THE VLASOV-POISSON-BOLTZMANN SYSTEM FOR A DISPARATE MASS BINARY MIXTURE

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Abstract. The Vlasov-Poisson-Boltzmann system is often used to govern the motion of plasmas consisting of electrons and ions with disparate masses when collisions of charged particles are described by the two-component Boltzmann collision operator. The perturbation theory of the system around global Maxwellians recently has been well established in [42]. It should be more interesting to further study the existence and stability of nontrivial large time asymptotic profiles for the system even with slab symmetry in space, particularly understanding the effect of the self-consistent potential on the non-trivial long-term dynamics of the binary system. In the paper, we consider the problem in the setting of rarefaction waves. The analytical tool is based on the macro-micro decomposition introduced in [59] that we can be able to develop into the case for the two-component Boltzmann equations around local bi-Maxwellians. Our focus is to explore how the disparate masses and charges of particles play a role in the analysis of the approach of the complex coupling system time-asymptotically toward a non-constant equilibrium state whose macroscopic quantities satisfy the quasineutral nonisentropic Euler system.

Contents

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1. Introduction

1.1. **Presentation of the problem.** In the paper we are concerned with the nontrivial long-time dynamics of the Vlasov-Poisson-Boltzmann (VPB for short) system used for describing the motion of charged particles in plasma (e.g., ions and electrons) when collisions between particles are taken into account, cf. [15, 56]. Compared to the close-to-constant-equilibrium framework (cf. [42]), the perturbation theory around the non-constant equilibrium state would be more interesting and difficult due to the appearance of disparate masses and charges for gas mixtures, cf. [1, 2, 73, 74]. In the case of three space dimensions with slab symmetry, the governing equations take the form of

$$\begin{aligned}\partial_t F_i + \xi_1\partial_x F_i - \frac{q_i}{m_i}\partial_x\phi\,\partial_{\xi_1}F_i &= Q_{ii}(F_i,F_i) + Q_{ie}(F_i,F_e),\\ \partial_t F_e + \xi_1\partial_x F_e - \frac{q_e}{m_e}\partial_x\phi\,\partial_{\xi_1}F_e &= Q_{ee}(F_e,F_e) + Q_{ei}(F_e,F_i).\end{aligned} \tag{1.1}$$

The self-consistent potential function $\phi = \phi(t,x)$ satisfies the Poisson equation

$$-\partial_x^2\phi = q_i\int_{\mathbb{R}^3} F_i\,d\xi + q_e\int_{\mathbb{R}^3} F_e\,d\xi. \tag{1.2}$$

Here $F_i(t,x,\xi)$ and $F_i(t,x,\xi)$ stand for the nonnegative number distribution functions for ions and electrons which have position $x\in\mathbb{R}$ and velocity $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$ at time $t\geq 0$. Ions and electrons are assumed to have masses $m_i>0$, $m_e>0$ and charges $q_i>0$, $q_e<0$, respectively. Without loss of generality we suppose $m_i\geq m_e$ which is consistent with the physical situation where ions are much heavier than electrons.

Regarding the binary collisions between like or unlike particles on the right-hand side of (1.1), we assume that they are described by the Boltzmann operator for the hard-sphere model whose exact form reads

$$Q_{\mathcal{AB}}(F_{\mathcal{A}},F_{\mathcal{B}}) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{\mathcal{AB}}(|\xi-\xi_*|,\omega)[F_{\mathcal{A}}(\xi')F_{\mathcal{B}}(\xi_*')-F_{\mathcal{A}}(\xi)F_{\mathcal{B}}(\xi_*)]\,d\xi_* d\omega, \tag{1.3}$$

for $\mathcal{A},\mathcal{B}\in\{i,e\}$. Here $\mathbb{S}^2$ is the unit sphere of $\mathbb{R}^3$. The collision kernel is given by

$$B_{\mathcal{AB}} = \frac{(\sigma_{\mathcal{A}}+\sigma_{\mathcal{B}})^2}{4}|(\xi-\xi_*)\cdot\omega|,$$

with $\sigma_{\mathcal{A}}>0$ denoting the dimeter of particles of $\mathcal{A}$ species, and through the paper we always take $\sigma_{\mathcal{A}}=\sigma_{\mathcal{B}}=\sigma$ without loss of generality. The pre-collisional velocity pair (ξ,ξ_*) and the post-collisional velocity pair (ξ',ξ_*') corresponding to the integrand of (1.3) satisfy the relationship

$$\begin{aligned}\xi' &= \xi - \frac{2m_{\mathcal{B}}}{m_{\mathcal{A}}+m_{\mathcal{B}}}\,[(\xi-\xi_*)\cdot\omega]\,\omega,\\ \xi_*' &= \xi_* + \frac{2m_{\mathcal{A}}}{m_{\mathcal{A}}+m_{\mathcal{B}}}\,[(\xi-\xi_*)\cdot\omega]\,\omega,\end{aligned}$$

which follows from conversation of momentum and energy

$$\begin{aligned}m_{\mathcal{A}}\xi + m_{\mathcal{B}}\xi_* &= m_{\mathcal{A}}\xi' + m_{\mathcal{B}}\xi_*',\\ m_{\mathcal{A}}|\xi|^2 + m_{\mathcal{B}}|\xi_*|^2 &= m_{\mathcal{A}}|\xi'|^2 + m_{\mathcal{B}}|\xi_*'|^2,\end{aligned}$$

for two colliding particles $\mathcal{A}$ and $\mathcal{B}$. Note that collisions between particles in plasma physics are often modelled by the long-range collision operator, for instance, the Boltzmann operator for soft potentials or the Landau operator for the Coulomb potential, cf. [80]. One may expect that the techniques of analysis developed in the paper together with the ones in [24, 31, 32, 44] could also be applied to those more physical situations.

For notational convenience, as in [41], we denote in the sequel

$$\mathbf{F}(t,x,\xi) = \begin{bmatrix} F_i(t,x,\xi)\\ F_e(t,x,\xi)\end{bmatrix}.$$

The system (1.1), (1.2) is supplemented with initial data

$$\mathbf{F}(0,x,\xi)=\mathbf{F}_0(x,\xi)=\left[\begin{array}{c} F_{0i}(x,\xi)\\ \\ F_{0e}(x,\xi)\end{array}\right], \tag{1.4}$$

and with boundary data at far fields

$$\lim_{x\to\pm\infty}\mathbf{F}_0(x,\xi)=\mathbf{F}_{0\pm\infty}(\xi), \tag{1.5}$$

and

$$\lim_{x\to\pm\infty}\phi(t,x)=\phi_\pm. \tag{1.6}$$

Through the paper, due to the basic property of the two-component Boltzmann collision operator as discussed in the next section, we assume that $\mathbf{F}_{0\pm\infty}(\xi)$ are the spatially homogeneous bi-Maxwellians whose exact definition will be introduced in (2.2) and (2.1), that is,

$$\mathbf{F}_{0\pm\infty}=\mathbf{M}_{\pm\infty}=\left[\begin{array}{c} M_{[n_{i\pm},u_\pm,\theta_\pm;m_i]}(\xi)\\ \\ M_{[n_{e\pm},u_\pm,\theta_\pm;m_e]}(\xi)\end{array}\right], \tag{1.7}$$

where $n_{i\pm}>0$, $n_{e\pm}$, $u_\pm=(u_{1\pm,0,0})$, $\theta_\pm>0$ are given constants, with the quasineutral assumption

$$q_in_{i\pm}+q_en_{e\pm}=0.$$

For later use, for brevity we always take

$$n_{e\pm}=n_\pm,\quad n_{i\pm}=-\frac{q_e}{q_i}n_\pm,$$

with given constants $n_\pm>0$.

A general question is to investigate the existence, uniqueness, regularity and large-time behavior of solutions to the Cauchy problem on the above VPB system in terms of given initial data with general far fields. Note that the far-field data at $x=\pm\infty$ could be distinct, and hence the long-term dynamics could be nontrivial with spatial variation along the direction of x variable.

1.2. **Literature and background.** In what follows we review some relevant literature. First of all, in general settings for large initial data, the Cauchy problem or the IBVP on the VPB system related to its one-dimensional version (1.1), (1.2) has been studied by many people. Among them, we would only mention Desvillettes-Dolbeault [22] for the long time asymptotics of the system, Bernis-Desvillettes [4] for the propagation of regularity of solutions, Mischler [66] for the initial boundary value problem, Bostan-Gamba-Goudon-Vasseur [9] for the stationary problem on the bounded domain, and Guo [43] for global existence of classical solutions near vacuum. Note that the existence of renormalised solutions of the much more complex Vlasov-Maxwell-Boltzmann system with a defect measure has been recently studied in Arsenio-Saint-Raymond [3].

In perturbation regime around global Maxwellians on the spatially periodic domain $\mathbb{T}^3$, a number of progresses have been made by Guo [41, 42, 44]. His approach is based on the robust energy method through constructing the appropriate energy functional and energy dissipation rate functional so that the nonlinear collision terms can be controlled along the linearised dynamics under smallness assumption, where the mathematical analysis strongly relies on both the structure of the system and the dissipative property of the linearised operator. A general technique in the proof is to design good velocity weight functions for closing the a priori estimates. In the case of the whole space, the Poincaré inequality fails to capture the dissipation of solutions over the low-frequency domain, and hence the energy method is also extent to further study the local stability and convergence rates of solutions around global Maxwellians in $\mathbb{R}^3$, for instance, Strain [70], Duan-Strain [29], Strain-Zhu [71], Wang [76], Duan-Liu [27], Duan-Lei-Yang-Zhao [24], and many references therein. Recently, the decay structure of the linearized system is characterized by the spectral analysis in Li-Yang-Zhong [57] and Huang [51] following the classical works by Ellis-Pinsky [33] and Ukai [75]; see also Glassey-Struass [36] for an early study of spectrum of the VPB system. We should point out that the appearance of the self-consistent electric field or the magnetic field makes the dissipative structure of system more complicated.

A common feature in most of works in perturbative regime mentioned above is that the large-time behavior of solutions to the VPB system is trivial, namely, $F_{i,e}(t,x,\xi)$ are global Maxwellians and $\phi(t,x)$ is a constant. Unfortunately, this property may not be true in the general situation where regarding the VPB system (1.1) and (1.2), either initial data $F_{0\alpha}(x,\xi)$ with $\alpha = i,e$ tend to two distinct global Maxwellians

$$M_{[n_{\alpha\pm},u_\pm,\theta_\pm;m_\alpha]}(\xi) = n_{\alpha\pm}\left(\frac{m_\alpha}{2\pi k_B\theta_\pm}\right)^{3/2}\exp\left\{-\frac{m_\alpha(|\xi_1-u_\pm|^2+|\xi_2|^2+|\xi_3|^2)}{2k_B\theta_\pm}\right\},$$

for a binary gas-mixture or $\phi(t,x)$ tends to two distinct constant states $\phi_\pm$, as x goes to $\pm\infty$, where $k_B>0$ is the Boltzmann constant. Here, as pointed out before, the fact that two Maxwellians have the same bulk velocities and the same temperatures is due to the Boltzmann's H-theorem in the two-component situation; see details in the next section. In such cases, from the local macroscopic balance laws, $F_\alpha(t,x,\xi)$ and $\phi(t,x)$ are no longer global Maxwellians and constant in large time, respectively. This is the situation considered in the paper, and our main objective is to construct the non-trivial rarefaction wave profile under certain compatibility conditions on far-field data, and further show the local time-asymptotic stability. As a byproduct, those results in the case of the constant-equilibrium state (cf. [42]) can be recovered when the strength of rarefaction wave reduces to zero.

We further recall a few literatures for the existence and stability of wave patterns in the content of the pure Boltzmann equation without any force as one may expect to extend them to the VPB system under consideration. These include the shock wave (cf., Caflisch-Nicolaenko [12], Liu-Yu [61], Yu [79], Liu-Yu [62]), rarefaction wave (cf., Liu-Yang-Yu-Zhao [60], Xin-Yang-Yu [77]), contact discontinuity (cf., Huang-Xin-Yang [53]); see also many other references therein. Note that the construction of solutions with a general BV data corresponding to the celebrated work Bianchini-Bressan [6] on the finite-dimensional conservation laws at the fluid level is a big open problem, cf. [67]. Regarding the rarefaction wave of the pure Boltzmann equation, one can take it as a local Maxwellian with the macroscopic fluid quantities solving the Riemann problem on the corresponding Euler system with initial data given by both far-field global Maxwellians. For (1.1), (1.2) we will explain later on how to construct the rarefaction wave through the quasineutral Euler equations. To study the local stability of such local Maxwellian, another type of energy method is proposed in Liu-Yu [61] and developed by Liu-Yang-Yu [59]. Here, different from the previous approach by setting perturbations around global Maxwellians, the key idea in [61, 59] is to make the macro-micro decomposition for the single-component Boltzmann equation

$$F(t,x,\xi) = M(t,x,\xi) + G(t,x,\xi),$$

with the local Maxwellian $M(t,x,\xi)$ determined by the solution $F(t,x,\xi)$ itself through conservation laws of mass, momentum and energy, and hence write the Boltzmann equation in the form of the compressible Navier-Stokes equations coupling to high-order moments of the microscopic part $G(t,x,\xi)$. A priori estimates can be made by a combination of the stability analysis of fluid dynamic equations and the kinetic dissipation of $G(t,x,\xi)$ from the H-theorem. We note that the nonlinear stability in large time of wave patterns for the viscous compressible fluid on the whole line has been well studied, for instance, Goodman [39], Matsumura-Nishihara [63, 64, 65], Huang-Xin-Yang [53], see also the monograph [19] for the general theory. Moreover, hydrodynamic limits of the Boltzmann equation to the classical Euler or Navier-Stokes equations have been also extensively studied by many people in different settings, for instance, see the recent works [46, 52] in perturbation framework and the monograph [67] in non perturbation framework.

When there is a self-consistent force, few results are known on the stability of wave patterns for the kinetic equation. A natural starting point is to look at the corresponding fluid dynamic approximate equations. In what follows, let us mainly focus on the rarefaction wave; the issue on the shock wave or contact discontinuity, even only regarding the existence, should be a completely different problem; see the Sone's book [68] and reference therein. In [30], Duan-Yang proposed to study the following

two-fluid system in the isothermal case

$$\begin{cases} \partial_t n_\alpha + \partial_x(n_\alpha u_\alpha) = 0, \\ m_\alpha n_\alpha(\partial_t u_\alpha + u_\alpha \partial_x u_\alpha) + T_\alpha \partial_x n_\alpha + q_\alpha n_\alpha \partial_x \phi = \mu_\alpha \partial_x^2 u_\alpha, \quad \alpha = i, e, \\ -\partial_x^2 \phi = q_i n_i + q_e n_e, \end{cases} \tag{1.8}$$

which is called the Navier-Stokes-Poisson system due to the appearance of diffusion terms. Here $T_\alpha > 0$, $\mu_\alpha > 0$ are constant temperatures and viscosity coefficients, respectively. Note, as pointed out in [17], that for a collisionless fluid plasma, the Euler-Poisson system is enough to describe the monition of charged particles, and the global existence of classical solutions close to constant steady state has been recently proved in Guo-Ionescu-Pausader [45] in the case of the whole space $\mathbb{R}^3$. Since we are interested in the study of (1.1), (1.2) in the context of collisional plasma, it could be a good way to make use of the theory of the viscous compressible fluid with self-consistent forces. We established in [30] the global-in-time stability of the rarefaction wave and the boundary layer for the outflow problem on (1.8) on the half line. A drawback of the result is that the large-time behavior of the electric field is zero, due to an artificial choice of physical constants, namely,

$$m_i = m_e,\ T_i = T_e,\ \mu_i = \mu_e,\ q_i + q_e = 0,$$

and hence the dynamics of the two-fluid NSP system is the same as the one of the single NS system. However, we recovered a good dissipative property of the electric field, that is, although $\partial_x \phi$ is not time-space integrable, it can be true for $(\partial_x u^r)^{1/2}\partial_x \phi$ by using the two-fluid coupling property, where $\partial_x u^r > 0$ has a good sign.

Recently, we removed in [25] the restrictions on those physical constants. Particularly, it is found that as long as initial data satisfy some compatibility conditions related to the construction of the rarefaction wave, the dynamics of system (1.8) can be described in large time by the corresponding quasineutral Euler system

$$\begin{cases} \partial_t n_\alpha + \partial_x(n_\alpha u_\alpha) = 0, \\ m_\alpha n_\alpha(\partial_t u + u \partial_x u) + T_\alpha \partial_x n_\alpha + q_\alpha n_\alpha \partial_x \phi = 0, \quad \alpha = i, e, \\ q_i n_i + q_e n_e = 0, \end{cases}$$

by formally assuming $u_i = u = u_e$ and ignoring all the second-order derivative terms. Note that by letting $n_e = n$ and $n_i = -\frac{q_e}{q_i} n$, the above quasineutral Euler system can further reduce to the form of

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ n(\partial_t u + u \partial_x u) + \dfrac{T_i|q_e| + T_e|q_i|}{m_i|q_e| + m_e|q_i|} \partial_x n = 0, \end{cases}$$

with the potential function ϕ given by

$$\phi = \frac{m_i T_e - m_e T_i}{m_i|q_e| + m_e|q_i|} \ln n.$$

The similar result has been also extent to the non-isentropic two-fluid case in [28] with some technical restriction on the ratio of masses of two fluids. Furthermore, in a parallel work [26] we also make use of the same idea to further have studied the stability of the rarefaction wave of the VPB system for ions' dynamics governed by the model of the form

$$\begin{cases} \partial_t F_i + \xi_1 \partial_x F_i - \dfrac{q_i}{m_i} \partial_x \phi\, \partial_{\xi_1} F_i = Q_{ii}(F_i, F_i), \\ -\partial_x^2 \phi = q_i \displaystyle\int_{\mathbb{R}^3} F_i \, d\xi + q_e n_e, \end{cases}$$

where compared to the two-component VPB system (1.1), the dynamical equation of electrons and the ions-electrons collisions have been omitted, and the number density $n_e = \int_{\mathbb{R}^3} f_e \, d\xi$ has been replaced

by an analogue of the classical Boltzmann relation

$$n_e = \exp\{\phi/T_e\},$$

or a general function depending on the potential function ϕ. We remark that the Boltzmann relation has been recently extensively visited in a lot of studies of kinetic and related fluid dynamic equations, for instance, [16, 47, 48, 49, 50, 72].

Inspired by our previous works [25, 26, 28], we expect in the paper to further consider the much more physical two-component VPB system, particularly extending the results in [42, 44] to the case of perturbations of the non-constant equilibrium state. In fact, besides its own importance in physics, the two-component collisional kinetic system enjoys more complex dissipation structure compared to either the modelling system studied in [28] or the single-component kinetic system, cf. [1, 18, 20, 21, 34, 35, 37, 69]. For the numerical and mathematical investigations on non-trivial profiles of a gas mixture with the Boltzmann collision, we would mention [2, 8, 11, 55, 73, 74] and reference therein; see also discussions in [5, 7, 23, 54] on the limit of the gas mixture kinetic equations to the fluid dynamical equations.

For the two-component VPB system (1.1), (1.2) under consideration, disparate masses and charges play a key role in the stability analysis of non-constant time-asymptotic profiles, which is different from the one for considering perturbations around constant equilibrium states where all physical constants can be normalised to be one without loss of generality, cf. [41]. Moreover, as discussed in [37], the dissipation by the two-component Boltzmann collisions behaves in a complex way, and the approach to equilibrium can be divided roughly into two processes: one is called the Maxwellization which occurs due to either self-collisions alone, or cross-collisions, or a combination of both, and the other is called equilibration of two species, i.e., the vanishing of differences in velocity and temperature in the species. In the paper, we expect to provide an analytical view to this issue by further developing the macro-micro decomposition in the two-component case.

1.3. **Main result.** We now begin to state the main result of the paper. Before doing that, we first introduce some notations. Let $[n^R(x/t), u^R(x/t), \theta^R(x/t)]$ and $[n^r(t,x), u^r(t,x), \theta^r(t,x)]$ with $u^R = (u_1^R, 0, 0)$ and $u^r = (u_1^r, 0, 0)$, where the far-field data at $x = \pm\infty$ are given by $[n_\pm, u_{1\pm}, \theta_\pm]$, be the 3-family centred rarefaction wave and the corresponding smooth rarefaction wave, respectively, in connection with the quasineutral Euler system

$$\begin{cases} \partial_t n + \partial_x(nu_1) = 0, \\ \partial_t u_1 + u_1\partial_x u_1 + \dfrac{2(|q_i| + |q_e|)}{3(m_e|q_i| + m_i|q_e|)}\dfrac{1}{n}\partial_x(n\theta) = 0, \\ \partial_t \theta + u_1\partial_x\theta + \dfrac{2}{3}\theta\partial_x u_1 = 0. \end{cases} \tag{1.9}$$

See Section 3 for more details to the derivation of the system (1.9). As in [78], we define

$$\begin{bmatrix} M_{*i} \\ M_{*e} \end{bmatrix} = \begin{bmatrix} M_{[n_{*i}, u_*, \theta_*; m_i]}(\xi) \\ M_{[n_{*e}, u_*, \theta_*; m_e]}(\xi) \end{bmatrix}, \tag{1.10}$$

with constants n_{*i}, n_{*e}, $u_* = [u_{*1}, 0, 0]$, θ_* satisfying

$$\begin{cases} \dfrac{1}{2}\sup\limits_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \theta^r(t,x) < \theta_* < \sup\limits_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \theta^r(t,x), \\ \sup\limits_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \left\{ \left| -\dfrac{q_e}{q_i} n^r(t,x) - n_{*i} \right| + |n^r(t,x) - n_{*e}| + |u^r(t,x) - u_*| + |\theta^r(t,x) - \theta_*| \right\} < \eta_0, \end{cases} \tag{1.11}$$

for a constant $\eta_0 > 0$ which is suitably small.

Then the main result of this paper can be stated as follows. More notations will be explained later on.

Theorem 1.1. *Consider the Cauchy problem on the VPB system* (1.1), (1.2), (1.4), (1.5), (1.6), (1.7). *Assume* $[n_+, u_{1+}, \theta_+] \in R_3(n_-, u_{1-}, \theta_-)$, $\phi_+ = \phi_-$, *and* $q_i \le 9|q_e|$, *where* $R_3(n_-, u_{1-}, \theta_-)$ *is defined in* (3.4) *denoting the set of right constant states connected with the left constant state* $[n_-, u_{1-}, \theta_-]$ *through the* 3*-family rarefaction wave of the quasineutral Euler system* (1.9). *Let*

$$\delta_r = |n_+ - n_-| + |u_{1+} - u_{1-}| + |\theta_+ - \theta_-|, \tag{1.12}$$

be the wave strength which is suitably small. There are constants $\epsilon_0 > 0$, $0 < \eta_0 \le \delta_r$ *and* $C_0 > 0$, *such that if* $F_{0i}(x,\xi) \ge 0$, $F_{0e}(x,\xi) \ge 0$, *and*

$$\begin{aligned}
&\sum_{\substack{|\alpha|+|\beta|\le 2\\ 0\le\alpha_0\le 1}} \left\| \partial^\alpha\partial^\beta \left(F_{0i}(x,\xi) - M_{[-\frac{q_e}{q_i}n^r, u^r, \theta^r; m_i](0,x)}(\xi) \right) \right\|^2_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{M_{*i}(\xi)}}\right)\right)} \\
&\quad + \sum_{\substack{|\alpha|+|\beta|\le 2\\ 0\le\alpha_0\le 1}} \left\| \partial^\alpha\partial^\beta \left(F_{0e}(x,\xi) - M_{[n^r, u^r, \theta^r; m_e](0,x)}(\xi) \right) \right\|^2_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{M_{*e}(\xi)}}\right)\right)} \\
&\quad + \sum_{|\alpha|\le 1} \|\partial^\alpha \partial_x \phi(0,x)\|^2_{H^1} + \delta_r \\
&\le \epsilon_0^2,
\end{aligned}$$

then the Cauchy problem admits a unique global solution $[F_i(t,x,\xi) \ge 0, F_e(t,x,\xi) \ge 0, \phi(t,x)]$ *satisfying*

$$\begin{aligned}
&\sup_{t\ge 0} \sum_{\substack{|\alpha|+|\beta|\le 2\\ 0\le\alpha_0\le 1}} \left\| \partial^\alpha\partial^\beta \left(F_i(t,x,\xi) - M_{[-\frac{q_e}{q_i}n^r, u^r, \theta^r; m_i](t,x)}(\xi) \right) \right\|^2_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{M_{*i}(\xi)}}\right)\right)} \\
&\quad + \sup_{t\ge 0} \sum_{\substack{|\alpha|+|\beta|\le 2\\ 0\le\alpha_0\le 1}} \left\| \partial^\alpha\partial^\beta \left(F_e(t,x,\xi) - M_{[n^r, u^r, \theta^r; m_e](t,x)}(\xi) \right) \right\|^2_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{M_{*e}(\xi)}}\right)\right)} \\
&\quad + \sup_{t\ge 0} \sum_{|\alpha|\le 1} \|\partial^\alpha \partial_x \phi(t,x)\|^2_{H^1} \\
&\le C_0\epsilon_0^2.
\end{aligned} \tag{1.13}$$

Moreover, it holds that

$$\begin{aligned}
\sup_{t\to+\infty}\sup_{x\in\mathbb{R}} \Bigg\{ &\left\| F_i(t,x,\xi) - M_{[-\frac{q_e}{q_i}n^R, u^R, \theta^R](x/t)}(\xi) \right\|_{L^2_\xi\left(\frac{1}{\sqrt{M_{*i}(\xi)}}\right)} \\
&+ \left\| F_e(t,x,\xi) - M_{[n^R, u^R, \theta^R](x/t)}(\xi) \right\|_{L^2_\xi\left(\frac{1}{\sqrt{M_{*e}(\xi)}}\right)} \Bigg\} = 0.
\end{aligned} \tag{1.14}$$

We give a few remarks on the above theorem. The estimate (1.14) indeed shows the convergence of the two-component VPB system (1.1), (1.2) to the quasineural Euler system (3.4) in the setting of rarefaction waves for well-prepared small and smooth initial data. Thus, the long-term dynamcis of the VPB system can be a non-trivial time-asymptotic profile connecting two distinct constant equilibrium states. As seen in (1.9), disparate masses and charges of particles also enter into the asymptotic profile and hence they can take the effect on the nontrivial large time behavior of the complex VPB system. The obtained result may be regarded as a generalisation of the existing perturbation theory for the VPB system in the cases either for initial data around constant equilibrium states in [42] or for the single-component Boltzmann collision in [60] and [26]. More importantly, although we may only provide a preliminary understanding of the stability of the rarefaction wave profile for the VPB system, it is expected that the analysis developed in the paper could be adopted to treat many other relevant problems in connection with those fluid-type systems derived in Section 2, cf. [40].

In the end we point out that the condition $q_i \le 9|q_e|$ is only a technical assumption used in the proof of the zero-order energy estimate; see (6.4) for its positivity in Section 6.1. On the other hand,

the condition $\phi_+ = \phi_-$ is essentially required in the proof of the theorem, and it is indeed unknown how to construct a non-trivial large-time profile of the potential function ϕ associated with $\phi_+ \neq \phi_-$ as we did in [25, 26, 28].

1.4. **Outline and key points of the proof.** The proof of Theorem 1.1 is based on the two-component decomposition as well as the refined energy method. First of all, H-theorem of the two-component Boltzmann equations implies that the large-time behavior of the VPB system should be in connection with a bi-Maxwellian $\mathbf{M}$ determined by six local fluid quantities n_i, n_e, $u = (u_1, u_2, u_3)$, and θ. This induces one to define $\mathbf{M}$ in terms of $\mathbf{F}$ such that they have the same average values with respect to all six two-component collision invariants, namely,

$$\int_{\mathbb{R}^3} \psi_j(\xi) \cdot \mathbf{F}\, d\xi = \int_{\mathbb{R}^3} \psi_j(\xi) \cdot \mathbf{M}\, d\xi, \quad j = 1, 2, ..., 6.$$

Therefore, the energy dissipation of the non-fluid part $\mathbf{G} := \mathbf{F} - \mathbf{M}$ can be obtained by the linearised H-theorem. See the coercivity estimate (4.5) in Lemma 4.2 whose proof is based on the compactness argument as in [41]. In most applications of (4.5), one has to vary the weight function such that the modified macroscopic quantities are sufficiently close to those of the background bi-Maxwellian, and this has been done in Lemma 4.3. Moreover, as in [78], it can not be direct to make the zero-order energy estimate on $\mathbf{G}$, because $\mathbf{M}^{-1/2}\mathbf{G}$ is not integrable in $L^2_{t,x,\xi}$. To treat this trouble, one has to construct a background non-fluid function $\overline{\mathbf{G}}$ in terms of the time-asymptotic fluid profile, see (5.8) for the exact formula. We would emphasise that as the large-time profile of $\phi(t, x)$ under the assumption $\phi_+ = \phi_-$ is expected to be constant, $\overline{\mathbf{G}}$ does not involve any term of the potential function, which is quite different from the previous work [26] in the single-component case. Due to this technique, it seems impossible for us to construct a non-trivial large-time potential function $\phi^r(t, x)$ accounting for some distinct far-field data $\phi_\pm$ similar to the two-fluid models considered in [25, 28].

The a priori estimates on the fluid part $\mathbf{M}$ of the solution $\mathbf{F}$ is much more technical. The key point is to find out the appropriate viscous fluid-type equations of the macroscopic quantities of $\mathbf{M}$ such that the energy estimates on the fluid part can be controled in terms of the non-fluid part in a good way; see Proposition 5.1. Considering the two-component moment equations with respect to all collision invariants, cf. (2.8) and (2.9), and using the two-component macro-micro decomposition, it is straightforward to obtain the two-fluid Euler-Poisson type system (2.10), (2.11), (2.12). To capture the diffusion and heat-conductivity, we essentially have used the dissipation effect of like-particle collisions. In fact, using the decomposition, one can rewrite $Q_{\mathcal{A}}(\mathbf{F}, \mathbf{F})$ in the way on the right-hand side of (2.15), and hence get the representation (2.16), where we note that the right-hand first term is exactly responsible for diffusion and heat-conductivity and the remaining term $\overline{R}_{\mathcal{A}}$ does not involve any linear term in $\phi(t, x)$. Therefore, by plugging (2.16) into the two-fluid Euler-Poisson type system, one can further obtain the two-fluid Navier-Stokes-Poisson type system (2.18), (2.19) and (2.12), which becomes the key step for making the energy estimates on the fluid part as in [28].

The rest of the paper is arranged as follows. In the following three sections we make some preparations for the proof of the main result. Particularly, in Section 2, we introduce the macro-micro decomposition for the two-component Boltzmann equation with disparate masses. In terms of the decomposition, we derive the zero-order and first-order approximate fluid-type systems, which is a crucial step for both the construction of large-time rarefaction wave profiles and the energy estimates on the stability of profiles. Note that we also make use of the single-component projections to find out the diffusion and heat-conductivity of the fluid part. In Section 3, we deduce the quasineutral Euler system as the time-asympotic equations of the VPB system, and further construct the corresponding rarefaction wave profile and study the basic properties of the profile. In Section 4, we consider the two-component Boltzmann collision operator and provide estimates on dissipation of the linearised operator and also upper bound estimates on the nonlinear term both with respect to some local bi-Maxwellians. In Section 5, we give a sketch of the proof of Theorem 1.1 basing on two main propositions whose proofs are postponed to Section 6 and Section 7, respectively.

Notations. Throughout the paper, C denotes some generic positive (generally large) constant and λ denotes some generic positive (generally small) constant, where both C and λ may take different values in different places. $D \lesssim E$ means that there is a generic constant $C > 0$ such that $D \le CE$. $D \sim E$ means $D \lesssim E$ and $E \lesssim D$. $\|\cdot\|_{L^p}$ $(1 \le p \le +\infty)$ stands for the L^p_x-norm. Sometimes, for convenience, we use $\|\cdot\|$ to denote L^2_x-norm, and use $(\cdot,\cdot)$ to denote the inner product in L^2_x or $L^2_{x,\xi}$. We also use H^k $(k \ge 0)$ to denote the usual Sobolev space with respect to x variable. We denote $\partial^\alpha\partial^\beta = \partial_t^{\alpha_0}\partial_x^{\alpha_1}\partial_\xi^\beta$ and $\partial_\xi^\beta = \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\xi_3}^{\beta_3}$, with $|\alpha| = \alpha_0+\alpha_1$ and $|\beta| = \beta_1+\beta_2+\beta_3$. We call $\beta' \le \beta$ if each component of β' is not greater than that of β. We also call $\beta' < \beta$ if $\beta' \le \beta$ and $|\beta'| < |\beta|$. For $\beta' \le \beta$, we also use $C^\beta_{\beta'}$ to denote the usual binomial coefficient. The same notations also apply to α and α'. For the notational simplicity, we use $\mathbf{M}_*^{-1}$ to denote the 2×2 diagonal matrix $\mathrm{diag}(1/M_{*i}, 1/M_{*e})$. Similarly, the 2×2 diagonal matrix $\mathrm{diag}(1/\sqrt{M_{*i}}, 1/\sqrt{M_{*e}})$ is denoted by $\mathbf{M}_*^{-1/2}$. The same notations also apply to all bi-Maxwellians used in the paper, for instance, $\mathbf{M}$, $\mathbf{M}_\sharp$ and $\widehat{\mathbf{M}}$, etc.

2. Two-component macro-micro decomposition

In this section we introduce the two-component macro-micro decomposition. First of all, we list the elementary properties of the collision operator, including the local equilibrium state, an identity, collision invariants, and the entropy inequality. An important and interesting concept is the bi-Maxwellian, cf. [2]. After that, we introduce the fluid quantities for a disparate mass binary mixture, define the macro-micro decomposition of the solution, and derive the zero-order macroscopic balance laws. In the end, we discuss how to capture the velocity diffusion and heat conductivity.

2.1. Elementary properties of collisions.

In what follows we list a few elementary properties of the two-component Boltzmann collision operator without any proof. Interested readers may refer to [2, 15]. To the end, we always denote

$$M_{[n(t,x),u(t,x),\theta(t,x);m]}(\xi) = n(t,x)\left(\frac{m}{2\pi k_B\theta(t,x)}\right)^{3/2}\exp\left(-\frac{m|\xi-u(t,x)|^2}{2k_B\theta(t,x)}\right), \tag{2.1}$$

to be a local Maxwellian with the fluid density $n(t,x)$, bulk velocity $u(t,x)$, and temperature $\theta(t,x)$ as well as the particle mass $m > 0$.

[P1]. For the like-particles collision ($\mathcal{A} = i$ or e),

$$Q_{\mathcal{A}\mathcal{A}}(F_\mathcal{A}, F_\mathcal{A}) = 0 \ \text{ iff } \ F_\mathcal{A} = \overline{M}_\mathcal{A},$$

where

$$\overline{M}_\mathcal{A} := M_{[n_\mathcal{A}(t,x),u_\mathcal{A}(t,x),\theta_\mathcal{A}(t,x);m_\mathcal{A}]}(\xi),$$

is a general local Maxwellian of $\mathcal{A}$-species. For later use it is also convenient to rewrite $\overline{M}_\mathcal{A}$ as

$$\overline{M}_\mathcal{A} = \frac{n_\mathcal{A}(t,x)}{\big(2\pi k_\mathcal{A}\theta_\mathcal{A}(t,x)\big)^{3/2}}\exp\left\{-\frac{|\xi-u_\mathcal{A}(t,x)|^2}{2k_\mathcal{A}\theta_\mathcal{A}(t,x)}\right\},$$

with $k_\mathcal{A} := \frac{k_B}{m_\mathcal{A}}$, and for brevity we always take $k_B = \frac{2}{3}$. For the unlike-particles collision ($\mathcal{A} \ne \mathcal{B}$),

$$Q_{\mathcal{A}\mathcal{B}}(\overline{M}_\mathcal{A}, \overline{M}_\mathcal{B}) = 0, \quad \text{provided that } u_\mathcal{A} = u_\mathcal{B} \text{ and } \theta_\mathcal{A} = \theta_\mathcal{B}.$$

[P2]. For $\mathbf{F} = [F_i, F_e]^T$, we set

$$\mathbf{Q}(\mathbf{F},\mathbf{F}) := \begin{bmatrix}\mathbf{Q}_i(\mathbf{F},\mathbf{F})\\ \mathbf{Q}_e(\mathbf{F},\mathbf{F})\end{bmatrix} = \begin{bmatrix}Q_{ii}(F_i,F_i)+Q_{ie}(F_i,F_e)\\ Q_{ee}(F_e,F_e)+Q_{ei}(F_e,F_i)\end{bmatrix}.$$

Then, for $\mathbf{G} = [G_i, G_e]^T$, one has

$$\big(\mathbf{Q}(\mathbf{F},\mathbf{F}),\mathbf{G}\big)_{L^2_\xi\times L^2_\xi} = -\frac{1}{4}I_{ii}(F_i,G_i) - \frac{1}{2}I_{ie}(F_i,F_e,G_i,G_e) - \frac{1}{4}I_{ee}(F_e,G_e),$$

with

$$I_{ii}(F_i,G_i)=\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}[F_i(\xi_*')F_i(\xi')-F_i(\xi_*)F_i(\xi)]\\ \times[G_i(\xi_*')+G_i(\xi')-G_i(\xi_*)-G_i(\xi)]B(|\xi-\xi_*|,\omega)\,d\xi d\xi_* d\omega,$$

$$I_{ee}(F_e,G_e)=\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}[F_e(\xi_*')F_e(\xi')-F_e(\xi_*)F_e(\xi)]\\ \times[G_e(\xi_*')+G_e(\xi')-G_e(\xi_*)-G_e(\xi)]B(|\xi-\xi_*|,\omega)\,d\xi d\xi_* d\omega,$$

$$I_{ie}(F_i,F_e,G_i,G_e)=\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2}[F_i(\xi_*')F_e(\xi')-F_i(\xi_*)F_e(\xi)]\\ \times[G_i(\xi_*')+G_e(\xi')-G_i(\xi_*)-G_e(\xi)]B(|\xi-\xi_*|,\omega)\,d\xi d\xi_* d\omega.$$

[P3]. Two-component Boltzmann collision operator $\mathbf{Q}$ has six collision invariants:

$$\psi_1=\begin{bmatrix}m_i\\0\end{bmatrix},\ \psi_2=\begin{bmatrix}0\\m_e\end{bmatrix},\ \psi_j=\begin{bmatrix}m_i\xi_j\\m_e\xi_j\end{bmatrix},\ j=3,4,5,\ \psi_6=\begin{bmatrix}\frac{1}{2}m_i|\xi|^2\\ \\ \frac{1}{2}m_e|\xi|^2\end{bmatrix},$$

satisfying

$$\int_{\mathbb{R}^3}\psi_j\cdot\mathbf{Q}(\mathbf{F},\mathbf{F})\,d\xi=0,\quad j=1,2,...,6.$$

Specifically, it holds that

$$\int_{\mathbb{R}^3}\psi_{1i}Q_{\mathcal{AB}}(F_{\mathcal{A}},F_{\mathcal{B}})\,d\xi=\int_{\mathbb{R}^3}\psi_{2e}Q_{\mathcal{AB}}(F_{\mathcal{A}},F_{\mathcal{B}})\,d\xi=0,\quad\text{for }\mathcal{A}\in\{i,e\},$$

$$\int_{\mathbb{R}^3}\psi_{j\mathcal{A}}Q_{\mathcal{AA}}(F_{\mathcal{A}},F_{\mathcal{A}})\,d\xi=0,\ j=3,4,5,6,\quad\text{for }\mathcal{A},\mathcal{B}\in\{i,e\},$$

and

$$\int_{\mathbb{R}^3}\psi_{j\mathcal{A}}Q_{\mathcal{AB}}(F_{\mathcal{A}},F_{\mathcal{B}})\,d\xi+\int_{\mathbb{R}^3}\psi_{j\mathcal{B}}Q_{\mathcal{BA}}(F_{\mathcal{B}},F_{\mathcal{A}})\,d\xi=0,\quad\text{for }\mathcal{A}\neq\mathcal{B}.$$

Here for $1\le j\le 6$, ψ_{ji} and ψ_{je} stand for the first and second component of the vector-valued function ψ_j.

[P4]. For any $\mathbf{F}=[F_i,F_e]^T$,

$$\big(\mathbf{Q}(\mathbf{F},\mathbf{F}),\ln\mathbf{F}\big)_{L^2_\xi\times L^2_\xi}:=\left(\mathbf{Q}(\mathbf{F},\mathbf{F}),\begin{bmatrix}\ln F_i\\ \ln F_e\end{bmatrix}\right)_{L^2_\xi\times L^2_\xi}=\sum_{\mathcal{A}=i,e}\int_{\mathbb{R}^3}Q_{\mathcal{A}}(\mathbf{F},\mathbf{F})\ln F_{\mathcal{A}}\,d\xi\le 0,$$

and "=" holds iff $\mathbf{F}=\mathbf{M}$ is a bi-Maxwellian defined by

$$\mathbf{M}=\begin{bmatrix}M_i\\ \\ M_e\end{bmatrix}=\begin{bmatrix}M_{[n_i,u,\theta;m_i]}(\xi)\\ \\ M_{[n_e,u,\theta;m_e]}(\xi)\end{bmatrix}.\tag{2.2}$$

Particularly, if $\mathbf{Q}(\mathbf{F},\mathbf{F})=0$ then $\mathbf{F}$ is a bi-Maxwellian. Here, we emphasise that for $\mathcal{A}=i$ or e, $M_{\mathcal{A}}$ is different from $\overline{M}_{\mathcal{A}}$. In fact, the bi-Maxwellian $\mathbf{M}$ is a two-component equilibrium state, with M_i, M_e being the first and second component of $\mathbf{M}$, and $\overline{M}_i$, $\overline{M}_e$ are Maxwellians of i-species and e-species, respectively, which are single-species equilibrium states.

2.2. **Decomposition around local bi-Maxwellian.** As in the single-component case [59], we introduce the two-component macro-micro decomposition around local bi-Maxwellians in the following way. Let $\mathbf{F}=\mathbf{F}(t,x,\xi)$ be a function satisfying the two-component VPB system (1.1). We decompose it as

$$\mathbf{F}(t,x,\xi)=\mathbf{M}(t,x,\xi)+\mathbf{G}(t,x,\xi).\tag{2.3}$$

Here $\mathbf{M}=\mathbf{M}(t,x,\xi)$ is the macroscopic (or fluid) part represented by the local bi-Maxwellian

$$\mathbf{M}=\begin{bmatrix}M_i\\ \\ M_e\end{bmatrix}=\begin{bmatrix}M_{[n_i(t,x),u(t,x),\theta(t,x);m_i]}(\xi)\\ \\ M_{[n_e(t,x),u(t,x),\theta(t,x);m_e]}(\xi)\end{bmatrix},$$

such that for all fix collision invariants,

$$\int_{\mathbb{R}^3} \psi_j(\xi)\cdot[\mathbf{F}(t,x,\xi)-\mathbf{M}(t,x,\xi)]\,d\xi=0,\quad j=1,2,...,6.$$

Note that $\mathbf{M}(t,x,\xi)$ involves the exact six macroscopic quantities

$$\Big[n_i(t,x),n_e(t,x),u(t,x)=\big(u_1(t,x),u_2(t,x),u_3(t,x)\big),\theta(t,x)\Big],$$

which can be determined by

$$\begin{cases} m_in_i\equiv\displaystyle\int_{\mathbb{R}^3}\psi_1\cdot\mathbf{F}(t,x,\xi)\,d\xi,\\ m_en_e\equiv\displaystyle\int_{\mathbb{R}^3}\psi_2\cdot\mathbf{F}(t,x,\xi)\,d\xi,\\ (m_in_i+m_en_e)u_j\equiv\displaystyle\int_{\mathbb{R}^3}\psi_j\cdot\mathbf{F}(t,x,\xi)\,d\xi,\quad j=3,4,5,\\ n_i\left(\theta+\dfrac{1}{2}m_i|u(t,x)|^2\right)+n_e\left(\theta+\dfrac{1}{2}m_e|u(t,x)|^2\right)\equiv\displaystyle\int_{\mathbb{R}^3}\psi_6\cdot\mathbf{F}(t,x,\xi)\,d\xi. \end{cases}$$

Therefore, $\mathbf{M}$ is well defined, and then $\mathbf{G}:=\mathbf{F}-\mathbf{M}$ is the microscopic (or non-fluid) part denoted by

$$\mathbf{G}=\mathbf{G}(t,x,\xi)=\begin{bmatrix}G_i(t,x,\xi)\\ G_e(t,x,\xi)\end{bmatrix}.$$

We remark that $\mathbf{F}$ also enjoys another kind of the decomposition with each component being around the single-species local equilibrium state

$$\mathbf{F}=\begin{bmatrix}M_i^{(1)}+G_i^{(1)}\\ M_e^{(1)}+G_e^{(1)}\end{bmatrix},\tag{2.4}$$

where for $\mathcal{A}=i$ or e, $M_{\mathcal{A}}^{(1)}:=M_{[n_{\mathcal{A}}(t,x),u_{\mathcal{A}}(t,x),\theta_{\mathcal{A}}(t,x);m_{\mathcal{A}}]}(\xi)$ is the local equilibrium state of collisions of like-particles with the fluid quantities determined by

$$\begin{cases} n_{\mathcal{A}}(t,x):=\displaystyle\int_{\mathbb{R}^3}F_{\mathcal{A}}(t,x,\xi)\,d\xi,\\ u_{\mathcal{A}j}(t,x):=\dfrac{1}{n_{\mathcal{A}}(t,x)}\displaystyle\int_{\mathbb{R}^3}\xi_jF_{\mathcal{A}}(t,x,\xi)\,d\xi,\quad j=1,2,3,\\ \theta_{\mathcal{A}}(t,x):=\dfrac{1}{3k_{\mathcal{A}}n_{\mathcal{A}}}\displaystyle\int_{\mathbb{R}^3}|\xi-u_{\mathcal{A}}(t,x)|^2F_{\mathcal{A}}(t,x,\xi)\,d\xi. \end{cases}$$

It should be pointed out that (2.4) is different from (2.3). One can also obtain the link of the two-component fluid quantities $[n_i,n_e,u,\theta]$ and the single-component fluid quantities $[n_{\mathcal{A}},u_{\mathcal{A}},\theta_{\mathcal{A}}]$ $(\mathcal{A}=i,e)$ in the way that

$$\begin{aligned} u&=\frac{m_in_iu_i+m_en_eu_e}{m_in_i+m_en_e},\\ \theta&=\frac{n_i\theta_i+n_e\theta_e}{n_i+n_e}+\frac{m_im_en_in_e}{3k_B(n_i+n_e)(m_in_i+m_en_e)}|u_i-u_e|^2. \end{aligned}$$

We further remark that though the single-component fluid quantities $u_i,u_e,\theta_i,\theta_e$ are not macroscopic in the two-component sense, the differences of them with the corresponding two-component fluid quantities u,θ turn out to be microscopic in the two-component sense. Namely, after direct computations, for $\mathcal{A}=i,e$,

$$\begin{aligned} u_{\mathcal{A}}-u&=\frac{1}{n_{\mathcal{A}}}\int_{\mathbb{R}^3}\xi G_{\mathcal{A}}\,d\xi,\\ \theta_{\mathcal{A}}-\theta&=\frac{1}{3k_{\mathcal{A}}}|u-u_{\mathcal{A}}|^2+\frac{1}{3k_{\mathcal{A}}n_{\mathcal{A}}}\int_{\mathbb{R}^3}|\xi-u_{\mathcal{A}}|^2G_{\mathcal{A}}\,d\xi. \end{aligned}$$

This observation is a key point for understanding the dissipation of macroscopic quantities of the two-component system.

We begin to introduce the two-component projection operators $\mathbf{P}_0^{\mathbf{M}}$ and $\mathbf{P}_1^{\mathbf{M}}$. For this purpose, one has to first introduce an orthonormal basis related to an arbitrary bi-Maxwellian

$$\widehat{\mathbf{M}} = \begin{bmatrix} \widehat{M}_i \\ \\ \widehat{M}_e \end{bmatrix}.$$

Associated with $\widehat{\mathbf{M}}$, we define an inner product in ξ variable as

$$\langle \mathbf{F}, \mathbf{H} \rangle_{\widehat{\mathbf{M}}} \equiv \int_{\mathbb{R}^3} \frac{F_i(\xi) H_i(\xi)}{\widehat{M}_i}\, d\xi + \int_{\mathbb{R}^3} \frac{F_e(\xi) H_e(\xi)}{\widehat{M}_e}\, d\xi,$$

for functions $\mathbf{F} = [F_i, F_e]^{\mathrm{T}}$ and $\mathbf{H} = [H_i, H_e]^{\mathrm{T}}$ such that the integrals above is well defined. Using the inner product with respect to the bi-Maxwellian $\widehat{\mathbf{M}}$, the following functions spanning the macroscopic subspace are pairwise orthogonal:

$$\chi_1^{\widehat{\mathbf{M}}}\left(\xi; \widehat{n}_i, \widehat{u}, \widehat{\theta}\right) \equiv \begin{bmatrix} \dfrac{1}{\sqrt{\widehat{n}_i}} M_i \\ 0 \end{bmatrix},$$

$$\chi_2^{\widehat{\mathbf{M}}}\left(\xi; \widehat{n}_e, \widehat{u}, \widehat{\theta}\right) \equiv \begin{bmatrix} 0 \\ \dfrac{1}{\sqrt{\widehat{n}_e}} M_e \end{bmatrix},$$

$$\chi_j^{\widehat{\mathbf{M}}}\left(\xi; \widehat{n}_i, \widehat{n}_e, \widehat{u}, \widehat{\theta}\right) \equiv \begin{bmatrix} \dfrac{\sqrt{m_i}}{\sqrt{m_i \widehat{n}_i + m_e \widehat{n}_e}} \dfrac{\xi_j - \widehat{u}_j}{\sqrt{k_i \widehat{\theta}}} \widehat{M}_i \\ \dfrac{\sqrt{m_e}}{\sqrt{m_i \widehat{n}_i + m_e \widehat{n}_e}} \dfrac{\xi_j - \widehat{u}_j}{\sqrt{k_e \widehat{\theta}}} \widehat{M}_e \end{bmatrix}, \quad j = 3, 4, 5,$$

$$\chi_6^{\widehat{\mathbf{M}}}\left(\xi; \widehat{n}_i, \widehat{n}_e, \widehat{u}, \widehat{\theta}\right) \equiv \begin{bmatrix} \dfrac{1}{\sqrt{6(\widehat{n}_i + \widehat{n}_e)}} \left(\dfrac{|\xi - \widehat{u}|^2}{k_i \widehat{\theta}} - 3 \right) \widehat{M}_i \\ \dfrac{1}{\sqrt{6(\widehat{n}_i + \widehat{n}_e)}} \left(\dfrac{|\xi - \widehat{u}|^2}{k_e \widehat{\theta}} - 3 \right) \widehat{M}_e \end{bmatrix},$$

$$\left\langle \chi_j^{\widehat{\mathbf{M}}}, \chi_k^{\widehat{\mathbf{M}}} \right\rangle_{\widehat{\mathbf{M}}} = \delta_{jk}, \quad \text{for} \quad j, k = 1, 2, \cdots, 6,$$

where δ_{jk} is the Kronecker delta. With the above orthonormal basis, the two-component macroscopic projection $\mathbf{P}_0^{\widehat{\mathbf{M}}}$ and the two-component microscopic projection $\mathbf{P}_1^{\widehat{\mathbf{M}}}$ can be defined as

$$\begin{cases} \mathbf{P}_0^{\widehat{\mathbf{M}}} \mathbf{F} \equiv \sum\limits_{j=1}^{6} \left\langle \mathbf{F}, \chi_j^{\widehat{\mathbf{M}}} \right\rangle_{\widehat{\mathbf{M}}} \chi_j^{\widehat{\mathbf{M}}}, \\ \mathbf{P}_1^{\widehat{\mathbf{M}}} \mathbf{F} \equiv \mathbf{F} - \mathbf{P}_0^{\widehat{\mathbf{M}}} \mathbf{F}. \end{cases}$$

Notice that the operators $\mathbf{P}_0^{\widehat{\mathbf{M}}}$ and $\mathbf{P}_1^{\widehat{\mathbf{M}}}$ are orthogonal (and thus self-adjoint) projections with respect to the inner product $\langle \cdot, \cdot \rangle_{\widehat{\mathbf{M}}}$, i.e.,

$$\mathbf{P}_0^{\widehat{\mathbf{M}}} \mathbf{P}_0^{\widehat{\mathbf{M}}} = \mathbf{P}_0^{\widehat{\mathbf{M}}}, \quad \mathbf{P}_1^{\widehat{\mathbf{M}}} \mathbf{P}_1^{\widehat{\mathbf{M}}} = \mathbf{P}_1^{\widehat{\mathbf{M}}}, \quad \mathbf{P}_0^{\widehat{\mathbf{M}}} \mathbf{P}_1^{\widehat{\mathbf{M}}} = \mathbf{P}_1^{\widehat{\mathbf{M}}} \mathbf{P}_0^{\widehat{\mathbf{M}}} = 0.$$

Moreover, it is straightforward to check that

$$\left\langle \mathbf{P}_0^{\widehat{\mathbf{M}}} \mathbf{F}, \mathbf{P}_1^{\widehat{\widehat{\mathbf{M}}}} \mathbf{F} \right\rangle_{\widehat{\mathbf{M}}} = \left\langle \mathbf{P}_0^{\widehat{\widehat{\mathbf{M}}}} \mathbf{F}, \mathbf{P}_1^{\widehat{\mathbf{M}}} \mathbf{F} \right\rangle_{\widehat{\widehat{\mathbf{M}}}} = 0$$

for any two bi-Maxwellians $\widehat{\mathbf{M}}$ and $\widehat{\widehat{\mathbf{M}}}$. Finally we remark that due to the definitions of $\mathbf{P}_0^{\widehat{\mathbf{M}}}$ and $\mathbf{P}_1^{\widehat{\mathbf{M}}}$, one has

$$\mathbf{P}_0^{\mathbf{M}}\mathbf{F} = \mathbf{M}, \quad \mathbf{P}_1^{\mathbf{M}}\mathbf{F} = \mathbf{G},$$

whenever the bi-Maxwellian $\widehat{\mathbf{M}} = \mathbf{M}$ is the macroscopic part of $\mathbf{F}$.

With the two-component macro-micro decomposition of the solution $\mathbf{F}$ to the VPB system (1.1), (1.2), one may derive the dynamical equations of the fluid part $\mathbf{M}$ and the non-fluid part $\mathbf{G}$. For this, in the sequel we denote

$$\mathbf{Q}(\mathbf{F},\mathbf{H}) = \begin{bmatrix} Q_i(\mathbf{F},\mathbf{H}) \\ Q_e(\mathbf{F},\mathbf{H}) \end{bmatrix} = \begin{bmatrix} Q_{ii}(F_i,H_i) + Q_{ie}(F_i,H_e) \\ Q_{ee}(F_e,H_e) + Q_{ei}(F_e,H_i) \end{bmatrix}.$$

For convenience, we rewrite (1.1) as the following vector form

$$\partial_t \mathbf{F} + \xi_1 \partial_x \mathbf{F} + q_0 \partial_x \phi \partial_{\xi_1} \mathbf{F} = \mathbf{Q}(\mathbf{F},\mathbf{F}), \tag{2.5}$$

where q_0 denotes the 2×2 diagonal matrix $\mathrm{diag}(-q_i/m_i, -q_e/m_e)$. Upon using the macro-micro decomposition (2.3), the VPB system (2.5) can be further rewritten as

$$\partial_t(\mathbf{M}+\mathbf{G}) + \xi_1\partial_x(\mathbf{M}+\mathbf{G}) + q_0\partial_x\phi(\mathbf{M}+\mathbf{G}) = \mathbf{L}_{\mathbf{M}}\mathbf{G} + \mathbf{Q}(\mathbf{G},\mathbf{G}).$$

Here, $\mathbf{L}_{\mathbf{M}}$ is the two-component linearized Boltzmann collision operator given by

$$\begin{aligned} \mathbf{L}_{\mathbf{M}}\mathbf{G} &= \begin{bmatrix} Q_i(\mathbf{M},\mathbf{G}) + Q_i(\mathbf{G},\mathbf{M}) \\ Q_e(\mathbf{M},\mathbf{G}) + Q_e(\mathbf{G},\mathbf{M}) \end{bmatrix} \\ &= \begin{bmatrix} Q_{ii}(M_i,G_i) + Q_{ii}(G_i,M_i) + Q_{ie}(M_i,G_e) + Q_{ie}(G_i,M_e) \\ Q_{ee}(M_e,G_e) + Q_{ee}(G_e,M_e) + Q_{ei}(M_e,G_i) + Q_{ei}(G_e,M_i) \end{bmatrix}, \end{aligned} \tag{2.6}$$

and the nonlinear part $\mathbf{Q}(\mathbf{G},\mathbf{G})$ is defined as

$$\mathbf{Q}(\mathbf{G},\mathbf{G}) = \begin{bmatrix} Q_i(\mathbf{G},\mathbf{G}) \\ Q_e(\mathbf{G},\mathbf{G}) \end{bmatrix} = \begin{bmatrix} Q_{ii}(G_i,G_i) + Q_{ie}(G_i,G_e) \\ Q_{ee}(G_e,G_e) + Q_{ei}(G_e,G_i) \end{bmatrix}.$$

Applying $\mathbf{P}_0^{\mathbf{M}}$ and $\mathbf{P}_1^{\mathbf{M}}$ to (2.5) respectively, one has

$$\partial_t\mathbf{M} + \mathbf{P}_0^{\mathbf{M}}(\xi_1\partial_x\mathbf{M}) + \mathbf{P}_0^{\mathbf{M}}(\xi_1\partial_x\mathbf{G}) + \mathbf{P}_0^{\mathbf{M}}(q_0\partial_x\phi\partial_{\xi_1}\mathbf{M}) + \mathbf{P}_0^{\mathbf{M}}(q_0\partial_x\phi\partial_{\xi_1}\mathbf{G}) = 0,$$

and

$$\begin{aligned} \partial_t\mathbf{G} + \mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{M}) + \mathbf{P}_1^{\mathbf{M}}(\xi_1\partial_x\mathbf{G}) + \mathbf{P}_1^{\mathbf{M}}(q_0\partial_x\phi\partial_{\xi_1}\mathbf{M}) \\ + \mathbf{P}_1^{\mathbf{M}}(q_0\partial_x\phi\partial_{\xi_1}\mathbf{G}) = \mathbf{L}_{\mathbf{M}}\mathbf{G} + \mathbf{Q}(\mathbf{G},\mathbf{G}). \end{aligned} \tag{2.7}$$

Moreover, one also may derive the fluid-type system of the macroscopic quantities of the fluid part $\mathbf{M}$ by using six two-component collision invariants $\psi_j(\xi)$ $(1\le j\le 6)$. For later use, we start from two component equations of (1.1). Taking the inner product of equations of $\mathcal{A}=i$ and $\mathcal{A}=e$ with $\psi_{j\mathcal{A}}$ over $\xi\in\mathbb{R}^3$ respectively, it follows that

$$\int_{\mathbb{R}^3}\psi_{ji}\left(\partial_t F_i + \xi_1\partial_x F_i - \frac{q_i\partial_x\phi}{m_i}\partial_{\xi_1}F_i\right)d\xi = \int_{\mathbb{R}^3}\psi_{ji}Q_i(\mathbf{F},\mathbf{F})\,d\xi, \quad j=1,3,4,5,6, \tag{2.8}$$

and

$$\int_{\mathbb{R}^3}\psi_{je}\left(\partial_t F_e + \xi_1\partial_x F_e - \frac{q_e\partial_x\phi}{m_e}\partial_{\xi_1}F_e\right)d\xi = \int_{\mathbb{R}^3}\psi_{je}Q_e(\mathbf{F},\mathbf{F})\,d\xi, \quad j=2,3,4,5,6. \tag{2.9}$$

Applying the component forms $F_{\mathcal{A}} = M_{\mathcal{A}} + G_{\mathcal{A}}$ $(\mathcal{A} = i, e)$ of the macro-micro decomposition $\mathbf{F} = \mathbf{M} + \mathbf{G}$ as well as the definition of the bi-Maxwellian $\mathbf{M}$, one further deduces

$$\begin{cases} \partial_t n_i + \partial_x(n_i u_1) = -\displaystyle\int_{\mathbb{R}^3} \xi_1 \partial_x G_i \, d\xi, \\ m_i n_i \partial_t u_1 + m_i n_i u_1 \partial_x u_1 + \dfrac{2}{3}\partial_x (n_i \theta) + q_i n_i \partial_x \phi \\ \quad = -\displaystyle\int_{\mathbb{R}^3} \psi_{3i} \partial_t G_i \, d\xi - \int_{\mathbb{R}^3} \psi_{3i} \xi_1 \partial_x G_i \, d\xi + \int_{\mathbb{R}^3} \psi_{3i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi + u_1 \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_i \, d\xi, \\ m_i n_i \partial_t u_j + m_i n_i u_1 \partial_x u_j = -\displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)i} \partial_t G_i \, d\xi - \int_{\mathbb{R}^3} \psi_{(j+2)i} \xi_1 \partial_x G_i \, d\xi \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi + u_j \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_i \, d\xi, \quad j = 2, 3, \\ n_i \partial_t \theta + n_i u_1 \partial_x \theta + \dfrac{2 n_i \theta}{3} \partial_x u_1 \\ \quad = -\displaystyle\int_{\mathbb{R}^3} \psi_{6i} \partial_t G_i \, d\xi - \int_{\mathbb{R}^3} \psi_{6i} \xi_1 \partial_x G_i \, d\xi + \int_{\mathbb{R}^3} \psi_{6i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi \\ \qquad + \displaystyle\sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} \xi_1 \partial_x G_i \, d\xi - \frac{1}{2} \sum_{j=1}^{3} u_j^2 \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_i \, d\xi + \sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} \partial_t G_i \, d\xi \\ \qquad - \displaystyle\sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi + \theta \int_{\mathbb{R}^3} \xi_1 \partial_x G_i \, d\xi + \frac{q_i \partial_x \phi}{m_i} \int_{\mathbb{R}^3} \psi_{6i} \partial_{\xi_1} G_i \, d\xi, \end{cases} \tag{2.10}$$

and

$$\begin{cases} \partial_t n_e + \partial_x(n_e u_1) = -\displaystyle\int_{\mathbb{R}^3} \xi_1 \partial_x G_e \, d\xi, \\ m_e n_e \partial_t u_1 + m_e n_e u_1 \partial_x u_1 + \dfrac{2}{3}\partial_x (n_e \theta) + q_e n_e \partial_x \phi \\ \quad = -\displaystyle\int_{\mathbb{R}^3} \psi_{3e} \partial_t G_e \, d\xi - \int_{\mathbb{R}^3} \psi_{3e} \xi_1 \partial_x G_e \, d\xi + \int_{\mathbb{R}^3} \psi_{3e} Q_e(\mathbf{F}, \mathbf{F}) \, d\xi + u_1 \int_{\mathbb{R}^3} m_e \xi_1 \partial_x G_e \, d\xi, \\ m_e n_e \partial_t u_j + m_e n_e u_1 \partial_x u_j = -\displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)e} \partial_t G_e \, d\xi - \int_{\mathbb{R}^3} \psi_{(j+2)e} \xi_1 \partial_x G_e \, d\xi \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)e} Q_e(\mathbf{F}, \mathbf{F}) d\xi + u_j \int_{\mathbb{R}^3} m_e \xi_1 \partial_x G_e \, d\xi, \quad j = 2, 3, \\ n_e \partial_t \theta + n_e u_1 \partial_x \theta + \dfrac{2 n_e \theta}{3} \partial_x u_1 \\ \quad = -\displaystyle\int_{\mathbb{R}^3} \psi_{6e} \partial_t G_e \, d\xi - \int_{\mathbb{R}^3} \psi_{6e} \xi_1 \partial_x G_e \, d\xi + \int_{\mathbb{R}^3} \psi_{6e} Q_e(\mathbf{F}, \mathbf{F}) \, d\xi \\ \qquad + \displaystyle\sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)e} \xi_1 \partial_x G_e \, d\xi - \frac{1}{2} \sum_{j=1}^{3} u_j^2 \int_{\mathbb{R}^3} m_e \xi_1 \partial_x G_e \, d\xi + \sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)e} \partial_t G_e \, d\xi \\ \qquad - \displaystyle\sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)e} Q_e(\mathbf{F}, \mathbf{F}) \, d\xi + \theta \int_{\mathbb{R}^3} \xi_1 \partial_x G_e \, d\xi + \frac{q_e \partial_x \phi}{m_e} \int_{\mathbb{R}^3} \psi_{6e} \partial_{\xi_1} G_e \, d\xi. \end{cases} \tag{2.11}$$

Here the self-consistent potential ϕ satisfies the Poisson equation

$$-\partial_x^2 \phi = q_i n_i + q_e n_e. \tag{2.12}$$

Note that if one only considers the macroscopic balance laws of (2.5) in terms of six collision invariants, one can obtain six dynamical equations of fluid quantities $n_i, n_e, u_1, u_2, u_3, \theta$ which correspond to the above two systems (2.10) and (2.11) after both the equations of momentums and the equations of

temperatures are taken summation, respectively. Namely, one has

$$\begin{cases} (m_in_i+m_en_e)(\partial_t u_1+u_1\partial_x u_1)+\dfrac{2}{3}\partial_x\left[(n_i+n_e)\theta\right]+(q_in_i+q_en_e)\partial_x\phi \\ \qquad =-\partial_x\displaystyle\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\mathbf{G}\,d\xi, \\ (m_in_i+m_en_e)(\partial_t u_j+u_1\partial_x u_j)=-\partial_x\displaystyle\int_{\mathbb{R}^3}\xi_1\psi_{(j+2)}\cdot\mathbf{G}\,d\xi,\quad j=2,3, \\ (n_i+n_e)(\partial_t\theta+u_1\partial_x\theta)+\dfrac{2}{3}(n_i+n_e)\theta\partial_x u_1 \\ \quad =-\partial_x\displaystyle\int_{\mathbb{R}^3}\xi_1\psi_6\cdot\mathbf{G}\,d\xi+\sum_{j=1}^{3}u_j\partial_x\int_{\mathbb{R}^3}\xi_1\psi_{(j+2)}\cdot\mathbf{G}\,d\xi \\ \qquad +\theta\displaystyle\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi+\partial_x\phi\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi. \end{cases} \tag{2.13}$$

Moreover, if one further ignores those terms involving the non-fluid part $\mathbf{G}$, one has the closed fluid-type system of six knowns $n_i, n_e, u_1, u_2, u_3, \theta$:

$$\begin{cases} \partial_t n_i+\partial_x(n_iu_1)=0, \\ \partial_t n_e+\partial_x(n_eu_1)=0, \\ (m_in_i+m_en_e)(\partial_t u_1+u_1\partial_x u_1)+\dfrac{2}{3}\partial_x\left[(n_i+n_e)\theta\right]+(q_in_i+q_en_e)\partial_x\phi=0, \\ (m_in_i+m_en_e)(\partial_t u_j+u_1\partial_x u_j)=0,\quad j=2,3, \\ (n_i+n_e)(\partial_t\theta+u_1\partial_x\theta)+\dfrac{2}{3}(n_i+n_e)\theta\partial_x u_1=0, \\ -\partial_x^2\phi=q_in_i+q_en_e. \end{cases} \tag{2.14}$$

Note that (2.14) could be thought to be the zero-order fluid dynamic approximation of the VPB system (1.1), (1.2).

2.3. **Diffusion and heat-conductivity.** As in [59], in order to further consider the first-order fluid dynamic approximation of the VPB system, one has to find out diffusion and heat-conductivity corresponding to velocity function u and temperature function θ, respectively. One way for that is to formally solve $\mathbf{G}$ through the microscopic equation (2.7) as

$$\mathbf{G}=\mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{M}\right)\right\}+\mathbf{R},$$

with

$$\begin{aligned}\mathbf{R}=&\mathbf{L}_{\mathbf{M}}^{-1}\left\{\partial_t\mathbf{G}+\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{G}\right)+\mathbf{P}_1^{\mathbf{M}}\left(q_0\partial_x\phi\partial_{\xi_1}\mathbf{G}\right)-\mathbf{Q}(\mathbf{G},\mathbf{G})\right\}\\&+\underbrace{\mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left(q_0\partial_x\phi\partial_{\xi_1}\mathbf{M}\right)\right\}}_{\mathbf{R}_\phi},\end{aligned}$$

and then plug it into (2.13) so that those terms related to diffusion and heat-conductivity could be obtained by computing

$$-\partial_x\int_{\mathbb{R}^3}\xi_1\psi_j\cdot\mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{M}\right)\right\}d\xi,\quad 3\le j\le 6.$$

We remark that such treatment may not be a good way because it is unknown whether or not the above integrals with $\mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{M}\right)\right\}$ replaced by $\mathbf{R}_\phi$ are vanishing, and thus the right-hand terms of (2.13) could involve ϕ in a linear way which should give much trouble to estimates on ϕ.

Therefore we turn to another way for obtaining the effect of diffusion and heat-conductivity on the basis of two single-component equations of the VPB system (1.1). The key point is to introduce single-component projection operators $P_0^{M_{\mathcal{A}}}$ and $P_1^{M_{\mathcal{A}}}$ for $\mathcal{A}=i$ and e, where we recall that M_i,

M_e are the component functions of the bi-Maxwellian $\mathbf{M}$ defined in the two-component macro-micro decomposition (2.3).

To do so, similarly as before, for any given local Maxwellian

$$\widehat{M}_{\mathcal{A}} = M_{[\widehat{n}_{\mathcal{A}},\widehat{u}_{\mathcal{A}},\widehat{\theta}_{\mathcal{A}}]},$$

we define an inner product in $\xi \in \mathbb{R}^3$ as

$$\langle f, g\rangle_{\widehat{M}_{\mathcal{A}}} \equiv \int_{\mathbb{R}^3} \frac{f(\xi)g(\xi)}{\widehat{M}_{\mathcal{A}}} d\xi,$$

for two scalar functions f and g such that the integral on the right is well defined. Applying the above inner product with respect to the single Maxwellian $\widehat{M}_{\mathcal{A}}$, the following five functions are also orthonormal:

$$\chi_0^{\widehat{M}_{\mathcal{A}}}\left(\xi; \widehat{n}_{\mathcal{A}}, \widehat{u}_{\mathcal{A}}, \widehat{\theta}_{\mathcal{A}}\right) \equiv \frac{1}{\sqrt{\widehat{n}_{\mathcal{A}}}} \widehat{M}_{\mathcal{A}},$$

$$\chi_i^{\widehat{M}_{\mathcal{A}}}\left(\xi; \widehat{n}_{\mathcal{A}}, \widehat{u}_{\mathcal{A}}, \widehat{\theta}_{\mathcal{A}}\right) \equiv \frac{\xi_j - \widehat{u}_j}{\sqrt{k_{\mathcal{A}} \widehat{n}_{\mathcal{A}}\, \widehat{\theta}_{\mathcal{A}}}} \widehat{M}_{\mathcal{A}}, \quad j = 1, 2, 3,$$

$$\chi_4^{\widehat{M}_{\mathcal{A}}}\left(\xi; \widehat{n}_{\mathcal{A}}, \widehat{u}_{\mathcal{A}}, \widehat{\theta}_{\mathcal{A}}\right) \equiv \frac{1}{\sqrt{6\widehat{n}_{\mathcal{A}}}} \left(\frac{|\xi - \widehat{u}|^2}{k_{\mathcal{A}} \widehat{\theta}_{\mathcal{A}}} - 3\right) \widehat{M}_{\mathcal{A}},$$

$$\left\langle \chi_j^{\widehat{M}_{\mathcal{A}}}, \chi_k^{\widehat{M}_{\mathcal{A}}} \right\rangle_{\widehat{M}_{\mathcal{A}}} = \delta_{jk}, \quad \text{for} \quad j, k = 0, 1, 2, 3, 4.$$

With the above orthonormal set, we can also define the macroscopic projection $P_0^{\widehat{M}_{\mathcal{A}}}$ and the microscopic projection $P_1^{\widehat{M}_{\mathcal{A}}}$ as follows

$$\begin{cases} P_0^{\widehat{M}_{\mathcal{A}}} h \equiv \sum\limits_{j=0}^{4} \left\langle h, \chi_j^{\widehat{M}_{\mathcal{A}}} \right\rangle_{\widehat{M}_{\mathcal{A}}} \chi_j^{\widehat{M}_{\mathcal{A}}}, \\ P_1^{\widehat{M}_{\mathcal{A}}} h \equiv h - P_0^{\widehat{M}_{\mathcal{A}}} h. \end{cases}$$

Note that the operators $P_0^{\widehat{M}_{\mathcal{A}}}$ and $P_1^{\widehat{M}_{\mathcal{A}}}$ enjoy the similar properties as $\mathbf{P}_0^{\widehat{\mathbf{M}}}$ and $\mathbf{P}_1^{\widehat{\mathbf{M}}}$ given in the previous subsection.

Using notations above and recalling the decomposition (2.3), the solution $F_{\mathcal{A}}(t, x, \xi)$ ($\mathcal{A} = i, e$) of (1.1) satisfies

$$P_0^{M_{\mathcal{A}}} F_{\mathcal{A}} = M_{\mathcal{A}} + P_0^{M_{\mathcal{A}}} G_{\mathcal{A}}, \qquad P_1^{M_{\mathcal{A}}} F_{\mathcal{A}} = P_1^{M_{\mathcal{A}}} G_{\mathcal{A}}.$$

Noticing that

$$P_1^{M_{\mathcal{A}}} \left\{ \frac{q_{\mathcal{A}} \partial_x \phi}{m_{\mathcal{A}}} \partial_{\xi_1} M_{\mathcal{A}} \right\} = 0.$$

Acting $P_1^{M_i}$ and $P_1^{M_e}$ to two equations of (1.1) respectively, one has that for $\mathcal{A} = i, e$,

$$\begin{aligned} &P_1^{M_{\mathcal{A}}} \partial_t G_{\mathcal{A}} + P_1^{M_{\mathcal{A}}} \left\{\xi_1 \partial_x M_{\mathcal{A}}\right\} + P_1^{M_{\mathcal{A}}} \left\{\xi_1 \partial_x G_{\mathcal{A}}\right\} - P_1^{M_{\mathcal{A}}} \left\{ \frac{q_{\mathcal{A}} \partial_x \phi}{m_{\mathcal{A}}} \partial_{\xi_1} G_{\mathcal{A}} \right\} \\ &\quad = L_{M_{\mathcal{A}}} P_1^{M_{\mathcal{A}}} G_{\mathcal{A}} + P_1^{M_{\mathcal{A}}} \overline{Q}_{\mathcal{A}}(\mathbf{G}, \mathbf{G}), \end{aligned} \tag{2.15}$$

where for the $\mathcal{A}$-component, we have defined the linearized Boltzmann collision operator around the local Maxwellian $M_{\mathcal{A}}$ by

$$L_{M_{\mathcal{A}}} P_1^{M_{\mathcal{A}}} G_{\mathcal{A}} = L_{M_{\mathcal{A}}} G_{\mathcal{A}} = Q_{\mathcal{A}\mathcal{A}}(M_{\mathcal{A}}, G_{\mathcal{A}}) + Q_{\mathcal{A}\mathcal{A}}(G_{\mathcal{A}}, M_{\mathcal{A}}),$$

and the remaining term by

$$\overline{Q}_{\mathcal{A}}(\mathbf{G}, \mathbf{G}) = Q_{\mathcal{A}\mathcal{B}}(M_{\mathcal{A}}, G_{\mathcal{B}}) + Q_{\mathcal{A}\mathcal{B}}(G_{\mathcal{A}}, M_{\mathcal{B}}) + Q_{\mathcal{A}\mathcal{A}}(G_{\mathcal{A}}, G_{\mathcal{A}}) + Q_{\mathcal{A}\mathcal{B}}(G_{\mathcal{A}}, G_{\mathcal{B}}),$$

with $\mathcal{A} \neq \mathcal{B}$. Moreover, from (2.15), it follows that

$$P_1^{M_\mathcal{A}} G_\mathcal{A} = L_{M_\mathcal{A}}^{-1} \left\{ P_1^{M_\mathcal{A}} \left\{ \xi_1 \partial_x M_\mathcal{A} \right\} \right\} + \overline{R}_\mathcal{A}, \tag{2.16}$$

with

$$\begin{aligned} \overline{R}_\mathcal{A} = L_{M_\mathcal{A}}^{-1} &\left\{ P_1^{M_\mathcal{A}} \partial_t G_\mathcal{A} + P_1^{M_\mathcal{A}} \left\{ \xi_1 \partial_x G_\mathcal{A} \right\} - P_1^{M_\mathcal{A}} \left\{ \frac{q_\mathcal{A} \partial_x \phi}{m_\mathcal{A}} \partial_{\xi_1} G_\mathcal{A} \right\} \right\} \\ &- L_{M_\mathcal{A}}^{-1} \left\{ P_1^{M_\mathcal{A}} \overline{Q}_\mathcal{A}(\mathbf{G}, \mathbf{G}) \right\}. \end{aligned} \tag{2.17}$$

Back to (2.10) and (2.11), we rewrite $G_\mathcal{A}$ in the right-hand second terms of momentum and temperature equations as

$$G_\mathcal{A} = P_0^{M_\mathcal{A}} G_\mathcal{A} + P_1^{M_\mathcal{A}} G_\mathcal{A},$$

and then use (2.16) to replace $P_1^{M_\mathcal{A}} G_\mathcal{A}$ so as to obtain by some further calculations:

$$\begin{cases} \partial_t n_i + \partial_x (n_i u_1) = -\displaystyle\int_{\mathbb{R}^3} \xi_1 \partial_x G_i \, d\xi, \\ m_i n_i (\partial_t u_1 + u_1 \partial_x u_1) + \dfrac{2}{3} \partial_x (n_i \theta) + q_i n_i \partial_x \phi \\ \quad = 3\partial_x \left(\mu_i(\theta) \partial_x u_1 \right) - \displaystyle\int_{\mathbb{R}^3} \psi_{3i} \partial_t G_i \, d\xi - \int_{\mathbb{R}^3} \psi_{3i} \xi_1 \partial_x (P_0^{M_i} G_i) \, d\xi \\ \qquad + \displaystyle\int_{\mathbb{R}^3} \psi_{3i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi + u_1 \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_i \, d\xi, - \int_{\mathbb{R}^3} \psi_{3i} \xi_1 \partial_x \overline{R}_i \, d\xi, \\ m_i n_i (\partial_t u_j + u_1 \partial_x u_j) \\ \quad = \partial_x \left(\mu_i(\theta) \partial_x u_j \right) - \displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)i} \partial_t G_i \, d\xi - \int_{\mathbb{R}^3} \psi_{(j+2)i} \xi_1 \partial_x (P_0^{M_i} G_i) \, d\xi \\ \qquad + \displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi + u_j \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_i \, d\xi, - \int_{\mathbb{R}^3} \psi_{(j+2)i} \xi_1 \partial_x \overline{R}_i \, d\xi, \ j = 2, 3, \\ n_i \partial_t \theta + n_i u_1 \partial_x \theta + \dfrac{2 n_i \theta}{3} \partial_x u_1 \\ \quad = \partial_x \left(\kappa_i(\theta) \partial_x \theta \right) + 3\mu_i(\theta) (\partial_x u_1)^2 + \displaystyle\sum_{j=2}^{3} \mu_i(\theta) (\partial_x u_j)^2 - \int_{\mathbb{R}^3} \xi_1 (\psi_{6i} - \sum_{j=1}^{3} u_j \psi_{(j+2)i}) \partial_x \overline{R}_i \, d\xi \\ \qquad - \displaystyle\int_{\mathbb{R}^3} \psi_{6i} \partial_t G_i \, d\xi - \int_{\mathbb{R}^3} \psi_{6i} \xi_1 \partial_x (P_0^{M_i} G_i) \, d\xi + \sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} \xi_1 \partial_x (P_0^{M_i} G_i) \, d\xi \\ \qquad - \dfrac{1}{2} \displaystyle\sum_{j=1}^{3} u_j^2 \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_i \, d\xi + \sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} \partial_t G_i \, d\xi + \int_{\mathbb{R}^3} \psi_{6i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi \\ \qquad - \displaystyle\sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} Q_i(\mathbf{F}, \mathbf{F}) \, d\xi + \theta \int_{\mathbb{R}^3} \xi_1 \partial_x G_i \, d\xi + q_i \partial_x \phi \int_{\mathbb{R}^3} \dfrac{|\xi|^2}{2} \partial_{\xi_1} G_i \, d\xi, \end{cases} \tag{2.18}$$

and

$$
\begin{cases}
\partial_t n_e + \partial_x(n_e u_1) = -\displaystyle\int_{\mathbb{R}^3} \xi_1 \partial_x G_e \, d\xi, \\
m_e n_e(\partial_t u_1 + u_1\partial_x u_1) + \dfrac{2}{3}\partial_x(n_e\theta) + q_e n_e \partial_x \phi \\
\quad = 3\partial_x\left(\mu_e(\theta)\partial_x u_1\right) - \displaystyle\int_{\mathbb{R}^3} \psi_{3e}\partial_t G_e \, d\xi - \int_{\mathbb{R}^3} \psi_{3e}\xi_1 \partial_x (P_0^{M_e} G_e)\, d\xi \\
\qquad + \displaystyle\int_{\mathbb{R}^3} \psi_{3e} Q_e(\mathbf{F},\mathbf{F})\, d\xi + u_1 \int_{\mathbb{R}^3} m_e \xi_1 \partial_x G_e \, d\xi, - \int_{\mathbb{R}^3} \psi_{3e}\xi_1 \partial_x \overline{R}_e \, d\xi, \\
m_e n_e(\partial_t u_j + u_1\partial_x u_j) \\
\quad = \partial_x\left(\mu_e(\theta)\partial_x u_j\right) - \displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)e}\partial_t G_e \, d\xi - \int_{\mathbb{R}^3} \psi_{(j+2)e}\xi_1 \partial_x (P_0^{M_e} G_e)\, d\xi \\
\qquad + \displaystyle\int_{\mathbb{R}^3} \psi_{(j+2)e} Q_e(\mathbf{F},\mathbf{F})\, d\xi + u_j \int_{\mathbb{R}^3} m_i \xi_1 \partial_x G_e \, d\xi, - \int_{\mathbb{R}^3} \psi_{(j+2)e}\xi_1 \partial_x \overline{R}_e \, d\xi, \ j = 2,3, \\
n_e\partial_t\theta + n_e u_1\partial_x\theta + \dfrac{2n_e\theta}{3}\partial_x u_1 \\
\quad = \partial_x\left(\kappa_e(\theta)\partial_x\theta\right) + 3\mu_e(\theta)(\partial_x u_1)^2 + \displaystyle\sum_{j=2}^{3} \mu_e(\theta)(\partial_x u_j)^2 - \int_{\mathbb{R}^3} \xi_1(\psi_{6e} - \sum_{j=1}^{3} u_j \psi_{(j+2)e})\partial_x \overline{R}_e \, d\xi \\
\qquad - \displaystyle\int_{\mathbb{R}^3} \psi_{6e}\partial_t G_e \, d\xi - \int_{\mathbb{R}^3} \psi_{6e}\xi_1 \partial_x (P_0^{M_e} G_e)\, d\xi + \sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)e}\xi_1 \partial_x (P_0^{M_e} G_e)\, d\xi \\
\qquad -\tfrac{1}{2}\displaystyle\sum_{j=1}^{3} u_j^2 \int_{\mathbb{R}^3} m_e \xi_1 \partial_x G_e \, d\xi + \sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)e}\partial_t G_e \, d\xi + \int_{\mathbb{R}^3} \psi_{6e} Q_e(\mathbf{F},\mathbf{F})\, d\xi \\
\qquad - \displaystyle\sum_{j=1}^{3} u_j \int_{\mathbb{R}^3} \psi_{(j+2)i} Q_e(\mathbf{F},\mathbf{F})\, d\xi + \theta \int_{\mathbb{R}^3} \xi_1 \partial_x G_e \, d\xi + q_e \partial_x \phi \int_{\mathbb{R}^3} \frac{|\xi|^2}{2} \partial_{\xi_1} G_e \, d\xi,
\end{cases}
\tag{2.19}
$$

where for $\mathcal{A} = i$ and e the viscosity coefficient $\mu_{\mathcal{A}}(\theta)$ and heat-conductivity coefficient $\kappa_{\mathcal{A}}(\theta)$ are represented by

$$
\begin{aligned}
\mu_{\mathcal{A}}(\theta) &= -\frac{1}{3k_{\mathcal{A}}\theta} \int_{\mathbb{R}^3} m_{\mathcal{A}} \xi_1^2 L^{-1}_{M_{[1,u,\theta;m_{\mathcal{A}}]}} \left(m_{\mathcal{A}}\xi_1^2 M_{[1,u,\theta;m_{\mathcal{A}}]}\right) \, d\xi \\
&= -\frac{1}{k_{\mathcal{A}}\theta} \int_{\mathbb{R}^3} m_{\mathcal{A}} \xi_1\xi_j L^{-1}_{M_{[1,u,\theta;m_{\mathcal{A}}]}} \left(m_{\mathcal{A}}\xi_1\xi_j M_{[1,u,\theta;m_{\mathcal{A}}]}\right) \, d\xi > 0, \ j = 2,3,
\end{aligned}
$$

and

$$
\kappa_{\mathcal{A}}(\theta) = -\frac{1}{4k_{\mathcal{A}}\theta^2} \int_{\mathbb{R}^3} m_{\mathcal{A}} |\xi - u|^2 \xi_j L^{-1}_{M_{[1,u,\theta;m_{\mathcal{A}}]}} \left(m_{\mathcal{A}}|\xi - u|^2\xi_j M_{[1,u,\theta;m_{\mathcal{A}}]}\right) \, d\xi > 0, \ j = 1,2,3,
$$

respectively. Here $L^{-1}_{M_{[1,u,\theta;m_{\mathcal{A}}]}}$ is defined in the same way as $L^{-1}_{M_{\mathcal{A}}}$.

Similarly for obtaining (2.13), from (2.18) and (2.19), one has the equations of momentum for $u=(u_1,u_2,u_3)$:

$$\begin{cases} (m_in_i+m_en_e)(\partial_t u_1+u_1\partial_x u_1)+\dfrac{2}{3}\partial_x[(n_i+n_e)\theta]+(q_in_i+q_en_e)\,\partial_x\phi \\ \quad =3\partial_x\left[(\mu_i(\theta)+\mu_e(\theta))\partial_x u_1\right] \\ \quad\quad -\displaystyle\int_{\mathbb{R}^3}\psi_{3i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi-\int_{\mathbb{R}^3}\psi_{3e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi-\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial_x\overline{\mathbf{R}}\,d\xi, \\ (m_in_i+m_en_e)(\partial_t u_j+u_1\partial_x u_j) \\ \quad =\partial_x\left[(\mu_i(\theta)+\mu_e(\theta))\partial_x u_j\right]-\displaystyle\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi \\ \quad\quad -\displaystyle\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi-\int_{\mathbb{R}^3}\xi_1\psi_{j+2}\cdot\partial_x\overline{\mathbf{R}}\,d\xi,\quad j=2,3, \end{cases} \tag{2.20}$$

and the equation of temperature for θ:

$$\begin{aligned} &(n_i+n_e)(\partial_t\theta+u_1\partial_x\theta)+\frac{2}{3}(n_i+n_e)\,\theta\partial_x u_1 \\ &=\partial_x\left((\kappa_i(\theta)+\kappa_e(\theta))\partial_x\theta\right)+3(\mu_i(\theta)+\mu_e(\theta))(\partial_x u_1)^2+\sum_{j=2}^{3}(\mu_i(\theta)+\mu_e(\theta))(\partial_x u_j)^2 \\ &\quad-\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum_{j=1}^{3}u_j\psi_{j+2}\right)\cdot\partial_x\overline{\mathbf{R}}\,d\xi-\int_{\mathbb{R}^3}\psi_{6i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi \\ &\quad-\int_{\mathbb{R}^3}\psi_{6e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi+\sum_{j=1}^{3}u_j\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi \\ &\quad+\sum_{j=1}^{3}u_j\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi+\theta\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi \\ &\quad+\partial_x\phi\int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\,[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi, \end{aligned} \tag{2.21}$$

where we have denoted $\overline{\mathbf{R}}=[\overline{R}_i,\overline{R}_e]^{\mathrm{T}}$. Note that n_i,n_e satisfy equations of mass conservation:

$$\begin{aligned} \partial_t n_i+\partial_x(n_iu_1)&=-\int_{\mathbb{R}^3}\xi_1\partial_x G_i\,d\xi, \\ \partial_t n_e+\partial_x(n_eu_1)&=-\int_{\mathbb{R}^3}\xi_1\partial_x G_e\,d\xi. \end{aligned} \tag{2.22}$$

Moreover, as for considering (2.14), if one further ignores those terms involving the non-fluid part $\mathbf{G}$, one has the closed viscous fluid-type system of six knowns $n_i,n_e,u_1,u_2,u_3,\theta$:

$$\begin{cases} \partial_t n_i+\partial_x(n_iu_1)=0, \\ \partial_t n_e+\partial_x(n_eu_1)=0, \\ (m_in_i+m_en_e)(\partial_t u_1+u_1\partial_x u_1)+\frac{2}{3}\partial_x\left[(n_i+n_e)\theta\right] \\ \qquad\qquad +(q_in_i+q_en_e)\partial_x\phi=3\partial_x\left[(\mu_i(\theta)+\mu_e(\theta))\partial_x u_1\right], \\ (m_in_i+m_en_e)(\partial_t u_j+u_1\partial_x u_j)=\partial_x\left[(\mu_i(\theta)+\mu_e(\theta))\partial_x u_j\right],\quad j=2,3, \\ (n_i+n_e)(\partial_t\theta+u_1\partial_x\theta)+\frac{2}{3}(n_i+n_e)\theta\partial_x u_1=\partial_x\left((\kappa_i(\theta)+\kappa_e(\theta))\partial_x\theta\right) \\ \qquad\qquad +3(\mu_i(\theta)+\mu_e(\theta))(\partial_x u_1)^2+\sum_{j=2}^{3}(\mu_i(\theta)+\mu_e(\theta))(\partial_x u_j)^2, \\ -\partial_x^2\phi=q_in_i+q_en_e. \end{cases} \tag{2.23}$$

Note that (2.23) could be thought to be the first-order fluid dynamic approximation of the VPB system (1.1), (1.2).

For later use, we also introduce the entropy quantity and the corresponding equation. For given densities n_i, n_e and temperature θ, we define the entropy S by

$$S=-\frac{2}{3}\ln(n_i+n_e)+\ln\left(\frac{4\pi}{3}\theta\right)+1. \tag{2.24}$$

According to (2.20) and (2.21) as well as (2.22), one deduces that S satisfies

$$\begin{aligned}\partial_t S+u_1\partial_x S=&\frac{1}{\overline{n}\theta}\partial_x\left((\kappa_i(\theta)+\kappa_e(\theta))\partial_x\theta\right)+\frac{3(\mu_i(\theta)+\mu_e(\theta))}{\overline{n}\theta}(\partial_x u_j)^2+\sum_{j=2}^{3}\frac{(\mu_i(\theta)+\mu_e(\theta))}{\overline{n}\theta}(\partial_x u_j)^2\\&+\left(1-\frac{2}{3\overline{n}}\right)\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi-\frac{1}{\overline{n}\theta}\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum_{j=1}^{3}u_j\psi_{j+2}\right)\cdot\partial_x\overline{\mathbf{R}}\,d\xi\\&-\frac{1}{\overline{n}\theta}\int_{\mathbb{R}^3}\psi_{6i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)\,d\xi-\frac{1}{\overline{n}\theta}\int_{\mathbb{R}^3}\psi_{6e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)\,d\xi\\&+\frac{1}{\overline{n}\theta}\sum_{j=1}^{3}u_j\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)\,d\xi+\frac{1}{\overline{n}\theta}\sum_{j=1}^{3}u_j\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)\,d\xi\\&+\frac{\partial_x\phi}{\overline{n}\theta}\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi,\end{aligned}$$

where we have denoted $\overline{n}=n_i+n_e$. For later use, we also introduce the pressure function P by

$$P=\frac{2}{3}\overline{n}\theta.$$

Note that from (2.24), one has

$$\theta=\frac{3}{2}ke^S\overline{n}^{2/3},\quad P=\frac{2}{3}\overline{n}\theta=ke^S\overline{n}^{5/3},$$

with the constant k given by $k:=\frac{1}{2\pi e}$.

3. Quasineutral Euler equations and rarefaction waves

Recall that (2.14) and (2.23) are thought to be the zero-order and first-order fluid dynamic approximation of the VPB system (1.1), (1.2), respectively, if the two-component non-fluid part $\mathbf{G}$ is ignored. Inspired by this, one may expect to justify in a rigorous way the large-time asymptotics of the VPB system (1.1), (1.2) toward (2.14) or (2.23). The goal of this paper is to treat this in the setting of rarefaction waves. Instead of directly using (2.14) and (2.23), the expected large-time asymptotic system is the quasineutral Euler system in the form of

$$\begin{cases}\partial_t n_i+\partial_x(n_iu_1)=0,\\ \partial_t n_e+\partial_x(n_eu_1)=0,\\ (m_in_i+m_en_e)(\partial_t u_1+u_1\partial_x u_1)+\dfrac{2}{3}\partial_x\left[(n_i+n_e)\theta\right]=0,\\ (n_i+n_e)(\partial_t\theta+u_1\partial_x\theta)+\dfrac{2}{3}(n_i+n_e)\theta\partial_x u_1=0,\\ q_in_i+q_en_e=0.\end{cases} \tag{3.1}$$

For simplicity, by letting

$$n_e=n,\quad n_i=-\frac{q_i}{q_e}n_e=-\frac{q_i}{q_e}n,$$

in terms of the quasineutral assumption, (3.1) reduces to

$$\begin{cases} \partial_t n + \partial_x(nu_1) = 0, \\ \partial_t u_1 + u_1\partial_x u_1 + \dfrac{1}{n}\partial_x P(n,S) = 0 \\ \partial_t S + u_1\partial_x S = 0, \end{cases} \tag{3.2}$$

where

$$P(n,S) = \frac{2(q_i-q_e)}{3(m_eq_i - m_iq_e)}n\theta, \quad S = -\frac{2}{3}\ln\left(\frac{q_i-q_e}{q_i}n\right) + \ln\left(\frac{4\pi R}{3}\theta\right) + 1.$$

To construct the large-time asymptotic rarefaction wave of the VPB system through (3.1) or (3.2), one has to assign some appropriate far-field data from (1.5), (1.6) and (1.7). Recall that we have set $n_{e\pm} = n_\pm$ and hence $n_{i\pm} = -\frac{q_e}{q_i}n_\pm$. In terms of $n_\pm$ and $\theta_\pm$, recalling (2.24), we define constants $S_\pm$ by

$$S_\pm = -\frac{2}{3}\ln\left(n_{i\pm} + n_{e\pm}\right) + \ln\left(\frac{4\pi R}{3}\theta_\pm\right) + 1,$$

that is,

$$S_\pm = -\frac{2}{3}\ln\left(\frac{q_i-q_e}{q_i}n_\pm\right) + \ln\left(\frac{4\pi R}{3}\theta_\pm\right) + 1,$$

To the end, we assume $S_+ = S_-$ or equivalently

$$\frac{\theta_+}{n_+^{2/3}} = \frac{\theta_-}{n_-^{2/3}} = \frac{3}{2}ke^{S_\pm}\left(\frac{q_i-q_e}{q_i}\right)^{2/3} := A.$$

Under the above settings on the far-field values of initial data (1.4) for the VPB system (1.1), (1.2), we then expect that the solution $\mathbf{F}(t,x,\xi)$ to the Cauchy problem tends time-asymptotically to the local bi-Maxwellian

$$\mathbf{M}_R = \begin{bmatrix} M_{Ri}(\xi) \\ M_{Re}(\xi) \end{bmatrix} = \begin{bmatrix} M_{[-\frac{q_e}{q_i}n^R, u^R, \theta^R; m_i]}(\xi) \\ M_{[n^R,u^R,\theta^R;m_e]}(\xi) \end{bmatrix},$$

where $[n^R, u^R, \theta^R]$ with $u^R = [u_1^R, 0, 0]$ is the rarefaction wave solution of the Riemann problem on the quasi-neutral Euler system (3.2) with Riemann initial data given by

$$[n,u_1,\theta](0,x) = \left[n_0^R, u_{1,0}^R, \theta_0\right](x) := \begin{cases} [n_-, u_{1-}, \theta_-], & x < 0, \\ [n_+, u_{1+}, \theta_+], & x > 0. \end{cases} \tag{3.3}$$

The Riemann problem can be solved in the usual way (cf. [58]). Indeed, the quasineutral Euler system (3.2) has three characteristics

$$\begin{cases} \lambda_1 = \lambda_1(n,u_1,S) := u_1 - \sqrt{\partial_n P(n,S)}, \\ \lambda_2 = \lambda_2(n,u_1,S) := u_1, \\ \lambda_3 = \lambda_3(n,u_1,S) := u_1 + \sqrt{\partial_n P(n,S)}. \end{cases}$$

In terms of two Riemann invariants of the third eigenvalue $\lambda_3(n,u_1,S)$, we define the set of right constant states $[n_+,u_{1+},\theta_+]$ to which a given left constant state $[n_-,u_{1-},\theta_-]$ with $n_- > 0$ and $\theta_- > 0$ is connected through the 3-rarefaction wave to be

$$R_3(n_-,u_{1-},\theta_-) \equiv \Bigg\{ [n,u_1,\theta] \in \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+ \;\Big|\; \frac{n^{2/3}}{\theta} = \frac{n_-^{2/3}}{\theta_-}, \\ u_1 - u_{1-} = \int_{n_-}^{n} \frac{\sqrt{\partial_n P(\eta, S_-)}}{\eta}d\eta,\ n > n_-,\ u_1 > u_{1-} \Bigg\}. \tag{3.4}$$

Now, letting $[n_+, u_{1+}, \theta_+] \in R_3(n_-, u_{1-}, \theta_-)$, the Riemann problem (3.2), (3.3) admits a self-similar solution, the 3-rarefaction wave $\left[n^R, u_1^R, \theta^R\right](z)$ with $z = x/t \in \mathbb{R}$, explicitly defined by

$$\begin{cases} \lambda_3\left(n^R(z), u_1^R(z), S_-\right) = \begin{cases} \lambda_3(n_-, u_{1-}, S_-) & \text{for } z < \lambda_3(n_-, u_{1-}, S_-), \\ z & \text{for } \lambda_3(n_-, u_{1-}, S_-) \le z \le \lambda_3(n_+, u_{1+}, S_-), \\ \lambda_3(n_+, u_{1+}, S_-) & \text{for } z > \lambda_3(n_+, u_{1+}, S_-), \end{cases} \\ u_1^R(z) = u_{1-} + \displaystyle\int_{n_-}^{n^R(z)} \frac{\sqrt{\partial_n P(\eta, S_-)}}{\eta} d\eta, \\ \theta^R(z) = A(n^R(z))^{2/3}, \quad A = \frac{3}{2} k e^{S_-} \left(\frac{q_i - q_e}{q_i}\right)^{2/3}. \end{cases}$$

Since $[n^R, u_1^R, \theta^R]$ is a weak solution of the Riemann problem (3.2) and (3.3) and lack of regularity, one has establish a smooth approximation to the rarefaction wave $[n^R, u_1^R, \theta^R]$. To do this, in the usual way, the smooth rarefaction wave $[n^r, u^r, \theta^r](t, x)$ with $u^r(t, x) = [u_1^r(t, x), 0, 0]$ is defined by

$$\begin{cases} \lambda_3(n^r(t,x), u_1^r, S_-) = w(t,x), \\ u_1^r(t,x) = u_{1-} + \displaystyle\int_{n_-}^{n^r(t,x)} \frac{\sqrt{\partial_n P(\eta, S_-)}}{\eta} d\eta, \\ \theta^r(t,x) = A(n^r(t,x))^{2/3}, \\ \lim\limits_{x \to \pm\infty} [n^r, u_1^r, \theta^r](t,x) = [n_\pm, u_{1\pm}, \theta_\pm], \quad [n_+, u_{1+}, \theta_+] \in R_3(n_-, u_{1-}, \theta_-), \end{cases} \tag{3.5}$$

with $w = w(t, x)$ being the solution to the Burgers' equation

$$\begin{cases} \partial_t w + w \partial_x w = 0, \\ w(0,x) = w_0(x) := \frac{1}{2}(w_+ + w_-) + \frac{1}{2}(w_+ - w_-) \tanh x, \quad w_\pm := \lambda_3(n_\pm, u_{1\pm}, S_-). \end{cases} \tag{3.6}$$

We remark that by letting $n_e^r = n_r$ and $n_i^r = -\frac{q_e}{q_i} n^r$, in view of the construction of the smooth rarefaction wave above, $[n_i^r, n_e^r, u_1^r, \theta^r]$ satisfies

$$\begin{cases} \partial_t n_i^r + \partial_x(n_i^r u_1^r) = 0, \\ \partial_t n_i^r + \partial_x(n_i^r u_1^r) = 0, \\ (m_i n_i^r + m_e n_e^r)(\partial_t u_1^r + u_1^r \partial_x u_1^r) + \dfrac{2}{3} \partial_x((n_i^r + n_e^r)\theta^r) = 0, \\ (q_i m_i n_i^r + q_e m_e n_e^r)(\partial_t u_1^r + u_1^r \partial_x u_1^r) \\ \qquad = -\dfrac{2\theta^r}{3} \dfrac{q_i m_i n_i^r + q_e m_e n_e^r}{m_i n_i^r + m_e n_e^r} \partial_x(n_i^r + n_e^r) - \dfrac{2}{3} \partial_x \theta^r \dfrac{q_i m_i n_i^r + q_e m_e n_e^r}{m_i n_i^r + m_e n_e^r}(n_i^r + n_e^r), \\ (n_i^r + n_e^r)(\partial_t \theta^r + u_1^r \partial_x \theta^r) + P^r \partial_x u_1^r = 0. \end{cases} \tag{3.7}$$

Here $P^r = \frac{2}{3}(n_i^r + n_e^r)\theta^r = \frac{2}{3} A \frac{q_i - q_e}{q_i} (n^r)^{5/3}$.

The next lemma is devoted to the study of the properties of the smooth rarefaction wave $[n^r, u_1^r, \theta^r]$ constructed in (3.5) and (3.6).

Lemma 3.1. *It holds that*
(i) $\partial_x u_1^r(t,x) > 0$ *and* $n_- < n^r(t,x) < n_+$, $u_{1-} < u_1^r(t,x) < u_{1+}$ *for* $x \in \mathbb{R}$ *and* $t \ge 0$.
(ii) For any $1 \le p \le +\infty$, *there exists a constant* C_p *such that for* $t > 0$,

$$\left\|\partial_x \left[n^r, u_1^r, \theta^r\right]\right\|_{L^p} \le C_p \min\left\{\delta_r, \delta_r^{1/p} t^{-1+1/p}\right\},$$

$$\left\|\partial_x^j \left[n^r, u_1^r, \theta^r\right]\right\|_{L^p} \le C_p \min\left\{\delta_r, t^{-1}\right\}, \quad j \ge 2,$$

where we recall that $\delta_r = |n_+ - n_-| + |u_{1+} - u_{1-}| + |\theta_+ - \theta_-|$ *is the wave strength.*

(iii) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|\left[n^r,u_1^r,\theta^r\right](t,x)-\left[n^R,u_1^R,\theta^R\right](x/t)\right|=0.$

4. Preliminary estimates on two-component collision operator

In this section, we list some basic inequalities on the two-component collision operator for later use. The first lemma is concerned with the nonlinear collision operators $Q_{\mathcal{AB}}(\cdot,\cdot)$, whose proof can be found in [38] when masses of the particles are normalised to be one.

Lemma 4.1. *Let $\mathcal{A},\mathcal{B}\in\{i,e\}$. There exists a positive constant $C>0$ such that*

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{(1+|\xi|)^{-1}\left|Q_{\mathcal{AB}}(F_{\mathcal{A}},F_{\mathcal{B}})\right|^2}{\widehat{M}_{\mathcal{A}}}\,d\xi\\&\leq C\left\{\int_{\mathbb{R}^3}\frac{(1+|\xi|)F_{\mathcal{A}}^2}{\widehat{M}_{\mathcal{A}}}\,d\xi\cdot\int_{\mathbb{R}^3}\frac{F_{\mathcal{B}}^2}{\widehat{M}_{\mathcal{B}}}\,d\xi+\int_{\mathbb{R}^3}\frac{F_{\mathcal{A}}^2}{\widehat{M}_{\mathcal{A}}}\,d\xi\cdot\int_{\mathbb{R}^3}\frac{(1+|\xi|)F_{\mathcal{B}}^2}{\widehat{M}_{\mathcal{B}}}\,d\xi\right\},\end{aligned}\tag{4.1}$$

where we have defined

$$\left[\widehat{M}_i,\widehat{M}_e\right]^{\mathrm{T}}\equiv\left[M_{[\widehat{n}_i,\widehat{u},\widehat{\theta};m_i]}(\xi),M_{[\widehat{n}_e,\widehat{u},\widehat{\theta};m_e]}(\xi)\right]^{\mathrm{T}},$$

to be any bi-Maxwellian such that the above integrals are well defined.

Proof. Note that one can rewrite (1.3) as

$$Q_{\mathcal{AB}}(F_{\mathcal{A}},F_{\mathcal{B}})=Q_{\mathcal{AB}}^{\mathrm{gain}}(F_{\mathcal{A}},F_{\mathcal{B}})+Q_{\mathcal{AB}}^{\mathrm{loss}}(F_{\mathcal{A}},F_{\mathcal{B}}),\tag{4.2}$$

with the normal meaning for the gain part and the loss part. To prove (4.1), we first consider the gain part and thus compute

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{(1+|\xi|)^{-1}\left|Q_{\mathcal{AB}}^{\mathrm{gain}}(F_{\mathcal{A}},F_{\mathcal{B}})\right|^2}{\widehat{M}_{\mathcal{A}}}\,d\xi\\&=\sigma^2\int_{\mathbb{R}^3}(1+|\xi|)^{-1}\widehat{M}_{\mathcal{A}}^{-1}\left(\int_{\mathbb{R}^3}|(\xi-\xi_*)\cdot\omega|F_{\mathcal{A}}(\xi')F_{\mathcal{B}}(\xi_*')d\xi_*\right)^2d\xi.\end{aligned}$$

We now set

$$F_{\mathcal{A}}=\sqrt{\widehat{M}_{\mathcal{A}}}f_{\mathcal{A}},\quad F_{\mathcal{B}}=\sqrt{\widehat{M}_{\mathcal{B}}}f_{\mathcal{B}},\tag{4.3}$$

and use the identity

$$\widehat{M}_{\mathcal{A}}(\xi)\widehat{M}_{\mathcal{B}}(\xi_*)=\widehat{M}_{\mathcal{A}}(\xi')\widehat{M}_{\mathcal{A}}(\xi_*'),$$

so as to derive

$$\begin{aligned}&\int_{\mathbb{R}^3}(1+|\xi|)^{-1}\widehat{M}_{\mathcal{A}}^{-1}\left(\int_{\mathbb{R}^3}|(\xi-\xi_*)\cdot\omega|F_{\mathcal{A}}(\xi')F_{\mathcal{B}}(\xi_*')d\xi_*\right)^2d\xi\\&=\int_{\mathbb{R}^3}(1+|\xi|)^{-1}\widehat{M}_{\mathcal{A}}^{-1}\left(\int_{\mathbb{R}^3}\sqrt{\widehat{M}_{\mathcal{A}}(\xi)\widehat{M}_{\mathcal{B}}(\xi_*)}|(\xi-\xi_*)\cdot\omega|f_{\mathcal{A}}(\xi')f_{\mathcal{B}}(\xi_*')d\xi_*\right)^2d\xi\\&\leq C\int_{\mathbb{R}^3}(1+|\xi|)^{-1}\left\{\int_{\mathbb{R}^3}\widehat{M}_{\mathcal{B}}(\xi_*)|(\xi-\xi_*)\cdot\omega|d\xi_*\int_{\mathbb{R}^3}|(\xi-\xi_*)\cdot\omega|\left|f_{\mathcal{A}}(\xi')f_{\mathcal{B}}(\xi_*')\right|^2d\xi_*\right\}d\xi\\&\leq C\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi-\xi_*|\left|f_{\mathcal{A}}(\xi')f_{\mathcal{B}}(\xi_*')\right|^2d\xi_*d\xi,\end{aligned}$$

where we have used the Hölder's inequality to obtain the first inequality above. In view of $|\xi-\xi_*|=|\xi'-\xi'_*|$ and by a change of variables $(\xi,\xi_*)\to(\xi',\xi'_*)$, one further has

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi-\xi_*|\left|f_{\mathcal{A}}(\xi')f_{\mathcal{B}}(\xi'_*)\right|^2\,d\xi_*d\xi\\
&\le C\int_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\xi|+|\xi_*|)|f_{\mathcal{A}}(\xi)f_{\mathcal{B}}(\xi_*)|^2\,d\xi_*d\xi\\
&\le C\int_{\mathbb{R}^3}(1+|\xi|)|f_{\mathcal{A}}(\xi)|^2\,d\xi\int_{\mathbb{R}^3}|f_{\mathcal{B}}(\xi_*)|^2\,d\xi_*+C\int_{\mathbb{R}^3}|f_{\mathcal{A}}(\xi)|^2\,d\xi\int_{\mathbb{R}^3}(1+|\xi_*|)|f_{\mathcal{B}}(\xi_*)|^2\,d\xi_*,
\end{aligned}\tag{4.4}$$

where the fact that

$$\left|\frac{\partial(\xi',\xi'_*)}{\partial(\xi,\xi_*)}\right|=1,$$

has been used. Rewriting (4.4) in terms of (4.3) gives (4.1) for the contribution from the gain part in (4.2). As to the loss part in (4.2), the proof is similar and details are omitted for brevity. This then completes the proof of Lemma 4.1. □

In order to obtain the energy estimates for the Boltzmann equation (2.5), for $\mathbf{P}_1^{\mathbf{M}}\mathbf{F}$ which means the microscopic projection of its solution $\mathbf{F}(t,x,\xi)$ with respect to a given bi-Maxwellian

$$\mathbf{M}=[M_i,M_e]^{\mathrm{T}}=[M_{[n_i(t,x),u(t,x),\theta(t,x);m_i]}(\xi),M_{[n_e(t,x),u(t,x),\theta(t,x);m_e]}(\xi)]^{\mathrm{T}},$$

one need to find out its dissipative effect through the microscopic H-theorem. Like the single-component case, the microscopic H-theorem states that the linearized collision operator $\mathbf{L}_{\mathbf{M}}$ around a fixed bi-Mawellian $\mathbf{M}$ is also negative definite on the non-fluid element $\mathbf{P}_1^{\mathbf{M}}\mathbf{F}$, cf. [10].

Lemma 4.2. *It holds that*

$$-\int_{\mathbb{R}^3}\mathbf{P}_1^{\mathbf{M}}\mathbf{F}\cdot\left\{\mathbf{M}^{-1}\left(\mathbf{L}_{\mathbf{M}}\mathbf{P}_1^{\mathbf{M}}\mathbf{F}\right)\right\}d\xi\ge\delta\int_{\mathbb{R}^3}(1+|\xi|\left|\mathbf{M}^{-1/2}\mathbf{P}_1^{\mathbf{M}}\mathbf{F}\right|^2\,d\xi,\tag{4.5}$$

for a positive constant $\delta>0$ depending on $[n_i,n_e,u,\theta]$. In fact, δ also depends on m_i and m_e, and in what follows we shall omit pointing out such dependence for brevity.

Proof. Recall (2.3). We denote

$$\mathbf{P}_1^{\mathbf{M}}\mathbf{F}=\mathbf{G}=\mathbf{G}(t,x,\xi)=\begin{bmatrix}G_i(\xi)\\G_e(\xi)\end{bmatrix}.$$

Further recall the definitions (2.6) and (1.3). Let us decompose $\mathbf{L}_{\mathbf{M}}\mathbf{G}$ as

$$\mathbf{L}_{\mathbf{M}}\mathbf{G}=-\nu\mathbf{G}+\mathbf{K}\mathbf{G},$$

with

$$-\nu\mathbf{G}=\begin{bmatrix}\nu_iG_i\\\nu_eG_e\end{bmatrix},\quad\mathbf{K}\mathbf{G}=\begin{bmatrix}\sqrt{M_i}K_i\mathbf{G}\\\sqrt{M_e}K_e\mathbf{G}\end{bmatrix}=\begin{bmatrix}\sqrt{M_i}K_i(\mathbf{M}^{-1/2}\mathbf{G})\\\sqrt{M_e}K_e(\mathbf{M}^{-1/2}\mathbf{G})\end{bmatrix},$$

and

$$\nu_{\mathcal{A}}=-\sum_{\mathcal{B}\in\{\mathcal{A},\mathcal{B}\}}Q^{\mathrm{loss}}_{\mathcal{A}\mathcal{B}}(1,M_{\mathcal{B}})=\int_{\mathbb{R}^3\times\mathbb{S}^2_+}B_{\mathcal{A}\mathcal{A}}M_{\mathcal{A}}(\xi_*)\,d\xi_*d\omega+\int_{\mathbb{R}^3\times\mathbb{S}^2_+}B_{\mathcal{A}\mathcal{B}}M_{\mathcal{B}}(\xi_*)\,d\xi_*d\omega,$$

$$K_{\mathcal{A}}=K^1_{\mathcal{A}}+K^2_{\mathcal{A}}+K^3_{\mathcal{A}}+K^4_{\mathcal{A}},\tag{4.6}$$

and

$$\begin{cases} K^1_{\mathcal{A}}\mathbf{G} =& M_{\mathcal{A}}^{-\frac{1}{2}} \displaystyle\sum_{\mathcal{B}\in\{\mathcal{A},\mathcal{B}\}} Q^{\rm loss}_{\mathcal{A}\mathcal{B}}(M_{\mathcal{A}}, G_{\mathcal{B}}) \\ =& -\displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B_{\mathcal{A}\mathcal{A}}\sqrt{M_{\mathcal{A}}(\xi)}\sqrt{M_{\mathcal{A}}(\xi_*)}\left(\frac{G_{\mathcal{A}}}{\sqrt{M_{\mathcal{A}}}}\right)(\xi_*)\,d\xi_* d\omega \\ & -\displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B_{\mathcal{A}\mathcal{B}}\sqrt{M_{\mathcal{A}}(\xi)}\sqrt{M_{\mathcal{B}}(\xi_*)}\left(\frac{G_{\mathcal{B}}}{\sqrt{M_{\mathcal{B}}}}\right)(\xi_*)\,d\xi_* d\omega, \quad \mathcal{A}\neq\mathcal{B}, \\ K^2_{\mathcal{A}}\mathbf{G} =& M_{\mathcal{A}}^{-\frac{1}{2}}\left\{Q^{\rm gain}_{\mathcal{A}\mathcal{A}}(M_{\mathcal{A}}, G_{\mathcal{A}}) + Q^{\rm gain}_{\mathcal{A}\mathcal{A}}(G_{\mathcal{A}}, M_{\mathcal{A}}\right\} \\ =& \displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B_{\mathcal{A}\mathcal{A}}\sqrt{M_{\mathcal{A}}(\xi_*)}\left[\sqrt{M_{\mathcal{A}}(\xi')}\left(\frac{G_{\mathcal{A}}}{\sqrt{M_{\mathcal{A}}}}\right)(\xi'_*) + \sqrt{M_{\mathcal{A}}(\xi'_*)}\left(\frac{G_{\mathcal{A}}}{\sqrt{M_{\mathcal{A}}}}\right)(\xi')\right] d\xi_* d\omega, \\ K^3_{\mathcal{A}}\mathbf{G} =& M_{\mathcal{A}}^{-\frac{1}{2}} Q^{\rm gain}_{\mathcal{A}\mathcal{B}}(M_{\mathcal{A}}, G_{\mathcal{B}}) \\ =& \displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B_{\mathcal{A}\mathcal{B}}\sqrt{M_{\mathcal{B}}(\xi_*)}\sqrt{M_{\mathcal{A}}(\xi')}\left(\frac{G_{\mathcal{B}}}{\sqrt{M_{\mathcal{B}}}}\right)(\xi'_*)\,d\xi_* d\omega, \quad \mathcal{A}\neq\mathcal{B}, \\ K^4_{\mathcal{A}}\mathbf{G} =& M_{\mathcal{A}}^{-\frac{1}{2}} Q^{\rm gain}_{\mathcal{A}\mathcal{B}}(G_{\mathcal{A}}, M_{\mathcal{B}}) \\ =& \displaystyle\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B_{\mathcal{A}\mathcal{B}}\sqrt{M_{\mathcal{B}}(\xi_*)}\sqrt{M_{\mathcal{B}}(\xi'_*)}\frac{G_{\mathcal{A}}}{\sqrt{M_{\mathcal{A}}}}(\xi')\,d\xi_* d\omega, \quad \mathcal{A}\neq\mathcal{B}. \end{cases} \tag{4.7}$$

One one hand, by performing the similar calculations as [10], one can see that K_i and K_e defined by (4.6) and (4.7) are compact from

$$L^2_\xi(\frac{1}{\sqrt{M_i}})\times L^2_\xi(\frac{1}{\sqrt{M_e}})$$

to itself. One the other hand, as $B_{\mathcal{A}\mathcal{B}} = B_{\mathcal{B}\mathcal{A}} = \sigma^2|(\xi-\xi_*)\cdot\omega|$, one can be able to show

$$\nu_i \sim (1+|\xi|), \quad \nu_e \sim (1+|\xi|).$$

Then the coercivity estimate (4.5) follows from the standard argument as [13, 14, 42]; see also [2]. This ends the proof of Lemma 4.2. □

Furthermore, one can vary the background for the linearisation and the weight function. In fact, basing on Lemma 4.1 as well as its proof, we also have the following result, cf. [60].

Lemma 4.3. *Let $\frac{\theta}{2} < \widehat{\theta}$. Then there exist two positive constants $\delta = \delta(n_i, n_e, u, \theta; \widehat{n}_i, \widehat{n}_e, \widehat{u}, \widehat{\theta})$ and $\eta_0 = \eta_0(n_i, n_e, u, \theta; \widehat{n}_i, \widehat{n}_e, \widehat{u}, \widehat{\theta})$ such that if*

$$|n_i - \widehat{n}_i| + |n_e - \widehat{n}_e| + |u - \widehat{u}| + |\theta - \widehat{\theta}| < \eta_0,$$

it holds that for $\mathbf{H}(\xi) = [H_i(\xi), H_e(\xi)]^{\rm T} \in \mathcal{N}^\perp$,

$$-\int_{\mathbb{R}^3} \mathbf{H}\cdot\left\{\widehat{\mathbf{M}}^{-1}(\mathbf{L}_{\mathbf{M}}\mathbf{H})\right\} d\xi \geq \delta\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{H}\right|^2 d\xi, \tag{4.8}$$

where we have denoted

$$\begin{aligned} &\mathbf{M} \equiv \left[M_{[n_i,u,\theta;m_i]}(\xi), M_{[n_e,u,\theta;m_e]}(\xi)\right]^{\rm T}, \\ &\widehat{\mathbf{M}} \equiv \left[\widehat{M}_i, \widehat{M}_e\right] \equiv \left[M_{[\widehat{n}_i,\widehat{u},\widehat{\theta};m_i]}(\xi), M_{[\widehat{n}_e,\widehat{u},\widehat{\theta};m_e]}(\xi)\right]^{\rm T}, \\ &\mathcal{N}^\perp = \left\{\mathbf{H}(\xi): \quad \int_{\mathbb{R}^3}\psi_j(\xi)\cdot\mathbf{H}(\xi)d\xi = 0,\ j = 1, 2, \cdots, 6\right\}. \end{aligned}$$

Proof. We first write

$$-\int_{\mathbb{R}^3}\mathbf{H}\cdot\left\{\widehat{\mathbf{M}}^{-1}\mathbf{L}_{\mathbf{M}}\mathbf{H}\right\}d\xi = -\int_{\mathbb{R}^3}\mathbf{H}\cdot\left\{\widehat{\mathbf{M}}^{-1}\mathbf{L}_{\widehat{\mathbf{M}}}\mathbf{H}\right\}d\xi - \int_{\mathbb{R}^3}\mathbf{H}\cdot\left\{\widehat{\mathbf{M}}^{-1}\mathbf{L}_{\mathbf{M}-\widehat{\mathbf{M}}}\mathbf{H}\right\}d\xi. \tag{4.9}$$

In light of Lemma 4.2 , one has

$$-\int_{\mathbb{R}^3}\mathbf{H}\cdot\left\{\widehat{\mathbf{M}}^{-1}\mathbf{L}_{\widehat{\mathbf{M}}}\mathbf{H}\right\}d\xi\geq\delta(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})\int_{\mathbb{R}^3}(1+|\xi|\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{H}\right|^2d\xi. \tag{4.10}$$

For the second term on the right hand side of (4.9), noticing

$$\mathbf{L}_{\mathbf{M}-\widehat{\mathbf{M}}}\mathbf{H}=\begin{bmatrix}Q_{ii}(M_i-\widehat{M}_i,H_i)+Q_{ii}(H_i,M_i-\widehat{M}_i)+Q_{ie}(M_i-\widehat{M}_i,H_e)+Q_{ie}(H_i,M_e-\widehat{M}_e)\\ Q_{ee}(M_e-\widehat{M}_e,H_e)+Q_{ee}(H_e,M_e-\widehat{M}_e)+Q_{ei}(M_e-\widehat{M}_e,H_i)+Q_{ei}(H_e,M_i-\widehat{M}_i)\end{bmatrix},$$

it follows from Cauchy-Schwarz inequality and Lemma 4.1 that

$$\begin{aligned}&\left|-\int_{\mathbb{R}^3}\mathbf{H}\cdot\left\{\widehat{\mathbf{M}}^{-1}\mathbf{L}_{\mathbf{M}-\widehat{\mathbf{M}}}\mathbf{H}\right\}d\xi\right|\\&\leq\frac{\delta(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})}{4}\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{H}\right|^2d\xi\\&\quad+\frac{4}{\delta(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})}\int_{\mathbb{R}^3}(1+|\xi|)^{-1}\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{L}_{\mathbf{M}-\widehat{\mathbf{M}}}\mathbf{H}\right|^2d\xi\\&\leq\frac{\delta(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})}{4}\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{H}\right|^2d\xi\\&\quad+\frac{C_1}{\delta(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})}\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{H}\right|^2d\xi\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}(\mathbf{M}-\widehat{\mathbf{M}})\right|^2d\xi.\end{aligned} \tag{4.11}$$

To treat the integral

$$\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}(\mathbf{M}-\widehat{\mathbf{M}})\right|^2d\xi,$$

we use $\frac{\theta}{2}<\widehat{\theta}$ and choose a large positive constant $C_2=C_2(n_i,n_e,u,\theta;\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})$ such that

$$\begin{aligned}\int_{|\xi|\geq C_2}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}(\mathbf{M}-\widehat{\mathbf{M}})\right|^2d\xi\leq&C_3\int_{|\xi|\geq C_2}(1+|\xi|)\left(\frac{M_i^2+\widehat{M}_i^2}{\widehat{M}_i}+\frac{M_e^2+\widehat{M}_e^2}{\widehat{M}_e}\right)d\xi\\\leq&\frac{\delta^2(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})}{16C_1}.\end{aligned} \tag{4.12}$$

For the integral in the remaining domain, it follows that

$$\int_{|\xi|<C_2}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}(\mathbf{M}-\widehat{\mathbf{M}})\right|^2d\xi\leq\frac{C_4}{4C_1}\left(|n_i-\widehat{n}_i|+|n_e-\widehat{n}_e|+|u-\widehat{u}|+|\theta-\widehat{\theta}|\right)^2, \tag{4.13}$$

for some constant $C_4=C_4(n_i,n_e,u,\theta;\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})$. Finally, by letting

$$\eta_0=\frac{\delta(\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})}{2C_4(n_i,n_e,u,\theta;\widehat{n}_i,\widehat{n}_e,\widehat{u},\widehat{\theta})},$$

and inserting (4.10), (4.11), (4.12) and (4.13) into (4.9), one sees that (4.8) holds true. This completes the proof of Lemma 4.3. □

A direct consequence of Lemma 4.3 with the help of the Cauchy inequality is the following corollary, cf. [60].

Corollary 4.1. *Under the assumptions in Lemma 4.3, it holds that for* $\mathbf{H}(\xi)\in\mathcal{N}^\perp$,

$$\int_{\mathbb{R}^3}(1+|\xi|)\left|\widehat{\mathbf{M}}^{-1/2}\mathbf{L}_{\mathbf{M}}^{-1}\mathbf{H}\right|^2d\xi\leq\delta^{-2}\int_{\mathbb{R}^3}(1+|\xi|)^{-1}|\widehat{\mathbf{M}}^{-1/2}\mathbf{H}|^2(\xi)\,d\xi.$$

5. Proof of the main result

With preparations in the previous sections, we begin to give the proof of Theorem 1.1. For later use we first introduce some notations. Recall that $[n^r, u_1^r, \theta^r]$ is the smooth 3-family rarefaction wave to the quasineutral Euler system (3.2) with far-field data $[n_\pm, u_{1\pm}, \theta_\pm]$ connected by $[n_+, u_{1+}, \theta_+] \in R_3(n_-, u_{1-}, \theta_-)$. We define the local bi-Maxwellian:

$$\mathbf{M}_r = \begin{bmatrix} M_{ri} \\ M_{re} \end{bmatrix} = \begin{bmatrix} M_{[n_i^r(t,x),u^r(t,x),\theta^r(t,x);m_i]}(\xi) \\ M_{[n_e^r(t,x),u^r(t,x),\theta^r(t,x);m_e]}(\xi) \end{bmatrix}.$$

with $n_e^r(t,x) = n^r(t,x)$ and $n_i^r(t,x) = -\frac{q_e}{q_i} n_e^r(t,x) = -\frac{q_e}{q_i} n^r(t,x)$. In terms of (1.10) and (1.11), we also define the global bi-Maxwellian:

$$\mathbf{M}_* = \begin{bmatrix} M_{*i} \\ M_{*e} \end{bmatrix} = \begin{bmatrix} M_{[n_{*i},u_*,\theta_*;m_i]}(\xi) \\ M_{[n_{*e},u_*,\theta_*;m_e]}(\xi) \end{bmatrix}.$$

For a vector-valued function $\mathbf{H} = [H_i, H_e]^{\mathrm{T}}$, we write that

$$\mathbf{H} \in L^2_\xi(\frac{1}{\sqrt{\mathbf{M}_*}}), \quad \text{if } \frac{H_i}{\sqrt{M_{*i}}} \in L^2_\xi \text{ and } \frac{H_e}{\sqrt{M_{*e}}} \in L^2_\xi.$$

Now we define the function space in which we seek the solutions of the VPB system (1.1), (1.2). For given $T \in (0, +\infty]$, we set

$$\widetilde{\mathcal{E}}([0,T]) = \{\mathbf{H}(t,x,\xi) | \frac{\partial^\alpha \partial^\beta H_{\mathcal{A}}(t,x,\xi)}{\sqrt{M_{*\mathcal{A}}(\xi)}} \in C\left([0,T]; L^2_{x,\xi}(\mathbb{R}\times\mathbb{R}^3)\right) \\ \text{for } |\alpha|+|\beta| \le 2, \alpha_0 \le 1, \ \mathcal{A} = i, e\},$$

associated with the norm $\widetilde{\mathcal{E}}_T(\cdot)$ defined by

$$\widetilde{\mathcal{E}}_T(\mathbf{H}) \equiv \sum_{\substack{|\alpha|+|\beta|\le 2 \\ 0\le\alpha_0\le 1}} \left\{ \sup_{0\le t\le T} \int_{\mathbb{R}\times\mathbb{R}^3} \frac{\left|\partial^\alpha\partial^\beta H_i(t,x,\xi)\right|^2}{M_{*i}} d\xi dx + \sup_{0\le t\le T} \int_{\mathbb{R}\times\mathbb{R}^3} \frac{\left|\partial^\alpha\partial^\beta H_e(t,x,\xi)\right|^2}{M_{*e}} d\xi dx \right\}.$$

The proof of Theorem 1.1 is based on the energy estimates on both the fluid and non-fluid part of the solution $\mathbf{F}(t,x,\xi)$. We first consider the fluid part. Recall that the macro quantities $[n_i, n_e, u, \theta]$ of the fluid part $\mathbf{M}(t,x,\xi)$ satisfy the two-fluid Navier-Stokes-Poisson-type system (2.18) and (2.19), and the macro quantities $[n_i^r, n_e^r, u^r, \theta^r]$ of the corresponding smooth approximate profile $\mathbf{M}^r(t,x,\xi)$ satisfy (3.7). We now define the perturbation

$$\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right](t,x) = \left[n_i - n_i^r, n_e - n_e^r, u - [u_1^r, 0, 0], \theta - \theta^r\right](t,x).$$

Then one can deduce the perturbed equations for $\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right]$ through (2.18), (2.19), (2.13) and (3.7) in the following way. For number densities $\widetilde{n}_i$ and $\widetilde{n}_e$, one has

$$\partial_t \widetilde{n}_i + \partial_x \left(n_i u_1 - n_i^r u_1^r\right) = -\int_{\mathbb{R}^3} \xi_1 \partial_x G_i \, d\xi, \tag{5.1}$$

$$\partial_t \widetilde{n}_e + \partial_x \left(n_e u_1 - n_e^r u_1^r\right) = -\int_{\mathbb{R}^3} \xi_1 \partial_x G_e \, d\xi. \tag{5.2}$$

For the momentum $\widetilde{u} = [\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3]$, one has

$$\begin{aligned} &(m_i n_i + m_e n_e)(\partial_t \widetilde{u}_1 + u_1 \partial_x \widetilde{u}_1 + \widetilde{u}_1 \partial_x u_1^r) + \partial_x (P - P^r) \\ &\quad + \left(1 - \frac{m_i n_i + m_e n_e}{m_i n_i^r + m_e n_e^r}\right) \partial_x P^r + (q_i n_i + q_e n_e)\partial_x \phi \\ &= 3\partial_x \left((\mu_i(\theta) + \mu_e(\theta))\partial_x \widetilde{u}_1\right) + \partial_x \left((\mu_i(\theta) + \mu_e(\theta))\partial_x u_1^r\right) \\ &\quad - \int_{\mathbb{R}^3} \xi_1 \psi_3 \cdot \partial_x \overline{\mathbf{R}} \, d\xi - \int_{\mathbb{R}^3} \psi_{3i} \xi_1 \partial_x \left(P_0^{M_i} G_i\right) d\xi - \int_{\mathbb{R}^3} \psi_{3e} \xi_1 \partial_x \left(P_0^{M_e} G_e\right) d\xi, \end{aligned} \tag{5.3}$$

and

$$(m_in_i+m_en_e)(\partial_t\widetilde{u}_j+u_1\partial_x\widetilde{u}_j)=\partial_x\left((\mu_i(\theta)+\mu_e(\theta))\partial_x\widetilde{u}_j\right)-\int_{\mathbb{R}^3}\xi_1\psi_{j+2}\cdot\partial_x\overline{\mathbf{R}}\,d\xi$$
$$-\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi-\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi,\quad j=2,3. \tag{5.4}$$

For the equation of temperature $\widetilde{\theta}$, one has

$$\begin{aligned}
&(n_i+n_e)(\partial_t\widetilde{\theta}+u_1\partial_x\widetilde{\theta}+\widetilde{u}_1\partial_x\theta^r)+P\partial_xu_1-P^r\partial_xu^r+\left(1-\frac{n_i+n_e}{n_i^r+n_e^r}\right)P^r\partial_xu^r\\
&=\partial_x\left((\kappa_i(\theta)+\kappa_e(\theta))\partial_x\widetilde{\theta}\right)+\partial_x\left((\kappa_i(\theta)+\kappa_e(\theta))\partial_x\theta^r\right)\\
&\quad+3(\mu_i(\theta)+\mu_e(\theta))(\partial_xu_1)^2+\sum_{j=2}^{3}(\mu_i(\theta)+\mu_e(\theta))(\partial_xu_j)^2\\
&\quad-\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum_{j=1}^{3}u_j\psi_{j+2}\right)\cdot\partial_x\overline{\mathbf{R}}d\xi-\int_{\mathbb{R}^3}\psi_{6i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi-\int_{\mathbb{R}^3}\psi_{6e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi\\
&\quad+\sum_{j=1}^{3}u_j\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi+\sum_{j=1}^{3}u_j\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi\\
&\quad+\theta\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi+\partial_x\phi\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi.
\end{aligned}\tag{5.5}$$

Note that ϕ is coupled to the Poisson equation

$$-\partial_x^2\phi=q_in_i+q_en_e=q_i\widetilde{n}_i+q_e\widetilde{n}_e. \tag{5.6}$$

The above reformulated Cauchy problem on $\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]$ is supplemented with initial data

$$\begin{aligned}
\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](0,x)&=\left[\widetilde{n}_{i0},\widetilde{n}_{e0},\widetilde{u}_0,\widetilde{\theta}_0\right](x)\\
&=\left[n_{i0}(x)-n_{i0}^r(x),n_{e0}(x)-n_{e0}^r(x),u_0(x)-[u_{1,0}^r(x),0,0],\theta_0(x)-\theta_0^r(x)\right].
\end{aligned}\tag{5.7}$$

Here we recall that $\overline{\mathbf{R}}$ is defined in (2.17). We also note that $\widetilde{u}_j=u_j$ for $j=2,3$. As $\phi_+=\phi_-$, we further assume $\phi_+=0=\phi_-$ without loss of generality and let $\phi(t,x)$ be determined by the elliptic equation (5.6) under the boundary condition that $\phi(t,x)\to0$ as $x\to\pm\infty$.

For the non-fluid part $\mathbf{G}(t,x,\xi)$, as in [78], we note that

$$\left\|\frac{G_{\mathcal{A}}}{\sqrt{M_{\mathcal{A}}}}\right\|_{L^2_{x,\xi}}^2$$

is not integrable with respect to the time variable, and hence it is necessary to consider the following perturbation

$$\widetilde{\mathbf{G}}=\left[\widetilde{G}_i,\widetilde{G}_e\right]^{\mathrm{T}}=\left[G_i-\overline{G}_i,G_e-\overline{G}_e\right]^{\mathrm{T}},$$

where

$$
\begin{aligned}
\overline{\mathbf{G}} &= \left[\overline{G}_i, \overline{G}_e\right]^{\mathrm{T}} \\
&= \frac{3}{2\theta}\mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left[\left[m_iM_i, m_eM_e\right]^{\mathrm{T}}\xi_1\left(\xi_1\partial_x u_1^r + \frac{|\xi-u|^2}{2\theta}\partial_x\theta^r\right)\right]\right\} \\
&\quad + \mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left[\left[n_i^{-1}M_i\partial_x n_i^r, n_e^{-1}M_e\partial_x n_e^r\right]^{\mathrm{T}}\xi_1\right]\right\} - \frac{3}{2\theta}\mathbf{L}_{\mathbf{M}}^{-1}\left\{\mathbf{P}_1^{\mathbf{M}}\left[\left[M_i, M_e\right]^{\mathrm{T}}\xi_1\partial_x\theta^r\right]\right\} \\
&= \frac{3}{2\theta}\partial_x u_1^r\mathbf{L}_{\mathbf{M}}^{-1}\left\{\left[m_iM_i, m_eM_e\right]^{\mathrm{T}}|\xi_1-u_1|^2 - \frac{1}{3}(|\xi-u|^2-3)\left[M_i, M_e\right]^{\mathrm{T}}\right\} \\
&\quad + \frac{3}{2\theta}\partial_x\theta^r\mathbf{L}_{\mathbf{M}}^{-1}\left\{(\xi_1-u_1)\left[\left[m_iM_i, m_eM_e\right]^{\mathrm{T}}\left(\frac{|\xi-u|^2}{2\theta} - \frac{5}{3}\frac{n_i+n_e}{m_in_i+m_en_e}\right)\right]\right\} \\
&\quad + \mathbf{L}_{\mathbf{M}}^{-1}\left\{(\xi_1-u_1)\left[\left[n_i^{-1}M_i\partial_x n_i^r, n_e^{-1}M_e\partial_x n_e^r\right]^{\mathrm{T}} - \left[m_iM_i, m_eM_e\right]^{\mathrm{T}}\frac{\partial_x(n_i^r+n_e^r)}{m_in_i+m_en_e}\right]\right\} \\
&\quad - \frac{3}{2\theta}\partial_x\theta^r\mathbf{L}_{\mathbf{M}}^{-1}\left\{(\xi_1-u_1)\left[\left[M_i, M_e\right]^{\mathrm{T}} - \left[m_iM_i, m_eM_e\right]^{\mathrm{T}}\frac{n_i+n_e}{m_in_i+m_en_e}\right]\right\}.
\end{aligned}
\tag{5.8}
$$

Now, to prove Theorem 1.1, the key point is to deduce the a priori energy estimates on the macroscopic part $\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}, \phi\right]$ and the microscopic part $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ based on the following a priori assumption:

$$
N^2(T) + \delta_r \le \epsilon_0^2, \tag{5.9}
$$

for an arbitrary positive time $T > 0$, where δ_r is the wave strength of the rarefaction wave given in (1.12), and $N(T)$ is defined by

$$
\begin{aligned}
N^2(T) := &\sup_{0\le t\le T}\left\|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right](t,x)\right\|^2 + \sup_{0\le t\le T}\sum_{|\alpha|\le 1}\left\|\partial^\alpha\partial_x[n_i, n_e, u, \theta](t,x)\right\|^2 \\
&+ \sup_{0\le t\le T}\sum_{\substack{1\le|\alpha|\le 2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{F}(t,x,\xi)\right|^2 d\xi dx + \sup_{0\le t\le T}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}(t,x,\xi)\right|^2 d\xi dx \\
&+ \sup_{0\le t\le T}\sum_{\substack{|\alpha|+|\beta|\le 2\\ |\beta|\ge 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial^\beta\widetilde{\mathbf{G}}(t,x,\xi)\right|^2 d\xi dx + \sup_{0\le t\le T}\sum_{|\alpha|\le 1}\left\|\partial^\alpha\partial_x\phi\right\|_{H^1}^2.
\end{aligned}
$$

Here we first claim that the a priori bound of $N(T)$ immediately yields

$$
\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\left\|\partial^\alpha[n_i, n_e, u, \theta]\right\|^2 + \sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{G}(t,x,\xi)\right|^2 d\xi dx \le C\epsilon_0^2, \tag{5.10}
$$

for a generic constant $C > 0$. Indeed, due to the decomposition $\mathbf{F} = \mathbf{M} + \mathbf{G}$, one may notice

$$
\begin{aligned}
&\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{M}\right|^2 d\xi dx + \sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}\right|^2 d\xi dx \\
&\qquad \le 2\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\left|\int_{\mathbb{R}\times\mathbb{R}^3}(\mathbf{M}^{-1}\partial^\alpha\mathbf{M})\cdot\partial^\alpha\mathbf{G}\, d\xi dx\right| + \sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{F}\right|^2 d\xi dx,
\end{aligned}
$$

where the right-hand first term is further bounded by

$$
C\sup_{0\le t\le T}\sum_{|\alpha'|=1,|\alpha''|=1}\left(\int_{\mathbb{R}}\left|\partial^{\alpha'}[u,\theta]\right|^2\left|\partial^{\alpha''}[u,\theta]\right|^2 dx\right)^{1/2}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\left(\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}(t,x,\xi)|^2 d\xi dx\right)^{1/2}.
$$

It then follows that

$$\begin{aligned}
&\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{M}\right|^2 d\xi dx+\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}\right|^2 d\xi dx\\
&\le C\epsilon_0\sup_{0\le t\le T}\sum_{\substack{1\le|\alpha|\le 2\\ \alpha_0\le 1}}\|\partial^\alpha[u,\theta]\|^2+C\epsilon_0\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2 d\xi dx\\
&\quad+\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{F}\right|^2 d\xi dx.
\end{aligned}\tag{5.11}$$

Moreover,

$$\begin{aligned}
&\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\|\partial^\alpha[n_i,n_e,u,\theta]\|^2\\
&\le C\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{M}(t,x,\xi)\right|^2 d\xi dx+C\sup_{0\le t\le T}\sum_{|\alpha'|=1}\left\|(\partial^{\alpha'}[n_i,n_e,u,\theta])^2\right\|^2\\
&\le C\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{M}(t,x,\xi)\right|^2 d\xi dx+C\epsilon_0^2\sup_{0\le t\le T}\sum_{\substack{1\le|\alpha|\le 2\\ \alpha_0\le 1}}\|\partial^\alpha[n_i,n_e,u,\theta]\|^2.
\end{aligned}\tag{5.12}$$

Therefore (5.12) together with (5.11) give (5.10). In addition, one can also see that the following a priori bound holds true:

$$\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\left\|\partial^\alpha\partial_x^2\phi(t)\right\|\le C\sup_{0\le t\le T}\sum_{\substack{|\alpha|=2\\ \alpha_0\le 1}}\|\partial^\alpha(q_in_i+q_en_e)\|^2\le C\epsilon_0^2.$$

This directly follows from the Poisson equation (5.6) as well as (5.9) and (5.10).

The a priori energy estimates under the assumption (5.9) are divided into two steps. The first step is concerned with the estimates on $\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\partial_x\phi\right](t,x)$ basing on equations (5.1), (5.2), (5.3), (5.4), (5.5), and (5.6).

Proposition 5.1. *Assume that all the conditions in Theorem 1.1 hold, and $\mathbf{F}\in\widetilde{\mathcal{E}}([0,T])$ for $T>0$. Let $\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\phi\right](t,x)$ be a smooth solution to the Cauchy problem (5.1), (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7) on $0\le t\le T$ and satisfy (5.9). Then the following energy estimate holds:*

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\frac{d}{dt}\|\partial_x\phi(t)\|^2+\kappa_0\frac{d}{dt}\sum_{|\alpha|=1}(\partial^\alpha\widetilde{u}_1,\partial_x\partial^\alpha(\widetilde{n}_i+\widetilde{n}_e))\\
&\quad+\lambda\left\|\sqrt{\partial_x u^r}\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\lambda\sum_{1\le|\alpha|\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\lambda\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\|^2\\
&\quad+\lambda\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2+\lambda\sum_{|\alpha|=2}\left\|\partial^\alpha\partial_x^2\phi\right\|^2\\
&\lesssim(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\delta_r^{1/6}(1+t)^{-6/7}+\sum_{1\le|\alpha|\le 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}\right|^2 d\xi dx\\
&\quad+\epsilon_0\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial_{\xi_1}\partial^\alpha\widetilde{\mathbf{G}}\right|^2 d\xi dx+\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx,
\end{aligned}\tag{5.13}$$

for all $0\le t\le T$, where κ_0 is a small positive constant. Here and in the sequel we also use $\mathbf{M}_\sharp$ to denote $\mathbf{M}_*$ or $\mathbf{M}$ for brevity, whenever there is no confusion.

The second step is to deduce the energy estimates on the microscopic part $\mathbf{G}$. Our method for that is much similar to the one of [26].

Proposition 5.2. *Under the conditions listed in Proposition 5.1, it holds that*

$$\begin{aligned}
&\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\sum_{\substack{|\alpha|\le 2\\ \alpha_0\le 1}}\|\partial^\alpha\partial_x\phi(t)\|^2+\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&+\sum_{\substack{1\le|\alpha|\le 2\\ \alpha_0\le 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{F}\right|^2d\xi dx+\sum_{\substack{|\alpha|+|\beta|\le 2\\ |\beta|\ge 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial^\beta\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&+\sum_{1\le|\alpha|\le 2}\int_0^T\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{G}\right|^2d\xi dxdt+\int_0^T\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dxdt\\
&+\sum_{\substack{|\alpha|+|\beta|\le 2\\ |\beta|\ge 1}}\int_0^T\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial^\beta\widetilde{\mathbf{G}}\right|^2d\xi dxdt+\int_0^T\int_{\mathbb{R}}\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]^2\partial_x u_1^r\,dxdt\\
&+\sum_{1\le|\alpha|\le 2}\int_0^T\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2dt+\int_0^T\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\|^2dt+\sum_{|\alpha|\le 1}\int_0^T\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2dt\\
&\le C_0N^2(0)+C_0\delta_r^{1/6},
\end{aligned}\tag{5.14}$$

for all $0\le t\le T$.

The proof of Proposition 5.1 and Proposition 5.2 will be given in Section 6 and Section 7, respectively. Assuming these two propositions, we are now in a position to complete

The proof of Theorem 1.1. The local existence of the solution $\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\phi,\mathbf{G}\right]$ of the system (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) can be obtained by the standard iteration method, cf. [26], and its proof is omitted for brevity.

The existence of the solution of (1.1), (1.4), (1.5) and (1.6) then follows from the standard continuation argument based on the local existence and the a priori estimate in Proposition 5.2. Moreover, one sees that

$$\begin{aligned}
&\sup_{t\ge 0}\sum_{\substack{|\alpha|+|\beta|\le 2\\ \alpha_0\le 1}}\left\|\partial^\alpha\partial^\beta\left(F_i(t,x,\xi)-\mathbf{M}_{[-\frac{q_e}{q_i}n^r,u^r,\theta^r](t,x)}(\xi)\right)\right\|_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{M_{*i}(\xi)}}\right)\right)}\\
&\quad+\sup_{t\ge 0}\sum_{\substack{|\alpha|+|\beta|\le 2\\ \alpha_0\le 1}}\left\|\partial^\alpha\partial^\beta\left(F_e(t,x,\xi)-\mathbf{M}_{[n^r,u^r,\theta^r](t,x)}(\xi)\right)\right\|_{L^2_x\left(L^2_\xi\left(\frac{1}{\sqrt{M_{*e}(\xi)}}\right)\right)}+\sup_{t\ge 0}\sum_{|\alpha|\le 1}\|\partial^\alpha\partial_x\phi(t,x)\|^2_{H^1}\\
&\lesssim\sup_{t\ge 0}\sum_{|\alpha|\le 2,\alpha_0\le 1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t,x)\right\|^2+\sup_{t\ge 0}\sum_{\substack{1\le|\alpha|\le 2\\ \alpha_0\le 1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{F}(t,x,\xi)\right|^2d\xi dx\\
&\quad+\sup_{t\ge 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}(t,x,\xi)\right|^2d\xi dx+\sup_{t\ge 0}\sum_{\substack{|\alpha|+|\beta|\le 2\\ |\beta|\ge 1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial^\beta\widetilde{\mathbf{G}}(t,x,\xi)\right|^2d\xi dx\\
&\quad+\delta_r^{1/6}.
\end{aligned}\tag{5.15}$$

Then (1.13) follows from (5.15) provided that $\delta\ll\epsilon_0$.

In order to obtain the large time behavior of solutions as in (1.14), one sees

$$\begin{aligned}
&\frac{d}{dt}\left\|\frac{\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right)}{\sqrt{M_{*\mathcal{A}}}}\right\|^2_{L^2_{x,\xi}}\\
&=2\left(M_{*\mathcal{A}}^{-1}\partial_t\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right),\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right)\right),
\end{aligned}$$

and thus it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\int_0^{+\infty}\left|\frac{d}{dt}\left\|\frac{\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right)}{\sqrt{M_{*\mathcal{A}}}}\right\|^2_{L^2_{x,\xi}}\right|dt\\
&\le\int_0^{+\infty}\left\|M_{*\mathcal{A}}^{-1/2}\partial_t\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right)\right\|^2_{L^2_{x,\xi}}dt\\
&\quad+\int_0^{+\infty}\left\|M_{*\mathcal{A}}^{-1/2}\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right)\right\|^2_{L^2_{x,\xi}}dt\\
&\le C\sum_{\substack{1\le|\alpha|\le2\\|\alpha_0|\le1}}\int_0^{+\infty}\left\|\partial^\alpha\left[\widetilde{n}_{\mathcal{A}},\widetilde{u},\widetilde{\theta}\right]\right\|^2dt+C\sum_{\substack{1\le|\alpha|\le2\\|\alpha_0|\le1}}\int_0^{+\infty}\left\|\frac{\partial^\alpha G_{\mathcal{A}}}{\sqrt{M_{*\mathcal{A}}}}\right\|^2_{L^2_{x,\xi}}dt\\
&<+\infty.
\end{aligned}\tag{5.16}$$

From (5.15) and (5.16), one sees that

$$\lim_{t\to+\infty}\left\|\frac{\partial_x\left(F_{\mathcal{A}}(t,x,\xi)-M_{[n^r_{\mathcal{A}},u^r,\theta^r;m_{\mathcal{A}}]}(\xi)\right)}{\sqrt{M_{*\mathcal{A}}}}\right\|^2_{L^2_{x,\xi}}=0.\tag{5.17}$$

Then (1.14) follows from Sobolev's inequality, (5.17) and Lemma 3.1. This completes the proof of Theorem 1.1. □

6. A priori estimates on the fluid part

This section is devoted to the proof of Proposition 5.1 on the energy estimates of the fluid part $\mathbf{M}(t,x,\xi)$. The proof is divided by three subsections.

6.1. **Estimate on zero-order energy.** We set $\Phi(y)=y-1-\ln y$, and define

$$\widetilde{\eta}=\frac{m_in_i+m_en_e}{2}\sum_{j=1}^3\widetilde{u}_j^2+\frac{2}{3}n_i\theta^r\Phi\left(\frac{n_i^r}{n_i}\right)+\frac{2}{3}n_e\theta^r\Phi\left(\frac{n_e^r}{n_e}\right)+(n_i+n_e)\theta^r\Phi\left(\frac{\theta}{\theta^r}\right).$$

By using (2.20), (2.21), (2.22), (3.7), (5.3), (5.4) and (5.5), direct computations give

$$\begin{aligned}
&\partial_t\widetilde{\eta}+3(\mu_i(\theta)+\mu_e(\theta))(\partial_x\widetilde{u}_1)^2+\sum_{j=2}^3(\mu_i(\theta)+\mu_e(\theta))(\partial_x\widetilde{u}_j)^2+\frac{\kappa_i(\theta)+\kappa_e(\theta)}{\theta}\left(\partial_x\widetilde{\theta}\right)^2\\
&\quad+\partial_x\mathcal{M}+\mathcal{N}_1=\sum_{l=2}^8\mathcal{N}_l,
\end{aligned}\tag{6.1}$$

where

$$\mathcal{M}=u_1\widetilde{\eta}+(P-P^r)\widetilde{u}_1-\sum_{j=1}^3\mu_j(\theta)\widetilde{u}_j\partial_x\widetilde{u}_j-\kappa(\theta)\frac{\partial_x\widetilde{\theta}\,\widetilde{\theta}}{\theta},$$

$$\begin{aligned}
\mathcal{N}_1=&\partial_xu_1^r\left[(m_in_i+m_en_e)\widetilde{u}_1^2+\frac{4}{9}\theta^r\left(n_i\Phi\left(\frac{n_i^r}{n_i}\right)+n_e\Phi\left(\frac{n_e^r}{n_e}\right)\right)+\frac{2}{3}(n_i+n_e)\theta^r\Phi\left(\frac{\theta}{\theta^r}\right)\right]\\
&+(n_i+n_e)\frac{\partial_x\theta^r}{\theta^r}\widetilde{u}_1\widetilde{\theta}-\frac{2}{3}\partial_x\theta^r\frac{m_i\widetilde{n}_i+m_e\widetilde{n}_e}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r)\widetilde{u}_1,
\end{aligned}$$

$$\mathcal{N}_2 =\frac{2}{3}\partial_x\theta^r\widetilde{u}_1\left[n_i\Phi\left(\frac{n_i^r}{n_i}\right)+n_e\Phi\left(\frac{n_e^r}{n_e}\right)\right]+(n_i+n_e)\partial_x\theta^r\widetilde{u}_1\Phi\left(\frac{\theta}{\theta^r}\right),$$

$$+\frac{2}{3}\partial_x n_i^r\frac{m_e(n_e^r\widetilde{n}_i-n_i^r\widetilde{n}_e)}{n_i^r(m_in_i^r+m_en_e^r)}\widetilde{u}_1+\frac{2}{3}\partial_x n_e^r\frac{m_i(n_i^r\widetilde{n}_e-n_e^r\widetilde{n}_i)}{n_e^r(m_in_i^r+m_en_e^r)}\widetilde{u}_1,$$

$$\mathcal{N}_3 =\partial_x((\mu_i(\theta)+\mu_e(\theta))\partial_x u_1^r)\widetilde{u}_1+\sum_{j=1}^3(\mu_i(\theta)+\mu_e(\theta))(\partial_x u_j)^2\frac{\widetilde{\theta}}{\theta}$$
$$+(\kappa_i(\theta)+\kappa_e(\theta))\frac{\widetilde{\theta}(\partial_x\widetilde{\theta})^2}{\theta^2}+(\kappa_i(\theta)+\kappa_e(\theta))\frac{\widetilde{\theta}\partial_x\widetilde{\theta}\partial_x\theta^r}{\theta^2}+\partial_x\left((\kappa_i(\theta)+\kappa_e(\theta))\partial_x\theta^r\right)\frac{\widetilde{\theta}}{\theta},$$

$$\mathcal{N}_4=-\widetilde{u}_1(q_in_i+q_en_e)\partial_x\phi,$$

$$\mathcal{N}_5 =\frac{2}{3}\theta^r\ln\left(\frac{n_i^r}{n_i}\right)\int_{\mathbb{R}^3}\xi_1\partial_xG_i\,d\xi+\frac{2}{3}\theta^r\ln\left(\frac{n_e^r}{n_e}\right)\int_{\mathbb{R}^3}\xi_1\partial_xG_e\,d\xi+\widetilde{\theta}\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi$$
$$-\sum_{j=1}^3\widetilde{u}_j\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi-\sum_{j=1}^3\widetilde{u}_j\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi$$
$$-\frac{\widetilde{\theta}}{\theta}\int_{\mathbb{R}^3}\psi_{6i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi-\frac{\widetilde{\theta}}{\theta}\int_{\mathbb{R}^3}\psi_{6e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi$$
$$+\sum_{j=1}^3\frac{u_j\widetilde{\theta}}{\theta}\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi+\sum_{j=1}^3\frac{u_j\widetilde{\theta}}{\theta}\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi,$$

$$\mathcal{N}_6=-\sum_{j=1}^3\widetilde{u}_j\int_{\mathbb{R}^3}\xi_1\psi_{j+2}\cdot\partial_x\overline{\mathbf{R}}\,d\xi-\frac{\widetilde{\theta}}{\theta}\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum_{j=1}^3u_j\psi_{j+2}\right)\cdot\partial_x\overline{\mathbf{R}}\,d\xi,$$

$$\mathcal{N}_7=\frac{\widetilde{\theta}}{\theta}\partial_x\phi\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi.$$

We now get by integrating (6.1) respect to x over $\mathbb{R}$ that

$$\frac{d}{dt}\int_{\mathbb{R}}\widetilde{\eta}\,dx+3\int_{\mathbb{R}}(\mu_i(\theta)+\mu_e(\theta))(\partial_x\widetilde{u}_1)^2\,dx+\sum_{j=2}^3\int_{\mathbb{R}}(\mu_i(\theta)+\mu_e(\theta))(\partial_x\widetilde{u}_j)^2\,dx$$
$$+\int_{\mathbb{R}}\frac{\kappa_i(\theta)+\kappa_e(\theta)}{\theta}\left(\partial_x\widetilde{\theta}\right)^2dx+\int_{\mathbb{R}}\mathcal{N}_1\,dx=\sum_{l=2}^7\int_{\mathbb{R}}\mathcal{N}_l\,dx.$$

To compute $\int_{\mathbb{R}}\mathcal{N}_1dx$, we first note that

$$\Phi\left(\frac{n_{\mathcal{A}}^r}{n_{\mathcal{A}}}\right)=\frac{\widetilde{n}_{\mathcal{A}}^2}{n_{\mathcal{A}}^2}+O(1)\widetilde{n}_{\mathcal{A}}^3,\ \mathcal{A}=i,e,\quad \Phi\left(\frac{\theta}{\theta^r}\right)=\frac{\widetilde{\theta}^2}{(\theta^r)^2}+O(1)\widetilde{\theta}^3,\tag{6.2}$$

and from (3.5), it follows that

$$\partial_x\theta^r=\frac{2A}{3B}(n^r)^{1/3}\partial_xu_1^r,\quad B:=\sqrt{\frac{10A}{9}\frac{q_i-q_e}{m_eq_i-m_iq_e}}.\tag{6.3}$$

With (6.2) and (6.3) in hand, we now write $\mathcal{N}_1$ as

$$\begin{aligned}
\mathcal{N}_1 =&\partial_x u_1^r\left[(m_in_i^r+m_en_e^r)\widetilde{u}_1^2+\frac{4}{9}\theta^r\left(\frac{1}{n_i^r}\widetilde{n}_i^2+\frac{1}{n_e^r}\widetilde{n}_e^2\right)+\frac{2}{3}\frac{(n_i^r+n_e^r)}{\theta^r}\widetilde{\theta}^2\right]\\
&+\frac{2}{3B}(n_i^r+n_e^r)(n^r)^{-1/3}\widetilde{u}_1\widetilde{\theta}\partial_x u_1^r-\frac{4A}{9B}(n^r)^{1/3}\widetilde{u}_1\partial_x u_1^r\frac{q_i-q_e}{m_eq_i-m_iq_e}(m_i\widetilde{n}_i+m_e\widetilde{n}_e)\\
&+\partial_x u_1^r\left[(m_i\widetilde{n}_i+m_e\widetilde{n}_e)\widetilde{u}_1^2-\frac{4}{9}\theta^r\left(\frac{\widetilde{n}_i^3}{n_in_i^r}+\frac{\widetilde{n}_e^3}{n_en_e^r}\right)+\frac{2}{3}\frac{(\widetilde{n}_i+\widetilde{n}_e)}{\theta^r}\widetilde{\theta}^2\right]\\
&+(\widetilde{n}_i+\widetilde{n}_e)\frac{\partial_x\theta^r}{\theta^r}\widetilde{u}_1\widetilde{\theta}+\partial_x u_1^r\left[O(1)\widetilde{n}_i^3+O(1)\widetilde{n}_e^3+O(1)\widetilde{\theta}^3\right]\\
=&\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\mathbb{M}\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]^{\mathrm{T}}\\
&+\partial_x u_1^r\left[(m_i\widetilde{n}_i+m_e\widetilde{n}_e)\widetilde{u}_1^2+\frac{4}{9}\theta^r\left(\frac{-\widetilde{n}_i^3}{n_in_i^r}+\frac{-\widetilde{n}_e^3}{n_en_e^r}\right)+\frac{2}{3}\frac{(\widetilde{n}_i+\widetilde{n}_e)}{\theta^r}\widetilde{\theta}^2\right]\\
&+(\widetilde{n}_i+\widetilde{n}_e)\frac{\partial_x\theta^r}{\theta^r}\widetilde{u}_1\widetilde{\theta}+\partial_x u_1^r\left[O(1)\widetilde{n}_i^3+O(1)\widetilde{n}_e^3+O(1)\widetilde{\theta}^3\right],
\end{aligned}$$

where $\mathbb{M}$ is a 4×4 symmetric matrix given by

$$\begin{bmatrix}
-\frac{4}{9}A\frac{q_i}{q_e}(n^r)^{-\frac{1}{3}} & 0 & -\frac{2A}{9B}\frac{m_i(q_i-q_e)}{m_eq_i-m_iq_e}(n^r)^{\frac{1}{3}} & 0\\
0 & \frac{4}{9}A(n^r)^{-\frac{1}{3}} & -\frac{2A}{9B}\frac{m_e(q_i-q_e)}{m_eq_i-m_iq_e}(n^r)^{\frac{1}{3}} & 0\\
* & * & \frac{m_eq_i-m_iq_e}{q_i}n^r & \frac{q_i-q_e}{3q_iB}(n^r)^{\frac{2}{3}}\\
0 & 0 & * & \frac{2(q_i-q_e)}{3q_iA}(n^r)^{\frac{1}{3}}
\end{bmatrix}. \tag{6.4}$$

We claim that $\mathbb{M}$ is positive-definite provided that $\frac{q_i}{|q_e|}\le 9$ and $m_i\ge m_e$. To see this, we compute its four leading principal minors as follows:

$$\begin{aligned}
&\Delta_{11}>0,\\
&\Delta_{22}=-\frac{16}{81}A^2\frac{q_i}{q_e}(n^r)^{-\frac{2}{3}}>0,\\
&\Delta_{33}=-\frac{16}{81q_e}A^2(n^r)^{\frac{1}{3}}(m_eq_i-m_iq_e)+\frac{16}{810q_e}A^2(n^r)^{\frac{1}{3}}\frac{(q_im_e^2-q_em_i^2)}{m_eq_i-m_iq_e}(q_i-q_e)\\
&\quad=-\frac{A^2}{q_e}\frac{(n^r)^{1/3}}{m_eq_i-m_iq_e}\left[\frac{16\times 9}{810}(m_e^2q_i^2+m_i^2q_e^2)-\frac{32}{81}m_im_eq_iq_e+\frac{16}{810}(m_i^2+m_e^2)q_iq_e\right],\\
&\Delta_{44}=\frac{16}{810q_iq_e}A(n^r)^{\frac{2}{3}}(q_i-q_e)(m_eq_i-m_iq_e)-\frac{2(q_i-q_e)}{3q_iA}(n^r)^{\frac{1}{3}}\Delta_{33}\\
&\quad=-\frac{q_i-q_e}{q_iq_e}\frac{A(n^r)^{2/3}}{m_eq_i-m_iq_e}\left[\frac{16\times 5}{810}(m_e^2q_i^2+m_i^2q_e^2)-\frac{32}{81}\times(\frac{2}{3}-\frac{1}{10})m_im_eq_iq_e\right.\\
&\qquad\qquad\left.+\frac{16}{810}\times\frac{2}{3}(m_i^2+m_e^2)q_iq_e\right].
\end{aligned}$$

One sees that whenever one has $\frac{q_i}{|q_e|}\le 9$ and $m_i\ge m_e$, it holds that $\Delta_{33}>0$ and $\Delta_{44}>0$ and hence $\mathbb{M}$ is positive-definite. As $\mathbb{M}$ is positive definite, we immediately get from (5.9) that there exists $\lambda>0$ such that

$$\int_{\mathbb{R}}\mathcal{N}_1\,dx\ge\lambda\int_{\mathbb{R}}\partial_x u_1^r\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]^2dx,$$

where we have also used the Cauchy-Schwarz inequality and Sobolev's inequality

$$\|h\|_{L^\infty} \le \sqrt{2}\|h\|^{1/2}\|\partial_x h\|^{1/2}, \text{ for } h \in H^1(\mathbb{R}). \tag{6.5}$$

Next, by applying (6.5) the a priori assumption (5.9) and Lemma 3.1, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{N}_2\, dx\right| \lesssim& \epsilon_0 \int_{\mathbb{R}} \partial_x u_1^r \left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}_1, \widetilde{\theta}\right]^2 dx + \delta_r^{1/2}\|q_i\widetilde{n}_i + q_e\widetilde{n}_e\|^2 + \delta_r^{-1/2}\|\partial_x n^r\|_{L^\infty}^2\|\widetilde{u}_1\|^2 \\ \lesssim& \epsilon_0 \int_{\mathbb{R}} \partial_x u_1^r \left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}_1, \widetilde{\theta}\right]^2 dx + \delta_r^{1/2}\|q_i\widetilde{n}_i + q_e\widetilde{n}_e\|^2 + (1+t)^{-3/2}\|\widetilde{u}_1\|^2.\end{aligned}$$

Likewise, one can see that

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{N}_4\, dx\right| \lesssim& \int_{\mathbb{R}} |q_i\widetilde{n}_i + q_e\widetilde{n}_e|\, |\partial_x\phi|\, |\widetilde{u}_1| dx \\ \lesssim& \epsilon_0 \|\partial_x\phi\|^2 + (\epsilon_0+\eta)\|q_i\widetilde{n}_i + q_e\widetilde{n}_e\|^2 + C_\eta(1+t)^{-2}\|\widetilde{u}_1\|^2.\end{aligned}$$

Utilizing Lemma 3.1, Cauchy-Schwarz inequality with $\eta > 0$ and Sobolev's inequality (6.5) repeatedly, we compute

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{N}_3\, dx\right| \lesssim& \int_{\mathbb{R}} \left|\left[\partial_x[u_1^r, \theta^r]\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{\theta}, n^r, \theta^r\right]\right]\left[\widetilde{u}_1, \widetilde{\theta}\right]\right| dx \\ &+ \int_{\mathbb{R}} \left|\left[\partial_x^2 u_1^r, \partial_x^2\theta^r\right]\left[\widetilde{u}_1, \widetilde{\theta}\right]\right| dx + \int_{\mathbb{R}} \left|\left[\partial_x\widetilde{u}, \partial_x\widetilde{\theta}\right]^2\left[\widetilde{u}_1, \widetilde{\theta}\right]\right| dx \\ \lesssim& (\epsilon_0 + \delta_r^{1/2})\left\|\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}_1, \widetilde{\theta}\right]\right\|^{1/2} + (1+t)^{-3/2}\left\|\left[\widetilde{u}_1, \widetilde{\theta}\right]\right\|^2 \\ &+ \left\|\left[\widetilde{u}_1, \widetilde{\theta}\right]\right\|^{1/2}\left\|\partial_x\left[\widetilde{u}_1, \widetilde{\theta}\right]\right\|^{1/2}\left\{\left\|\partial_x^2[u_1^r, \theta^r]\right\|_{L^1} + \|\partial_x[u_1^r, \theta^r]\partial_x[n^r, \theta^r]\|_{L^1}\right\} \\ \lesssim& (\epsilon_0 + \delta_r^{1/2})\left\|\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}_1, \widetilde{\theta}\right]\right\|^{1/2} + (1+t)^{-3/2}\left\|\left[\widetilde{u}_1, \widetilde{\theta}\right]\right\|^2 + \delta_r^{1/6}(1+t)^{-7/6}.\end{aligned}$$

We now turn to estimate the terms involving $\mathcal{N}_5$, $\mathcal{N}_6$ and $\mathcal{N}_7$. Let us first consider $\int_{\mathbb{R}} \mathcal{N}_5\, dx$ and $\int_{\mathbb{R}} \mathcal{N}_7\, dx$. Recalling $\mathbf{G} = \widetilde{\mathbf{G}} + \overline{\mathbf{G}}$ with $\overline{\mathbf{G}}$ given by (5.8), we get from integration by parts, Cauchy-Schwarz inequality and Lemma 3.1 that

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{N}_5\, dx\right| \le& (\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{\theta}\right]\right\|^2 + \int_{\mathbb{R}} \left|\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right]\right| |\partial_x[n^r, u_1^r, \theta^r]| \left|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{\theta}\right]\right| dx \\ &+ \int_{\mathbb{R}} |\partial_x[n^r, u_1^r, \theta^r]|^2 \left|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{\theta}\right]\right| dx + \int_{\mathbb{R}} \left|\partial_x^2[n^r, u_1^r, \theta^r]\right| \left|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{\theta}\right]\right| dx \\ &+ C_\eta \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx \\ \le& (\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right]\right\|^2 + C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{\theta}\right]\right\|^2 + \delta_r^{1/6}(1+t)^{-7/6} \\ &+ C_\eta \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx.\end{aligned}$$

Next, by Cauchy-Schwarz inequality, one has

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{N}_7\, dx\right| \lesssim& (\epsilon_0+\eta)\|\partial_x\phi\|^2 + C_\eta(1+t)^{-2}\left\|\widetilde{\theta}\right\|^2 + \epsilon_0\left\|\partial_x\widetilde{\theta}\right\|^2 \\ &+ \delta_r^{1/6}(1+t)^{-7/6} + (\epsilon_0+\eta)\int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_{\xi_1}\widetilde{\mathbf{G}}\right|^2 d\xi dx.\end{aligned}$$

It now remains to consider $\int_{\mathbb{R}} \mathcal{N}_6\, dx$. Notice that

$$\partial^\alpha\partial^\beta\left\{L_{M_\mathcal{A}}^{-1} h\right\} = L_{M_\mathcal{A}}^{-1}(\partial^\alpha\partial^\beta h) - \sum_{j=1}^{\alpha+|\beta|}\sum_{|\alpha'|+|\beta'|=j} C_{\alpha',\beta'}^{\alpha,\beta} L_{M_\mathcal{A}}^{-1}\mathcal{M}_{\alpha\beta}, \tag{6.6}$$

where $\mathcal{M}_{\alpha\beta}$ is given by

$$\mathcal{M}_{\alpha\beta}=Q_{\mathcal{A}\mathcal{A}}\left(\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\left(L^{-1}_{M_{\mathcal{A}}}h\right),\partial^{\alpha'}\partial^{\beta'}M_{\mathcal{A}}\right)+Q_{\mathcal{A}\mathcal{A}}\left(\partial^{\alpha'}\partial^{\beta'}M_{\mathcal{A}},\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\left(L^{-1}_{M_{\mathcal{A}}}h\right)\right).$$

Utilizing (6.6), Lemma 4.1, Lemma 3.1 and Corollary 4.1 again, we now arrive at

$$\begin{aligned}\left|\int_{\mathbb{R}}\mathcal{N}_6\,dx\right|\lesssim&\left|\sum_{j=1}^{3}\left(\int_{\mathbb{R}^3}\xi_1\psi_{j+2}\cdot\partial_xL^{-1}_{M_{\mathcal{A}}}\overline{Q}_{\mathcal{A}}(\mathbf{G},\mathbf{G})\,d\xi,\widetilde{u}_j\right)\right|\\&+\left|\left(\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum_{j=1}^{3}u_j\cdot\psi_{j+2}\right)\cdot\partial_xL^{-1}_{M_{\mathcal{A}}}\overline{Q}_{\mathcal{A}}(\mathbf{G},\mathbf{G})\,d\xi,\theta^{-1}\widetilde{\theta}\right)\right|\\&+C_\eta\sum_{\mathcal{A}\in\{i,e\}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|M_{\mathcal{A}}^{-1/2}\left(\overline{R}_{\mathcal{A}}-L^{-1}_{M_{\mathcal{A}}}\overline{Q}_{\mathcal{A}}(\mathbf{G},\mathbf{G})\right)\right|^2d\xi dx\\&+\eta\sum_{j=1}^{3}\int_{\mathbb{R}\times\mathbb{R}^3}|\xi|^4|\mathbf{M}||\partial_x\widetilde{u}_j|^2d\xi dx\\&+\eta\int_{\mathbb{R}\times\mathbb{R}^3}|\xi|^6|\mathbf{M}|\left|\partial_x\left(\frac{\widetilde{\theta}}{\theta}\right)\right|^2d\xi dx+\eta\sum_{i=j}^{3}\int_{\mathbb{R}\times\mathbb{R}^3}|\xi|^4|\mathbf{M}|\left|\partial_x\left(\frac{\widetilde{\theta}}{\theta}\widetilde{u}_j\right)\right|^2d\xi dx\\&+\eta\int_{\mathbb{R}\times\mathbb{R}^3}|\xi|^4|\mathbf{M}|\left|\partial_x\left(\frac{\widetilde{\theta}}{\theta}u_1^r\right)\right|^2d\xi dx,\end{aligned}$$

which further gives

$$\begin{aligned}\left|\int_{\mathbb{R}}\mathcal{N}_6\,dx\right|\leq&(\eta+\epsilon_0)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+C(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\&+\epsilon_0\|\partial_x\phi\|^2+C_\eta\delta_r^{1/6}(1+t)^{-7/6}\\&+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\&+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}|\partial_x\phi|^2|\mathbf{M}^{-1/2}\partial_{\xi_1}(\widetilde{\mathbf{G}}+\overline{\mathbf{G}})|^2d\xi dx\\&+C_\eta\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)|\mathbf{M}_\sharp^{-1/2}\mathbf{G}|^2d\xi\right)\left(\int_{\mathbb{R}^3}|\mathbf{M}_\sharp^{-1/2}\mathbf{G}|^2d\xi\right)dx.\end{aligned}$$

It then follows that

$$\begin{aligned}\left|\int_{\mathbb{R}}\mathcal{N}_6\,dx\right|\leq&(\eta+\epsilon_0)\left\|\partial_x\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\delta_r\left\|\partial_x\phi\right\|^2+C(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\delta_r^{1/6}(1+t)^{-7/6}\\&+C_\eta\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\&+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx.\end{aligned}$$

Here the following estimate has been used:

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)|\mathbf{M}_\sharp^{-1/2}\mathbf{G}|^2d\xi\right)\left(\int_{\mathbb{R}^3}|\mathbf{M}_\sharp^{-1/2}\mathbf{G}|^2d\xi\right)dx\\
\leq& C\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}(\widetilde{\mathbf{G}}+\overline{\mathbf{G}})\right|^2d\xi\right)\left(\int_{\mathbb{R}^3}\left|\mathbf{M}_\sharp^{-1/2}(\widetilde{\mathbf{G}}+\overline{\mathbf{G}})\right|^2d\xi\right)dx\\
\leq&\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+C\int_{\mathbb{R}}|\partial_x[u^r,\theta^r]|^4\,dx\\
\leq&\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+C\delta_r(1+t)^{-2}.
\end{aligned}\tag{6.7}$$

Moreover, Corollary 4.1 and Lemma 4.1 as well as similar calculations for obtaining the formula (6.6) are also applied to deduce

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|\mathbf{M}_\sharp^{-1/2}\partial_{\xi_1}\overline{\mathbf{G}}\right|^2d\xi\\
\leq& C\sum_{|\beta|\leq1}\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\mathbf{L}_{\mathbf{M}}^{-1}\partial^\beta\left\{\mathbf{P}_1^{\mathbf{M}}\left[[m_iM_i,m_eM_e]^{\mathrm{T}}\,\xi_1\left(\xi_1\partial_xu_1^r+\frac{|\xi-u|^2}{2\theta}\partial_x\theta^r\right)\right]\right\}\right|^2d\xi\\
&+C\sum_{|\beta|\leq1}\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\mathbf{L}_{\mathbf{M}}^{-1}\partial^\beta\left\{\mathbf{P}_1^{\mathbf{M}}\left[\left[n_i^{-1}M_i\partial_xn_i^r,n_e^{-1}M_e\partial_xn_e^r\right]^{\mathrm{T}}\,\xi_1\right]\right\}\right|^2d\xi\\
&+C\sum_{|\beta|\leq1}\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\mathbf{L}_{\mathbf{M}}^{-1}\partial^\beta\left\{\mathbf{P}_1^{\mathbf{M}}\left[[M_i,M_e]^{\mathrm{T}}\,\xi_1\partial_x\theta^r\right]\right\}\right|^2d\xi\\
\leq& C\left|\partial_x\left[n^r,u_1^r,\theta^r\right]\right|^2.
\end{aligned}$$

Finally, by substituting the above estimates for $\int_{\mathbb{R}}\mathcal{N}_l\,dx$ $(1\leq l\leq7)$ into (6.1), we conclude that

$$\begin{aligned}
&\frac{d}{dt}\widetilde{\eta}+\lambda\left\|\partial_x\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+\lambda\int_{\mathbb{R}}\partial_xu_1^r\left|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right|^2dx\\
\leq&(\eta+\epsilon_0)\left\|\partial_t\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+(\eta+\epsilon_0)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\phi\right]\right\|^2+(\epsilon_0+\eta)\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2\\
&+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\delta_r^{1/6}(1+t)^{-7/6}+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\
&+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+C_\eta\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned}\tag{6.8}$$

6.2. **Estimate on first-order dissipation.** One has to further consider the dissipation terms involving $\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\phi,\partial_x\phi\right]$ and $\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]$. The computations are divided into three steps.

Step 1. Dissipation of $\partial_x(\widetilde{n}_i+\widetilde{n}_e)$: We first differentiate (5.1) and (5.2) with respect to x, respectively, to obtain

$$\partial_t\partial_x\widetilde{n}_i+\partial_x^2\left(n_iu_1-n_i^ru_1^r\right)=-\partial_x^2\int_{\mathbb{R}^3}\xi_1G_i\,d\xi,\tag{6.9}$$

and

$$\partial_t\partial_x\widetilde{n}_e+\partial_x^2\left(n_eu_1-n_e^ru_1^r\right)=-\partial_x^2\int_{\mathbb{R}^3}\xi_1G_e\,d\xi.\tag{6.10}$$

Then taking the inner products of (5.3), (6.9) and (6.10) with terms

$$\partial_x(\widetilde{n}_i+\widetilde{n}_e),\quad\frac{3(\mu_i(\theta)+\mu_e(\theta))}{n_i^r}\partial_x\widetilde{n}_i,\quad\text{and}\quad\frac{3(\mu_i(\theta)+\mu_e(\theta))}{n_e^r}\partial_x\widetilde{n}_e,$$

with respect to x over $\mathbb{R}$, respectively, one has

$$\begin{aligned}
&\left((m_in_i+m_en_e)\left(\partial_t\widetilde{u}_1+u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r\right),\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+(\partial_xP-\partial_xP^r,\partial_x(\widetilde{n}_i+\widetilde{n}_e))\\
&\quad+\left(\left(1-\frac{m_in_i+m_en_e}{m_in_i^r+m_en_e^r}\right)\partial_xP^r,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+((q_in_i+q_en_e)\partial_x\phi,\partial_x(\widetilde{n}_i+\widetilde{n}_e))\\
&=3\left((\mu_i(\theta)+\mu_e(\theta))\partial_x^2\widetilde{u}_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+3\left((\mu_i(\theta)+\mu_e(\theta))\partial_x^2u_1^r,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)\\
&\quad+3\left(\partial_x(\mu_i(\theta)+\mu_e(\theta))\partial_xu_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)-\left(\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial_x\overline{\mathbf{R}}\,d\xi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)\\
&\quad-\left(\int_{\mathbb{R}^3}\psi_{3i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)-\left(\int_{\mathbb{R}^3}\psi_{3e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),
\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}
&\left(\partial_t\partial_x\widetilde{n}_i+\partial_x^2(\widetilde{n}_i\widetilde{u}_1+\widetilde{n}_iu_1^r)+\partial_x^2n_i^r\widetilde{u}_1+\partial_xn_i^r\partial_x\widetilde{u}_1+n_i^r\partial_x^2\widetilde{u}_1,\frac{3(\mu_i(\theta)+\mu_e(\theta))}{n_i^r}\partial_x\widetilde{n}_i\right)\\
&=-\left(\partial_x^2\int_{\mathbb{R}^3}\xi_1G_i\,d\xi,\frac{3(\mu_i(\theta)+\mu_e(\theta))}{n_i^r}\partial_x\widetilde{n}_i\right),
\end{aligned}\tag{6.12}$$

and

$$\begin{aligned}
&\left(\partial_t\partial_x\widetilde{n}_e+\partial_x^2(\widetilde{n}_e\widetilde{u}_1+\widetilde{n}_eu_1^r)+\partial_x^2n_e^r\widetilde{u}_1+\partial_xn_e^r\partial_x\widetilde{u}_1+n_e^r\partial_x^2\widetilde{u}_1,\frac{3(\mu_i(\theta)+\mu_e(\theta))}{n_e^r}\partial_x\widetilde{n}_e\right)\\
&=-\left(\partial_x^2\int_{\mathbb{R}^3}\xi_1G_e\,d\xi,\frac{3(\mu_i(\theta)+\mu_e(\theta))}{n_e^r}\partial_x\widetilde{n}_e\right).
\end{aligned}\tag{6.13}$$

Notice that

$$P-P^r=\frac{2}{3}\left((\widetilde{n}_i+\widetilde{n}_e)\widetilde{\theta}+(\widetilde{n}_i+\widetilde{n}_e)\theta^r+(n_i^r+n_e^r)\widetilde{\theta}\right).\tag{6.14}$$

We get from the summation of (6.11), (6.12) and (6.13) that

$$\begin{aligned}
&-\frac{d}{dt}\left((m_in_i+m_en_e)\widetilde{u}_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+\frac{3}{2}\frac{d}{dt}\left(\partial_x\widetilde{n}_i,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_i^r}\partial_x\widetilde{n}_i\right)\\
&\quad+\frac{3}{2}\frac{d}{dt}\left(\partial_x\widetilde{n}_e,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_e^r}\partial_x\widetilde{n}_e\right)+\frac{2}{3}\left(\theta^r\partial_x(\widetilde{n}_i+\widetilde{n}_e),\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)\\
&=\sum_{l=1}^{9}\mathcal{I}_l,
\end{aligned}\tag{6.15}$$

with

$$\begin{cases}
\mathcal{I}_1=-\left((m_in_i+m_en_e)(u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r),\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),\\
\mathcal{I}_2=-\dfrac{2}{3}\left(\partial_x\theta^r(\widetilde{n}_i+\widetilde{n}_e),\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+\dfrac{2}{3}\left(\partial_x\left(\widetilde{\theta}\,(\widetilde{n}_i+\widetilde{n}_e)\right),\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),\\
\mathcal{I}_3=\dfrac{2}{3}\left(\partial_x\left(\widetilde{\theta}(n_i^r+n_e^r)\right),\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),\quad \mathcal{I}_4=\left((q_in_i+q_en_e)\partial_x\phi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),\\
\mathcal{I}_5=\left((m_in_i+m_en_e)\widetilde{u}_1,\partial_t\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+\left((m_i\partial_tn_i+m_e\partial_tn_e)\widetilde{u}_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),\\
\mathcal{I}_6=\dfrac{3}{2}\left(\partial_t\left(\dfrac{\mu_i(\theta)+\mu_e(\theta)}{n_i^r}\right)\partial_x\widetilde{n}_i,\partial_x\widetilde{n}_i\right)+\dfrac{3}{2}\left(\partial_t\left(\dfrac{\mu_i(\theta)+\mu_e(\theta)}{n_e^r}\right)\partial_x\widetilde{n}_e,\partial_x\widetilde{n}_e\right),\\
\mathcal{I}_7=3\left((\mu_i(\theta)+\mu_e(\theta))\partial_x^2u_1^r,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+3\left(\partial_x(\mu_i(\theta)+\mu_e(\theta))\partial_xu_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right),
\end{cases}$$

and

$$\begin{cases} \mathcal{I}_8 = -\left(\partial_x^2(\widetilde{n}_i\widetilde{u}_1+\widetilde{n}_iu_1^r)+\partial_x^2n_i^r\widetilde{u}_1+\partial_xn_i^r\partial_x\widetilde{u}_1,\dfrac{3(\mu_i(\theta)+\mu_e(\theta))}{n_i^r}\partial_x\widetilde{n}_i\right)\\ \qquad -\left(\partial_x^2(\widetilde{n}_e\widetilde{u}_1+\widetilde{n}_eu_1^r)+\partial_x^2n_e^r\widetilde{u}_1+\partial_xn_e^r\partial_x\widetilde{u}_1,\dfrac{3(\mu_i(\theta)+\mu_e(\theta))}{n_e^r}\partial_x\widetilde{n}_e\right),\\ \mathcal{I}_9 = -\left(\displaystyle\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial_x\overline{\mathbf{R}}\,d\xi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)-\left(\displaystyle\int_{\mathbb{R}^3}\psi_{3i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)\\ \qquad -\left(\displaystyle\int_{\mathbb{R}^3}\psi_{3e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)-\left(\partial_x^2\displaystyle\int_{\mathbb{R}^3}\xi_1G_i\,d\xi,\dfrac{3(\mu_i(\theta)+\mu_e(\theta))}{n_i^r}\partial_x\widetilde{n}_i\right)\\ \qquad -\left(\partial_x^2\displaystyle\int_{\mathbb{R}^3}\xi_1G_e\,d\xi,\dfrac{3(\mu_i(\theta)+\mu_e(\theta))}{n_e^r}\partial_x\widetilde{n}_e\right). \end{cases}$$

We now turn to estimate $\mathcal{I}_l$ $(1\le l\le 9)$ term by term. It follows from Lemma 3.1, Sobolev's inequality and Cauchy-Schwarz inequality with $0<\eta<1$ that

$$|\mathcal{I}_1|,\ |\mathcal{I}_3|\lesssim\eta\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+C_\eta\left\|\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2,$$

$$|\mathcal{I}_2|\lesssim(\eta+\epsilon_0)\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\|^2+\epsilon_0\left\|\partial_x\widetilde{\theta}\right\|^2+C_\eta(1+t)^{-2}\|[\widetilde{n}_i,\widetilde{n}_e]\|^2,$$

$$|\mathcal{I}_4|\lesssim\epsilon_0\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\|^2+\epsilon_0\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\|^2.$$

By integration by parts and the a priori estimates (5.9), we see that

$$|\mathcal{I}_5|\lesssim(\eta+\epsilon_0)\|\partial_x[\widetilde{n}_i,\widetilde{n}_e]\|^2+(\eta+\epsilon_0)\|\partial_t[\widetilde{n}_i,\widetilde{n}_e]\|^2+C_\eta(1+t)^{-2}\|\widetilde{u}_1\|^2+C_\eta\left\|\partial_x\widetilde{u}_1\right\|^2.$$

For the estimate on $\mathcal{I}_6$, from (5.9), it follows that

$$|\mathcal{I}_6|\lesssim\epsilon_0\|\partial_x[\widetilde{n}_i,\widetilde{n}_e]\|^2.$$

For $\mathcal{I}_7$, Lemma 3.1 and Cauchy-Schwarz inequality imply

$$|\mathcal{I}_7|\lesssim\eta\left\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right\|^2+C_\eta(\delta_r+\epsilon_0)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+C_\eta\delta_r^{1/2}(1+t)^{-3/2}.$$

As to I_8, one has

$$|\mathcal{I}_8|\lesssim(\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\partial_x\widetilde{u}_1\right]\right\|^2+\eta\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e\right]\right\|^2+C_\eta(1+t)^{-2}\left\|[\widetilde{n}_i,\widetilde{n}_e]\right\|^2,$$

according to Lemma 3.1, Sobolev's inequality and Cauchy-Schwarz inequality with $0<\eta<1$ again.

Finally, for $\mathcal{I}_9$, by performing the similar calculations as for $\int_{\mathbb{R}}\mathcal{N}_6\,dx$ in the previous step, we have

$$\begin{aligned}|\mathcal{I}_9|\lesssim&(\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\phi,\partial_x\phi\right]\right\|^2+\delta_r(1+t)^{-2}+C_\eta\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\&+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+C_\eta\epsilon_0\sum_{|\alpha|\le1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx.\end{aligned}$$

We insert the above estimations for $\mathcal{I}_l$ $(1 \le l \le 9)$ into (6.15), choose ϵ_0 and η to be suitably small and then obtain that

$$\begin{aligned}
&\frac{d}{dt}\left((m_in_i+m_en_e)\widetilde{u}_1, \partial_x(\widetilde{n}_i+\widetilde{n}_e)\right) + \frac{3}{2}\frac{d}{dt}\left(\partial_x\widetilde{n}_i, \frac{\mu_i(\theta)+\mu_e(\theta)}{n_i^r}\partial_x\widetilde{n}_i\right)\\
&\quad + \frac{3}{2}\frac{d}{dt}\left(\partial_x\widetilde{n}_e, \frac{\mu_i(\theta)+\mu_e(\theta)}{n_e^r}\partial_x\widetilde{n}_e\right) + \lambda\left\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right\|^2\\
&\lesssim C_\eta\left\|\partial_x\left[\widetilde{u}_1, \widetilde{\theta}\right]\right\|^2 + (\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i, \widetilde{n}_e, \phi, \partial_x\widetilde{u}_1, \partial_x\phi\right]\right\|^2 + \delta_r^{1/2}(1+t)^{-3/2}\\
&\quad + (\epsilon_0+\eta)\left\|\partial_t\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right]\right\|^2 + (\epsilon_0+\eta)\left\|\partial_x^2\left[\widetilde{n}_i, \widetilde{n}_e\right]\right\|^2\\
&\quad + C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}_1, \widetilde{\theta}\right]\right\|^2 + C_\eta\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\
&\quad + C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx + C_\eta\epsilon_0\sum_{|\alpha|\le1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned} \tag{6.16}$$

Step 2. Dissipation of $\partial_x[\phi, \partial_x\phi]$: To do this, we shall make use of the equations (2.10), (2.11) and (3.7). Specifically, we get from the summation of the second equation of (2.10) multiplied by q_i and the second equation of (2.11) multiplied by q_e that

$$\begin{aligned}
&(q_im_in_i+q_em_en_e)(\partial_tu_1+u_1\partial_xu_1) + \frac{2}{3}\partial_x\left((q_in_i+q_en_e)\theta\right) + (q_i^2n_i+q_e^2n_e)\partial_x\phi\\
&= -q_i\int_{\mathbb{R}^3}\psi_{3i}\partial_tG_i\,d\xi - q_i\int_{\mathbb{R}^3}\psi_{3i}\xi\partial_xG_i\,d\xi + q_i\int_{\mathbb{R}^3}\psi_{3i}Q_i(\mathbf{F},\mathbf{F})\,d\xi + q_iu_1\int_{\mathbb{R}^3}\psi_{1i}\xi_1\partial_xG_i\,d\xi\\
&\quad - q_e\int_{\mathbb{R}^3}\psi_{3e}\partial_tG_e\,d\xi - q_e\int_{\mathbb{R}^3}\psi_{3e}\xi\partial_xG_e\,d\xi + q_e\int_{\mathbb{R}^3}\psi_{3e}Q_e(\mathbf{F},\mathbf{F})\,d\xi + q_eu_1\int_{\mathbb{R}^3}\psi_{2e}\xi_1\partial_xG_e\,d\xi.
\end{aligned} \tag{6.17}$$

Then the difference of (6.17) and the third equation of (3.7) yields

$$\begin{aligned}
&(q_im_in_i+q_em_en_e)(\partial_t\widetilde{u}_1+u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r) + (q_im_i\widetilde{n}_i+q_em_e\widetilde{n}_e)(\partial_tu_1^r+u_1^r\partial_xu_1^r)\\
&\quad + \frac{2}{3}\partial_x\left(q_in_i\theta+q_en_e\theta\right) + (q_i^2n_i+q_e^2n_e)\partial_x\phi\\
&= -q_i\int_{\mathbb{R}^3}\psi_{3i}\partial_tG_i\,d\xi - q_i\int_{\mathbb{R}^3}\psi_{3i}\xi\partial_xG_i\,d\xi + q_i\int_{\mathbb{R}^3}\psi_{3i}Q_i(\mathbf{F},\mathbf{F})\,d\xi + q_iu_1\int_{\mathbb{R}^3}\psi_{1i}\xi_1\partial_xG_i\,d\xi\\
&\quad - q_e\int_{\mathbb{R}^3}\psi_{3e}\partial_tG_e\,d\xi - q_e\int_{\mathbb{R}^3}\psi_{3e}\xi\partial_xG_e\,d\xi + q_e\int_{\mathbb{R}^3}\psi_{3e}Q_e(\mathbf{F},\mathbf{F})\,d\xi + q_eu_1\int_{\mathbb{R}^3}\psi_{2e}\xi_1\partial_xG_e\,d\xi\\
&\quad + \frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\partial_x(n_i^r+n_e^r) + \frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r).
\end{aligned} \tag{6.18}$$

Taking the inner product of (6.18) with $\partial_x\phi$ with respect to x over $\mathbb{R}$, one has

$$\begin{aligned}
&((q_im_in_i+q_em_en_e)(\partial_t\widetilde{u}_1+u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r),\partial_x\phi)\\
&\quad+((q_im_i\widetilde{n}_i+q_em_e\widetilde{n}_e)(\partial_tu_1^r+u_1^r\partial_xu_1^r),\partial_x\phi)+\left((q_i^2n_i+q_e^2n_e)\partial_x\phi,\partial_x\phi\right)\\
&\quad-\frac{2}{3}\left(\theta\left(q_in_i+q_en_e\right),\partial_x^2\phi\right)\\
&=-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\partial_tG_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial_tG_e\,d\xi,\partial_x\phi\right)\\
&\quad-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\xi\partial_xG_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\xi\partial_xG_e\,d\xi,\partial_x\phi\right)\\
&\quad+\left(q_iu_1\int_{\mathbb{R}^3}\psi_{1i}\xi_1\partial_xG_i\,d\xi+q_eu_1\int_{\mathbb{R}^3}\psi_{2e}\xi_1\partial_xG_e\,d\xi,\partial_x\phi\right)\\
&\quad+\left(q_i\int_{\mathbb{R}^3}\psi_{3i}Q_i(\mathbf{F},\mathbf{F})\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}Q_e(\mathbf{F},\mathbf{F})\,d\xi,\partial_x\phi\right)\\
&\quad+\left(\frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\partial_x(n_i^r+n_e^r),\partial_x\phi\right)\\
&\quad+\left(\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r),\partial_x\phi\right).
\end{aligned}\tag{6.19}$$

Thanks to (2.10), (2.11) and (3.7), we have

$$\begin{aligned}
&(m_in_i+m_en_e)(\partial_t\widetilde{u}_1+u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r)\\
&\quad+\partial_xP-\partial_xP^r+\left(1-\frac{m_in_i+m_en_e}{m_in_i^r+m_en_e^r}\right)\partial_xP^r+\left(q_in_i+q_en_e\right)\partial_x\phi\\
&=-\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial_x\mathbf{G}\,d\xi.
\end{aligned}\tag{6.20}$$

In view of (6.20) and (5.9), we get from Cauchy-Schwarz inequality with $\eta>0$ that

$$\begin{aligned}
&|((q_im_in_i+q_em_en_e)\partial_t\widetilde{u}_1,\partial_x\phi)|\\
&\lesssim(\epsilon_0+\delta_r+\eta)\left\|\partial_x\phi\right\|^2+(\epsilon_0+\delta_r)\left\|q_in_i+q_en_e\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\\
&\quad+C_\eta\left\{\left\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right\|^2+\left\|\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\right\}\\
&\quad+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx.
\end{aligned}$$

Next, by Cauchy-Schwarz inequality with $\eta>0$ and Lemma 3.1, one has

$$|((q_im_in_i+q_em_en_e)(u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r),\partial_x\phi)|\lesssim\eta\left\|\partial_x\phi\right\|^2+C_\eta(1+t)^{-2}\left\|\widetilde{u}_1\right\|^2+C_\eta\left\|\partial_x\widetilde{u}_1\right\|^2,$$

and

$$|((q_im_i\widetilde{n}_i+q_em_e\widetilde{n}_e)(\partial_tu_1^r+u_1^r\partial_xu_1^r),\partial_x\phi)|\lesssim\eta\left\|\partial_x\phi\right\|^2+C_\eta(1+t)^{-2}\left\|[\widetilde{n}_i,\widetilde{n}_e]\right\|^2.$$

For the terms on the right-hand side of (6.19), we only present the exact computations of those terms involving $Q_{\mathcal{A}}(\mathbf{F},\mathbf{F})$. Note that

$$Q_{\mathcal{A}}(\mathbf{F},\mathbf{F})=(\mathbf{L}_{\mathbf{M}}\mathbf{G})_{\mathcal{A}}+Q_{\mathcal{A}}(\mathbf{G},\mathbf{G})=(\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}})_{\mathcal{A}}+(\mathbf{L}_{\mathbf{M}}\overline{\mathbf{G}})_{\mathcal{A}}+Q_{\mathcal{A}}(\mathbf{G},\mathbf{G}).$$

With this we write

$$\begin{aligned}
&\left(q_i\int_{\mathbb{R}^3}\psi_{3i}Q_i(\mathbf{F},\mathbf{F})\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}Q_e(\mathbf{F},\mathbf{F})\,d\xi,\partial_x\phi\right)\\
&=\left(q_i\int_{\mathbb{R}^3}\psi_{3i}(\mathbf{L_M}\widetilde{\mathbf{G}})_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}(\mathbf{L_M}\widetilde{\mathbf{G}})_e\,d\xi,\partial_x\phi\right)\\
&\quad+\left(q_i\int_{\mathbb{R}^3}\psi_{3i}(\mathbf{L_M}\overline{\mathbf{G}})_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}(\mathbf{L_M}\overline{\mathbf{G}})_e\,d\xi,\partial_x\phi\right)\\
&\quad+\left(q_i\int_{\mathbb{R}^3}\psi_{3i}Q_i(\mathbf{G},\mathbf{G})\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}Q_e(\mathbf{G},\mathbf{G})\,d\xi,\partial_x\phi\right).
\end{aligned}\tag{6.21}$$

Substituting (5.8) into the second term on the right-hand side of (6.21), we find

$$\begin{aligned}
&\left(q_i\int_{\mathbb{R}^3}\psi_{3i}(\mathbf{L_M}\overline{\mathbf{G}})_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}(\mathbf{L_M}\overline{\mathbf{G}})_e\,d\xi,\partial_x\phi\right)\\
&\quad+\left(\frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\partial_x(n_i^r+n_e^r),\partial_x\phi\right)+\left(\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r),\partial_x\phi\right)\\
&=\left(\left\{-\frac{2\theta}{3}\frac{q_im_in_i+q_em_en_e}{m_in_i+m_en_e}+\frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\right\}\partial_x(n_i^r+n_e^r),\partial_x\phi\right)\\
&\quad+\left(-\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i+q_em_en_e}{m_in_i+m_en_e}(n_i+n_e)+\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r),\partial_x\phi\right),
\end{aligned}$$

whose absolute value can be bounded by

$$\eta\left\|\partial_x\phi\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\right\|^2.$$

The remaining terms on the right-hand side of (6.19) and (6.21) are bounded by

$$\begin{aligned}
&(\epsilon_0+\eta)\left\|\partial_x\phi\right\|^2+\epsilon_0\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2+C_\eta\delta_r(1+t)^{-2}+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&\qquad+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+\epsilon_0\sum_{|\alpha|\le1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx,
\end{aligned}$$

according to Cauchy-Schwarz inequality and the estimate (6.7). Finally, substituting the above estimates into (6.19) and applying (5.6), we arrive at

$$\begin{aligned}
&\lambda\left(\partial_x\phi,\partial_x\phi\right)+\lambda\left(\partial_x^2\phi,\partial_x^2\phi\right)\\
&\lesssim(\epsilon_0+\eta)\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+C_\eta\left\{\left\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right\|^2+\left\|\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\right\}+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\
&\quad+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx.
\end{aligned}\tag{6.22}$$

Step 3. Dissipation of $\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)$: To deduce this, we take the inner product of (6.18) with $\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)$ with respect to x over $\mathbb{R}$ to obtain

$$
\begin{aligned}
&((q_im_in_i+q_em_en_e)(\partial_t\widetilde{u}_1+u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right))\\
&\quad+\left((q_i^2n_i+q_e^2n_e)\partial_x\phi,\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\frac{2}{3}\left(\theta\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)+\frac{2}{3}\left(\partial_x\theta\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+(q_im_i\widetilde{n}_i+q_em_e\widetilde{n}_e)(\partial_tu_1^r+u_1^r\partial_xu_1^r),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right))\\
&=-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\partial_tG_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial_tG_e\,d\xi,\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\xi_1\partial_xG_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\xi_1\partial_xG_e\,d\xi,\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\left(q_iu_1\int_{\mathbb{R}^3}\psi_{1i}\xi_1\partial_xG_i\,d\xi+q_eu_1\int_{\mathbb{R}^3}\psi_{2e}\xi_1\partial_xG_e\,d\xi,\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\left(q_i\int_{\mathbb{R}^3}\psi_{3i}Q_i(\mathbf{F},\mathbf{F})\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}Q_e(\mathbf{F},\mathbf{F})\,d\xi,\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\left(\frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\partial_x(n_i^r+n_e^r),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\left(\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right).
\end{aligned}
\tag{6.23}
$$

With (6.23) in hand, by performing the similar calculations as for obtaining (6.22), one has

$$
\begin{aligned}
&\lambda\left(\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)+\lambda\left(\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\lesssim(\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\phi\right]\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\right\|^2+C_\eta\left\{\|\partial_x(\widetilde{n}_i+\widetilde{n}_e)\|^2+\left\|\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\right\}\\
&\quad+C_\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&\quad+\delta_r^{1/2}(1+t)^{-3/2}.
\end{aligned}
\tag{6.24}
$$

We now get from (6.16), (6.22) and (6.24) that

$$
\begin{aligned}
&\frac{d}{dt}\left((m_in_i+m_en_e)\widetilde{u}_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)+\frac{3}{2}\frac{d}{dt}\left(\partial_x\widetilde{n}_i,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_i^r}\partial_x\widetilde{n}_i\right)\\
&\quad+\frac{3}{2}\frac{d}{dt}\left(\partial_x\widetilde{n}_e,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_e^r}\partial_x\widetilde{n}_e\right)+\lambda\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\phi,\partial_x\phi\right]\right\|^2+\lambda\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2\\
&\lesssim C_\eta\left\|\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+(\epsilon_0+\eta)\|\partial_x^2\widetilde{u}_1\|^2+\delta_r^{1/2}(1+t)^{-3/2}+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\\
&\quad+C_\eta\sum_{1\le\alpha\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&\quad+C_\eta\epsilon_0\sum_{|\alpha|\le1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned}
\tag{6.25}
$$

Having obtained (6.25), one can see that $\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]$ also enjoys the similar estimate. In fact, we get from (2.10), (2.11) and (3.7) that

$$\begin{cases}
\partial_t\widetilde{n}_i+\partial_x(n_iu_1-n_i^ru_1^r)=-\displaystyle\int_{\mathbb{R}^3}\xi_1\partial_xG_i\,d\xi,\\
\partial_t\widetilde{n}_e+\partial_x(n_eu_1-n_e^ru_1^r)=-\displaystyle\int_{\mathbb{R}^3}\xi_1\partial_xG_e\,d\xi,\\
(m_in_i+m_en_e)\partial_t\widetilde{u}_1+(m_in_i+m_en_e)(u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r)+\partial_xP-\partial_xP^r+(q_i\widetilde{n}_i+q_e\widetilde{n}_e)\,\partial_x\phi\\
\qquad+\left(1-\dfrac{m_in_i+m_en_e}{m_in_i^r+m_en_e^r}\right)\partial_xP^r=-\displaystyle\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial_x\mathbf{G}d\xi,\\
(m_in_i+m_en_e)\partial_t\widetilde{u}_j+(m_in_i+m_en_e)u_1\partial_x\widetilde{u}_j=-\displaystyle\int_{\mathbb{R}^3}\xi_1\psi_{j+2}\partial_x\mathbf{G}d\xi,\quad j=2,\ 3,\\
(n_i+n_e)\partial_t\widetilde{\theta}+(n_i+n_e)(u_1\partial_x\widetilde{\theta}+\widetilde{u}_1\partial_x\theta^r)+P\partial_xu_1-P^r\partial_xu_1^r+\left(1-\dfrac{n_i+n_e}{n_i^r+n_e^r}\right)P^r\partial_xu^r\\
\qquad=-\displaystyle\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum_{j=1}^3u_j\cdot\psi_{j+2}\right)\cdot\partial_x\mathbf{G}\,d\xi+\theta\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi\\
\qquad\quad+\partial_x\phi\displaystyle\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}\,[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi.
\end{cases}\tag{6.26}$$

This yields that

$$\begin{aligned}
&\left\|\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&\lesssim\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\phi\right]\right\|^2+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\delta_r(1+t)^{-2}\\
&\quad+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial_x\mathbf{G}|^2d\xi dx+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&\quad+C_\eta\epsilon_0\sum_{|\alpha|\le1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned}\tag{6.27}$$

Letting $1\gg\kappa_1\gg\kappa_2>0$ and taking the summation of (6.8), (6.25) $\times\,\kappa_1$ and (6.27) $\times\,\kappa_2$, one has that for suitably small constants $\epsilon_0>0$, $\delta_r>0$ and $\eta>0$,

$$\begin{aligned}
&\frac{d}{dt}\widetilde{\eta}+\kappa_1\frac{d}{dt}\left((m_in_i+m_en_e)\widetilde{u}_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e)\right)\\
&\quad+\kappa_1\frac{3}{2}\frac{d}{dt}\left\{\left(\partial_x\widetilde{n}_i,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_i^r}\partial_x\widetilde{n}_i\right)+\left(\partial_x\widetilde{n}_e,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_e^r}\partial_x\widetilde{n}_e\right)\right\}\\
&\quad+\lambda\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\lambda\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2+\lambda\left\|\partial_x\left[\phi,\partial_x\phi\right]\right\|^2+\lambda\int_{\mathbb{R}}\partial_xu_1^r\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]^2dx\\
&\le(\epsilon_0+\eta)\left\|\partial_x^2\widetilde{u}_1\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\delta_r^{1/6}(1+t)^{-7/6}\\
&\quad+C_\eta\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&\quad+C_\eta\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned}\tag{6.28}$$

6.3. **Estimate on first-order energy.** Let $|\alpha|=1$. Taking the inner product of ∂^α(5.1), ∂^α(5.2), ∂^α(5.3), ∂^α(5.4) and ∂^α(5.5) with $\partial^\alpha\widetilde{n}_i$, $\partial^\alpha\widetilde{n}_e$, $\partial^\alpha\widetilde{u}_1$, $\partial^\alpha\widetilde{u}_j$ $(2\le j\le3)$ and $\partial^\alpha\widetilde{\theta}$, respectively, and then

taking the summation of the resulting equations, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\partial^\alpha\widetilde{n}_i\|^2+\|\partial^\alpha\widetilde{n}_e\|^2+\sum_{j=1}^{3}\left\|\sqrt{m_in_i+m_en_e}\partial^\alpha\widetilde{u}_j\right\|^2+\left\|\sqrt{n_i+n_e}\partial^\alpha\widetilde{\theta}\right\|^2\right\}\\
&\quad+3\left((\mu_i(\theta)+\mu_e(\theta))\partial_x\partial^\alpha\widetilde{u}_1,\partial_x\partial^\alpha\widetilde{u}_1\right)+\sum_{j=2}^{3}\left((\mu_i(\theta)+\mu_e(\theta))\partial_x\partial^\alpha\widetilde{u}_j,\partial_x\partial^\alpha\widetilde{u}_j\right)\\
&\quad+\left((\kappa_i(\theta)+\kappa_i(\theta))\partial_x\partial^\alpha\widetilde{\theta},\partial_x\partial^\alpha\widetilde{\theta}\right)=\sum_{l=1}^{12}J_l,
\end{aligned}\tag{6.29}$$

where the right-hand terms are given by

$$\begin{cases}
J_1=-\left(\partial^\alpha\partial_x(n_iu_1-n_i^ru_1^r),\partial^\alpha\widetilde{n}_i\right)+\left(\partial^\alpha\partial_x(n_eu_1-n_e^ru_1^r),\partial^\alpha\widetilde{n}_e\right),\\
J_2=\dfrac{1}{2}\displaystyle\sum_{j=1}^{3}\left(\partial_t(m_in_i+m_en_e)\partial^\alpha\widetilde{u}_j,\partial^\alpha\widetilde{u}_j\right)+\frac{1}{2}\left(\partial_t(n_i+n_e)\partial^\alpha\widetilde{\theta},\partial^\alpha\widetilde{\theta}\right),\\
J_3=-\left(\partial^\alpha\left\{(m_in_i+m_en_e)(u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r)\right\},\partial^\alpha\widetilde{u}_1\right)-\displaystyle\sum_{j=2}^{3}\left(\partial^\alpha\left\{(m_in_i+m_en_e)u_1\partial_x\widetilde{u}_j\right\},\partial^\alpha\widetilde{u}_j\right),\\
J_4=-\left(\partial^\alpha\left((q_i\widetilde{n}_i+q_e\widetilde{n}_e)\,\partial_x\phi\right),\partial^\alpha\widetilde{u}_1\right),\\
J_5=\left(\partial^\alpha(P-P^r),\partial^\alpha\partial_x\widetilde{u}_1\right)-\left(\partial^\alpha\left\{\left(1-\dfrac{m_in_i+m_en_e}{m_in_i^r+m_en_e^r}\right)\partial_xP^r\right\},\partial^\alpha\widetilde{u}_1\right),\\
J_6=-\left(\partial^\alpha\left\{(n_i+n_e)(u_1\partial_x\widetilde{\theta}+\widetilde{u}_1\partial_x\theta^r)\right\},\partial^\alpha\widetilde{\theta}\right)-\left(\partial^\alpha\left(P\partial_xu_1-P^r\partial_xu_1^r\right),\partial^\alpha\widetilde{\theta}\right),\\
\qquad-\left(\partial^\alpha\left\{\left(1-\dfrac{n_i+n_e}{n_i^r+n_e^r}\right)P^r\partial_xu^r\right\},\partial^\alpha\widetilde{\theta}\right),
\end{cases}$$

$$\begin{cases}
J_7=-3\left((\mu_i(\theta)+\mu_e(\theta))\partial^\alpha\partial_xu_1^r,\partial^\alpha\partial_x\widetilde{u}_1\right)-\left((\kappa_i(\theta)+\kappa_e(\theta))\partial^\alpha\partial_x\theta^r,\partial^\alpha\partial_x\widetilde{\theta}\right),\\
J_8=-3\left(\partial^\alpha(\mu_i(\theta)+\mu_e(\theta))\partial_xu_1^r,\partial^\alpha\partial_x\widetilde{u}_1\right)-\displaystyle\sum_{j=2}^{3}\left(\partial^\alpha(\mu_i(\theta)+\mu_e(\theta))\partial_xu_j,\partial^\alpha\partial_x\widetilde{u}_j\right)\\
\qquad-\left(\partial^\alpha(\kappa_i(\theta)+\kappa_e(\theta))\partial_x\theta,\partial^\alpha\partial_x\widetilde{\theta}\right),\\
J_9=3\left(\partial^\alpha\left((\mu_i(\theta)+\mu_e(\theta))(\partial_xu_1)^2\right),\partial^\alpha\widetilde{\theta}\right)+\displaystyle\sum_{j=2}^{3}\left(\partial^\alpha\left((\mu_i(\theta)+\mu_e(\theta))(\partial_x\widetilde{u}_j)^2\right),\partial^\alpha\widetilde{\theta}\right),
\end{cases}$$

$$\begin{aligned}
J_{10}=&-\left(\int_{\mathbb{R}^3}\xi_1\partial^\alpha\partial_xG_i\,d\xi,\partial^\alpha\widetilde{n}_i\right)-\left(\int_{\mathbb{R}^3}\xi_1\partial^\alpha\partial_xG_e\,d\xi,\partial^\alpha\widetilde{n}_e\right)\\
&-\sum_{j=1}^{3}\left(\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial^\alpha\partial_x\left(P_0^{M_i}G_i\right)d\xi,\partial^\alpha\partial_x\widetilde{u}_j\right)\\
&-\sum_{j=1}^{3}\left(\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial^\alpha\partial_x\left(P_0^{M_e}G_e\right)d\xi,\partial^\alpha\partial_x\widetilde{u}_j\right)\\
&+\sum_{j=1}^{3}\left(\partial^\alpha\left\{u_j\int_{\mathbb{R}^3}\psi_{(j+2)i}\xi_1\partial_x\left(P_0^{M_i}G_i\right)d\xi\right\},\partial^\alpha\widetilde{\theta}\right)\\
&+\sum_{j=1}^{3}\left(\partial^\alpha\left\{u_j\int_{\mathbb{R}^3}\psi_{(j+2)e}\xi_1\partial_x\left(P_0^{M_e}G_e\right)d\xi\right\},\partial^\alpha\widetilde{\theta}\right),
\end{aligned}$$

and

$$\begin{cases} J_{11}=\sum\limits_{j=1}^{3}\left(\int_{\mathbb{R}^3}\xi_1\psi_{j+2}\cdot\partial^\alpha\overline{\mathbf{R}}\,d\xi,\partial^\alpha\partial_x\widetilde{u}_j\right)+\left(\int_{\mathbb{R}^3}\xi_1\left(\psi_6-\sum\limits_{j=1}^{3}u_j\psi_{j+2}\right)\cdot\partial^\alpha\overline{\mathbf{R}}\,d\xi,\partial_x\partial^\alpha\widetilde{\theta}\right),\\ J_{12}=-\sum\limits_{j=1}^{3}\left(\int_{\mathbb{R}^3}\xi_1\partial_x u_j\psi_{j+2}\cdot\partial^\alpha\overline{\mathbf{R}}\,d\xi,\partial^\alpha\widetilde{\theta}\right)+\sum\limits_{j=1}^{3}\left(\int_{\mathbb{R}^3}\xi_1\partial^\alpha u_j\psi_{j+2}\cdot\partial_x\overline{\mathbf{R}}\,d\xi,\partial^\alpha\widetilde{\theta}\right)\\ \qquad+\left(\partial^\alpha\left(\theta\int_{\mathbb{R}^3}[\xi_1,\xi_1]^{\mathrm{T}}\cdot\partial_x\mathbf{G}\,d\xi\right),\partial^\alpha\widetilde{\theta}\right)+\left(\partial^\alpha\left(\partial_x\phi\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}[q_i,q_e]^{\mathrm{T}}\cdot\partial_{\xi_1}\mathbf{G}\,d\xi\right),\partial^\alpha\widetilde{\theta}\right).\end{cases}$$

We now turn to estimate J_l $(1\le l\le 12)$ term by term. For brevity, we give straightforward calculations as follows:

$$|J_1|\lesssim\eta\sum_{|\alpha|=1}\|\partial_x\partial^\alpha[\widetilde{n}_i,\widetilde{n}_e]\|^2+C_\eta\sum_{|\alpha|=1}\|\partial^\alpha[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1]\|^2+C_\eta(1+t)^{-2}\|[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1]\|^2,$$

$$|J_2|\lesssim\epsilon_0\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2,\quad |J_3|\lesssim\eta\sum_{|\alpha|=1}\|\partial_x\partial^\alpha\widetilde{u}\|^2+C_\eta\sum_{|\alpha|=1}\|\partial^\alpha\widetilde{u}\|^2+C_\eta(1+t)^{-2}\|\widetilde{u}_1\|^2,$$

$$|J_4|\lesssim\epsilon_0\sum_{|\alpha|=1}\|\partial^\alpha\widetilde{u}_1\|^2+\epsilon_0\sum_{|\alpha|\le1}\|\partial^\alpha(q_i\widetilde{n}_i+q_e\widetilde{n}_e)\|^2,$$

$$|J_5|\le\eta\sum_{|\alpha|=1}\|\partial_x\partial^\alpha\widetilde{u}_1\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\right\|^2,$$

$$|J_6|\lesssim\eta\sum_{|\alpha|=1}\left\|\partial_x\partial^\alpha\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+C_\eta(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2,$$

$$|J_7|\lesssim\eta\sum_{|\alpha|=1}\left\|\partial_x\partial^\alpha\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\delta_r^{1/2}(1+t)^{-3/2},$$

$$\begin{aligned}|J_8|\le&\eta\sum_{|\alpha|=1}\left\|\partial_x\partial^\alpha\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\partial_x\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha[n^r,\theta^r]\partial_x\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2\\&+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\partial_x[u^r,\theta^r]\right\|^2+C_\eta\sum_{|\alpha|=1}\|\partial^\alpha[n^r,\theta^r]\partial_x[u_1^r,\theta^r]\|^2\\\le&\eta\sum_{|\alpha|=1}\left\|\partial_x\partial^\alpha\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2++C_\eta\delta_r^2(1+t)^{-2},\end{aligned}$$

$$\begin{aligned}|J_9|\lesssim&\left|\left(\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right](\partial_x u)^2,\partial^\alpha\widetilde{\theta}\right)\right|+\left|\left(\partial^\alpha[n^r,\theta^r](\partial_x\widetilde{u})^2,\partial^\alpha\widetilde{\theta}\right)\right|+\left|\left(\partial^\alpha[n^r,\theta^r](\partial_x u_1^r)^2,\partial^\alpha\widetilde{\theta}\right)\right|\\&+\left|\left(\partial^\alpha\partial_x\widetilde{u}\partial_x\widetilde{u},\partial^\alpha\widetilde{\theta}\right)\right|+\left|\left(\partial^\alpha\partial_x u_1^r\partial_x u_1^r,\partial^\alpha\widetilde{\theta}\right)\right|\\\lesssim&\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\epsilon_0\sum_{|\alpha|=1}\|\partial_x\partial^\alpha\widetilde{u}\|^2+\delta_r^2(1+t)^{-2},\end{aligned}$$

and

$$
\begin{aligned}
|J_{10}|+|J_{11}|+|J_{12}| \leq & \eta \sum_{|\alpha|=1}\left\|\partial_x\partial^\alpha\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&+C_\eta\sum_{1\leq|\alpha|\leq 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\
&+\epsilon_0\sum_{|\alpha|\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\partial^\alpha\widetilde{\mathbf{G}}|^2d\xi dx\\
&+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+\delta_r^{1/2}(1+t)^{-3/2}.
\end{aligned}
$$

Plugging the above estimates for J_l $(1\leq l\leq 12)$ into (6.29), one thus has

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha=1|}\left\{\|\partial^\alpha\widetilde{n}_i\|^2+\|\partial^\alpha\widetilde{n}_e\|^2+\sum_{j=1}^3\left\|\sqrt{m_in_i+m_en_e}\partial^\alpha\widetilde{u}_j\right\|^2+\left\|\sqrt{n_i+n_e}\partial^\alpha\widetilde{\theta}\right\|^2\right\}\\
&\quad+\sum_{|\alpha|=1}\lambda\left\|\partial_x\partial^\alpha\left[\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&\lesssim\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\|^2+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{\theta}\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+\sum_{1\leq|\alpha|\leq 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&\quad+\epsilon_0\sum_{\gamma\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\partial^\alpha\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned}
\tag{6.30}
$$

Let us now deduce the second-order dissipation of $\widetilde{n}_i$ and $\widetilde{n}_e$. As it has been shown in the previous subsection, it may not be direct to obtain the second-order dissipation of $\widetilde{n}_i$ and $\widetilde{n}_e$ in a separate way, and instead one has to consider $\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)$ and $\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)$ $(|\alpha|=1)$ in an equivalent way. In what follows, we shall turn to derive these two kinds of dissipations by using different equations. In fact, one can first take the inner product of $\partial^\alpha(6.20)$ with $\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)$ $(|\alpha|=1)$ to obtain

$$
\begin{aligned}
&((m_in_i+m_en_e)\partial_t\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right))+(\partial^\alpha(m_in_i+m_en_e)\partial_t\widetilde{u}_1,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right))\\
&\quad+\left(\partial^\alpha\left\{(m_in_i+m_en_e)(u_1\partial_x\widetilde{u}_1+\widetilde{u}_1\partial_xu_1^r)\right\},\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&\quad+\left(\partial^\alpha\partial_xP-\partial^\alpha\partial_xP^r,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)+\left(\partial^\alpha\left(\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\partial_x\phi\right),\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&=-\left(\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial^\alpha\partial_x\mathbf{G}\,d\xi,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right),
\end{aligned}
$$

from which as well as (6.14), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\left((m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)+\frac{2}{3}\left(\theta^r\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right),\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&=\left((m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\partial_t\left\{\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right\}\right)+\left(\partial_t(m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\left\{\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right\}\right)\\
&\quad-\left(\partial^\alpha(m_in_i+m_en_e)\partial_t\widetilde{u}_1,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)-\frac{2}{3}\left(\partial^\alpha\theta^r\partial_x(\widetilde{n}_i+\partial_x\widetilde{n}_e),\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&\quad-\frac{2}{3}\left(\partial^\alpha\left\{\partial_x\theta^r(\widetilde{n}_i+\widetilde{n}_e)\right\},\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)-\frac{2}{3}\left(\partial^\alpha\partial_x\left\{\widetilde{\theta}(\widetilde{n}_i+\widetilde{n}_e)\right\},\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&\quad-\left(\partial^\alpha\partial_x\left(\widetilde{\theta}(n_i^r+n_e^r)\right),\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)-\left(\partial^\alpha\left(\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\partial_x\phi\right),\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&\quad-\left(\int_{\mathbb{R}^3}\xi_1\psi_3\cdot\partial^\alpha\partial_x\mathbf{G}\,d\xi,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right).
\end{aligned}
\tag{6.31}
$$

Using integration by parts and applying (5.1) and (5.2), one can deduce

$$\begin{aligned}
&|((m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\partial_t\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right))|\\
&=|(\partial_x\left\{(m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1\right\},\partial_t\partial^\alpha\left(\widetilde{n}_i+\widetilde{n}_e\right))|\\
&\lesssim|(\partial_x\left\{(m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1\right\},\partial^\alpha\partial_x\left(n_iu_1-n_i^ru_1^r+n_eu_1-n_e^ru_1^r\right))|\\
&\quad+\left|\left(\partial_x\left\{(m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1\right\},\int_{\mathbb{R}^3}\xi_1\partial^\alpha\partial_x(G_i+G_e)d\xi\right)\right|\\
&\lesssim C_\eta\sum_{|\alpha|=1}\|\partial^\alpha\partial_x\widetilde{u}_1\|^2+\eta\sum_{|\alpha|=1}\|\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\|^2+\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\\
&\quad+(1+t)^{-2}\left\|[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1]\right\|^2+\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx.
\end{aligned}\tag{6.32}$$

Furthermore, it is straightforward to show that the remaining terms on the right-hand side of (6.31) are bounded by

$$\begin{aligned}
&(\epsilon_0+\eta)\left\|\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\widetilde{\theta}\right\|^2\\
&\quad+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\epsilon_0\sum_{|\alpha|\le1}\|\partial^\alpha(q_i\widetilde{n}_i+q_e\widetilde{n}_e)\|^2+C_\eta\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+C_\eta\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx.
\end{aligned}\tag{6.33}$$

Substituting (6.32) and (6.33) into (6.31), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=1}\left((m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)+\lambda\sum_{|\alpha|=1}\left(\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right),\partial^\alpha\partial_x\left(\widetilde{n}_i+\widetilde{n}_e\right)\right)\\
&\lesssim\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\\
&\quad+\sum_{|\alpha|\le1}\|\partial^\alpha(q_i\widetilde{n}_i+q_e\widetilde{n}_e)\|^2+\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx.
\end{aligned}\tag{6.34}$$

One the other hand, taking the inner product of ∂^α(6.18) with $\partial^\alpha\partial_x\left(q_in_i+q_en_e\right)$ with respect to x over $\mathbb{R}$, one has that

$$\begin{aligned}
&((q_im_in_i+q_em_en_e)\partial_t\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right))+\left((q_i^2n_i+q_e^2n_e)\partial^\alpha\partial_x\phi,\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\frac{2}{3}\left(\theta\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)+\left(\partial^\alpha\left(q_i^2n_i+q_e^2n_e\right)\partial_x\phi,\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\frac{2}{3}\left(\partial^\alpha\theta\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)+\frac{2}{3}\left(\partial^\alpha\left(\partial_x\theta\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\quad+\left(\partial^\alpha\left(q_im_i\widetilde{n}_i+q_em_e\widetilde{n}_e\right)(\partial_tu_1^r+u_1^r\partial_xu_1^r)\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right))
\end{aligned}$$

is equal to

$$
\begin{aligned}
&-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\partial^\alpha\partial_t G_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial^\alpha\partial_t G_e\,d\xi,\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\xi_1\partial^\alpha\partial_x G_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial^\alpha\xi_1\partial_x G_e\,d\xi,\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&+\left(q_i\partial^\alpha\left(u_1\int_{\mathbb{R}^3}\psi_{1i}\xi_1\partial_x G_i\,d\xi\right)+q_e\partial^\alpha\left(u_1\int_{\mathbb{R}^3}\psi_{2e}\xi_1\partial_x G_e\,d\xi\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&+\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\partial^\alpha Q_i(\mathbf{F},\mathbf{F})\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial^\alpha Q_e(\mathbf{F},\mathbf{F})\,d\xi,\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&+\left(\partial^\alpha\left(\frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\partial_x(n_i^r+n_e^r)\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&+\left(\partial^\alpha\left(\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r)\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right).
\end{aligned}
$$

Therefore, by the similar argument as for obtaining (6.22), it follows that

$$
\begin{aligned}
&\lambda\sum_{|\alpha|=1}\left(\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)+\lambda\sum_{|\alpha|=1}\left(\partial^\alpha\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial^\alpha\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right)\\
&\lesssim\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\epsilon_0\sum_{|\alpha|\le1}\|\partial^\alpha(q_i\widetilde{n}_i+q_e\widetilde{n}_e)\|^2\\
&\quad+\epsilon_0\sum_{|\alpha|\le1}\|\partial^\alpha\partial_x\phi\|^2+\delta_r^{1/2}(1+t)^{-3/2}+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2\\
&\quad+\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx.
\end{aligned}
\tag{6.35}
$$

We are now in a position to derive the dissipation of $\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]$ with $|\alpha|=1$. For this, we take the inner product of ∂^α(6.18) with $\partial^\alpha\partial_x\phi$ with respect to x over $\mathbb{R}$ to obtain that

$$
\begin{aligned}
&((q_im_in_i+q_em_en_e)\partial_t\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\phi)+\left((q_i^2n_i+q_e^2n_e)\partial^\alpha\partial_x\phi,\partial^\alpha\partial_x\phi\right)\\
&\quad+\frac{2}{3}\left(\theta\partial^\alpha\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial^\alpha\partial_x\phi\right)+\left(\partial^\alpha\left(q_i^2n_i+q_e^2n_e\right)\partial_x\phi,\partial^\alpha\partial_x\phi\right)\\
&\quad+\frac{2}{3}\left(\partial^\alpha\theta\partial_x\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right),\partial^\alpha\partial_x\phi\right)+\frac{2}{3}\left(\partial^\alpha\left(\partial_x\theta\left(q_i\widetilde{n}_i+q_e\widetilde{n}_e\right)\right),\partial^\alpha\partial_x\phi\right)\\
&\quad+\left(\partial^\alpha\left(q_im_i\widetilde{n}_i+q_em_e\widetilde{n}_e\right)(\partial_tu_1^r+u_1^r\partial_xu_1^r)\right),\partial^\alpha\partial_x\phi)
\end{aligned}
$$

is equal to

$$
\begin{aligned}
&-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\partial^\alpha\partial_t G_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial^\alpha\partial_t G_e\,d\xi,\partial^\alpha\partial_x\phi\right)\\
&-\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\xi_1\partial^\alpha\partial_x G_i\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial^\alpha\xi_1\partial_x G_e\,d\xi,\partial^\alpha\partial_x\phi\right)\\
&+\left(q_i\partial^\alpha\left(u_1\int_{\mathbb{R}^3}\psi_{1i}\xi_1\partial_x G_i\,d\xi\right)+q_e\partial^\alpha\left(u_1\int_{\mathbb{R}^3}\psi_{2e}\xi_1\partial_x G_e\,d\xi\right),\partial^\alpha\partial_x\phi\right)\\
&+\left(q_i\int_{\mathbb{R}^3}\psi_{3i}\partial^\alpha Q_i(\mathbf{F},\mathbf{F})\,d\xi+q_e\int_{\mathbb{R}^3}\psi_{3e}\partial^\alpha Q_e(\mathbf{F},\mathbf{F})\,d\xi,\partial^\alpha\partial_x\phi\right)\\
&+\left(\partial^\alpha\left(\frac{2\theta^r}{3}\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}\partial_x(n_i^r+n_e^r)\right),\partial^\alpha\partial_x\phi\right)\\
&+\left(\partial^\alpha\left(\frac{2}{3}\partial_x\theta^r\frac{q_im_in_i^r+q_em_en_e^r}{m_in_i^r+m_en_e^r}(n_i^r+n_e^r)\right),\partial^\alpha\partial_x\phi\right).
\end{aligned}
$$

In almost the same way as for obtaining (6.22), one can further derive that

$$\begin{aligned}
&\lambda\sum_{|\alpha|=1}(\partial^\alpha\partial_x\phi,\partial^\alpha\partial_x\phi)+\lambda\sum_{|\alpha|=1}\left(\partial^\alpha\partial_x^2\phi,\partial^\alpha\partial_x^2\phi\right)\\
&\lesssim\sum_{|\alpha|\le1}\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\|^2+\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\|\partial_x\phi\|^2\\
&\quad+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\mathbf{G}\right|^2d\xi dx.
\end{aligned}\tag{6.36}$$

Taking the suitable linear combination of (6.34), (6.35) and (6.36), we conclude that

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=1}((m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\partial_x\partial^\alpha(\widetilde{n}_i+\widetilde{n}_e))+\lambda\sum_{|\alpha|=1}\|\partial^\alpha\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\phi,\partial_x\phi\right]\|^2\\
&\lesssim\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\|\partial_x\phi\|^2+\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\left[\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\sum_{|\alpha|\le1}\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\|^2\\
&\quad+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+C_\eta\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\
&\quad+C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx.
\end{aligned}\tag{6.37}$$

As to the second-order time derivative of $\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]$, one has by (6.26) that

$$\begin{aligned}
&\left\|\partial_t^2\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&\lesssim\sum_{|\alpha|=1}\left\|\partial^\alpha\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta},\phi\right]\right\|^2+(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u}_1,\widetilde{\theta}\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}\\
&\quad+\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx+\epsilon_0\sum_{|\alpha|\le1}\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\partial^\alpha\widetilde{\mathbf{G}}|^2d\xi dx.
\end{aligned}\tag{6.38}$$

In addition, in light of (5.6), one can see that ϕ enjoys much higher order dissipative property, namely,

$$\sum_{|\alpha|=2}\left\|\partial^\alpha\partial_x^2\phi\right\|^2\lesssim\sum_{|\alpha|=2}\left\|\partial^\alpha\left[q_i\widetilde{n}_i+q_e\widetilde{n}_e\right]\right\|^2.\tag{6.39}$$

Finally, letting $\kappa_2 \gg \kappa_3 \gg \kappa_4 \gg \kappa_5 \gg \kappa_6 > 0$, we get from the summation of (6.28), (6.30)$\times\kappa_3$, (6.37)$\times\kappa_4$, (6.38)$\times\kappa_5$ and (6.39)$\times\kappa_6$ that

$$\begin{aligned}
&\frac{d}{dt}\widetilde{\eta}+\kappa_1\frac{d}{dt}\left\{((m_in_i+m_en_e)\widetilde{u}_1,\partial_x(\widetilde{n}_i+\widetilde{n}_e))+\frac{3}{2}\left(\partial_x\widetilde{n}_i,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_i^r}\partial_x\widetilde{n}_i\right)\right.\\
&\qquad\left.+\frac{3}{2}\left(\partial_x\widetilde{n}_e,\frac{\mu_i(\theta)+\mu_e(\theta)}{n_e^r}\partial_x\widetilde{n}_e\right)\right\}\\
&+\kappa_3\frac{d}{dt}\sum_{|\alpha|=1}\left\{\|\partial^\alpha\widetilde{n}_i\|^2+\|\partial^\alpha\widetilde{n}_e\|^2+\sum_{j=1}^3\left\|\sqrt{m_in_i+m_en_e}\partial^\alpha\widetilde{u}_j\right\|^2+\left\|\sqrt{n_i+n_e}\partial^\alpha\widetilde{\theta}\right\|^2\right\}\\
&+\kappa_4\frac{d}{dt}\sum_{|\alpha|=1}((m_in_i+m_en_e)\partial^\alpha\widetilde{u}_1,\partial_x\partial^\alpha(\widetilde{n}_i+\widetilde{n}_e))\\
&+\lambda\sum_{1\le|\alpha|\le2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\lambda\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2+\lambda\sum_{|\alpha|\le1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2\\
&+\lambda\sum_{|\alpha|=2}\left\|\partial^\alpha\partial_x^2\phi\right\|^2+\lambda\int_{\mathbb{R}}\partial_xu_1^r\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]^2dx\\
&\le C(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+C\delta_r^{1/6}(1+t)^{-7/6}+C_\eta\sum_{1\le|\alpha|\le2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}|^2d\xi dx\\
&+C\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_\sharp^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx+C\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx. \qquad (6.40)
\end{aligned}$$

Noticing that

$$\widetilde{\eta}\sim\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2,\quad\left|\frac{d}{dt}\|\partial_x\phi\|^2\right|\lesssim\left|(\partial_t\partial_x\phi,\partial_x\phi)\right|,$$

we see that (5.13) follows from (6.40). This concludes the proof of Proposition 5.1. □

7. A priori estimates on the non-fluid part

With estimates on the fluid part in Proposition 5.1, this section is further devoted to the proof of Proposition 5.2 on the non-fluid part. In a way similar to the previous section, the proof is divided by three subsections.

7.1. **Estimate on zero-order dissipation.** The goal of this subsection is to obtain the dissipation of $\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}$. Notice that $\widetilde{\mathbf{G}}$ solves

$$\begin{aligned}
&\partial_t\widetilde{\mathbf{G}}+\frac{3\partial_x\phi(\xi_1-u_1)(q_im_e-q_em_i)}{2\theta(m_in_i+m_en_e)}\left[n_eM_i,-n_iM_e\right]^{\mathrm{T}}-\mathbf{L}_{\mathbf{M}}\widetilde{\mathbf{G}}\\
&=-\frac{3}{2\theta}\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[m_iM_i,m_eM_e\right]^{\mathrm{T}}\left(\xi\cdot\partial_x\widetilde{u}+\frac{|\xi-u|^2}{2\theta}\partial_x\widetilde{\theta}\right)\right\}-\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[n_i^{-1}M_i\partial_x\widetilde{n}_i,n_e^{-1}M_e\partial_x\widetilde{n}_e\right]^{\mathrm{T}}\right\}\\
&\quad+\frac{3}{2\theta}\mathbf{P}_1^{\mathbf{M}}\left\{\left[M_i,M_e\right]^{\mathrm{T}}\xi_1\right\}\partial_x\widetilde{\theta}-\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{G}\right)-\mathbf{P}_1^{\mathbf{M}}\left(q_0\partial_x\phi\partial_{\xi_1}\mathbf{G}\right)+\mathbf{Q}(\mathbf{G},\mathbf{G})-\partial_t\overline{\mathbf{G}}, \qquad (7.1)
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{M}\right)-L_{\mathbf{M}}\overline{\mathbf{G}}=&\frac{3}{2\theta}\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[m_iM_i,m_eM_e\right]^{\mathrm{T}}\left(\xi\cdot\partial_x\widetilde{u}+\frac{|\xi-u|^2}{2\theta}\partial_x\widetilde{\theta}\right)\right\}\\
&+\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[n_i^{-1}M_i\partial_x\widetilde{n}_i,n_e^{-1}M_e\partial_x\widetilde{n}_e\right]^{\mathrm{T}}\right\}+\frac{3}{2\theta}\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[M_i,M_e\right]^{\mathrm{T}}\right\}\partial_x\widetilde{\theta},
\end{aligned}$$

and

$$\mathbf{P}_1^{\mathbf{M}}\left(q_0\partial_x\phi\partial_{\xi_1}\mathbf{M}\right)=\frac{3\partial_x\phi(\xi_1-u_1)(q_im_e-q_em_i)}{2\theta(m_in_i+m_en_e)}\left[n_eM_i,-n_iM_e\right]^{\mathrm{T}}.$$

Let $\alpha_0=0$ or 1. Taking the inner product of $\partial_t^{\alpha_0}(7.1)$ with $(n^r)^{-1}\theta(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}$ over $\mathbb{R}\times\mathbb{R}^3$, one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}^3}(n^r)^{-1}\theta(m_in_i+m_en_e)\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\cdot\left(\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)d\xi dx+\mathcal{J}_1+\mathcal{J}_2=\sum_{l=3}^{10}\mathcal{J}_l, \tag{7.2}$$

where $\mathcal{J}_l$ $(1\le\mathcal{J}\le10)$ are given by

$$\begin{aligned}
\mathcal{J}_1&=-\int_{\mathbb{R}\times\mathbb{R}^3}(n^r)^{-1}\theta(m_in_i+m_en_e)\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\cdot\left(\mathbf{M}_*^{-1}\mathbf{L}_{\mathbf{M}}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)d\xi dx,\\
\mathcal{J}_2&=\frac{3}{2}\left(\partial_t^{\alpha_0}\partial_x\phi(\xi_1-u_1)(q_im_e-q_em_i)\left[n_eM_i,-n_iM_e\right]^{\mathrm{T}},(n^r)^{-1}\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_3&=-\chi_{\alpha_0}\left(\partial_t\widetilde{\mathbf{G}}\partial_t^{\alpha_0}\left((n^r)^{-1}\theta(m_in_i+m_en_e)\right),\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)\\
&\quad+\frac{1}{2}\left(\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\partial_t\left((n^r)^{-1}\theta(m_in_i+m_en_e)\right),\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_4&=-\frac{3}{2}\left(\partial_x\phi(q_im_e-q_em_i)\partial_t^{\alpha_0}\left\{\frac{\xi_1-u_1}{m_in_i+m_en_e}\left[n_eM_i,-n_iM_e\right]^{\mathrm{T}}\right\},\frac{\theta}{n^r}(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_5&=-\frac{3}{2}\left(\partial_t^{\alpha_0}\left\{\frac{1}{\theta}\mathbf{P}_1^{\mathbf{M}}\left[\xi_1\left[m_iM_i,m_eM_e\right]^{\mathrm{T}}\left(\xi\cdot\partial_x\widetilde{u}+\frac{|\xi-u|^2}{2\theta}\partial_x\widetilde{\theta}\right)\right]\right\},\frac{\theta}{n^r}(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_6&=\left(\partial_t^{\alpha_0}\left\{\mathbf{P}_1^{\mathbf{M}}\left[\xi_1\left[\frac{M_i}{n_i}\partial_x\widetilde{n}_i,n_e^{-1}M_e\partial_x\widetilde{n}_e\right]^{\mathrm{T}}+\frac{3}{2\theta}\left[M_i,M_e\right]^{\mathrm{T}}\xi_1\partial_x\widetilde{\theta}\right]\right\},\frac{\theta}{n^r}(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_7&=-\left(\partial_t^{\alpha_0}\mathbf{P}_1^{\mathbf{M}}\left[q_0\partial_x\phi\partial_{\xi_1}\mathbf{G}\right],(n^r)^{-1}\theta(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_8&=-\left(\partial_t^{\alpha_0}\partial_t\overline{\mathbf{G}},(n^r)^{-1}\theta(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)\\
&\quad-\left(\partial_t^{\alpha_0}\mathbf{P}_1^{\mathbf{M}}\left[\xi_1\partial_x\mathbf{G}\right],(n^r)^{-1}\theta(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_9&=\chi_{\alpha_0}\left(\mathbf{Q}(\partial_t\mathbf{M},\mathbf{G})+\mathbf{Q}(\mathbf{G},\partial_t\mathbf{M}),(n^r)^{-1}\theta(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t\widetilde{\mathbf{G}}\right),\\
\mathcal{J}_{10}&=\left(\partial_t^{\alpha_0}\mathbf{Q}(\mathbf{G},\mathbf{G}),(n^r)^{-1}\theta(m_in_i+m_en_e)\mathbf{M}_*^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right).
\end{aligned}$$

Here we have used the notation

$$\chi_{\alpha_0}=\begin{cases}0, & \alpha_0=0,\\ 1, & \alpha_0>0.\end{cases}$$

From Lemma 4.3, we see that

$$\mathcal{J}_1\gtrsim\delta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2d\xi dx.$$

For $\mathcal{J}_2$, if $\alpha_0=0$, it is bounded by

$$\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi+C_\eta\|\partial_x\phi\|^2.$$

If $\alpha_0 = 1$, we first rewrite $\mathcal{J}_2$ as

$$\begin{aligned}
\mathcal{J}_2 =&\frac{3}{2}\left(\partial_t\partial_x\phi\xi_1(q_im_e - q_em_i)\left[n_eM_i, -n_iM_e\right]^{\mathrm{T}},\ (n^r)^{-1}\mathbf{M}^{-1}\partial_t\widetilde{\mathbf{G}}\right)\\
&+\frac{3}{2}\left(\partial_t\partial_x\phi\xi_1(q_im_e - q_em_i)\left[n_eM_i, -n_iM_e\right]^{\mathrm{T}},\ (n^r)^{-1}\left(\mathbf{M}_*^{-1} - \mathbf{M}^{-1}\right)\partial_t\widetilde{\mathbf{G}}\right)\\
=&\frac{3}{2}\left(\partial_t\partial_x\phi\xi_1(q_im_e - q_em_i)\left[(n_e - n_e^r)M_i, -(n_i - n_i^r)M_e\right]^{\mathrm{T}},\ (n^r)^{-1}\mathbf{M}^{-1}\partial_t\widetilde{\mathbf{G}}\right)\\
&+\frac{3}{2}\left(\partial_t\partial_x\phi\xi_1(q_im_e - q_em_i)\left[M_i, \frac{q_e}{q_i}M_e\right]^{\mathrm{T}},\ \mathbf{M}^{-1}\partial_t\widetilde{\mathbf{G}}\right)\\
&+\frac{3}{2}\left(\partial_t\partial_x\phi\xi_1(q_im_e - q_em_i)\left[n_eM_i, -n_iM_e\right]^{\mathrm{T}},\ (n^r)^{-1}\left(\mathbf{M}_*^{-1} - \mathbf{M}^{-1}\right)\partial_t\widetilde{\mathbf{G}}\right)\\
:=&\mathcal{J}_{2,1} + \mathcal{J}_{2,2} + \mathcal{J}_{2,3}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&|n_i(t,x) - n_{*i}| + |n_e(t,x) - n_{*e}| + |u(t,x) - u_*| + |\theta(t,x) - \theta_*|\\
&\le |n_i^r(t,x) - n_i(t,x)| + |n_e^r(t,x) - n_e(t,x)| + |u^r(t,x) - u(t,x)| + |\theta^r(t,x) - \theta(t,x)|\\
&\quad + |n_i^r(t,x) - n_{*i}| + |n_e^r(t,x) - n_{*e}| + |u^r(t,x) - u_*| + |\theta^r(t,x) - \theta_*|\\
&\lesssim \left|\left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta}\right]\right| + \eta_0.
\end{aligned}$$

From this together with the Cauchy-Schwarz inequality, it follows that

$$|\mathcal{J}_{2,3}| \lesssim (\epsilon_0 + \eta_0)\left\|\partial_t\partial_x\phi\right\|^2 + (\epsilon_0 + \eta_0)\int_{\mathbb{R}\times\mathbb{R}^3}(1 + |\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t\widetilde{\mathbf{G}}\right|^2 d\xi.$$

Moreover, one can see that $\mathcal{J}_{2,1}$ also enjoys the same upper bound as $\mathcal{J}_{2,3}$.

As to $\mathcal{J}_{2,2}$, from integration by parts and using the first equations of (2.10) and (2.11) as well as (5.6), one has

$$\begin{aligned}
\mathcal{J}_{2,1} =& -\frac{3}{2q_i}\left(\partial_t\phi\xi_1(q_im_e - q_em_i)\left[q_i, q_e\right]^{\mathrm{T}},\ \partial_t\partial_x\widetilde{\mathbf{G}}\right)\\
=& -\frac{3}{2q_i}\left(\partial_t\phi\xi_1(q_im_e - q_em_i)\left[q_i, q_e\right]^{\mathrm{T}},\ \partial_t\partial_x\mathbf{G}\right)\\
&+\frac{3}{2q_i}\left(\partial_x\partial_t\phi\xi_1(q_im_e - q_em_i)\left[q_i, q_e\right]^{\mathrm{T}},\ \partial_t\overline{\mathbf{G}}\right)\\
=&\frac{3(q_im_e - q_em_i)}{2q_i}\left(\partial_t\phi, \left(\partial_t(q_in_i + q_en_e) + \partial_x((q_in_i + q_en_e)u_1)\right)\right)\\
&+\frac{3}{2q_i}\left(\partial_x\partial_t\phi\xi_1(q_im_e - q_em_i)\left[q_i, q_e\right]^{\mathrm{T}},\ \partial_t\overline{\mathbf{G}}\right)\\
=& -\frac{3(q_im_e - q_em_i)}{4q_i}\frac{d}{dt}\|\partial_t\partial_x\phi\|^2 + \frac{3(q_im_e - q_em_i)}{2q_i}\left(\partial_x\partial_t\phi, (q_in_i + q_en_e)u_1\right)\\
&+\frac{3}{2q_i}\left(\partial_x\partial_t\phi\xi_1(q_im_e - q_em_i)\left[q_i, q_e\right]^{\mathrm{T}},\ \partial_t\overline{\mathbf{G}}\right).
\end{aligned}$$

Thus it holds that

$$\left|\mathcal{J}_{2,1} + \frac{3(q_im_e - q_em_i)}{4q_i}\frac{d}{dt}\|\partial_t\partial_x\phi\|^2\right| \lesssim \|\partial_x\partial_t\phi\|^2 + \|q_in_i + q_en_e\|^2 + \delta_r^{1/2}(1+t)^{-3/2}.$$

Next, we get from (5.9) that

$$|\mathcal{J}_3| \le \epsilon_0\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1 + |\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx.$$

By applying Lemma 3.1 together with (5.9), one can see that $\mathcal{J}_4$, $\mathcal{J}_5$ and $\mathcal{J}_6$ can be bounded as follows:

$$|\mathcal{J}_4|+|\mathcal{J}_5|+|\mathcal{J}_6|\leq \eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx+C_\eta\sum_{\alpha_0\leq 1}\left\|\partial_x\partial_t^{\alpha_0}\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\phi\right]\right\|^2,$$

$$\begin{aligned}|\mathcal{J}_7|\leq&\sum_{\alpha_0\leq 1}\left|\left(\partial_x\partial_t^{\alpha_0}\phi\partial_{\xi_1}\widetilde{\mathbf{G}},\mathbf{M}^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)\right|+\sum_{\alpha_0\leq 1}\left|\left(\partial_x\partial_t^{\alpha_0}\phi\partial_{\xi_1}\overline{\mathbf{G}},\mathbf{M}^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)\right|\\&+\sum_{\alpha_0\leq 1}\left|\left(\partial_x\phi\partial_{\xi_1}\widetilde{\mathbf{G}},\mathbf{M}^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)\right|+\left|\left(\partial_x\phi\partial_{\xi_1}\overline{\mathbf{G}},\mathbf{M}^{-1}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right)\right|\\&+\sum_{\alpha_0\leq 1}\eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx+C_\eta\sum_{\alpha_0\leq 1}\|\partial_x\partial_t^{\alpha_0}\phi\|^2\\\leq&(\epsilon_0+\eta)\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx+C_\eta\sum_{\alpha_0\leq 1}\|\partial_x\partial_t^{\alpha_0}\phi\|^2,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{J}_8|\leq&\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx+C_\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx\\&+C_\eta\delta_r^{1/2}(1+t)^{-3/2}.\end{aligned}$$

As to $\mathcal{J}_9$ and $\mathcal{J}_{10}$, it follows from (6.7), Lemma 4.1 and Cauchy-Schwarz's inequality with η that

$$\begin{aligned}|\mathcal{J}_9|\leq&\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t\widetilde{\mathbf{G}}\right|^2 d\xi dx\\&+C_\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t\mathbf{M}\right|^2 d\xi\right)\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi\right)dx\\&+C_\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|\left|\partial_t\mathbf{M}_*^{-1/2}\mathbf{M}\right|^2 d\xi\right)\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\overline{\mathbf{G}}\right|^2 d\xi\right)dx\\\leq&(\epsilon_0+\eta)\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx\\&+C_\eta\delta_r\left\|\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\delta_r^{1/2}(1+t)^{-3/2},\end{aligned}$$

and

$$\begin{aligned}|\mathcal{J}_{10}|\leq&\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx\\&+C_\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\mathbf{G}\right|^2 d\xi\right)\left(\int_{\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\mathbf{G}\right|^2 d\xi\right)dx\\&+C_\eta\sum_{\alpha_0\leq 1}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|\left|\mathbf{M}_*^{-1/2}\mathbf{G}\right|^2 d\xi\right)\left(\int_{\mathbb{R}^3})\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\mathbf{G}\right|^2 d\xi\right)dx\\\leq&(\epsilon_0+\eta)\sum_{\alpha_0\leq 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx+C\delta_r(1+t)^{-2}.\end{aligned}$$

Now substituting all the above estimates into (7.2), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx+\frac{d}{dt}\|\partial_t\partial_x\phi\|^2\\
&\quad+\lambda\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx\\
&\lesssim\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\frac{(1+|\xi|)\left|\partial_x\partial_t^{\alpha_0}\mathbf{G}\right|^2}{\mathbf{M}_*}d\xi dx+\sum_{1\le|\alpha|\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&\quad+\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}.
\end{aligned}\tag{7.3}$$

Furthermore, it follows from (7.1) that

$$\begin{aligned}
\partial_t^2\widetilde{\mathbf{G}}=&\partial_t\left\{\frac{3\partial_x\phi(\xi_1-u_1)(q_im_e-q_em_i)}{2\theta(m_in_i+m_en_e)}\left[n_eM_i,-n_iM_e\right]^{\mathrm{T}}\right\}\\
&-\partial_t\left\{\frac{3}{2\theta}\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[m_iM_i,m_eM_e\right]^{\mathrm{T}}\left(\xi\cdot\partial_x\widetilde{u}+\frac{|\xi-u|^2}{2\theta}\partial_x\widetilde{\theta}\right)\right\}\right\}\\
&-\partial_t\left\{\mathbf{P}_1^{\mathbf{M}}\left\{\xi_1\left[n_i^{-1}M_i\partial_x\widetilde{n}_i,n_e^{-1}M_e\partial_x\widetilde{n}_e\right]^{\mathrm{T}}\right\}\right\}\\
&+\partial_t\left\{\frac{3}{2\theta}\mathbf{P}_1^{\mathbf{M}}\left\{\left[M_i,M_e\right]^{\mathrm{T}}\xi_1\right\}\partial_x\widetilde{\theta}\right\}-\partial_t\left\{\mathbf{P}_1^{\mathbf{M}}\left(\xi_1\partial_x\mathbf{G}\right)+\mathbf{P}_1^{\mathbf{M}}\left(q_0\partial_x\phi\partial_{\xi_1}\mathbf{G}\right)\right\}\\
&+\partial_t(\mathbf{Q}(\mathbf{M},\mathbf{G})+\mathbf{Q}(\mathbf{G},\mathbf{M}))+\partial_t\mathbf{Q}(\mathbf{G},\mathbf{G})-\partial_t^2\overline{\mathbf{G}}.
\end{aligned}\tag{7.4}$$

Then (7.3) and (7.4) give rise to

$$\begin{aligned}
&\frac{d}{dt}\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx+\frac{d}{dt}\left\|\partial_t\partial_x\phi\right\|^2\\
&\quad+\lambda\sum_{\alpha_0\le 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2 d\xi dx\\
&\lesssim\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\frac{(1+|\xi|)\left|\partial_x\partial_t^{\alpha_0}\mathbf{G}\right|^2}{\mathbf{M}_*}d\xi dx+\sum_{1\le|\alpha|\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&\quad+\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2+\delta_r^{1/2}(1+t)^{-3/2}.
\end{aligned}\tag{7.5}$$

7.2. **Estimate on high-order energy.** In this subsection, let us now deduce estimates on the higher order energy of $\mathbf{F}$. The desired estimates will be obtained by the interplay of two kinds of weighted energy estimates. Let $|\alpha|\le 1$. Taking the $L^2\times L^2$ inner product of (2.5) with $k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}$ with respect to x and ξ over $\mathbb{R}\times\mathbb{R}^3$, one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}^3}k_B\theta\partial_x\partial^\alpha\mathbf{F}\cdot\left(\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\right)d\xi dx+\mathcal{K}_1+\mathcal{K}_2=\sum_{l=3}^{9}\mathcal{K}_l,\tag{7.6}$$

where all terms $\mathcal{K}_l$ $(1\le l\le 9)$ are given by

$$\begin{aligned}
\mathcal{K}_1=&-\left(\mathbf{L}_{\mathbf{M}}\partial_x\partial^\alpha\mathbf{G},k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{G}\right),\\
\mathcal{K}_2=&\left(q_0\partial^\alpha\partial_x^2\phi\partial_{\xi_1}\mathbf{M},k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_3 &= \frac{1}{2}\left(\partial_x\partial^\alpha \mathbf{F}, k_B\theta\partial_t\left(\mathbf{M}^{-1}\right)\partial_x\partial^\alpha\mathbf{F}\right) + \frac{1}{2}\left(\partial_x\partial^\alpha\mathbf{F}, k_B\partial_t\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\right),\\
\mathcal{K}_4 &= \sum_{\alpha'\le\alpha}\Big(\mathbf{Q}(\partial^{\alpha'}\partial_x\mathbf{M}, \partial^{\alpha-\alpha'}\mathbf{G}) + \mathbf{Q}(\partial^{\alpha-\alpha'}\mathbf{G}, \partial^{\alpha'}\partial_x\mathbf{M}), k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\Big),\\
\mathcal{K}_5 &= \left(\mathbf{L}_{\mathbf{M}}\partial_x\partial^\alpha\mathbf{G}, k_B\theta\mathbf{P}_1^{\mathbf{M}}\left(\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{M}\right)\right),\\
\mathcal{K}_6 &= -\left(\xi_1\partial_x^2\partial^\alpha\mathbf{F}, k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\right),\\
\mathcal{K}_7 &= -\sum_{\alpha'\le\alpha}C_{\alpha'}^{\alpha}\Big(q_0\partial^{\alpha-\alpha'}\partial_x\phi\partial^{\alpha'}\partial_x\partial_{\xi_1}\mathbf{F}, k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\Big),\\
\mathcal{K}_8 &= -\left(q_0\partial^\alpha\partial_x^2\phi\partial_{\xi_1}\mathbf{G}, k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\right),\\
\mathcal{K}_9 &= \left(\partial_x\partial^\alpha\mathbf{Q}(\mathbf{G},\mathbf{G}), k_B\theta\mathbf{M}^{-1}\partial_x\partial^\alpha\mathbf{F}\right).
\end{aligned}$$

Here we have used the decomposition $\mathbf{F}=\mathbf{M}+\mathbf{G}$. First of all, for $\mathcal{K}_1$, Lemma 4.3 implies that

$$\mathcal{K}_1 \ge \delta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\partial^\alpha\mathbf{G}\right|^2 d\xi dx.$$

For $\mathcal{K}_2$, from the first equations of (2.10) and (2.11), we claim that

$$\left|\mathcal{K}_2 - \frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 1}\left\|\partial^\alpha\partial_x^2\phi\right\|^2\right| \lesssim \epsilon_0\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi, \partial_x^2\phi\right]\right\|^2. \tag{7.7}$$

In fact, to show (7.7), we notice

$$\begin{aligned}
\mathcal{K}_2 =& -\left(\partial^\alpha\partial_x^2\phi[q_i,q_e]^{\mathrm{T}}, (\xi_1-u_1)\partial^\alpha\partial_x\mathbf{M}\right) - \left(\partial^\alpha\partial_x^2\phi[q_i,q_e]^{\mathrm{T}}, (\xi_1-u_1)\partial^\alpha\partial_x\mathbf{G}\right)\\
&-\left(\partial^\alpha\partial_x^2\phi[q_i,q_e]^{\mathrm{T}}, (\xi_1-u_1)\partial^\alpha\partial_x\mathbf{M}\right) - \left(\partial^\alpha\partial_x^2\phi[q_i,q_e]^{\mathrm{T}}, \xi_1\partial^\alpha\partial_x\mathbf{G}\right),
\end{aligned}$$

with $|\alpha|\le 1$. Here, by direct computations, it holds that

$$\partial_x M_{\mathcal{A}} = \frac{\partial_x n_{\mathcal{A}}}{n_{\mathcal{A}}}M_{\mathcal{A}} + \frac{\xi-u}{k_{\mathcal{A}}\theta}\cdot\partial_x u M_{\mathcal{A}} + \left(\frac{|\xi-u|^2}{2k_{\mathcal{A}}\theta}-\frac{3}{2}\right)\frac{\partial_x\theta}{\theta}M_{\mathcal{A}},$$

for $\mathcal{A}=i,e$. Then, from the first equations of (2.10) and (2.11), it follows that for $|\alpha|=0$,

$$\begin{aligned}
\mathcal{K}_2 &= -\left(\partial_x^2\phi, (q_in_i+q_en_e)\partial_x u_1 + [q_i,q_e]^{\mathrm{T}}\cdot\xi_1\partial_x\mathbf{G}\right)\\
&= \left(\partial_x^2\phi, \partial_t(q_in_i+q_en_e) + \partial_x(q_in_i+q_en_e)u_1\right)\\
&= -\left(\partial_x^2\phi, \partial_t\partial_x^2\phi\right) - \left(\partial_x^2\phi, \partial_x^3\phi u_1\right),
\end{aligned}$$

and hence one has from integration by parts and (5.9) that

$$\left|\mathcal{K}_2 + \frac{d}{d}\|\partial_x^2\phi\|^2\right| \lesssim \epsilon_0\|\partial_x^2\phi\|^2, \tag{7.8}$$

for $|\alpha=0|$. Furthermore, for $|\alpha|=1$, one can also obtain from direct calculations that

$$\begin{aligned}
\mathcal{K}_2 =& -\left(\partial^\alpha\partial_x^2\phi, (q_in_i+q_en_e)\partial^\alpha\partial_x u_1\right)\\
&-\left(\partial^\alpha\partial_x^2\phi, (q_i\partial^\alpha n_i+q_e\partial^\alpha n_e)\partial_x u_1 + (q_i\partial_x n_i+q_e\partial_x n_e)\partial^\alpha u_1\right)\\
&-\left(\partial^\alpha\partial_x^2\phi, [q_i,q_e]^{\mathrm{T}}\cdot\xi_1\partial^\alpha\partial_x\mathbf{G}\right)\\
=&\left(\partial^\alpha\partial_x^2\phi, \partial^\alpha\partial_t(q_in_i+q_en_e) + (q_i\partial^\alpha\partial_x n_i+q_e\partial^\alpha\partial_x n_e)u_1\right)\\
=& -\left(\partial^\alpha\partial_x^2\phi, \partial_t\partial^\alpha\partial_x^2\phi + \partial^\alpha\partial_x^3\phi u_1\right),
\end{aligned}$$

which implies

$$\left|\mathcal{K}_2 + \frac{d}{d}\|\partial^\alpha\partial_x^2\phi\|^2\right| \lesssim \epsilon_0\|\partial^\alpha\partial_x^2\phi\|^2, \tag{7.9}$$

for $|\alpha|=1$. Therefore (7.7) follows from (7.8) and (7.9). This completes the estimate on $\mathcal{K}_2$.

For the remaining terms in (7.6), we only give estimates in the case of $|\alpha|=0$ as the proof in the case of $|\alpha|=1$ is similar. For this, by applying Lemma 3.1, Sobolev's inequality and Cauchy Schwarz inequality, we have that for $|\alpha|=0$,

$$\begin{aligned}
|\mathcal{K}_3| \lesssim & \int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)|\partial_t[n_i,n_e,u,\theta]|\left(\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{M}\right|^2+\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{G}\right|^2\right)d\xi dx\\
\lesssim & \int_{\mathbb{R}}\left|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right|^2|\partial_t\left[n_i,n_e,u,\theta\right]|\,dx+\int_{\mathbb{R}}|\partial_x[n^r,u^r,\theta^r]|^2|\partial_t[n^r,u^r,\theta^r]|\,dx\\
& +\int_{\mathbb{R}}|\partial_x[n^r,u^r,\theta^r]|^2\left|\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right|dx+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx\\
\lesssim & \left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\|\partial_t[n^r,u^r,\theta^r]\|_{L^\infty}+\|\partial_t[n^r,u^r,\theta^r]\|_{L^\infty}\|\partial_x[n^r,u^r,\theta^r]\|^2\\
& +\left\|\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^{1/2}\left\|\partial_x\partial_t\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^{1/2}\|\partial_x[n^r,u^r,\theta^r]\|^2\\
& +\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx\\
\lesssim & (\epsilon_0+\delta_r)\sum_{|\alpha|=1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx\\
& +\delta_r^{1/6}(1+t)^{-7/6}.
\end{aligned}$$

For $\mathcal{K}_4$, one sees that for $|\alpha|=0$, $\mathcal{K}_4$ reduces to

$$\left(\mathbf{Q}(\partial_x\mathbf{M},\mathbf{G})+\mathbf{Q}(\mathbf{G},\partial_x\mathbf{M}),\mathbf{M}^{-1}\partial_x\mathbf{G}\right),$$

and hence we have

$$\begin{aligned}
|\mathcal{K}_4| \lesssim & \eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx\\
& +C_\eta\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\partial_x\mathbf{M}|^2d\xi\right)\left(\int_{\mathbb{R}^3}|\mathbf{M}^{-1/2}(\widetilde{\mathbf{G}}+\overline{\mathbf{G}})|^2d\xi\right)dx\\
& +C_\eta\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}|\mathbf{M}^{-1/2}\partial_x\mathbf{M}|^2d\xi\right)\left(\int_{\mathbb{R}^3}(1+|\xi|)|\mathbf{M}^{-1/2}\widetilde{\mathbf{G}}+\overline{\mathbf{G}}|^2d\xi\right)dx\\
\lesssim & \eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
& +\epsilon_0\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\delta_r(1+t)^{-2}.
\end{aligned}$$

For $\mathcal{K}_5$, it should vanish for $|\alpha|=0$. For $\mathcal{K}_6$, by using integration by parts and performing the similar calculations as for $\mathcal{K}_3$, one sees that for $|\alpha|=0$, $|\mathcal{K}_6|$ is bounded by

$$\begin{aligned}
& \int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\partial_x\left[n_i,n_e,u,\theta\right]\right|\left(\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{M}\right|^2+\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{G}\right|^2\right)d\xi dx\\
\lesssim & \sum_{|\alpha|=1}(\epsilon_0+\delta_r)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx+\delta_r^{1/6}(1+t)^{-7/6}.
\end{aligned}$$

For $\mathcal{K}_7$ with $|\alpha|=0$, using $\mathbf{F}=\mathbf{M}+\mathbf{G}$ again, one has

$$
\begin{aligned}
|\mathcal{K}_7| \lesssim & \left|\left(q_0\partial_x\phi\partial_x\partial_{\xi_1}\mathbf{M},\mathbf{M}^{-1}\partial_x\mathbf{M}\right)\right|+\left|\left(q_0\partial_x\phi\partial_x\partial_{\xi_1}\mathbf{G},\mathbf{M}^{-1}\partial_x\mathbf{G}\right)\right| \\
& +\left|\left(q_0\partial_x\phi\partial_x\partial_{\xi_1}\mathbf{M},\mathbf{M}^{-1}\partial_x\mathbf{G}\right)\right| \\
\lesssim & \int_{\mathbb{R}}|\partial_x\phi||\partial_x[n_i,n_e,u,\theta]|^2dx+\int_{\mathbb{R}\times\mathbb{R}^3}|\partial_x\phi||\mathbf{M}^{-1/2}\partial_x\partial_{\xi_1}\overline{\mathbf{G}}|^2d\xi dx \\
& +\int_{\mathbb{R}\times\mathbb{R}^3}|\partial_x\phi||\mathbf{M}^{-1/2}\partial_x\partial_{\xi_1}\widetilde{\mathbf{G}}|^2d\xi dx+\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}|\partial_x\phi|\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx \\
& +C_\eta\sum_{|\alpha|=1}\|\partial_x\phi\partial_x[n_i,n_e,u,\theta]\|^2+\eta\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx \\
\lesssim & \sum_{|\alpha|=1}(\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\epsilon_0\|\partial_x\phi\|^2+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\partial_{\xi_1}\widetilde{\mathbf{G}}\right|^2d\xi dx \\
& +(\epsilon_0+\eta)\sum_{|\alpha|=1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx+C_\eta\delta_r^{1/6}(1+t)^{-7/6}.
\end{aligned}
$$

Likewise, for $\mathcal{K}_8$ with $|\alpha|=0$, it follows that

$$
\begin{aligned}
|\mathcal{K}_8| \le & \left|-\left(q_0\partial_x^2\phi\partial_{\xi_1}\mathbf{G},\mathbf{M}^{-1}\partial_x\mathbf{G}\right)\right|+\left|-\left(q_0\partial_x^2\phi\partial_{\xi_1}\mathbf{G},\mathbf{M}^{-1}\partial_x\mathbf{M}\right)\right| \\
\lesssim & C_\eta\int_{\mathbb{R}\times\mathbb{R}^3}|\partial_x^2\phi|^2\left|\mathbf{M}^{-1/2}\partial_{\xi_1}\overline{\mathbf{G}}\right|^2d\xi dx+(\epsilon_0+\eta)\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_{\xi_1}\widetilde{\mathbf{G}}\right|^2d\xi dx \\
& +(\epsilon_0+\eta)\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx+C_\eta\int_{\mathbb{R}}|\partial_x^2\phi|^2|\partial_x[n_i,n_e,u,\theta]|^2dx \\
& +\int_{\mathbb{R}\times\mathbb{R}^3}|\partial_x^2\phi||\partial_x[n_i,n_e,u,\theta](1+|\xi|)|\partial_{\xi_1}\overline{\mathbf{G}}|d\xi dx \\
\lesssim & (\epsilon_0+\eta)\left\|\partial_x\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\partial_x\phi\right]\right\|^2+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\partial_{\xi_1}\widetilde{\mathbf{G}}\right|^2d\xi dx \\
& +(\epsilon_0+\eta)\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx+\delta_r(1+t)^{-2}.
\end{aligned}
$$

As to the last term $\mathcal{K}_9$ with $\alpha=0$, we get from Lemma 4.1 and Cauchy Schwarz inequality that

$$
\begin{aligned}
\mathcal{K}_9 \lesssim & \eta\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx \\
& +\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi\right)\left(\int_{\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\mathbf{G}\right|^2d\xi\right)dx \\
& +\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi\right)\left(\int_{\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\mathbf{G}\right|^2d\xi\right)dx \\
\lesssim & (\eta+\epsilon_0)\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\mathbf{G}\right|^2d\xi dx+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx.
\end{aligned}
$$

Substituting all the above estimates for $\mathcal{K}_l$ $(1 \le l \le 9)$ into (7.6) and performing the similar calculation as above for the case $|\alpha| = 1$, one sees that

$$\begin{aligned}
&\frac{d}{dt}\left\{\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial_x\partial^\alpha\mathbf{F}\right|^2 d\xi dx + \sum_{|\alpha|\le 1}\left\|\partial^\alpha\partial_x^2\phi\right\|\right\}\\
&\quad+\lambda\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\partial^\alpha\mathbf{G}\right|^2 d\xi dx\\
&\lesssim(\epsilon_0+\eta)\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\partial^\alpha\mathbf{G}\right|^2 d\xi dx+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx \qquad (7.10)\\
&\quad+(\epsilon_0+\eta)\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}\right|^2 d\xi dx+(\epsilon_0+\eta)\sum_{1\le\gamma\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2\\
&\quad+(\epsilon_0+\eta)\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2+\delta_r^{1/6}(1+t)^{-7/6}.
\end{aligned}$$

Similarly, one can obtain the following energy estimates for $\partial_x\partial^\alpha\mathbf{F}$ $(|\alpha| \le 1)$ with respect to the global Maxwellian $\mathbf{M}_*$:

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial_x\partial^\alpha F\right|^2 d\xi dx+\lambda\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_x\partial^\alpha\mathbf{G}\right|^2 d\xi dx\\
&\lesssim(\epsilon_0+\eta)\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial_{\xi_1}\widetilde{\mathbf{G}}\right|^2 d\xi dx+\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx \qquad (7.11)\\
&\quad+C_\eta\sum_{1\le|\alpha|\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+C_\eta\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2+\delta_r^{1/6}(1+t)^{-7/6}.
\end{aligned}$$

Note that one may not require the smallness of the coefficient of

$$\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta},\partial_x\phi,\partial_x^2\phi\right]\right\|^2,$$

and this implies that the derivation of (7.11) is much simpler than the one of (7.10). Due to this we would omit details of the proof of (7.11) for brevity.

With (7.10) in hand, by letting $1 \gg \kappa_7 > 0$, we get from the summation of (7.10) and (5.13) $\times\,\kappa_7$ that

$$\begin{aligned}
&\kappa_7\frac{d}{dt}\left\{\sum_{\gamma\le 1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\|\partial_x\phi(t)\|^2\right\}-\kappa_7\kappa_0\frac{d}{dt}\sum_{|\alpha|=1}(\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\widetilde{v}_i+\partial^\alpha\partial_x\widetilde{v}_e)\\
&\quad+\frac{d}{dt}\sum_{|\alpha|\le 1}\left\{\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial_x\partial^\alpha\mathbf{F}\right|^2 d\xi dx+\left\|\partial^\alpha\partial_x^2\phi\right\|^2\right\}\\
&\quad+\lambda\left\|\sqrt{\partial_x u^r}\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\lambda\sum_{1\le\gamma\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\lambda\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2 \qquad (7.12)\\
&\quad+\lambda\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2+\lambda\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial_x\partial^\alpha\mathbf{G}\right|^2 d\xi dx\\
&\lesssim(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\delta_r^{1/6}(1+t)^{-7/6}+\sum_{|\alpha|\le 1}\epsilon_0\int_{\mathbb{R}\times\mathbb{R}^3}|\mathbf{M}_*^{-1/2}\partial_{\xi_1}\partial^\alpha\widetilde{\mathbf{G}}|^2 d\xi dx\\
&\quad+\kappa_7\sum_{1\le|\alpha_0|\le 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\mathbf{G}\right|^2 d\xi dx+\kappa_7\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2 d\xi dx.
\end{aligned}$$

On the other hand, by choosing $1 \gg \kappa_8 \gg \kappa_9 > 0$, it follows from the summation of (7.5) $\times\, \kappa_9$ and (7.11) $\times\, \kappa_8$ that

$$\begin{aligned}
&\kappa_9 \frac{d}{dt} \sum_{\alpha_0 \le 1} \int_{\mathbb{R}\times\mathbb{R}^3} \left| \mathbf{M}_*^{-1/2} \partial_t^{\alpha_0} \widetilde{\mathbf{G}} \right|^2 d\xi dx + \kappa_9 \frac{d}{dt} \left\| \partial_t \partial_x \phi \right\|^2 + \kappa_8 \frac{d}{dt} \sum_{1\le|\alpha|\le 2} \int_{\mathbb{R}\times\mathbb{R}^3} \left| \mathbf{M}_*^{-1/2} \partial^\alpha \mathbf{F} \right|^2 d\xi dx \\
&\quad + \lambda \sum_{1\le|\alpha|\le 2} \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \partial^\alpha \mathbf{G} \right|^2 d\xi dx + \lambda \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \widetilde{\mathbf{G}} \right|^2 d\xi dx \\
&\lesssim (\kappa_8+\kappa_9) \sum_{1\le\alpha\le 2} \left\| \partial^\alpha \left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta} \right] \right\|^2 + (\kappa_8+\kappa_9) \sum_{|\alpha|\le 1} \left\| \partial^\alpha \left[\partial_x \phi, \partial_x^2 \phi \right] \right\|^2 \\
&\quad + \delta_r^{1/6} (1+t)^{-7/6} + \sum_{|\alpha|\le 1} \epsilon_0 \int_{\mathbb{R}\times\mathbb{R}^3} |\mathbf{M}^{-1/2} \partial_{\xi_1} \partial^\alpha \widetilde{\mathbf{G}}|^2 d\xi dx,
\end{aligned} \tag{7.13}$$

where we have used the fact that

$$\begin{aligned}
&\sum_{1\le|\alpha|\le 2} \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \partial^\alpha \mathbf{G} \right|^2 d\xi dx \\
&\lesssim \sum_{1\le|\alpha|\le 2} \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \partial^\alpha \widetilde{\mathbf{G}} \right|^2 d\xi dx + \delta_r \sum_{1\le|\alpha|\le 2} \left\| \partial^\alpha \left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta} \right] \right\|^2 + \delta_r^{1/2} (1+t)^{-3/2}.
\end{aligned}$$

7.3. **Estimate on energy with mixed derivatives.** In what follows, we deduce the energy estimates on the mixed derivative terms $\partial^\alpha \partial^\beta \widetilde{\mathbf{G}}$. To do so, let $|\beta| \ge 1$ and $|\alpha| + |\beta| \le 2$. Acting $\partial^\alpha \partial^\beta$ to (7.1) and taking the inner product of the resulting equation with $\mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}}$ over $\mathbb{R}\times\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}\times\mathbb{R}^3} \left| \mathbf{M}_*^{-1/2} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right|^2 d\xi dx - \int_{\mathbb{R}\times\mathbb{R}^3} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \cdot \left(\mathbf{M}_*^{-1} \mathbf{L}_{\mathbf{M}} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) d\xi dx \\
&= \sum_{\substack{1\le|\alpha'|+|\beta'| \\ \alpha'\le\alpha, \beta'\le\beta}} \left(\mathbf{Q}(\partial^{\alpha'} \partial^{\beta'} \mathbf{M}, \partial^{\alpha-\alpha'} \partial^{\beta-\beta'} \widetilde{\mathbf{G}}), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) \\
&\quad + \sum_{\substack{1\le|\alpha'|+|\beta'| \\ \alpha'\le\alpha, \beta'\le\beta}} \left(\mathbf{Q}(\partial^{\alpha'} \partial^{\beta'} \widetilde{\mathbf{G}}, \partial^{\alpha-\alpha'} \partial^{\beta-\beta'} \mathbf{M}), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) \\
&\quad - \left(\partial^\alpha \partial^\beta \mathbf{P}_1^{\mathbf{M}} \left(\frac{3}{2\theta} \xi_1 \left[m_i M_i, m_e M_e \right]^{\mathrm{T}} \left(\xi \cdot \partial_x \widetilde{u} + \frac{|\xi-u|^2}{2\theta} \partial_x \widetilde{\theta} \right) \right), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) \\
&\quad + \left(\partial^\alpha \partial^\beta \mathbf{P}_1^{\mathbf{M}} \left(\xi_1 \left[n_i^{-1} M_i \partial_x \widetilde{n}_i, n_e^{-1} M_e \partial_x \widetilde{n}_e \right]^{\mathrm{T}} + \frac{3}{2\theta} \left[\left[M_i, M_e \right]^{\mathrm{T}} \xi_1 \partial_x \widetilde{\theta} \right] \right), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) \\
&\quad - \left(\partial^\alpha \partial^\beta \left(\mathbf{P}_1^{\mathbf{M}} \left(\xi_1 \partial_x \mathbf{G} \right) \right), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) - \left(\partial^\alpha \partial^\beta \left(q_0 \partial_x \phi \partial_{\xi_1} \mathbf{M} \right), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) \\
&\quad - \left(\partial^\alpha \partial^\beta \left(q_0 \partial_x \phi \partial_{\xi_1} \mathbf{G} \right), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) + \left(\partial^\alpha \partial^\beta \mathbf{Q}(\mathbf{G}, \mathbf{G}), \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right) \\
&\quad - \left(\partial_t \partial^\alpha \partial^\beta \overline{\mathbf{G}}, \mathbf{M}_*^{-1} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right).
\end{aligned}$$

Similar to those calculations in the previous subsection, it holds that

$$\begin{aligned}
&\frac{d}{dt} \sum_{\substack{|\alpha|+|\beta|\le 2 \\ |\beta|\ge 1}} \int_{\mathbb{R}\times\mathbb{R}^3} \left| \mathbf{M}_*^{-1/2} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right|^2 d\xi dx + \lambda \sum_{\substack{|\alpha|+|\beta|\le 2 \\ |\beta|\ge 1}} \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \partial^\alpha \partial^\beta \widetilde{\mathbf{G}} \right|^2 d\xi dx \\
&\lesssim \sum_{1\le|\alpha|\le 2} \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \partial^\alpha \mathbf{G} \right|^2 d\xi dx + \int_{\mathbb{R}\times\mathbb{R}^3} (1+|\xi|) \left| \mathbf{M}_*^{-1/2} \widetilde{\mathbf{G}} \right|^2 d\xi dx \\
&\quad + \sum_{1\le|\alpha|\le 2} \left\| \partial^\alpha \left[\widetilde{n}_i, \widetilde{n}_e, \widetilde{u}, \widetilde{\theta} \right] \right\|^2 + \sum_{|\alpha|\le 1} \left\| \partial_x \phi \right\|^2 + \delta_r^{1/6} (1+t)^{-7/6}.
\end{aligned} \tag{7.14}$$

Consequently, it follows from (7.12), (7.13) and (7.14) that

$$
\begin{aligned}
&K_0\kappa_7\frac{d}{dt}\left\{\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\|\partial_x\phi(t)\|^2\right\}+K_0\kappa_7\kappa_0\frac{d}{dt}\sum_{|\alpha|=1}(\partial^\alpha\widetilde{u}_1,\partial^\alpha\partial_x\widetilde{n}_i+\partial^\alpha\partial_x\widetilde{n}_e)\\
&+K_0\frac{d}{dt}\sum_{|\alpha|\le 1}\left\{\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}^{-1/2}\partial_x\partial^\alpha\mathbf{F}\right|^2d\xi dx+\left\|\partial^\alpha\partial_x^2\phi\right\|^2\right\}\\
&+\kappa_9\frac{d}{dt}\sum_{\alpha_0\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial_t^{\alpha_0}\widetilde{\mathbf{G}}\right|^2d\xi dx+\kappa_9\frac{d}{dt}\left\|\partial_t\partial_x\phi\right\|^2\\
&+\kappa_8\frac{d}{dt}\sum_{|\alpha|\le 1}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial_x\partial^\alpha\mathbf{F}\right|^2d\xi dx+\kappa_{10}\frac{d}{dt}\sum_{\substack{|\alpha|+|\beta|\le 2\\|\beta|\ge 1}}\int_{\mathbb{R}\times\mathbb{R}^3}\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial^\beta\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&+\lambda\sum_{1\le|\alpha|\le 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial^\alpha\mathbf{G}\right|^2d\xi dx+\lambda\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&+\lambda\sum_{1\le|\alpha|\le 2}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}^{-1/2}\partial^\alpha\mathbf{G}\right|^2d\xi dx+\lambda\sum_{\substack{|\alpha|+|\beta|\le 2\\|\beta|\ge 1}}\int_{\mathbb{R}\times\mathbb{R}^3}(1+|\xi|)\left|\mathbf{M}_*^{-1/2}\partial^\alpha\partial^\beta\widetilde{\mathbf{G}}\right|^2d\xi dx\\
&+\lambda\left\|\sqrt{\partial_x u^r}\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2+\lambda\sum_{1\le|\alpha|\le 2}\left\|\partial^\alpha\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right](t)\right\|^2\\
&+\lambda\left\|q_i\widetilde{n}_i+q_e\widetilde{n}_e\right\|^2+\lambda\sum_{|\alpha|\le 1}\left\|\partial^\alpha\left[\partial_x\phi,\partial_x^2\phi\right]\right\|^2\\
&\lesssim(1+t)^{-2}\left\|\left[\widetilde{n}_i,\widetilde{n}_e,\widetilde{u},\widetilde{\theta}\right]\right\|^2+\delta_r^{1/6}(1+t)^{-7/6},
\end{aligned}
\tag{7.15}
$$

where K_0 is a positive large constant and κ_{10} is also a positive but suitably small constant. Therefore (5.14) follows from (7.15) with the help of the Gronwall's inequality. This completes the proof of Proposition 5.2. □

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