arXiv:1601.01047v1 [hep-ph] 6 Jan 2016

Annihilation Rates of $^3D_2(2^{--})$ and $^3D_3(3^{--})$ Heavy Quarkonia

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January 7, 2016

Abstract

We calculate the annihilation decay rates of the $^3D_2(2^{--})$ and $^3D_3(3^{--})$ charmonia and bottomonia by using the instantaneous Bethe-Salpeter method. The wave functions of states with quantum numbers $J^{PC} = 2^{--}$ and 3^{--} are constructed. By solving the corresponding instantaneous Bethe-Salpeter equations, we obtain the mass spectra and wave functions of the quarkonia. The annihilation amplitude is written within Mandelstam formalism and the relativistic corrections are taken into account properly. This is important, especially for high excited states, since their relativistic corrections are very large. The results for the $3g$ channel are as follows: $\Gamma_{^3D_2(c\bar{c})\to ggg} = 3.71$ keV, $\Gamma_{^3D_3(c\bar{c})\to ggg} = 38.2$ keV, $\Gamma_{^3D_2(b\bar{b})\to ggg} = 0.140$ keV, and $\Gamma_{^3D_3(b\bar{b})\to ggg} = 1.01$ keV.

1 Introduction

The $1^3D_2(2^{--})$ charmonium has been found in B decays by the Belle Collaboration [1]. It was confirmed very recently by the BESIII Collaboration through the e^+e^- annihilation process with a statistical significance of 6.2σ [2]. The mass of this particle is measured to be $3821.7 \pm 1.3 \pm 0.7$ MeV, and the decay width is less than 16 MeV. The discovery of this triplet D-wave charmonium is important for checking the validity of phenomenological models, such as the quark potential models, which have predicted abundant heavy quarkonium spectra [3].

These experimental results have attracted some theoretical attention to the production properties of this particle, such as the possibility to find this particle through B_c decays [4] or e^+e^- annihilation with soft pion limit [5]. For the decay properties of this particle, since

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the mass of this particle is below the $D\bar{D}^*$ threshold, and the $D\bar{D}$ channel is forbidden, there is no OZI-allowed channel. As a result, one-photon radiation processes [6] and decays to light hadrons [7] are important. The later one is closely related to the three-gluon annihilation process. This channel is expected to have a relatively small partial width as it is in order of α_s^3. Nevertheless, it still deserves a careful investigation, as it gives useful information to understand the formalism of quark-antiquark interaction and provides a testing ground for the non-perturbative properties of QCD.

For similar reasons, annihilation processes of D-wave quarkonia with $J^{PC} = 3^{--}$ also need investigations. The $1^3D_3(3^{--})$ charmonium has not been found experimentally, and its mass is predicted to be $3812 \sim 3903$ MeV by potential models [8]. Although the $D\bar{D}$ channel of this particle is opened, the high partial wave contribution makes it suppressed. In the bottomonium sector, only the 1^3D_2 state has been found [9, 10]. The mass of 1^3D_3 state is predicted to be 10.181 GeV by Lattice QCD [11], and 10.16 GeV [3] by potential models. Both states are below the open-flavor-decay threshold.

The annihilation processes of ${}^3D_2$ and ${}^3D_3$ quarkonium states have been investigated only in a few works. Refs. [12, 13, 14] employed non-relativistic models to calculate the annihilation amplitudes, which, for D-wave states, are only related to the second derivative of the wave functions at the origin. Ref. [7] used the NRQCD method to calculate annihilation decay widths. Since the relativistic corrections to the three-gluon annihilation processes of quarkonia are large [15, 16], especially, the non-original parts of the wave functions give considerable contributions, it is important at this stage to investigate the three-gluon annihilation processes of 2^{--} and 3^{--} D-wave quarkonia with relativistic corrections taken into account. In our previous work [15], the three-gluon (photon) annihilation process of ${}^3S_1(1^{--})$ charmonia and bottomonia have been calculated with an instantaneous Bethe-Salpeter (BS) method [17, 18], and the obtained decay widths are within the limits of experimental error [19]. So in this work, we use the same framework as the one used in Ref. [15] to calculate annihilations of ${}^3D_2$ and ${}^3D_3$ charmonium and bottomonium states, that is, we construct the Salpeter wave functions for these mesons and write the decay amplitude within Mandelstam formalism [20].

The remaining of this paper is organized as follows. In Section 2, we present the details of the theoretical formalism including the wave functions and the decay amplitude. Numerical results and discussions for the the annihilation processes of ${}^3D_2$ and ${}^3D_3$ heavy quarkonia are presented in Section 3. Section 4 is devoted to a summary. The eigenvalue equations fulfilled by ${}^3D_2$ and ${}^3D_3$ heavy mesons are given in the Appendix.

2 Theoretical calculations

The ggg and γgg decay widths of the ${}^3D_2$ and ${}^3D_3$ mesons are related to that of the three-photon channel just by a parameter. So here we first calculate the later case. According to the Mandelstam formalism [20], the three-photon annihilation amplitude (see Fig. 1) is written as

$$T_{3\gamma} = \sqrt{3}(iee_q)^3 \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\left[\chi_P(q)\left(\not{\epsilon}_3 \frac{1}{\not{k}_3 - \not{p}_2 - m_q + i\epsilon}\not{\epsilon}_2 \frac{1}{\not{p}_1 - \not{k}_1 - m_q + i\epsilon}\not{\epsilon}_1 \right.\right.$$
$$\left.\left. + \text{all other permutations of } 1,2,3\right)\right], \tag{1}$$

where $\sqrt{3}$ is the color factor; ee_q is the electric charge of the heavy quark in unit of e (for charmonium $e_q = \frac{2}{3}$ and for bottomonium $e_q = -\frac{1}{3}$); $\chi_P(q)$ is the Bethe-Salpeter wave function of the meson with mass M and momentum P, and q is the relative momentum of the inner quark and antiquark (with mass m_q and momentum p_i); $k_1 \sim k_3$ are momenta of final photons (gluons) with polarizations $\epsilon_1 \sim \epsilon_3$, respectively.

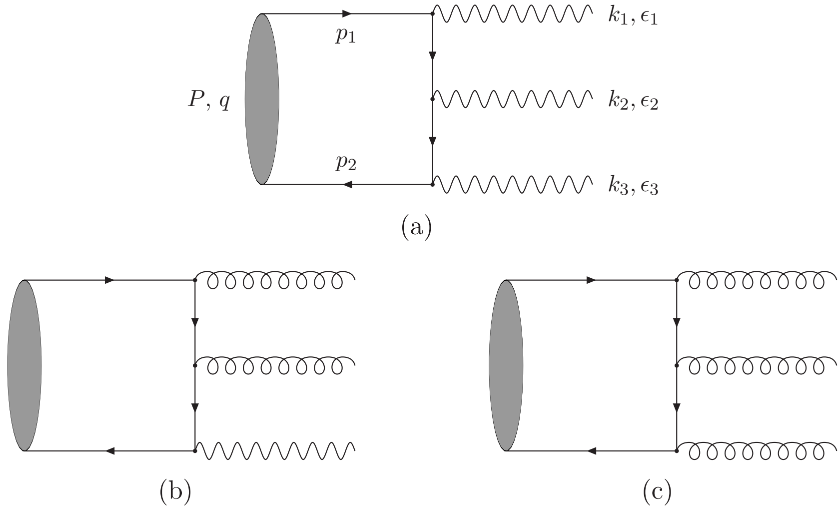


Figure 1: Feynman diagrams for the annihilation processes: (a) $\gamma\gamma\gamma$; (b)γgg; (c)ggg. For each case, there are also five other diagrams with permutations of photons and gluons.

To do the integration in Eq. (1), we take the approximation $p_1 \to \widetilde{p}_1 = \frac{1}{2}P + q_\perp$ and $p_2 \to \widetilde{p}_2 = \frac{1}{2}P - q_\perp$ ($q_\perp$ is defined as $q - \frac{P\cdot q}{\sqrt{P^2}}P$), which is reasonable when $p_1^0 + p_2^0 \approx M$. By doing so, the heavy quark propagators will only depend on $\vec{q}$, while q^0 is only included in the wave function. By using the definition

$$\varphi_P(q_\perp) = i \int \frac{dq^0}{2\pi} \chi_P(q), \tag{2}$$

we could get the three-dimensional form of the amplitude

$$T_{3\gamma} = \sqrt{3}(iee_q)^3 \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left\{\varphi_P(q_\perp)\left[\not{\epsilon}_3 \frac{\not{k}_3 - \widetilde{\not{p}}_2 + m_q}{(k_3 - \widetilde{p}_2)^2 - m_q^2 + i\epsilon}\not{\epsilon}_2 \frac{\widetilde{\not{p}}_1 - \not{k}_1 + m_q}{(\widetilde{p}_1 - k_1)^2 - m_q^2 + i\epsilon}\not{\epsilon}_1 \right.\right.$$
$$\left.\left. + \text{all other permutations of } 1,2,3\right]\right\}. \tag{3}$$

Here we give the explicit expressions for the D-wave mesons. Following Refs. [21, 22], with the instantaneous approximation (set $q^0 = 0$), the general wave function of $^3D_2$ meson is constructed to have the following form

$$\varphi_{2^{--}}(q_\perp) = i\epsilon_{\mu\nu\alpha\beta}\frac{P^\nu}{M}q_\perp^\alpha\epsilon^{\beta\delta}q_{\perp\delta}\gamma^\mu\left(g_1 + \frac{\not{P}}{M}g_2 + \frac{\not{q}_\perp}{M}g_3 + \frac{\not{P}\not{q}_\perp}{M^2}g_4\right), \tag{4}$$

where $\epsilon^{\mu\nu}$ is the polarization tensor of the meson and $\epsilon_{\mu\nu\alpha\beta}$ is the *Levi-Civita* simbol; g_is are functions of $q_\perp^2$, one can check that this wave function has the quantum number of $J^{PC} = 2^{--}$. For the $^3D_3$ state, according to the quantum number 3^{--}, its wave function is given as follows

$$\begin{aligned}\varphi_{3^{--}}(q_\perp) = \epsilon_{\mu\nu\alpha}q_\perp^\mu q_\perp^\nu\Bigg[q_\perp^\alpha\left(f_1 + \frac{\not{P}}{M}f_2 + \frac{\not{q}_\perp}{M}f_3 + \frac{\not{P}\not{q}_\perp}{M^2}f_4\right) + M\gamma^\alpha\Bigg(f_5 + \frac{\not{P}}{M}f_6 \\ + \frac{\not{q}_\perp}{M}f_7 + \frac{\not{P}\not{q}_\perp}{M^2}f_8\Bigg)\Bigg],\end{aligned} \tag{5}$$

where $\epsilon^{\mu\nu\alpha}$ is the third-order polarization tensor of the meson. As there are constrained conditions (see Appendix), not all the g_is and f_is are independent. For the 2^{--} state, only g_1 and g_2 are independent, while for the 3^{--} state, $f_3 \sim f_6$ are independent. The numerical values of these independent wave functions can be obtained by solving the Salpeter equations, and the corresponding eigenvalue equations and normalization conditions are given in the Appendix.

The three-photon decay width is given by

$$\Gamma_{3\gamma} = \frac{1}{3!}\frac{1}{8M(2\pi)^3}\int_0^{\frac{M}{2}}dk_1\int_{\frac{M}{2}-k_1}^{\frac{M}{2}}dk_2\frac{1}{2J+1}\sum_{\text{pol}}|T_{3\gamma}|^2, \tag{6}$$

where J is the spin of the meson. To sum the meson polarization, we have used the complete relation of polarization tensors [23]. First we difine

$$\mathcal{P}_{\mu\nu} \equiv -g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}. \tag{7}$$

For the 2^{--} state, the relation is

$$\sum_\lambda \epsilon_{\mu\nu}^{(\lambda)}\epsilon_{\mu'\nu'}^{*(\lambda)} = \frac{1}{2}\left(\mathcal{P}_{\mu\mu'}\mathcal{P}_{\nu\nu'} + \mathcal{P}_{\mu\nu'}\mathcal{P}_{\nu\mu'}\right) - \frac{1}{3}\mathcal{P}_{\mu\nu}\mathcal{P}_{\mu'\nu'}, \tag{8}$$

and for the 3^{--} state, it has the form

$$\begin{aligned}\sum_\lambda \epsilon_{abc}^{(\lambda)}\epsilon_{xyz}^{*(\lambda)} = \frac{1}{6}(&\mathcal{P}_{ax}\mathcal{P}_{by}\mathcal{P}_{cz} + \mathcal{P}_{ax}\mathcal{P}_{bz}\mathcal{P}_{cy} + \mathcal{P}_{ay}\mathcal{P}_{bx}\mathcal{P}_{cz} \\ &+ \mathcal{P}_{ay}\mathcal{P}_{bz}\mathcal{P}_{cx} + \mathcal{P}_{az}\mathcal{P}_{by}\mathcal{P}_{cx} + \mathcal{P}_{az}\mathcal{P}_{bx}\mathcal{P}_{cy}) \\ &- \frac{1}{15}(\mathcal{P}_{ab}\mathcal{P}_{cz}\mathcal{P}_{xy} + \mathcal{P}_{ab}\mathcal{P}_{cy}\mathcal{P}_{xz} + \mathcal{P}_{ab}\mathcal{P}_{cx}\mathcal{P}_{yz} \\ &+ \mathcal{P}_{ac}\mathcal{P}_{bz}\mathcal{P}_{xy} + \mathcal{P}_{ac}\mathcal{P}_{by}\mathcal{P}_{xz} + \mathcal{P}_{ac}\mathcal{P}_{bx}\mathcal{P}_{yz} \\ &+ \mathcal{P}_{bc}\mathcal{P}_{az}\mathcal{P}_{xy} + \mathcal{P}_{bc}\mathcal{P}_{ay}\mathcal{P}_{xz} + \mathcal{P}_{bc}\mathcal{P}_{ax}\mathcal{P}_{yz}).\end{aligned} \tag{9}$$

For the decay channels $^3D_2(^3D_3) \to \gamma gg$ and $^3D_2(^3D_3) \to ggg$, the decay widths are [15]

$$\Gamma_{\gamma gg} = \frac{2}{3}\frac{\alpha_s^2}{\alpha^2 e_q^4}\Gamma_{3\gamma}, \tag{10}$$

and [24]

$$\Gamma_{ggg} = \frac{5}{54}\frac{\alpha_s^3}{\alpha^3 e_q^6}\Gamma_{3\gamma}, \tag{11}$$

respectively.

3 Results and Discussions

When solving the Salpeter equation, we use the instantaneously approximated potential which in the momentum space has the following form

$$\begin{aligned} V(\vec{q}) &= (2\pi)^3 V_s(\vec{q}) + \gamma_0 \otimes \gamma^0 (2\pi)^3 V_v(\vec{q}), \\ V_s(\vec{q}) &= -\left(\frac{\lambda}{\alpha} + V_0\right)\delta^3(\vec{q}) + \frac{\lambda}{\pi^2}\frac{1}{(\vec{q}^2+\alpha^2)^2}, \\ V_v(\vec{q}) &= -\frac{2}{3\pi^2}\frac{\alpha_s(\vec{q})}{\vec{q}^2+\alpha^2}, \\ \alpha_s(\vec{q}) &= \frac{12\pi}{(33-2N_f)}\frac{1}{\ln\left(a+\frac{\vec{q}^2}{\Lambda^2_{QCD}}\right)}. \end{aligned} \tag{12}$$

Parameters in the above equations have the values [15, 22]: $a = e = 2.71828$, $\alpha = 0.06$ GeV, $\lambda = 0.21$ GeV^2, $\Lambda_{QCD} = 0.27$ GeV (0.20 GeV for $b\bar{b}$), $m_b = 4.96$ GeV, $m_c = 1.62$ GeV. We set the flavor number $N_f = 3$ for charmonia and $N_f = 4$ for bottomonia. By using the fourth equation above we get $\alpha_s(m_c) = 0.38$ and $\alpha_s(m_b) = 0.23$. We choose appropriate values of V_0 to get the mass spectra and wave functions. The results are as follows: $M_{1^3D_2(c\bar{c})} = 3.8217$ GeV, $M_{1^3D_3(c\bar{c})} = 3.830$ GeV, $M_{1^3D_2(b\bar{b})} = 10.1637$ GeV and $M_{1^3D_3(b\bar{b})} = 10.165$ GeV.

The wave functions are plotted in Fig. 2. For the 2^{--} state, g_1 and g_2 are independent functions. The numerical result shows that this two functions are very close to each other. So here we just plot g_1 as an example. For the 3^{--} state, there are four independent functions, while f_3 is close to f_4 and f_5 is close to f_6. So we only plot the f_3 and f_5. To make the wave functions to be dimensionless, we have rescaled them by a factor. Because the normalization condition is different, this factor for 2^{--} and 3^{--} is different. One can see the wave functions of $b\bar{b}$ are quite large than those of $c\bar{c}$. This is mainly because we have used different scale factors. Actually, for 2^{--} states, $g_i(b\bar{b})$ is more than two times smaller than $g_i(c\bar{c})$. It should be mentioned that the position of the peak value for the former is at the right of the later. This means that the contribution coming from non-zero $|\vec{q}|$ for the $b\bar{b}$ state is larger than that for the $c\bar{c}$ state.

In Fig. 3, we present the three-photon differential decay widths of 2^{--} and 3^{--} charmonia against k_1 and k_2 (here we use k_i to represent the photon energy; the projection to the k_1-k_2 plane is the Dalitz plot). For bottomonia, the results are plotted in Fig. 4. One notices that as the kinematics (k_1, k_2) goes toward the points $(0, \frac{M}{2})$, $(\frac{M}{2}, 0)$, and $(0, 0)$,

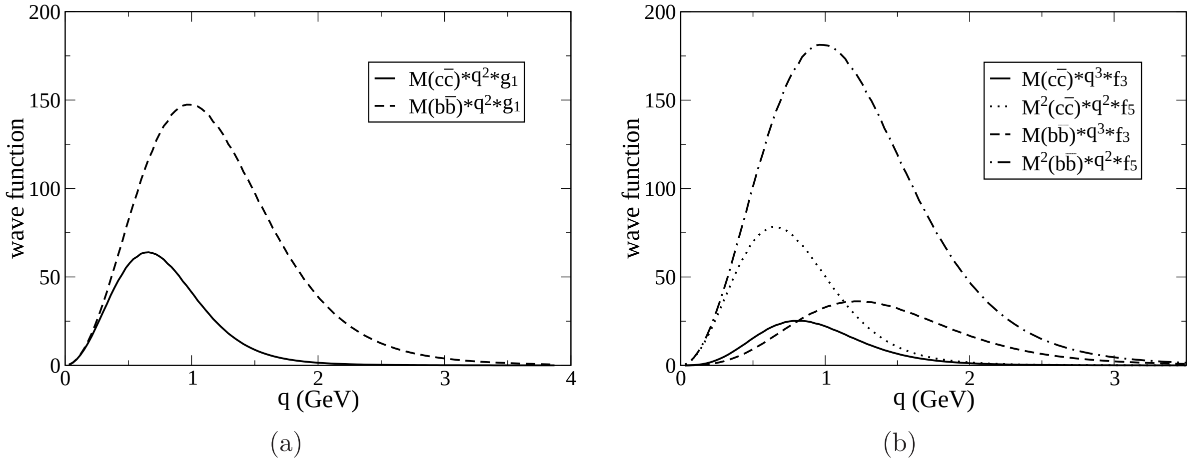


Figure 2: Wave functions of 2^{--} and 3^{--} mesons. (a) for the 2^{--} state, only f_1 is plotted both for charmonium and bottomonium. (b) for the 3^{--} states, the f_3 and f_5 are presented. All the wave functions are rescaled to be dimensionless.

respectively, the differential width gets larger and larger. In Ref. [12], a similar diagram was given, while there k_2 was integrated out. The differential decay width of the 2^{--} $c\bar{c}$ ($b\bar{b}$) state is generally smaller than that of the 3^{--} $c\bar{c}$ ($b\bar{b}$) state at the same kinematic point. Compared to the charmonium states, the differential widths of the bottomonia are more flat in the central region.

The decay widths for the chamonia are presented in Table 1. One can see that the three-photon results are tiny which is hard to be detected in the future. For the $^3D_2 \to ggg$ channel, our result is 3.71 keV, which is about 5 times smaller than that of Ref. [13] and 3 times smaller than that of Ref. [12]. This indicates a large suppression due to relativistic corrections in this channel. For the $^3D_3 \to ggg$ channel, we get 38.2 keV, which is about 3 times smaller than that of Ref. [13] and 2 times smaller than that of Ref. [12]. These comparisons show that the relativistic corrections to the three-gluon annihilations of $^3D_2$ and $^3D_3$ are considerably large. Ref. [7] gave the result which is more than 10 times larger than ours for the $^3D_2 \to ggg$ channel and 6 times for the $^3D_3 \to ggg$ channel. The results of Ref. [7] we cited here is calculated at $\mu = m_c$. The authors there also gave the widths at $\mu = 2m_c$, which are 50 keV and 172 keV for $^3D_2$ and $^3D_3$, respectively.

The decay widths of $^3D_2$ and $^3D_3$ bottomonia are listed in Table 2. For the three-photon decay channels, they are smaller than that of the charmonia almost by three orders of magnitude, while for the other two channels, they are about two orders and one order of magnitude smaller, respectively. The reason for this is that the wave function (f_i or g_i) of bottomonia is smaller than that of charmonia, which cause the differential decay width to be small. Besides that, the electric charges of heavy quarks differs by a factor of 2, and the strong coupling constant α_s has different value at different energy scale. Our results for $\Gamma_{^3D_2 \to ggg}$ of the bottomonium is roughly 2 (4, 4) times smaller than that of Ref. [14] ([12], [7]), while for the $\Gamma_{^3D_3 \to ggg}$ channel, it is 1 (3, 3) times smaller.

Our result for the ratio of the γgg channel and the ggg channel is

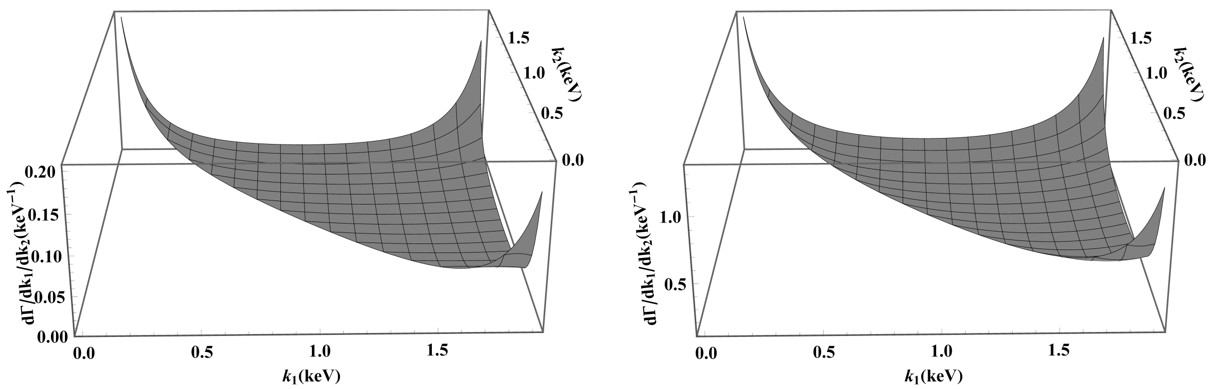
$$\frac{\Gamma(^3D_J \to \gamma gg)}{\Gamma(^3D_J \to ggg)} = 6.2\% \quad (J = 2, 3) \tag{13}$$

for the charmonium, which is close to 7% given in Ref. [12]. This ratio is totally determined by some basic parameters (the fine structure constant, the strong coupling at the relevant scale, etc.), so it is irrelevant to the model employed and could be used to measure the strong coupling at the corresponding scale. For the bottomonium, the corresponding result is 2.5% which is about half of the ratio above. In Ref. [12], this ratio is 3% which is also close to ours. As for the ratios of $3g$ channel for different D-waves, we get

$$\frac{\Gamma(^3D_3 \to ggg)}{\Gamma(^3D_2 \to ggg)} = 10.3 \ \ (\text{for } c\bar{c}) \text{ and } 7.2 \ (\text{for } b\bar{b}). \tag{14}$$

The ratio is irrelevant to the strong coupling and only reflects the difference in wave functions between 2^{--} and 3^{--} states. Our results of this ratio are larger than those of other models ($5 \sim 6$ for $c\bar{c}$ and $4 \sim 5$ for $b\bar{b}$), which indicates that the relativistic corrections to the 2^{--} and 3^{--} states are different.

In conclusion, we have calculated the three-photon (gluon) decay widths with the Bethe-Salpeter method with which the relativistic corrections are taken into account properly. Our results show that three photon decay channels have very small decay widths, especially for the bottomonium state. For the three-gluon processes we get: $\Gamma_{3g}[^3D_2,\ ^3D_3] = (3.71,\ 38.2)$ keV for the charmonia and $(0.140,\ 1.01)$ keV for the bottomonia. Compared to the results given by the non-relativistic models, our results are considerably suppressed due to relativistic corrections, which indicates that the three-gluon (photon) annihilation processes of heavy quarkonia suffer large relativistic corrections for the $^3D_2$ and $^3D_3$ states.

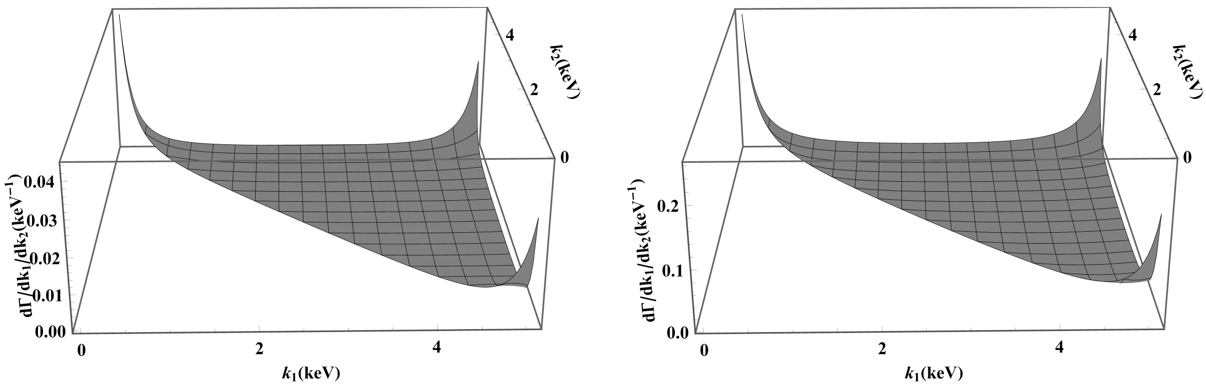


(a) (b)

Figure 3: The differential width $\frac{d\Gamma}{dk_1 dk_2}$ of three-photon decay changes with respect to k_1 and k_2. (a) for $1^3D_2(c\bar{c})$ and (b) for $1^3D_3(c\bar{c})$.

4 Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (NSFC) under Grant No. 11405037, No. 11575048 and No. 11505039, and in part by PIRS of HIT No.T201405, No.A201409, and No.B201506.



(a) (b)

Figure 4: The differential width $\frac{d\Gamma}{dk_1 dk_2}$ of three-photon decay changes with respect to k_1 and k_2. (a) for $1^3D_2(b\bar{b})$ and (b) for $1^3D_3(b\bar{b})$.

Table 1: Partial decay widths (keV) of $^3D_2$ and $^3D_3$ charmonia.

Decay Channel	Ours	Ref. [13]	Ref. [12]	Ref. [7]
$\Gamma_{^3D_2\to\gamma\gamma\gamma}$	2.49×10^{-5}			
$\Gamma_{^3D_3\to\gamma\gamma\gamma}$	2.57×10^{-4}			
$\Gamma_{^3D_2\to\gamma gg}$	0.228		0.84	
$\Gamma_{^3D_3\to\gamma gg}$	2.35		4.76	
$\Gamma_{^3D_2\to ggg}$	3.71	19 ± 3	12	42
$\Gamma_{^3D_3\to ggg}$	38.2	121	68	223

5 Appendix

The Bethe-Salpeter equation which describes two-body bound state relativistically has the following form [17]

$$S_1^{-1}(p_1)\chi_{_P}(q)S_2^{-1}(-p_2) = i\int\frac{d^4k}{(2\pi)^4}V(P;q,k)\chi_{_P}(k), \tag{15}$$

where $p_1 = \frac{1}{2}P + q_\perp$ and $p_2 = \frac{1}{2}P - q_\perp$ are respectively the momenta of quark and antiquark in the bound state; $\chi_{_P}(q)$ is the BS wave function of the bound state; $V(P;q,k)$ is the interaction potential between quark and antiquark. The fermion propagator $S_i(Jp_i)$ ($J=(-1)^{i+1}$, $i=1$ for quark and $i=2$ for antiquark) is defined as

$$-iJS_i(Jp_i) = \frac{\Lambda_i^+}{p_i - \omega_i + i\epsilon} + \frac{\Lambda_i^-}{p_i + \omega_i - i\epsilon}, \tag{16}$$

where we have used the projector $\Lambda_i^\pm(p_{i\perp}^\mu) = \frac{1}{2\omega_i}[\frac{\not{P}}{M}\omega_i \pm (\not{p}_{i\perp} + Jm_i)]$. m_i is the (anti)quark mass and ω_i has the form $\sqrt{m_i^2 - q_\perp^2}$. With instantaneous approximation, $V(P;q,k)\approx V(q_\perp,k_\perp)$, we write the integral in Eq. (15) as $\eta_{_P}(q_\perp) = \int\frac{d^3\vec{k}}{(2\pi)^3}V(q_\perp,k_\perp)\varphi_{_P}(k_\perp)$. By

Table 2: Partial decay widths (keV) of $^3D_2$ and $^3D_3$ bottomonia.

Decay Channel	Ours	Ref. [14]	Ref. [12]	Ref. [7]
$\Gamma_{^3D_2\to\gamma\gamma\gamma}$	6.62×10^{-8}			
$\Gamma_{^3D_3\to\gamma\gamma\gamma}$	4.76×10^{-7}			
$\Gamma_{^3D_2\to\gamma gg}$	3.55×10^{-3}		1.53×10^{-2}	
$\Gamma_{^3D_3\to\gamma gg}$	2.55×10^{-2}		8.1×10^{-2}	
$\Gamma_{^3D_2\to ggg}$	0.140	0.26	0.51	0.60
$\Gamma_{^3D_3\to ggg}$	1.01	1.1	2.7	2.85

introducing the notation $\varphi^{\pm\pm}\equiv\Lambda_1^{\pm}\frac{\not P}{M}\varphi(q_\perp)\frac{\not P}{M}\Lambda_2^{\pm}$, we get the instantaneous form of BS equation, which is also called full Salpeter equation [18]

$$(M-\omega_1-\omega_2)\varphi_P^{++}(q_\perp)=\Lambda_1^+(q_\perp)\eta_P(q_\perp)\Lambda_2^+(q_\perp), \tag{17a}$$

$$(M+\omega_1+\omega_2)\varphi_P^{--}(q_\perp)=-\Lambda_1^-(q_\perp)\eta_P(q_\perp)\Lambda_2^-(q_\perp), \tag{17b}$$

$$\varphi_P^{+-}(q_\perp)=\varphi_P^{-+}(q_\perp)=0. \tag{17c}$$

Here Eq. (17c) are the constrained conditions, which result in relations between g_is or f_is (see below). The normalization condition for Salpeter wave functions is [18]

$$\int\frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr}\left[\overline{\varphi}^{++}\frac{\not P}{M}\varphi^{++}\frac{\not P}{M}-\overline{\varphi}^{--}\frac{\not P}{M}\varphi^{--}\frac{\not P}{M}\right]=2P^0, \tag{18}$$

where $\overline{\varphi}_P(q_\perp)$ is defined as $\gamma^0\varphi_P^\dagger(q_\perp)\gamma^0$.

Eq. (17b) describes the negative energy part of the wave function which gives small contributions. So it is neglected by many authors in literatures. Here we solve the full Salpeter equation. By inserting Eq. (4) into Eq. (17c), we get the constrained conditions

$$g_3=\frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1}g_1,\quad g_4=\frac{M(\omega_1+\omega_2)}{m_1\omega_2+m_2\omega_1}g_2. \tag{19}$$

From Eq. (17a) and Eq. (17b) we get the eigenvalue equations fulfilled by 2^{--} states

$$\begin{aligned}(M-\omega_1-\omega_2)\left(g_1-\frac{\omega_1+\omega_2}{m_1+m_2}g_2\right)=\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{4\omega_1\omega_2\vec{q}^4}\left[A_1\left(g_1-\frac{m_1+m_2}{\omega_1+\omega_2}g_2\right)\right.\\ \left.+A_2\left(g_1-\frac{\omega_1+\omega_2}{m_1+m_2}g_2\right)\right],\end{aligned} \tag{20a}$$

$$\begin{aligned}(M+\omega_1+\omega_2)\left(g_1+\frac{\omega_1+\omega_2}{m_1+m_2}g_2\right)=\int\frac{d\vec{k}}{(2\pi)^3}\frac{-1}{4\omega_1\omega_2\vec{q}^4}\left[A_1\left(g_1+\frac{m_1+m_2}{\omega_1+\omega_2}g_2\right)\right.\\ \left.+A_2\left(g_1+\frac{\omega_1+\omega_2}{m_1+m_2}g_2\right)\right],\end{aligned} \tag{20b}$$

where g_is on the left side of the equation are functions of $-q_\perp^2$, while those on the right side are functions of $-k_\perp^2$ ($k_\perp \equiv k - \frac{P\cdot k}{\sqrt{P^2}}P$). A_is are defined as

$$
\begin{aligned}
A_1 &= (m_1 m_2 + \vec{q}^2 + \omega_1\omega_2)\left[\vec{k}^2\vec{q}^2 - 3(\vec{k}\cdot\vec{q})^2\right](V_s - V_v)\,, \\
A_2 &= \frac{(E_1 - E_2)(m_1 - m_2)}{m_1 E_2 + m_2 E_1} 2(\vec{k}\cdot\vec{q})^3 (V_s + V_v),
\end{aligned}
\tag{21}
$$

where we have used the definition $E_i = \sqrt{m_i^2 - k_\perp^2}$. By solving the eigenvalue equation (Eq. (20)) numerically, we obtain the eigenvalue M and wave functions $g_i s$ with the normalization condition (Eq.(18))

$$
\int \frac{d\vec{q}}{(2\pi)^3} \frac{8\omega_1\omega_2\vec{q}^4}{m_1\omega_2 + m_2\omega_1} g_1 g_2 = -5M. \tag{22}
$$

For the 3^{--} state, the constrained conditions are

$$
\begin{aligned}
f_1 &= \frac{-\vec{q}^2 f_3(\omega_1+\omega_2) + 2M^2 f_5\omega_2}{M(m_1\omega_2 + m_2\omega_1)}, \qquad f_2 = \frac{-\vec{q}^2 f_4(\omega_1-\omega_2) + 2M^2 f_6\omega_2}{M(m_1\omega_2 + m_2\omega_1)} \\
f_7 &= \frac{M(\omega_1-\omega_2)}{m_1\omega_2 + m_2\omega_1} f_5, \qquad f_8 = \frac{M(\omega_1+\omega_2)}{m_1\omega_2 + m_2\omega_1} f_6.
\end{aligned}
\tag{23}
$$

And the eigenvalue equations are

$$
\begin{aligned}
&(M - \omega_1 - \omega_2)\left[-\frac{\vec{q}^2}{M^2}\left(f_3 + \frac{m_1+m_2}{\omega_1+\omega_2} f_4\right) + \left(f_5 - \frac{m_1+m_2}{\omega_1+\omega_2} f_6\right)\right] = \\
&\int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{4\omega_1\omega_2\vec{q}^4}\left\{B_1\left[f_3 + \frac{(E_1-E_2)(\omega_1-\omega_2)}{(E_1+E_2)(m_1+m_2)} f_4\right] + B_2\left(f_3 + \frac{\omega_1+\omega_2}{m_1+m_2} f_4\right)\right. \\
&\left. + B_3\left[f_5 - \frac{(E_1-E_2)(\omega_1-\omega_2)}{(E_1+E_2)(m_1+m_2)} f_6\right] + B_4\left(f_5 - \frac{\omega_1+\omega_2}{m_1+m_2} f_6\right)\right\},
\end{aligned}
\tag{24a}
$$

$$
\begin{aligned}
&(M + \omega_1 + \omega_2)\left[-\frac{\vec{q}^2}{M^2}\left(f_3 - \frac{m_1+m_2}{\omega_1+\omega_2} f_4\right) + \left(f_5 + \frac{m_1+m_2}{\omega_1+\omega_2} f_6\right)\right] = \\
&\int \frac{d\vec{k}}{(2\pi)^3} \frac{-1}{4\omega_1\omega_2\vec{q}^4}\left\{B_1\left[f_3 - \frac{(E_1-E_2)(\omega_1-\omega_2)}{(E_1+E_2)(m_1+m_2)} f_4\right] + B_2\left(f_3 - \frac{\omega_1+\omega_2}{m_1+m_2} f_4\right)\right. \\
&\left. + B_3\left[f_5 + \frac{(E_1-E_2)(\omega_1-\omega_2)}{(E_1+E_2)(m_1+m_2)} f_6\right] + B_4\left(f_5 + \frac{\omega_1+\omega_2}{m_1+m_2} f_6\right)\right\},
\end{aligned}
\tag{24b}
$$

$$
\begin{aligned}
&(M - \omega_1 - \omega_2)\left(f_5 - \frac{\omega_1+\omega_2}{m_1+m_2} f_6\right) = \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{4\omega_1\omega_2\vec{q}^4}\left\{C_1\left(f_3 + \frac{m_1+m_2}{\omega_1+\omega_2} f_4\right)\right. \\
&\left. + C_2\left[f_5 - \frac{(\omega_1+\omega_2)(E_1+E_2)}{(m_1-m_2)(E_1-E_2)} f_6\right] + C_3\left(f_5 - \frac{m_1+m_2}{\omega_1+\omega_2} f_6\right)\right\},
\end{aligned}
\tag{24c}
$$

$$
\begin{aligned}
&(M + \omega_1 + \omega_2)\left(f_5 + \frac{\omega_1+\omega_2}{m_1+m_2} f_6\right) = \int \frac{d\vec{k}}{(2\pi)^3} \frac{-1}{4\omega_1\omega_2\vec{q}^4}\left\{C_1\left(f_3 - \frac{m_1+m_2}{\omega_1+\omega_2} f_4\right)\right. \\
&\left. + C_2\left[f_5 + \frac{(\omega_1+\omega_2)(E_1+E_2)}{(m_1-m_2)(E_1-E_2)} f_6\right] + C_3\left(f_5 + \frac{m_1+m_2}{\omega_1+\omega_2} f_6\right)\right\},
\end{aligned}
\tag{24d}
$$

where we have defined

$$
\begin{aligned}
B_1 &= \frac{(E_1+E_2)(m_1+m_2)}{M^2(m_1E_2+m_2E_1)}\vec{k}\cdot\vec{q}\vec{k}^2\left[3\vec{q}^2\vec{k}^2-5\left(\vec{k}\cdot\vec{q}\right)^2\right](V_s+V_v),\\
B_2 &= -\frac{1}{M^2\vec{q}^2}(m_1\omega_2+m_2\omega_1)(\vec{k}\cdot\vec{q})^2\left[3\vec{q}^2\vec{k}^2-5\left(\vec{k}\cdot\vec{q}\right)^2\right]\frac{m_1+m_2}{\omega_1+\omega_2}(V_s-V_v),\\
B_3 &= -\frac{(E_1+E_2)(m_1+m_2)}{m_1E_2+m_2E_1}\vec{k}\cdot\vec{q}\left[3\vec{q}^2\vec{k}^2-5\left(\vec{k}\cdot\vec{q}\right)^2\right](V_s+V_v),\\
B_4 &= (m_1\omega_2+m_2\omega_1)\left[\vec{q}^2\vec{k}^2-3\left(\vec{k}\cdot\vec{q}\right)^2\right]\frac{m_1+m_2}{\omega_1+\omega_2}(V_s-V_v),
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
C_1 &= -\frac{3}{4M^2\vec{q}^2}\left(\omega_1\omega_2+m_1m_2+\vec{q}^2\right)\left[\vec{k}^4\vec{q}^4-6\vec{k}^2\vec{q}^2\left(\vec{k}\cdot\vec{q}\right)^2+5\left(\vec{k}\cdot\vec{q}\right)^4\right](V_s-V_v),\\
C_2 &= -\frac{(m_1-m_2)(E_1-E_2)}{m_1E_2+m_2E_1}\vec{k}\cdot\vec{q}\left[3\vec{k}^2\vec{q}^2-5\left(\vec{k}\cdot\vec{q}\right)^2\right](V_s-V_v),\\
C_3 &= \left(\omega_1\omega_2+m_1m_2+\vec{q}^2\right)\left[\vec{k}^2\vec{q}^2-3\left(\vec{k}\cdot\vec{q}\right)^2\right](V_s-V_v).
\end{aligned}
\tag{26}
$$

In Eq. (24), f_is on the left side and right side are functions of $-q_\perp^2$ and $-k_\perp^2$, respectively. And the normalization condition is

$$
\int\frac{d\vec{q}}{(2\pi)^3}\frac{16\omega_1\omega_2\vec{q}^4}{15(m_1\omega_2+m_2\omega_1)}\left(-\frac{3\vec{q}^4}{M^2}f_3f_4-3\vec{q}^2f_3f_6+3\vec{q}^2f_4f_5+7M^2f_5f_6\right)=7M. \tag{27}
$$

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