Non-compactness of the space of minimal hypersurfaces

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1 Introduction

In [9] F. C. Marques and A. Neves have shown the existence of infinitely many embedded minimal hypersurfaces in a closed manifold with positive Ricci curvature. Their result is divided in two cases: when $\omega_p < \omega_{p+1}$ for all p or the equality case $\omega_p = \omega_{p+1}$, for some p.

In the first case the minimal hypersurfaces they obtain are geometrically distinct because they must have different areas. However, nothing is known about their topological types, a priori they could all be the same surface with distinct embeddings. For example, in the 3-torus it is possible to find a sequence of embedded 2-tori with area tending to infinity.

In the second case the hypersurfaces given by their proof actually have constant area, so they could all be the same embedding under isometries. Take the round 3-sphere as an example. As it is known, in this case $\omega_1 = \omega_2$, so their construction is actually giving us the 3-parameter of S^2 in the equator, all of which are isometric.

It would be interesting to know whether in the second case the minimal hypersurfaces in [9] are isometrically distinct. To answer this one could analyse either how the index or the area changes along the space of minimal hypersurfaces. It turns out that a bound on both the index and the area is sufficient to have compactness, as it was proven by B. Sharp in [14]. With that in mind, a non-compactness result would imply that either the index or the area of minimal hypersurfaces must be unbounded, thus yielding geometrical distinctness.

In this paper we are interested in showing that the space of minimal hypersurfaces is non-compact when the metric is analytic with positive Ricci curvature. The idea of the proof is the following. First we show that if we have compactness then there exists $N > 0$ so that $\omega_p < \omega_{p+N}$. Now the result follows as in [9] because we are able to obtain an increasing subsequence of the width spectrum with the number of parameters growing linearly.

The first step is based on the ideas of Lusternik-Schnirelmann category theory. In their context they are able to obtain results on the topology of the critical set whenever one has equality $\omega_p = \omega_q$. However, their method only works for smooth functions in Banach manifolds so we need a careful adaptation to our setting. The second step follows from the asymptotic behaviour of the width proved first by M. Gromov (see [5]).

This work is divided as follows. In section 2 we establish notation and cover some preliminaries to make this sufficiently self-contained. All of the results and definitions in this section are taken from [9]. In section 3 we introduce the concept of 1-category and we prove the topological theorem about the critical set under the equality case $\omega_p^{(m)} = \omega_{p+N}^{(m)}$. In section 4 we apply the result of the previous section to some specific cases and we prove the main non-compactness result.

2 Preliminaries

Throughout this section we assume that (M,g) is a Riemannian manifold ismetrically embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. We will establish notations and definitions that are not standard in the literature.

Varifolds and Currents

Denote by $\mathcal{I}_k(M;\mathbb{Z}_2)$ and $\mathcal{Z}_k(M;\mathbb{Z}_2)$ the spaces of k-currents modulo 2 and k-cycles in M, respectively. Let $\mathrm{R}\mathcal{V}_k(M)$ be the space of k-dimensional rectifiable varifolds in $\mathbb{R}^N$ whose support lies in M with the weak topology (we agree with the definition in [11, §2]). The subspace of k-dimensional integral varifolds is denoted by $\mathrm{I}\mathcal{V}_k(M) \subset \mathrm{R}\mathcal{V}_k(M)$.

Given $V \in \mathrm{R}\mathcal{V}_k(M)$ we denote by $\|V\|$ the Radon measure in M associated with V, we call $\|V\|(M)$ the mass of V. Now, given a k-current $T \in \mathcal{I}_k(M;\mathbb{Z}_2)$ we denote $|T| \in \mathrm{I}\mathcal{V}_k(M)$ the integral varifold associated to T and to simplify notation we write $\|T\|$ its associated Radon measure in M. Reversely, if $V \in \mathrm{I}\mathcal{V}_k(M)$ then $[V] \in \mathcal{I}_k(M;\mathbb{Z}_2)$ denotes the unique k-current satisfying $\Theta^k([V],x) = \Theta^k(V,x) \, mod \, 2$ for all $x \in M$ (see [15]).

The weak topology in $\mathrm{R}\mathcal{V}_k(M)$ is induced by the $\underline{\underline{\mathbf{F}}}$-metric, denoted by $\underline{\underline{\mathbf{F}}}$ (see [11, §2]). On the space of currents we will work with three different topologies induced by the flat metric $\mathcal{F}$, the mass $\underline{\underline{\mathbf{M}}}$ and the $\underline{\underline{\mathbf{F}}}$-metric for currents also denoted by $\underline{\underline{\mathbf{F}}}$. For the definition of the first two see [4, §4.2.26], the latter is defined as

$$\underline{\underline{\mathbf{F}}}(T,S) = \mathcal{F}(T-S) + \underline{\underline{\mathbf{F}}}(|T|,|S|),$$

for all $T,S \in \mathcal{I}_k(M;\mathbb{Z}_2)$. We will always assume $\mathcal{I}_k(M;\mathbb{Z}_2)$ and $\mathcal{Z}_k(M;\mathbb{Z}_2)$ to be endowed with the flat topology unless otherwise specified.

Almost-minimising Varifolds

For our purposes it will be sufficient to only consider $\mathbb{Z}_2$-almost-minimising varifolds, the definition is the same for a different group G (see [11, §3.1]). We also remark that our definition is slightly different from [11] but all the results therein contained remain true.

Definition 2.1. Let $U \subset M$ be an open set, $\varepsilon > 0$ and $\delta > 0$. We define

$$\mathfrak{A}_k(U;\varepsilon,\delta) \subset \mathcal{Z}_k(M;\mathbb{Z}_2)$$

to be the set of cycles $T \in \mathcal{Z}_k(M;\mathbb{Z}_2)$ such that any finite sequence $T_1, \dots, T_m \in \mathcal{Z}_k(M;\mathbb{Z}_2)$ satisfying

(a) $\operatorname{supp}(T - T_i) \subset U$ for all $i = 1, \dots, m$;

(b) $\mathcal{F}(T_i, T_{i-1}) \le \delta$ for all $i = 1, \dots, m$ and

(c) $\underline{\underline{\mathbf{M}}}(T_i) \le \underline{\underline{\mathbf{M}}}(T) + \delta$

must also satisfy

$$\underline{\underline{\mathbf{M}}}(T_m) \ge \underline{\underline{\mathbf{M}}}(T) - \varepsilon.$$

We say that a varifold $V \in \mathcal{V}_k(M)$ is almost-minimising in U if for every $\varepsilon > 0$ there exists $\delta > 0$ and $T \in \mathfrak{A}_k(U;\varepsilon;\delta)$ such that

$$\underline{\underline{\mathbf{F}}}(V, |T|) < \varepsilon.$$

Furthermore, we say that V is almost-minimising in annuli if for every $p \in \operatorname{supp}\|V\|$ there exists $r > 0$ such that V is almost-minimising in the annulus $A(p;s,r) = B(p,r) \setminus B(p,s)$ for all positive $s < r$.

The following is a well known regularity theorem for stationary varifolds of codimension 1. This was originally proven in by Pitts, up to dimension $n+1 \le 6$ and later extended by Schoen-Simon to $n+1 \le 7$.

Theorem 2.2 ([11, §7], [12, §4]). *Let M^{n+1} be a closed manifold of dimension $n+1$ with $2 \le n \le 6$. If $V \in \mathbf{I}\mathcal{V}_n(M)$ is stationary and almost-minimising in annuli, then* $\operatorname{supp}\|V\|$ *is a smooth embedded minimal hypersurface.*

<u>Remark</u> 1. If $n \ge 7$ then it was also proven that $\operatorname{supp}\|V\|$ has a singular set of Hausdorff dimension at most $n-7$.

Almgren-Pitts Min-max Theory

We want to present the appropriate modification of the Almgren-Pitts Min-max Theory that will be necessary. All of the results and definitions are taken from [8] where one can find detailed proofs. Henceforth we restrict ourselves to the codimension one case, that is, $k = n$ and M has dimension $n+1$.

Firstly, given a cell complex X and $l \in \mathbb{Z}_{\ge 0}$ we denote by $X_{(l)}$ the set of l-cells. Let $I^m = [0,1]^m$ denote the m-dimensional cube. For each $j \in \mathbb{N}$ we denote by $I(1,j)$ the cell decomposition of $I = I^1$ whose 0-cells and 1-cells are given by

$$I(1,j)_{(0)} = \{[0], [3^{-j}], \dots, [1-3^{-j}], [1]\},$$
$$I(1,j)_{(1)} = \{[0,3^{-j}], \dots, [1-3^{-j}, 1]\}.$$

Now, if $m > 1$ then, for each $j \in \mathbb{N}$, the standard cell complex of I^m is defined as

$$I(m,j) = \underbrace{I(1,j) \otimes \ldots \otimes I(1,j)}_{m \text{ times}}.$$

Definition 2.3. A set $X \subset I^m$ is said to be a *cubical subcomplex* of I^m (according to our standard chosen cell decomposition) if X is a subcomplex of $I(m,j)$ for some $j \in \mathbb{N}$.

By abuse of notation we write X for both the cell decomposition and its support. Note that the dimension of X is not required to be m.

If X is a cubical subcomplex of $I(m,j)$ and $l \geq j$ we write $X(l)$ for the union of all cells in $I(m,l)$ whose support is contained in X.

Definition 2.4. Let $\{m_i\} \subset \mathbb{N}$ be positive integers, $X_i \subset I^{m_i}$ cubical subcomplexes and $S = \{\Phi_i : X_i \to \mathcal{Z}_n(M;\mathbb{Z}_2)\}$ a sequence of flat continuous maps. We define the width of a sequence of maps as

$$\mathbf{L}(S) = \limsup_{i\to\infty} \sup\{\underline{\underline{\mathbf{M}}}(\Phi_i(x)) : x \in X_i\}$$

and the following compact set of critical varifolds

$$\mathbf{C}(S) = \{V \in \mathrm{R}\mathcal{V}_n(M) : V = \lim_{j\to\infty} |\Phi_{i_j}(x_j)| \text{ for some increasing sequence } \{i_j\}_{j\in\mathbb{N}},\ x_j \in X_{i_j} \text{ and } \|V\|(M) = \mathbf{L}(S)\}.$$

In case we have a fixed map $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ it defines an homotopy class (with free boundary) $[\Phi] \in [X : \mathcal{Z}_n(M;\mathbb{Z}_2)]$ and its width is given by

$$\mathbf{L}[\Phi] = \inf_{\Psi\in[\Phi]} \sup_{x\in X} \underline{\underline{\mathbf{M}}}(\Psi(x)).$$

Remark 2. Although the nomenclature is the same it will always be clear when we refer to the width of a sequence, width of an homotopy class.

Definition 2.5. We say that a map $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ has *no concentration of mass* if

$$\lim_{r\to 0} \sup\{\|\Phi(x)\|(B(q,r)) : x \in X \text{ and } q \in M\} = 0.$$

Remark 3. One can show that mass continuous maps have no concentration of mass (see [9, Lemma 3.8])

The following theorem is a consquence of the interpolation theorems in [9].

Theorem 2.6. *Let $X \subset I^m$ be a cubical subcomplex of $I(m,j)$ and $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ be a flat continuous map with no concentration of mass. There exist $l \geq j$, $\tilde{X} = X(l)$ cubical subcomplex and $\tilde{\Phi} : \tilde{X} \to \mathcal{Z}_n(M;\underline{\underline{\mathbf{M}}};\mathbb{Z}_2)$ a mass continuous map satisfying:*

(i) $\Phi|_{\tilde{X}}$ is homotopic to $\tilde{\Phi}$ in the flat topology;

(ii) $\boldsymbol{L}[\tilde{\Phi}] \leq \boldsymbol{L}[\Phi]$

The critical set of a sequence is the set of candidates to be critical min-max varifolds. However, it is not even true that they are stationary in general. Applying a pull-tight procedure and using the interpolation results in [9] we can always refine a sequence of flat continuous maps such that its critical set contains only stationary varifolds.

Theorem 2.7 ([8, 9],[11, §4.3])**.** *Let $X_i \subset I^{m_i}$ be cubical subcomplexes of $I(m_i, j_i)$ and $S = \{\Phi_i : X_i \to \mathcal{Z}_n(M;\mathbb{Z}_2)\}$ be a sequence of flat continuous maps with no concentration of mass. There exist $l_i \geq j_i$, $\tilde{X}_i = X_i(l_i)$ cubical subcomplexes and $\tilde{S} = \{\tilde{\Phi}_i : \tilde{X}_i \to \mathcal{Z}_n(M;\underline{\underline{\boldsymbol{M}}};\mathbb{Z}_2)\}$ sequence of mass continuous maps such that:*

(i) $\Phi_i|_{\tilde{X}_i}$ *is homotopic to* $\tilde{\Phi}_i$ *in the flat topology;*

(ii) if $V \in \boldsymbol{C}(\tilde{S})$ *then* V *is stationary.*

(iii) $\boldsymbol{L}(\tilde{S}) \leq \boldsymbol{L}(S)$;

Furthermore, if $\boldsymbol{L}(\tilde{S}) = \boldsymbol{L}(S)$ *then*

$$\boldsymbol{C}(\tilde{S}) \subset \boldsymbol{C}(S) \cap \{V \in \mathrm{R}\mathcal{V}_n(M) : V \text{ is stationary}\}$$

The following theorem shows the existence of almost-minimising varifolds and it was originally proven by Pitts for maps with cubical domain and a boundary condition. However, it remains true for a cubical subcomplex and allowing homotopies with free boundary (see [9]).

Theorem 2.8. *Let $X \subset I^m$ be a cubical subcomplex and $\Phi : X \to \mathcal{Z}_n(M;\underline{\underline{\boldsymbol{F}}};\mathbb{Z}_2)$ a $\underline{\underline{\boldsymbol{F}}}$-continuous map. If $\boldsymbol{L}[\Phi] > 0$ then there exists $V \in \mathrm{I}\mathcal{V}_n(M)$ satisfying*

(i) V *is stationary;*

(ii) V *is almost-minimising in annuli;*

(iii) $\|V\|(M) = \boldsymbol{L}[\Phi]$.

From the proof of the previous theorem we extract a result that follows from Pitts' combinatorial arguments [11, §4.10]. To obtain the version that we state here it is necessary to further apply the interpolation theorems in [9].

Theorem 2.9. *Fix $m \in \mathbb{N}$ and let $X_i \subset I^m$ be cubical subcomplexes of $I(m, j_i)$ and $S = \{\Phi_i : X_i \to \mathcal{Z}_n(M;\mathbb{Z}_2)\}$ be a sequence of flat continuous maps with no concentration of mass such that every $V \in \boldsymbol{C}(S)$ is stationary.*

If no element of $\boldsymbol{C}(S)$ is almost-minimising in annuli then there exist $l_i \geq j_i$, $X_i^* = X_i(l_i)$ cubical subcomplexes and $S^* = \{\Phi_i^* : X_i^* \to \mathcal{Z}_n(M;\underline{\underline{\boldsymbol{M}}};\mathbb{Z}_2)\}$ sequence of mass continuous maps such that

(i) $\Phi_i|_{X_i^*}$ *is homotopic to* Φ_i^* *in the flat topology;*

(ii) $\boldsymbol{L}(S^*) < \boldsymbol{L}(S)$.

In [1] F.J. Almgren Jr. shows, in particular, the existence of an isomorphism $\boldsymbol{F}_M : \pi_q(\mathcal{Z}_n(M;\mathbb{Z}_2),\{0\}) \to H_{q+n}(M;\mathbb{Z}_2)$ for all $q \in \mathbb{N}$ which is called the Almgren isomorphism.

Definition 2.10. We say that a flat continuous map $\Phi : S^1 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a *sweepout* if $\boldsymbol{F}_M([\Phi]) \neq 0$, where $[\Phi] \in \pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2))$.

It is possible to show the existence of a fundamental cohomology class $\bar{\lambda} \in H^1(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2)$ such that the p-th cup product is non-zero for all $p \in \mathbb{N}$, $\bar{\lambda}^p \neq 0$. In particular it means that the cohomology ring $H^*(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2)$ contain the polynomial ring $\mathbb{Z}_2[\bar{\lambda}]$ generated by $\bar{\lambda} \in H^1$. In fact we have that they are isomorphic, that is, $H^*(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2) = \mathbb{Z}_2[\bar{\lambda}]$. For further details see [5, §1].

Definition 2.11. Let $X \subset I^m$ be a cubical subcomplex for some $m \in \mathbb{N}$, $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ a flat continuous map and $p \in \mathbb{N}$. We say that Φ is a *p-sweepout* if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X;\mathbb{Z}_2),$$

where $\bar{\lambda}^p$ is the p-th cup product of $\bar{\lambda}$. This is equivalent to saying that there exists $\lambda \in H^1(X;\mathbb{Z}_2)$ such that

(a) given any map $\gamma : S^1 \to X$, we have $\lambda(\gamma) \neq 0$ if, and only if, $\Phi \circ \gamma$ is a sweepout (as in Definition 2.10) and

(b) $\lambda^p \neq 0$ in $H^p(X;\mathbb{Z}_2)$.

We denote by $\mathcal{P}_p(M)$ the set of p-sweepouts in M with no concentration of mass:

$$\begin{aligned}\mathcal{P}_p(M) = \{(\Phi, X) : X \subset I^m &\text{ is a cubical subcomplex for some } m \in \mathbb{N} \\ &\text{and } \Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2) \text{ is a } p\text{-sweepout} \\ &\text{with no concentration of mass}\}\end{aligned}$$

Given a fixed $m \in \mathbb{N}$ we denote $\mathcal{P}_p^{(m)}(M) = \{(\Phi, X) \in \mathcal{P}_p(M) : X \subset I^m\}$, that is, the p-sweepouts with no concentration of mass whose domain is contained in a cube I^m of fixed dimension.

Note that a nullhomotopic map is not a sweepout. It is easy to see that $\mathcal{P}_p^{(m)}(M) \subset \mathcal{P}_p^{(m+1)}(M)$ and $\mathcal{P}_p(M) = \cup_{m\in\mathbb{N}}\mathcal{P}_p^{(m)}(M)$.

The following is an adaptation of an elementary result and is often referred to as Vanishing Lemma (see [5] or [9, Claim 6.3]).

Lemma 2.12 (Vanishing Lemma)**.** *Let $p, l \in \mathbb{N}$, $X, Y \subset I^m$ two cubical subcomplexes and $Z = X \cup Y$. If $\Phi : Z \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a $(p+l)$-sweepout and $\Phi_{|_Y}$ is **not** a l-sweepout then $\Phi_{|_X}$ must be a p-sweepout.*

Proof. Take $\lambda \in H^1(Z;\mathbb{Z}_2)$ so that condition (a) of Definition 2.11 is satisfied in Z and $\lambda^{p+l} \neq 0$. Define $\lambda_X = i_X^*\lambda$ and $\lambda_Y = i_Y^*\lambda$, where i_X, i_Y denote the respective inclusion maps onto Z. Since every 1-cycle in X or Y is also in Z, then condition (a) with respect to λ_X and λ_Y is satisfied for both spaces. We can assume that $(\lambda_Y)^l = 0$ and we want to prove that $(\lambda_X)^p \neq 0$.

Consider the exact sequence of the pair (Z,Y):

$$H^l(Z,Y;\mathbb{Z}_2) \xrightarrow{j_Y^*} H^l(Z;\mathbb{Z}_2) \xrightarrow{i_Y^*} H^l(Y;\mathbb{Z}_2).$$

Because $i_Y^*(\lambda^l) = 0$, there exists $\lambda_1 \in H^l(Z,Y;\mathbb{Z}_2)$ so that $j_Y^*\lambda_1 = \lambda^l$.

Now, suppose $(\lambda_X)^p = (i_X^*\lambda)^p = i_X^*(\lambda^p) = 0$ and consider the exact sequence for the pair (Z,X):

$$H^p(Z,X;\mathbb{Z}_2) \xrightarrow{j_X^*} H^p(Z;\mathbb{Z}_2) \xrightarrow{i_X^*} H^p(X;\mathbb{Z}_2).$$

If we chose $\lambda_2 \in H^p(Z,X;\mathbb{Z}_2)$ such that $j_X^*\lambda_2 = \lambda^p$, then we will have

$$j_Y^*\lambda_1 \cup j_X^*\lambda_2 = \lambda^{p+l} \in H^{(p+l)}(Z;\mathbb{Z}_2).$$

However, $X \cup Y = Z$, hence $H^*(Z, X\cup Y;\mathbb{Z}_2) = 0$. By the definition of cup product on relative cohomology we must have $\lambda_1 \cup \lambda_2 \in H^{(p+l)}(Z, X\cup Y;\mathbb{Z}_2)$, that is, $\lambda_1 \cup \lambda_2 = 0$(see [6, §3.2]).

On the other hand, we have

$$\lambda^{p+l} = j_Y^*\lambda_1 \cup j_X^*\lambda_2 = j_{X\cup Y}^*(\lambda_1 \cup \lambda_2) = 0,$$

which is a contradiction. We conclude that $(\lambda_X)^p \neq 0$, hence Φ_X is a p-sweepout. □

Definition 2.13. Given $p \in \mathbb{N}$, the *p-width* of (M,g) is defined as

$$\omega_p(M,g) = \inf_{(\Phi,X)\in\mathcal{P}_p(M)} \sup\{\underline{\underline{\mathbf{M}}}(\Phi(x)) : x \in X\}.$$

For a fixed $m \in \mathbb{N}$ we define the *restricted p-width* as

$$\omega_p^{(m)}(M,g) = \inf_{(\Phi,X)\in\mathcal{P}_p^{(m)}(M,g)} \sup\{\underline{\underline{\mathbf{M}}}(\Phi(x)) : x \in X\},$$

where we only consider p-sweepouts whose domain is contained in a cube I^m of fixed dimension m.

<u>*Remark*</u> 4. Note that for any p-sweepout Φ it is true that $\omega_p \leq \mathbf{L}[\Phi]$. However, it is not known in general whether it is always possible to have equality for some sweepout. It is trivial from the definition that we can always find a sequence of p-sweepouts S that satisfies $\mathbf{L}(S) = \omega_p$. Nevertheless that is not very useful because we must allow the ambient cubical domain I^{m_i} to vary and in this case Pitts combinatorial construction do not work (see [11, §4.10]).

3 Category of a Critical Set

In this section we are going to explain the notion of $\mathcal{A}$-category of a set, which is a generalization of the Lusternik-Schnirelmann Category (see [2]). We will use this alternate notion of category to study the topology of the space of min-max minimal hypersurfaces.

Let us briefly explain the reason for not using the Lusternik-Shcnirelmann Category. We are working with the space $\mathcal{Z}_n(M;\mathbb{Z}_2)$ and we want to obtain a lower bound on the category of the set of minimal hypersurfaces. Since $\mathcal{Z}_n(M;\mathbb{Z}_2)$ might not be locally contractible this result could be useless as a covering of the critical set by contractible sets might not exist.

Definition 3.1 ([2, 1.1])**.** Let X be a topological space and $\mathcal{A}$ a non-empty collection of non-empty subsets of X. We say that a subset $U \subset X$ is *deformable* to $\mathcal{A}$ if there exists $A \in \mathcal{A}$ and an homotopy $h_t : U \to X$, $t \in [0,1]$, such that $h_0 = \iota_U$ is the inclusion map and $h_1(U) \subset A$.

A finite covering $\{U_1, \dots, U_k\}$ of open sets such that each U_j is deformable to $\mathcal{A}$ is called a *$\mathcal{A}$-categorical covering*. Given a subspace $Y \subset X$ we define the $\mathcal{A}$-category of Y as the smallest cardinality k of such covering and we write $\mathcal{A}\text{-cat}(Y) = k$. If no such covering exists we put $\mathcal{A}\text{-cat}(Y) = \infty$.

Remark 5. The $\mathcal{A}$-category of a subset $Y \subset X$ is relative to the ambient space X. In general the relative category is different from the intrinsic category (seeing Y as a subset of itself). This happens for the Lusternik-Schnirelmann category as well.

In our case we consider the collection

$$\mathcal{N}_1 = \{N \subset \mathcal{Z}_n(M;\mathbb{Z}_2) : U \text{ is open in the flat topology and } (\iota_U)_* : \pi_l(U) \to \pi_l(\mathcal{Z}_n(M;\mathbb{Z}_2)) \text{ is trivial } \}.$$

It follows from [1, Theorem 8.2] that for every neighborhood of $0 \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ contains an element $U \in \mathcal{N}_1$ such that $0 \in U$.

Remark 6. Since $\pi_l(\mathcal{Z}_n(M;\mathbb{Z}_2)) = 0$ for all $l > 1$, it follows that the induced map $(\iota_U)_*$ is trivial for all $l \in \mathbb{N}$.

We only summarize some trivial properties that we will be necessary for our applications.

Proposition 3.2. *Let $\mathcal{N}_1$ be defined as above. For any subset $Y \subset \mathcal{Z}_n(M;\mathbb{Z}_2)$ the following holds:*

(i) $\mathcal{N}_1\text{-cat}(Y) = 1$ *if and only if Y is contained in an open set U such that $(\iota_U)_* : H_1(U) \to H_1(X)$ is zero;*

(ii) *if $W \subset Y$ then* $\mathcal{N}_1\text{-cat}(W) \le \mathcal{N}_1\text{-cat}(Y)$*;*

(iii) *if $K \subset X$ is compact then* $\mathcal{N}_1\text{-cat}(K) < \infty$.

Proof. **(i)**:It follows from the definition that there must be a set U such that the maps induced by the inclusion on the fundamental group is trivial. Simply note that the Hurewicz homomorphism is surjective in dimension 1 and natural, so the induced map in homology must also be trivial.
(ii) and **(iii)** are straightforward from the definition and the fact that $\mathcal{N}_1$ defines a local neighborhood system in $\mathcal{Z}_n(M;\mathbb{Z}_2)$. □

The motivation is to try to obtain a result similar to [2, 2.3(iii)] in our weaker setting, where we don't have Banach manifolds or a smooth functional. One could hope to mimic their proof but it is not clear that the critical values c_i, defined in their paper, correspond to the width ω_i. It might be possible to show that c_i corresponds to a critical value even in our setting, but even so, nothing is known about its asymptotic behavior, which is a crucial property of ω_i. Nevertheless, we have found that it is possible to obtain information about the topology of the critical set measured by its $\mathcal{N}_1$-category. To do so we must know how the existence of sweepouts contribute to the 1-category of a set. The main property that establishes this relation is given by the next lemma.

Lemma 3.3. *Let $K \subset \mathcal{Z}_n(M;\mathbb{Z}_2)$ be a set with $\mathcal{N}_1$-cat$(K) \le N$. There exists an open set $U \subset \mathcal{Z}_n(M;\mathbb{Z}_2)$ with $K \subset U$ satisfying the following property:*

If X is a cubical subcomplex and $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a flat continuous map with $\Phi(X) \subset U$ then Φ is not a N-sweepout.

Proof. We prove it by induction. If $N = 1$ then K is contained in an open set $U \subset \mathcal{Z}_n(M;\mathbb{Z}_2)$ such that every map $f : S^1 \to U$ is nullhomotopic in $\mathcal{Z}_n(M;\mathbb{Z}_2)$. So it cannot be a 1-sweepout.

Assume the result is valid for $N-1$ and suppose $\mathcal{N}_1$-cat$(K) \le N$. There exists $U_1, \dots, U_N$ each of which does not contain 1-sweepouts and $K \subset U_1 \cup \dots \cup U_N$. It is clear that $K' = K \setminus U_N$ has $\mathcal{N}_1$-cat$(K') \le N - 1$ so we can take U' with $K' \subset U'$ that doesn't contain $(N-1)$-sweepouts. We can also assume that $\bar{U}' \subset U_1 \cup \dots \cup U_{N-1}$. Let $U = U' \cup U'_N$, where U'_N is such that $K \setminus U' \subset U'_N$ and $\overline{U'_N} \subset U_N$.

Now, for $\Phi : X \to U$ let $X_1 = \overline{\{x \in X : \Phi(x) \in U'\}}$ and $X_2 = \overline{X \setminus X_1}$. Note that if either X_1 or X_2 are empty then the result follows simply because a N-sweepout must also be a q-sweepout for all $q < N$. By the induction hypothesis $\Phi|_{X_1}$ is not a $(N-1)$-sweepout and, as in the first step, $\Phi|_{X_2}$ is not a 1-sweepout. Thus the Vanishing Lemma 2.12 implies that Φ cannot be a N-sweepout. □

Let us denote the set of min-max minimal hypersurfaces as

$$\Lambda(M,g) = \{V \in \mathbf{I}\mathcal{V}_n(M) : \text{supp}\|V\| \text{ is a smooth embedded minimal hypersurface and } V \in \mathbf{C}(S) \text{ for some sequence of flat continuous maps with no concentration of mass}\}$$

and its associated cycles

$$\mathcal{T} = \{T \in \mathcal{Z}_k(M;\mathbb{Z}_2) : \text{supp} T \text{ is a smooth embedded minimal hypersurface or } T = 0\}.$$

For $\beta > 0$ we denote $\Lambda_\beta = \{V \in \Lambda : \|V\|(M) \leq \beta\}$ and similarly $\mathcal{T}_\beta = \{T \in \mathcal{T} : \underline{\underline{\mathbf{M}}}(T) \leq \beta\}$

The next Lemma is a direct application of the Constancy Theorem and lower semicontinuity of the mass (see [9, Claim 6.2]).

Lemma 3.4. *Fix $m \in \mathbb{N}$ and $\beta > 0$. For every open set $U \subset \mathcal{Z}_n(M;\mathbb{Z}_2)$, with $\mathcal{T} \subset U$, there exists $\delta > 0$ such that for any $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$*

$$\underline{\underline{\mathbf{F}}}(|T|, \Lambda_\beta) < \delta \Rightarrow T \in U.$$

We are now ready to prove the main theorem of this section. The proof follows the exact same ideas of [9, Theorem 6.1] with the appropriate modifications.

Theorem 3.5. *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $n+1$, with $2 \leq n \leq 6$, and $m, p, N \in \mathbb{N}$ such that $p + N \leq m$. If $\omega_p^{(m)} = \omega_{p+N}^{(m)}$ then $\mathcal{N}_1$-cat$(\mathcal{T}_{\omega_{p+N}^{(m)}}) \geq N + 1$.*

Proof. To simplify notation, put $\omega = \omega_p^{(m)} = \omega_{p+N}^{(m)}$.

Suppose by contradiction that $\mathcal{N}_1$-cat$(\mathcal{T}_\omega) \leq N$. By Lemma 3.3 there exists an open set $U \subset \mathcal{Z}_n(M;\mathbb{Z}_2)$ with $\mathcal{T}_\omega \subset U$ that does not contain N-sweepouts. It follows from Lemma 3.4 that there exists $\varepsilon_0 > 0$ such that

$$\underline{\underline{\mathbf{F}}}(|T|, \Lambda_\omega) < 2\varepsilon_0 \Rightarrow T \in U.$$

Let $S = \{\Phi_i : X_i \to \mathcal{Z}_n(M;\mathbb{Z}_2)\}_{i\in\mathbb{N}}$, with $X_i \subset I^m$ cubical subcomplexes, be a sequence of $(p+N)$-sweepouts such that $\mathbf{L}(S) = \omega_{p+N}^{(m)}$. By Theorem 2.7 there exist $X_i' \subset X_i$ cubical subcomplexes and a sequence of mass continuous (in particular $\underline{\underline{\mathbf{F}}}$-continuous) $(p+N)$-sweepouts $S' = \{\Phi_i' : X_i' \to \mathcal{Z}_n(M;\underline{\underline{\mathbf{M}}};\mathbb{Z}_2)\}$ such that $\mathbf{L}(S') \leq \mathbf{L}(S)$.

We claim that $\mathbf{L}(S') = \mathbf{L}(S)$. Indeed, if we had $\mathbf{L}(S') < \mathbf{L}(S)$ then for i sufficiently large Φ_i' would be a $(p+N)$-sweepout such that $\sup\{\underline{\underline{\mathbf{M}}}(\Phi_i'(x')) : x' \in X_i'\} < \mathbf{L}(S) = \omega_{p+N}^{(m)}$, which is a contradiction.

For each $i \in \mathbb{N}$ define Y_i to be the cubical subcomplex of I^m consisting of all cells $\alpha \subset X_i'$ such that

$$\sup\{\underline{\underline{\mathbf{F}}}(|\Phi_i'(x')|, \Lambda_\omega) : x' \in \alpha\} \geq \varepsilon_0.$$

It follows that $\underline{\underline{\mathbf{F}}}(|\Phi_i'(x')|, \Lambda_\omega^{(m)}) < 2\varepsilon_0$ for all $x' \in \overline{X_i' \setminus Y_i}$, that is,

$$\Phi_i'(\overline{X_i' \setminus Y_i}) \subset U.$$

Hence $\Phi_i'|_{\overline{X_i' \setminus Y_i}}$ is not a N-sweepout. Since Φ_i' is a $(N+p)$-sweepout, it follows that Y_i must be non-empty and from the Vanishing Lemma 2.12 we get that $\Phi_i'|_{Y_i}$ is a p-sweepout.

Applying Theorem 2.7 for the sequence $\{\Phi_i'|_{Y_i}\}_{i\in\mathbb{N}}$ we obtain $\tilde{Y}_i \subset Y_i$ cubical subcomplexes and another sequence of mass continuous p-sweepouts $\tilde{S} =$

$\{\tilde{\Phi}_i : \tilde{Y}_i \to \mathcal{Z}_n(M; \underline{\underline{\mathbf{M}}}; \mathbb{Z}_2)\}_{i\in\mathbb{N}}$. Since $\mathbf{L}(\{\Phi'_i|_{Y_i}\}_{i\in\mathbb{N}}) \leq \mathbf{L}(S') = \omega^{(m)}_{p+N} = \omega^{(m)}_p$, we conclude as before that $\mathbf{L}(\tilde{S}) = \mathbf{L}(\{\Phi'_i|_{Y_i}\}_{i\in\mathbb{N}})$. Thus, $\mathbf{L}(\tilde{S}) = \omega$ and $\mathbf{C}(\tilde{S}) \subset \mathbf{C}(\{\Phi'_i|_{Y_i}\}_{i\in\mathbb{N}}) \cap \{V \in \mathrm{R}\mathcal{V}_n(M) : V \text{ is stationary}\}$. It follows that $\underline{\underline{\mathbf{F}}}(V, \Lambda^{(m)}_\omega) \geq \varepsilon_0$ for all $V \in \mathbf{C}(\tilde{S})$. In particular no element of $\mathbf{C}(\tilde{S})$ has smooth embedded support.

Applying Theorem 2.9 we obtain a sequence of p-sweepouts S^* such that $\mathbf{L}(S^*) < \mathbf{L}(\tilde{S}) = \omega^{(m)}_p$ which contradicts our initial hypothesis and concludes the proof. □

4 Applications

We will use the result in the previous section together with Sharp's Compactness Theorem [14, Theorem 2.3] to derive a non-compactness theorem for the space of all minimal hypersurfaces in a manifold with positive Ricci curvature.

Thourghout this section (M^{n+1}, g) denotes a closed Riemannian manifold of dimension $3 \leq n+1 \leq 7$ and $\Lambda = \Lambda(M, g)$.

Before proceeding, let us first state a characterization of convergence of minimal hypersurfaces. Given a minimal hypersurface $\Sigma \subset M$ we denote by L_Σ the Jacobi operator acting either on smooth functions (when Σ is two-sided) or on normal vectorfields (when it is one-sided). The following is proved in Claims $4-6$ in [14].

Proposition 4.1. *Let M^{n+1} be a closed Riemannian manifold of dimension $3 \leq n+1 \leq 7$, $\{\Sigma_i\}_{i\in\mathbb{N}}$, Σ_∞ be a sequence of minimal embedded smooth hypersurfaces and $\mathcal{S} \subset \Sigma_\infty$ a finite set of points. Suppose $\Sigma_i \to \Sigma_\infty$ in the $\mathcal{C}^\infty_{loc}(M \setminus \mathcal{S})$ graphical sense (see [14]). We have the following characterization of Σ_∞:*

*(i) if the convergence is **one-sheeted** then $\mathcal{S} = \emptyset$;*

(ii) if Σ_∞ is <u>two-sided</u> then there exists $u \in \mathcal{C}^\infty(\Sigma_\infty)$ such that

$$\begin{cases} u \geq 0 \\ L_{\Sigma_\infty}(u) = 0. \end{cases}$$

*Furthermore, if the convergence is at least **two-sheeted** or $\Sigma_i \cap \Sigma_\infty = \emptyset$ for all i sufficiently large then $u > 0$ everywhere and Σ_∞ is stable. In case the convergence is **one-sheeted** and $\Sigma_i \cap \Sigma_\infty \neq \emptyset$ for all i sufficiently large then we can further conclude that index$(\Sigma_\infty) \geq 1$.*

*(iii) if Σ_∞ is <u>one-sided</u> and the convergence is **one-sheeted** then, in addition to (i) we have a normal vectorfield $J \in \mathcal{C}^\infty(\Sigma_\infty, T^\perp\Sigma_\infty)$ such that*

$$\begin{cases} J \not\equiv 0 \\ L_{\Sigma_\infty}(J) = 0. \end{cases}$$

That is, J is a non-trivial Jacobi field.

*(iv) if Σ_∞ is <u>one-sided</u> and the convergence is at least **two-sheeted** then we must have $\lambda_1(L_{\Sigma_\infty}) > 0$. In addition, if $\tilde{\Sigma}_\infty$ denotes the oriented double covering of Σ_∞ then $\lambda_1(L_{\tilde{\Sigma}_\infty}) = 0$. That is, $\tilde{\Sigma}_\infty$ is a <u>two-sided</u> immersed minimal hypersurface with a non-trivial Jacobi field.*

For $\Omega \subset \Lambda$ we define

$$\begin{aligned}\mathcal{I}(\Omega) &= \{\text{index}(\text{supp}\|V\|) \in \mathbb{Z}_{\geq 0} : V \in \Omega\},\\ \mathcal{A}(\Omega) &= \{\text{area}(\text{supp}\|V\|) \in \mathbb{R}_{\leq 0} : V \in \Omega\}.\end{aligned}$$

We know by Sharp's Compactness Theorem that $\sup \mathcal{I}(\Omega) + \sup \mathcal{A}(\Omega) < \infty$ implies that Ω is compact in the weak topology. Furthermore, the convergence is as described in Proposition 4.1.

Our goal is to prove that the space Λ is non-compact, then it is sufficient to show that $\sup \mathcal{I}(\Lambda) + \sup \mathcal{A}(\Lambda) = \infty$. However, in the general case we are not able to show this. We managed to overcome this by considering the quantity $\sup \mathcal{I}(\Lambda) + \#\mathcal{A}(\Lambda)$ instead.

Theorem 4.2. *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq n+1 \leq 7$.*

Fix $p \in \mathbb{N}$, if $\#\mathcal{A}(\Lambda_{\omega_p+1}) < \infty$ then there exists $m \in \mathbb{N}$ such that

$$\omega_p^{(m)}(M) = \omega_p(M).$$

Proof. Suppose false, that is, we have a strictly decreasing sequence $\{\omega_p^{(m)}\}_{m\in\mathbb{N}}$ converging from above to ω_p. In particular we obtain a sequence of p-sweepouts with no concentration of mass $\{\Phi_m : X_m \subset I^m \to \mathcal{Z}_n(M;\mathbb{Z}_2)\}_{m\in\mathbb{N}}$ satisfying

$$\omega_p^{(m+1)} \leq \mathbf{L}[\Phi_{m+1}] < \omega_p^{(m)} \leq \mathbf{L}[\Phi_m].$$

We can further assume that $\mathbf{L}[\Phi_m] < \omega_p + 1$ for all $m \in \mathbb{N}$.

First we apply Theorem 2.6 to each Φ_m and obtain $\tilde{\Phi}_m$ a mass continuous p-sweepout. In particular is is $\underline{\underline{\mathbf{F}}}$-continuous and has no concentration of mass. Now, from Theorem 2.8 and the Regularity Theorem 2.2 we obtain a sequence $\{V_m\}_{m\in\mathbb{N}}$ of stationary varifolds with smooth embedded support such that $\|V_{m+1}\|(M) < \|V_m\|(M)$. We can write for each m

$$V_m = \sum_{i=1}^{l} n_i \cdot \Sigma_i,$$

where Σ_i are minimal hypersurfaces such that $\text{area}(\Sigma_i) \in \mathcal{A}(\Lambda_{\omega_p+1})$ and $n_i \in \mathbb{N}$. Since $\mathcal{A}(\Lambda_{\omega_p+1})$ is finite there are only finitely many possible values of $\|V_m\|(M) = \sum_{i=1}^{l} n_i \text{area}(\Sigma_i) \leq \omega_p + 1$ for all m, which is a contradiction. ☐

Now we use the result from the previous section to prove a non-compactness theorem.

Theorem 4.3. *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \le n+1 \le 7$ with $Ric(g) > 0$. If the metric g is analytic then the space $\Lambda(M, g)$ of minimal hypersurfaces (with multiplicity) is non-compact.*

Proof. Suppose false, then there exists $C > 0$ such that $\sup \mathcal{I}(\Lambda) + \sup \mathcal{A}(\Lambda) < C$.

`Claim` **1.** $\#\mathcal{A}(\Lambda) < \infty$

Indeed, if it is not true then there exists a sequence $\{\Sigma_i\}_{i \in \mathbb{N}}$ of multiplicity one minimal hypersurfaces with distinct area. Without loss of generality we can assume that area(Σ_i) is increasing. By hypothesis we have $\Sigma_i) < C$ for all i. From Sharp's Compactness Theorem we can take a convergent subsequence, still denoted by $\{\Sigma_i\}_{i \in \mathbb{N}}$, with limit Σ_∞.

We claim that the convergence must **one-sheeted**, hence smooth everywhere. In fact, suppose the convergence is **two-sheeted**, then we can divide in two cases. That is to say, whether Σ_∞ is <u>one-sided</u> or <u>two-sided</u>. If Σ_∞ is <u>one-sided</u>, then we are in case 4.1(iv). In this case the oriented double covering gives us a stable minimal hypersurface $\tilde{\Sigma}_\infty$, which is a contradiction because Ric$(g) > 0$. If Σ_∞ is <u>two-sided</u>, then we are in case 4.1(ii) with **two-sheeted** convergence, which give us a contradiction for the same reason. We conclude that the convergence is graphically smooth everywhere.

Now, a Theorem by L.Simon [13, §2 Theorem 3] says that, in a manifold with analytic metric, there exists a $\mathcal{C}^\infty$-neighbourhood of a minimal hypersurface such that any other minimal hypersurface in that neighbourhood has constant area. Since we have smooth convergence this shows that area(Σ_i) must be constant and equal to area(Σ_∞) for all i sufficiently large. This is a contradiction and it finishes the proof of our first claim.

`Claim` **2.** *There exists a constant $N \in \mathbb{N}$ so that $\omega_p < \omega_{p+N}$ for all $p \in \mathbb{N}$.*

Suppose false, then we can find a sequence $\{p_i\}_{i \in \mathbb{N}}$ such that

$$\omega_{p_i} = \omega_{p_i + i}.$$

We already know that $\#\mathcal{A}(\Lambda) < \infty$, thus Theorem 4.2 tells us that for each i there exists $m_i \in \mathbb{N}$ so that $\omega_{p_i}^{(m_i)} = \omega_{p_i}$ and $\omega_{p_i+i}^{(m_i)} = \omega_{p_i}$. Hence,

$$\omega_{p_i}^{(m_i)} = \omega_{p_i+i}^{(m_i)}.$$

Finally, it follows from Theorem 3.5 that $\mathcal{N}_1$-cat$(\mathcal{T}_{\omega_{p_i+i}}) \ge i$. By the monotonicity property of $\mathcal{N}_1$-cat, Proposition 3.2(ii), this implies that $\mathcal{N}_1$-cat$(\mathcal{T}) = \infty$. However, we are supposing that Λ is compact, which implies that so is $\mathcal{T}$. This is a contradiction because compact sets must have finite 1-category, thus proving our second claim.

Now, for each $i \in \mathbb{N}$ we can find a $(1 + iN)$-sweepout Φ_i such that

$$\omega_{1+iN} \le \mathbf{L}[\Phi_i] < \omega_{1+(i+1)N}.$$

For each such sweepout we obtain by Theorem 2.8 a stationary varifold V_i whose support is smooth and embedded and $\|V_i\|(M) = \mathbf{L}[\Phi_i]$. By Frankel's

Theorem for manifolds with $\mathrm{Ric}(g) > 0$ any two minimal hypersurfaces must intersect, then the support of V_i can only have one connected component, that is, $V_i = n_i \cdot \Sigma_i$ where Σ_i is a multiplicity one minimal hypersurface and $n_i \in \mathbb{N}$.

We already know that $\mathrm{area}(\Sigma_i)$ can only assume finitely many values, from which follows that $\|V_i\|(M) = n_i \mathrm{area}(\Sigma_i)$ must have at least linear growth in i. However, it is known that ω_p has sublinear growth in p (see [5, Theorem 1] or [9, Theorems 5.1 and 8.1]). Thus ω_{1+iN} has sublinear growth, which implies that so does $\mathbf{L}[\Phi_i]$. We arrive to a contradiction and this concludes the proof of the theorem. □

Corollary 4.4. *Let S^n denote the n-sphere with the round metric and $3 \leq n \leq 7$. Then S^n admits infinitely many non-isometric minimal hypersurfaces.*

Proof. Since the round metric in S^n is analytic we can apply the previous theorem. From Sharp's Compactness Theorem it follows that $\sup \mathcal{I}(\Lambda(S^n)) + \sup \mathcal{A}(\Lambda(S^n)) = \infty$, so it must contain a sequence of minimal hypersurfaces with either the index going to infinity or the area. □

We can also change the analyticity hypothesis by a bumpy metric. In this case it extends a result by H. Li and X. Zhou to higher dimensions (see [7, Corollary 1.5]).

We say that a metric is bumpy if no immersed minimal hypersurface has a non-trivial Jacobi field. In [16] B. White showed that bumpy metrics for embedded minimal hypersurfaces are generic and recently the author extended the same result for bumpy metric for immersed minimal hypersurfaces (see [17]).

To prove this we also have to use a recent result shown by Marques-Neves in [10]. The authors show that for a given $\underline{\underline{\mathbf{F}}}$-continuous k-sweepout we can always find a varifold that realizes the width of its homotopy class and has index $\leq k$.

Theorem 4.5. *Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq n+1 \leq 7$ with $Ric(g) > 0$. If the metric g is bumpy, then*

$$\omega_p < \omega_{p+1}$$

and $\sup \mathcal{I}(\Lambda(M,g)) + \sup \mathcal{A}(\Lambda(M,g)) = \infty$.

Proof. The proof is very similar to the previous theorem. First we show that for every p there exists m such that $\omega_p = \omega_p^{(m)}$.

`Claim` **1.** *For a fixed p, we have $\#\mathcal{A}(\Lambda_{\omega_p+1}) < \infty$*

Suppose it is false. Arguing exactly as in the previous theorem we obtain a sequence of varifolds $\{V_m\}_{m\in\mathbb{N}} \subset \Lambda_{\omega_p+1}$ such that $\mathrm{index}(\mathrm{supp}\|V\|) \leq p$ (see [10, Theorem 1.2]). By Sharp's Compactness Theorem we know that $V_m \to V_\infty$ for some $V_\infty \in \Lambda_{\omega_p+1}$ and the convergence is classified by proposition 4.1. Now, in any situation described in 4.1 it is possible to construct a non-trivial Jacobi field over $\mathrm{supp}\|V_\infty\|$ or its immersed double covering. In any case, that is a contradiction.

Suppose now that $\omega_p = \omega_{p+1}$ that is, $\omega_p^{(m)} = \omega_{p+1}^{(m)}$ for some $m \in \mathbb{N}$. In particular the set $\Omega = V \in \Lambda : \|V\|(M) = \omega_p$ is infinite and $\text{index}(\text{supp}\|V\|) \leq p+1$ for all $V \in \Omega$. Arguing as before, these varifolds must accumulate on a minimal hypersurface (possibly immersed) with a non-trivial Jacobi field, which is a contradiction.

The remaining statement follows directly from Sharp's Compactness Theorem. ☐

References

[1] F. J. Almgren Jr. - *The Homotopy Groups of the Integral Cycle Groups*, Topology 1, p.257-299 (1962)

[2] M. Clapp, D. Puppe - *Invariants of the Lusternik-Schnirelmann Type and the Topology of Critical Sets*, Trans. of the AMS v.298 No. 2, p.603-620 (1986)

[3] O. Cornea et al - *Lusternik-Schnirelmann Category*, Math. Surveys and Monographs 103, AMS (2003)

[4] H. Federer - *Geometric measure theory*, Classics in Mathematics, Springer-Verlag (1969)

[5] L. Guth - *Minimax Problems Related to Cup Powers and Steenrod Squares*, arxiv:0702066v2 (2008)

[6] A. Hatcher - *Algebraic Topology*, Cambridge University Press (2002)

[7] H. Li, X. Zhou - *Existence of minimal surfaces of arbitrary large Morse index*, arxiv:1504.00970v1 (2015)

[8] F. C. Marques, A. Neves - *Min-Max theory and the Willmore conjecture*, Ann. of Math. 179, No. 2, p.683-782 (2014)

[9] F. C. Marques, A. Neves - *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*, arXiv:1311.6501 (2013)

[10] F. C. Marques, A. Neves - *Morse index and multiplicity of min-max minimal hypersurfaces*, arXiv:1512.06460 (2015)

[11] J. T. Pitts - *Existence and regularity of minimal surfaces on Riemannian manifolds*, Mathematical Notes 27, Princeton University Press, Princeton (1981)

[12] R. Schoen, L. Simon - *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. 34, p.741-797 (1981)

[13] L. Simon - *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. 118, p.525-571 (1983)

[14] B. Sharp - *Compactness of minimal hypersurfaces with bounded index*, arxiv: arXiv:1501.02703v1 (2015)

[15] B. White - *Currents and Flat Chains Associated to Varifolds, with an Application to Mean Curvature Flow*, Duke Math. J. 148, p.213-228 (2009)

[16] B. White - *The Space of Minimal Submanifolds for Varying Riemannian Metrics*, Indiana Univ. Math. Journal 40, No.1, p.161-200 (1991)

[17] B. White - *On the Bumpy Metrics Theorem for Minimal Submanifolds*, arXiv:1503.01803 (2015)