximal forest; hence $\beta \leq |E(G)|$.

Since every edge of G is in a cycle, by the axiom of choice we can choose a cycle $C(e)$ containing e for each edge e of G. Let $\mathfrak{C} = \{C(e) : e \in E(G)\}$. The total number of pairs (f, C) such that $f \in C \in \mathfrak{C}$ is no more than $\aleph_0.|\mathfrak{C}| \leq \aleph_0.|\mathfrak{C}(G)| = \aleph_0.\beta = \beta$. This number of pairs is not less than the number of edges, so $|E(G)| \leq \beta$. It follows that G has exactly β edges.

iii) Let F be a maximal forest of G. By part (i), $|E(G) \setminus E(F)| = \beta$. By adding any edge e from $E(G) \setminus E(F)$ to F we get a cycle C. Removing any edge other than e from the cycle C gives a new maximal forest which differs by exactly one edge with F. The number of maximal forests we get in this way is $\beta_1(G)$ because there are $\beta_1(G)$ ways to choose e and a finite number of edges of C to choose to remove, and $\beta_1(G)$ is infinite. Thus we get β maximal forests of G, each of which differs by exactly one edge with F. Every such maximal forest is generated by this construction. Therefore, the degree of any vertex in $\mathbf{F}(G)$ is β.

iv) Let A be a connected component of $\mathbf{F}(G)$. As $\mathbf{F}(G)$ is β-regular by part (iii), it follows that $|V(A)| \geq \beta$. Fix a vertex v in A and define the n^{th} neighborhood $D_n = \{v' : d(v, v') = n\}$ for each n in $\mathbb{N}$. Since every vertex has degree β, $|D_0| = 1$, $|D_1| = \beta$ and $|D_k| \leq \beta|D_{k-1}|$. Thus, by induction on n, $|D_n| \leq \beta$ for $n > 0$.

Since A is connected, it follows that $V(A) = \bigcup_{i \in \mathbb{N} \cup \{0\}} D_i$, i.e., $V(A)$ is the countable union of sets of order β. Therefore $|A| = \beta$, as $|\mathbb{N}|.\beta' = \beta'$. Hence any connected component of $\mathbf{F}(G)$ has β vertices. By Lemma 2.6 it has β edges.

v) By parts (ii, iv) the order of $\mathbf{F}(G)$ is 2^β and the order of each component of $\mathbf{F}(G)$ is β. Since $|\mathbf{F}(G)| = 2^\beta$, $\mathbf{F}(G)$ has at most 2^β components. Suppose that $\mathbf{F}(G)$ has β' components where $\beta' < 2^\beta$. As each component has β vertices, it follows that $\mathbf{F}(G)$ has order at most $\beta'.\beta = \max\{\beta', \beta\}$. This is a contradiction to part (ii). Therefore $\mathbf{F}(G)$ has exactly 2^β components.

vi) Let A be a component of $\mathbf{F}(G)$. Since it is infinite, by part (iv) it has exactly β edges. Suppose that $|\mathfrak{C}(A)| = \beta'$. Then β' is at most the number of finite subsets of $E(A)$, which is β since $|E(A)| = \beta$ is infinite; that is, $\beta' \leq \beta$. By the argument in part (iii) every edge of $\mathbf{F}(G)$ lies on a cycle. The length of each cycle is finite. Thus A has at most $\aleph_0.\beta' = \max\{\beta', \aleph_0\} = \beta'$ edges if β' is infinite and it has a finite number of edges if β' is finite. Since $|E(A)| = \beta$, which is infinite, $\beta' \geq \beta$. We conclude that $\beta' = \beta$.

vii) By parts (v, vi) $\mathbf{F}(G)$ has 2^β components and each component has β cycles. Since every cycle is contained in a component, $|\mathfrak{C}(\mathbf{F}(G))| = \beta.2^\beta = 2^\beta$. ■

From the above proposition it follows that an infinite graph cannot be a forest graph unless every component has the same infinite order β and there are 2^β components. A consequence is that the infinite graph itself must have order 2^β. Hence,

Corollary 3.2. *Any infinite graph whose order is not a power of 2, including $\aleph_0$ and all other limit cardinals, is not a forest graph.*

Corollary 3.3. *For a graph G the following statements are equivalent.*

i) $\mathbf{F}(G)$ *is connected.*

ii) $\mathbf{F}(G)$ *is finite.*

iii) *The union of all cycles in G is a finite graph.*

Proof: (i) $\Longrightarrow$ (iii). Suppose that $\mathbf{F}(G)$ is connected. If G has infinitely many cycles then by Proposition 3.1(v) $\mathbf{F}(G)$ is disconnected. Therefore G has finitely many cycles. Let $A = \{e \in E(G) : \text{edge } e \text{ lies on a cycle in } G\}$. Then $|A|$ is finite because the length of each cycle is finite. That proves (iii).

(iii) $\Longrightarrow$ (ii). As every maximal forest of G consists of a maximal forest of A and all the edges of G which are not in A, G has at most 2^n maximal forests where $n = |A|$. Hence $\mathbf{F}(G)$ has a finite number of vertices and consequently is finite.

(ii) $\Longrightarrow$ (i). By identifying vertices in different components (Whitney vertex identification; see Section 4) we can assume G is connected so $\mathbf{F}(G) = \mathbf{T}(G)$. Cummins [2] proved that the tree graph of a finite graph is Hamiltonian; therefore it is connected. ■

4 F-Roots

In this section we establish properties of **F**-roots of graphs. We begin with the question of what an **F**-root should be.

Since any graph H' that is isomorphic to an **F**-root H of G is immediately also an **F**-root, the number of non-isomorphic **F**-roots is a better question than the number of labeled **F**-roots. We now show in some detail that a still better question is the number of non-isomorphic **F**-roots without isthmi.

Let t_β be the number of non-isomorphic rooted trees of order β. We note that $t_{\aleph_0} \geq 2^{\aleph_0}$, by a construction of Reinhard Diestel (personal communication, July 10, 2015). (We do not know a corresponding lower bound on t_β for $\beta > \aleph_0$.) Let P be a one-way infinite path whose vertices are labelled by natural numbers, with root 1; choose any subset S of $\mathbb{N}$ and attach two edges at every vertex in S, forming a rooted tree T_S (rooted at 1). Then S is determined by T_S because the vertices in S are those of degree at least 3 in T_S. (If $2 \in S$ but $1 \notin S$, then vertex 1 is determined only up to isomorphism by T_S, but S itself is determined uniquely.) The number of sets S is $2^{\aleph_0}$, hence $t_{\aleph_0} \geq 2^{\aleph_0}$.

Proposition 4.1. *Let G be a graph with an* **F***-root of order α. If α is finite, then G has infinitely many non-isomorphic finite* **F***-roots. If α is finite or infinite, then G has at least t_β non-isomorphic* **F***-roots of order β for every infinite $\beta \geq \alpha$.*

Proof: Let G be a graph which has an **F**-root H, i.e., $\mathbf{F}(H) \cong G$, and let α be the order of H. We may assume H has no isthmi and no isolated vertices unless it is K_1.

Suppose α is finite; then let T be a tree, disjoint from H, of any finite order n. Identify any vertex v of H with any vertex w of T. The resulting graph H_T also has G as its forest graph since T is contained in every maximal forest of H_T. As the order of H_T is $\alpha + n - 1$ and n can be any natural number, the graphs H_T are an infinite number of non-isomorphic finite graphs with the same forest graph up to isomorphism.

Suppose α is finite or infinite and $\beta \geq \alpha$ is infinite. Let T be a rooted tree of order β with root vertex w; for instance, T can be a star rooted at the star center. Attach T to a vertex v of H by identifying v with the root vertex w. Denote the resulting graph by H_T; it is an **F**-root of G and it has order β because it has order $\alpha + \beta$, which equals β because β is infinite and $\beta \geq \alpha$. As H has no isthmi, T and w are determined by H_T; therefore, if we have a non-isomorphic rooted tree T' with root w' (that means there is no isomorphism of T with T' in which w corresponds to w'), $H_{T'}$ is not isomorphic to H_T. (The one exception is when $H = K_1$, which is easy to treat separately.) The number of non-isomorphic **F**-roots of G of order β is therefore at least the number of non-isomorphic rooted trees of order β, i.e., t_β. ∎

Proposition 4.1 still does not capture the essence of the number of **F**-roots. Whitney's *2-operations* on a graph G are the following [17]:

(1) *Whitney vertex identification.* Identify a vertex in one component of G with a vertex in a another component of G, thereby reducing the number of components by 1. For an infinite graph we modify this by allowing an infinite number of vertex identifications; specifically, let W be a set of vertices with at most one from each component of G, and let $\{W_i : i \in I\}$ be a partition of W into $|I|$ sets (where I is any index set); then for each $i \in I$ we identify all the vertices in W_i with each other.

(2) *Whitney vertex splitting.* The reverse of vertex identification.

(3) *Whitney twist.* If u, v are two vertices that separate G—that is, $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{u, v\}$ and $|V(G_1)|, |V(G_2)| > 2$, then reverse the names u and v in G_2 and then take the union $G_1 \cup G_2$ (so vertex u in G_1 is identified with the former vertex v in G_2 and v with the former vertex u). Call the new graph G'. For an infinite graph we allow an infinite number of Whitney twists.

It is easy to see that the edge sets of maximal forests in G and G' are identical, hence $\mathbf{F}(G)$ and $\mathbf{F}(G')$ are naturally isomorphic. It follows by Whitney vertex identification that every graph with an **F**-root has a connected **F**-root, and it follows from Whitney vertex splitting that every graph with an F-root has an **F**-root without cut vertices.

We may conclude from Proposition 4.1 that the most interesting question about the number of **F**-roots of a graph G that has an **F**-root is not the total number of non-isomorphic **F**-roots (which by Proposition 4.1 cannot be assigned any cardinality); it is not the number of a given order; it is not even the number that have no isthmi; it is the number of non-2-isomorphic, connected **F**-roots with no isthmi and (except when $G = K_1$) no isolated vertices.

We do not know which graphs have **F**-roots, but we do know two large classes that cannot have **F**-roots.

Theorem 4.2. *No infinite connected graph has an* **F***-root.*

Proof: This follows by Corollary 3.3. ■

Theorem 4.3. *No bipartite graph G has an* **F***-root.*

Proof: Let G be a bipartite graph of order p ($p \geq 2$) and let H be a root of G, i.e., $\mathbf{F}(H) \cong G$. Suppose H has no cycle; then $\mathbf{F}(H)$ is K_1, which is a contradiction. Therefore H has a cycle of length ≥ 3. It follows by Lemma 2.8 that $\mathbf{F}(H)$ contains K_3, a contradiction. Hence no bipartite graph G has a root. ■

5 F-Convergence and F-Divergence

In this section we establish the necessary and sufficient conditions for **F**-convergence of a graph.

Lemma 5.1. *Let G be a finite graph that contains a C_n (for $n \geq 4$) or at least two edge disjoint triangles; then G is* **F***-divergent.*

Proof: Let G be a finite graph. By Lemma 2.11, $\mathbf{F}^m(G)$ contains K_{m^2} as a subgraph. Therefore, as m increases the clique size of $\mathbf{F}^m(G)$ increases. Hence G is **F**-divergent. ■

Lemma 5.2. *If $|\mathfrak{C}(G)| = \beta$ where β is infinite, then G is* **F***-divergent.*

Proof: Assume $|\mathfrak{C}(G)| = \beta$ (β infinite). By Proposition 3.1(vii), as $2^\beta < 2^{2^\beta} < 2^{2^{2^\beta}} < \cdots$, it follows that $|\mathfrak{C}(\mathbf{F}(G))| < |\mathfrak{C}(\mathbf{F}^2(G))| < |\mathfrak{C}(\mathbf{F}^3(G))| < \cdots$. Therefore, as n increases $|\mathfrak{C}(\mathbf{F}^n(G))|$ increases. Hence G is **F**-divergent. ■

Theorem 5.3. *Let G be a graph. Then,*

i) *G is* **F***-convergent if and only if either G is acyclic or G has only one cycle, which is of length* 3.

ii) *If G is* **F***-convergent, then it converges in at most two steps.*

Proof: i) If G has no cycle, then it is a forest and $\mathbf{F}(G)$ is K_1. If G has only one cycle and that cycle has length 3, then $\mathbf{F}(G)$ is K_3. Therefore in each case G is **F**-convergent.

Conversely, suppose that G has a cycle of length greater than 3 or has at least two triangles. If G has infinitely many cycles, then it follows by Lemma 5.2 that G is **F**-divergent. Therefore we may assume that G has a finite number of cycles. If G has a finite number of vertices, then it is finite and by Lemma 5.1 it is **F**-divergent. Therefore G has an infinite number of vertices. However, it can have only a finite number of edges that are not isthmi, because each cycle is finite. Thus G consists of a finite graph G_0 and any number of isthmi and isolated vertices. Since $\mathbf{F}(G)$ depends only on the edges that are not isthmi and the vertices that are not isolated, $\mathbf{F}(G) = \mathbf{F}(G_0)$ (under the natural identification of maximal forests in G_0 with their extensions in G by adding all isthmi of G). Therefore, G is **F**-divergent.

ii) If G has no cycle, then G is a forest and $\mathbf{F}(G) \cong \mathbf{F}^2(G) \cong K_1$. If G has only one cycle, which is of length 3, then $\mathbf{F}(G) \cong \mathbf{F}^2(G) \cong K_3$. Therefore G converges in at most 2 steps. ∎

Corollary 5.4. *A graph G is* **F***-stable if and only if $G = K_1$ or K_3.*

6 F-Depth

In this section we establish results about the **F**-depth of a graph.

Theorem 6.1. *Let G be a finite graph. The* **F***-depth of G is infinite if and only if G is K_1 or K_3.*

Proof: Let G be a finite graph. Suppose that G is K_1 or K_3. Then by Corollary 5.4, it follows that G is **F**-stable. Therefore, the **F**-depth of G is infinite.

Conversely, suppose that G is different from K_1 and K_3.

Case 1: Let $|V| < 4$. Then G has no **F**-root so its **F**-depth is zero.

Case 2: Let $|V| = 4$. Suppose G has an **F**-root H (i.e., $\mathbf{F}(H) \cong G$). Then H should have exactly 4 maximal forests. That is possible only when H has only one cycle, which is of length 4. By Lemma 2.8 it follows that $\mathbf{F}(H)$ contains K_4, hence it is K_4. Therefore G has an **F**-root if and only if it is K_4. Hence the **F**-depth of G is zero, except that the depth of K_4 is 1.

Case 3: Let $|V| = n$ where $n > 4$. Suppose that G has infinite **F**-depth. Then for every m there is a graph H_m such that $\mathbf{F}^m(H_m) = G$. If H_m does not have two triangles or a cycle of length greater than 3, then H_m has only one cycle which is of length 3, or no cycle and H_m converges to K_1 or K_3 in at most two steps, a contradiction. Therefore H_m has two triangles or a cycle of length greater than 3. By Lemma 2.11 it follows that $\mathbf{F}^m(H_m)$ contains K_{m^2} for each $m \geq 2$, so that in particular $\mathbf{F}^n(H_n)$ contains K_{n^2}. That is, G contains K_{n^2}. This is impossible as G has order n. Hence the **F**-depth of G is finite. ∎

Theorem 6.2. *The* **F***-depth of any infinite graph is finite.*

Proof: Let G be a graph of infinite order α. If G has an **F**-root, then G is without isthmi or isolated vertices.

If G is connected, Theorem 4.2 implies that G has no root. Therefore its **F**-depth is zero.

If G is disconnected, assume it has infinite depth. Then for each natural number n there exists a graph H_n such that $G \cong \mathbf{F}^n(H_n)$. Let β_n denote the order of H_n. Since $\mathbf{F}(H_1) \cong G$, by Proposition 3.1(ii) $\alpha = 2^{\beta_1}$, from which we infer that $\beta_1 < \alpha$. This is independent of which root H_1 is, so in particular we can take $H_1 = \mathbf{F}(H_2)$ and conclude that $\beta_1 = 2^{\beta_2}$, hence that $\beta_2 < \beta_1$. Continuing in like manner we get an infinite decreasing sequence of cardinal numbers starting with α. The cardinal numbers are well ordered [9], so they cannot contain such an infinite sequence. It follows that the **F**-depth of G must be finite. ■

Acknowledgements

This work was supported by DST-CMS project Lr.No.SR/S4/MS:516/07, Dt.21-04-2008.

T. Zaslavsky also thanks the C.R. Rao Advanced Institute of Mathematics, Statistics and Computer Science for its hospitality during the initiation of this paper.

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