A Comparative Study of Different Entropies in Fractal Universe

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Here we make an attempt to extend the idea of generalized Hawking temperature and modified Bekenstein entropy at event horizon in fractal universe. The modified Hawking temperature and Bekenstein entropy is considered in the governing Friedmann equations, which is modified in the background of a fractal universe. The validity of the Generalised second law of thermodynamics (GSLT) and Thermodynamic Equilibrium (TE) have been examined in a fractal universe filled with perfect fluid having constant equation of state in four different generalized Bekenstein system. Finally both laws are examined and compared numerically in all four cases.

Keywords: Fractal Universe, fractal parameter, Bekenstein entropy, Hawking temperature, Non-equilibrium Thermodynamics

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I. INTRODUCTION

In 1970s Hawking and Bekenstein started the discovery of Black holes (BH) thermodynamics. Since then there is a general opinion that there is a deep connection between gravity and thermodynamics [1–5]. Eventually the BH are regarded as a black body emitting thermal radiations known as Hawking radiation [4, 5]. It was realized that laws of BH physics and thermodynamical laws are equivalent. Furthermore, Jacobson and Padmanabhan did a pioneer work in this direction by establishing a relationship between first law of thermodynamics and Einstein field

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equations [6, 7].

Subsequently these results are generalized in the cosmological system and there have been lot of studies dealing with universe as a thermodynamical system [9].

The cosmological event horizon usually does not exist in standard Big Bang Cosmology. Due to the recent observations [10–15], the universe is in an accelerating phase dominated by dark energy. So the event horizon is assured to exist in this phase and is distinct from the apparent horizon [16–18]. In this context Wang *et al* [19, 20] in 2006 investigated the laws of thermodynamics in an accelerating universe with a time dependent equation of state. They showed that the first and second laws of thermodynamics are satisfied on apparent horizon while the thermodynamical laws break down on cosmological event horizon. As a result, they concluded that the universe bounded by apparent horizon as a Bekenstein system (perfect thermodynamical system) and termed the universe bounded by event horizon as a non Bekenstein system (an unphysical system).

Later it has been shown that the generalized second law of thermodynamics holds (GSLT) (in any gravity theory) with some reasonable restrictions for the Universe bounded by an event horizon under the assumption that the first law holds for Einstein gravity [21, 22] and in other gravity theories [21–23] and for different fluid systems [21, 22, 24] (including dark energy [22, 24]). Subsequently by generalizing the Hawking temperature or modifying the Bekenstein entropy [25, 26] it has been possible to show the validity of both the first and GSLT for the Universe bounded by an event horizon in the Einstein gravity [27].

In the modern era, new observational data [16–18] have suggested that modified theories of gravity may be relevant in order to explain, for instance, the accelerated expansion of the Universe and certain instabilities observed in galaxies. Among all these theories, the brane cosmology received lot of attention in recent years [28–33]. Another theoretical approach was made by Calcagni [34, 35] for a power-counting renormalizable field theory living in a fractal space-time and consequently fractal cosmology was developed. Historically the first appearance of fractal cosmology was in Andrei Linde's paper [36]. For an overview of fractal cosmology one can see the ref [37]. The action in this model is Lorentz covariant and the metric space-time $(\mathcal{M}, \varrho)$ is equipped with a Stieltjes measure ϱ . Very recently it was shown that in a fractal universe, the Friedmann equations can be transformed to Clausius relation, but a treatment with non-equilibrium thermodynamics of space-time is needed [39]. Furthermore, Sheykhi *et al* examined GSLT in a fractal universe on apparent horizon and found that GSLT is valid for particular choice of fractal parameter [38].

In this paper, we shall make an attempt to see the validity of thermodynamical laws in fractal universe filled with perfect fluid having constant equation of state, using modified entropy and temperature. In a sense our main focus is to extend the idea of generalized Hawking temperature and modified Bekenstein entropy in fractal universe. The paper is organized as follows : Section II deals with basic concepts related to earlier works. Brief review and basic equations of fractal cosmology are presented in section III while section IV deals with thermodynamical analysis in this context. Finally, summary of the work and possible conclusions are presented in section V.

II. BASIC EQUATIONS OF UNIVERSAL THERMODYNAMICS

Let us consider the homogeneous and isotropic FRW model of universe expressed by the metric

$$ds^2 = h_{ij}(x^i)dx^i dx^j + R^2 d\Omega_2^2 \tag{1}$$

where $h_{ij} = \text{diag}\left(-1, \frac{a^2}{1-kr^2}\right)$ is the two-dimensional metric tensor, known as normal metric. Here $x^0 = t, x^1 = r$ *i.e.* i, j can take values 0 and 1, $R = ar$ being the area radius considered in the normal 2-D space. On this normal space another relevant scalar quantity is defined as

$$\chi(x) = h_{ij}\partial_i R \partial_j R = 1 - \left(H^2 + \frac{k}{a^2}\right) R^2 \tag{2}$$

where $k = 0, +1, -1$ stands for flat, closed or open model, and $H = \frac{\dot{a}}{a}$ is the Hubble parameter.

The apparent horizon is given by the vanishing of the scalar $\chi(x)$ as

$$R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} \tag{3}$$

which becomes $\frac{1}{H}$ for a flat space (*i.e.* $k = 0$). Again, the definition for event horizon (EH) is $R_E = a\int_t^\infty \frac{dt}{a(t)}$ which exists only in the present accelerating era.

According to the generalized second law of thermodynamics (GSLT), the entropy of an isolated macroscopic physical system should be a non-decreasing function. Also such a system always evolves towards thermodynamic equilibrium (TE). So the following inequalities can be used to verify the validity of GSLT and TE :

$$\begin{aligned} &\text{for GSLT :} \quad \frac{dS_T}{dt} \geq 0 \\ &\text{and for TE :} \quad \frac{d^2 S_T}{dt^2} < 0 \end{aligned} \tag{4}$$

where $S_T = S_h + S_f$, with S_h and S_f denoting respectively the horizon entropy and the entropy of the fluid bounded by the horizon. One can use Gibb's relation to calculate S_f,

$$T_f dS_f = dE_f + p dV_h \tag{5}$$

where V_h is the volume of the fluid, $E_f = \rho V_h$ is the total energy of the fluid and T_f is the temperature of the fluid.

In the present context, it is assumed that the temperature T_f of the cosmic fluid inside the horizon is same as that of the bounding horizon, unless there is a spontaneous flow of energy between the horizon and the fluid which is not consistent with FRW model. So it is assumed that $T_f \propto T^h$ or $T_f = bT^h$, which is widely taken as $T_f = T^h$ to avoid mathematical complexity of non-equilibrium thermodynamics.

III. ANALYSIS IN A FRACTAL UNIVERSE

The fractal properties of quantum gravity theories in n-dimensions have been explored in several contexts. Assuming that matter is minimally coupled with gravity, the total action of Einstein gravity in a fractal space-time is given by,

$$S = S_G + S_m \tag{6}$$

where

$$S_G = \frac{M_P^2}{2} \int d^n x \sqrt{-g}\,(R - 2\Lambda - w\,\partial_\mu v\,\partial^\mu v) \tag{7}$$

is the gravitational part of the action and

$$S_m = \int d^n x \sqrt{-g} \mathcal{L}_m \tag{8}$$

is the matter part of this action. Here g is the determinant of the metric $g_{\mu\nu}$, $M_P = (8\pi G)^{-1/2}$ is reduced Planck mass, Λ is the cosmological constant and R is the Ricci scalar. Also v and w respectively denote the fractional function (plays the role of a weight function of the integral in eq. (8)) and the fractal parameter respectively.

The standard measure in the action is replaced by a nontrivial measure which appears in Lebesgue-Stieltjes integral

$$d^n x = d\varrho(x).$$

The scaling dimension of ϱ is $[\varrho] = -n\alpha \neq -n$, where $\alpha > 0$ is a positive parameter.

Taking the variation of the action (6) with respect to the FRW metric $g_{\mu\nu}$, the Friedmann equations can be obtained in a fractal universe as

$$3\left(H^2+\frac{k}{a^2}\right)=8\pi G\rho-3H\frac{\dot{v}}{v}+\frac{w}{2}\dot{v}^2+\Lambda \tag{9}$$

and

$$6(\dot{H}+H^2)=-8\pi G(\rho+3p)+6H\frac{\dot{v}}{v}-2w\dot{v}^2+3\frac{\Box v}{v}+2\Lambda \tag{10}$$

where k is the curvature constant mentioned as before. Here we evaluate the quantity $\Box v$ by simple tensorial calculation as

$$\Box v=g^{\mu\nu}v_{;\mu\nu}=-\ddot{v}-3H\dot{v}$$

For a flat FRW metric and non-static universe we consider $k=0$ and $\Lambda=0$. Also to avoid mathematical complexity we assume $8\pi G$ to be unity, which yields the following equations

$$3H^2=\rho+\rho_e \tag{11}$$

and

$$2\dot{H}=-(\rho+p)-(\rho_e+p_e) \tag{12}$$

where ρ_e and p_e denote the effective energy density and the effective pressure respectively. These are given by

$$\rho_e=\frac{w}{2}\dot{v}^2-3H\frac{\dot{v}}{v} \tag{13}$$

and

$$p_e=\frac{w}{2}\dot{v}^2+2H\frac{\dot{v}}{v}+\frac{\ddot{v}}{v} \tag{14}$$

To proceed further we need to specify fractional function v. In what follows in order to remain general, we choose following two types of fractional functions [38].

Type I:

First we assume a power law form of the fractional function v as,

$$v=v_0t^{-\beta} \tag{15}$$

where v_0 is an arbitrary constant and β is the fractal dimension. The parameters α and β are related as $\beta = n(1-\alpha)$. Note that for an ultraviolet nontrivial fixed point $\alpha = \frac{2}{n}$ while $\alpha = \frac{4}{n}$ for infraded fixed point [34]. So, in a four-dimensional space ($n=4$), α ranges as $0 < \alpha \leq 1$. Subsequently the equations (13) and (14) take the forms

$$\rho_e = \frac{w}{2}\beta^2 v_0^2 t^{-2(\beta+1)} + 3H\frac{\beta}{t} \tag{16}$$

and

$$p_e = \frac{w}{2}\beta^2 v_0^2 t^{-2(\beta+1)} - 2H\frac{\beta}{t} + \frac{\beta(\beta+1)}{t^2} \tag{17}$$

Type II:

Here we have considered an exponential form to the fractional function v given by

$$v = v_0 e^{-\beta t} \tag{18}$$

Hence equations (13) and (14) become

$$\rho_e = \frac{w}{2}\beta^2 v_0^2 e^{-2\beta t} + 3H\beta \tag{19}$$

and

$$p_e = \frac{w}{2}\beta^2 v_0^2 e^{-2\beta t} - 2H\beta + \beta^2 \tag{20}$$

IV. THERMODYNAMICAL ANALYSIS

In this section we extend the idea of generalized Hawking temperature and modified entropy to fractal universe at event horizon. In what follows we study the four different generalized Bekenstein formulation at event horizon.

From the equations (13) and (14) we found

$$\frac{\partial}{\partial t}(\rho_e + p_e) = \frac{\dot{v}}{v^2}(H\dot{v} - \ddot{v}) - \frac{1}{v}(\dot{H}\dot{v} + H\ddot{v}) + 2w\dot{v}\ddot{v} + \frac{\dddot{v}}{v} \tag{21}$$

using which in eq. (12), one can get

$$\frac{\partial}{\partial t}(\rho + p) = 2f_A H^2 + 4v_A H\dot{H} - \frac{\dot{v}}{v^2}(H\dot{v} - \ddot{v}) + \frac{1}{v}(\dot{H}\dot{v} + H\ddot{v}) - \left(2w\dot{v}\ddot{v} + \frac{\dddot{v}}{v}\right) \tag{22}$$

where $v_A = \dot{R}_A,\ f_A = \dot{v}_A$.

Case-1

Here we consider Bekenstein entropy and the generalized Hawking temperature at the event horizon [27] i.e.,

$$S_h = \frac{\pi R_E^2}{G} \tag{23}$$

$$T^h = T^m = \frac{\alpha R_E}{2\pi R_A^2} \tag{24}$$

where $\alpha = \frac{\dot{R}_A/R_A}{\dot{R}_E/R_E}$ is the reciprocal of the relative growth rate of the radius of the event horizon to that of the apparent horizon.

We now use the equation of continuity but in a modified form due to a fractal universe [38] as

$$\dot{\rho} + \left(3H + \frac{\dot{v}}{v}\right)(\rho + p) = 0 \tag{25}$$

Clearly we can see, for $v = 1$, the equations (9) and (10) reduces to the standard Friedmann equations and eq. (25) to the standard equation of continuity. Here the fractal is taken to be time-like only, so that the fractional function $v = v(t)$ depends only on time. Therefore, considering the Gibb's relation (5), we obtain

$$dS_f = \frac{1}{T^m} 4\pi R_E^2 (\rho + p)\left(1 + \frac{\dot{v} R_E}{3v}\right) dt \tag{26}$$

where we have considered the bounded fluid distribution with spherical volume $V_h = \frac{4}{3}\pi R_E^3$. Using the equations (25) and (26), the time variation of the total entropy is given by

$$\dot{S}_T = \frac{2\pi R_E \dot{R}_E}{G} - \frac{8\pi^2 v_E (\rho + p)}{v_A H^3}\left(1 + \frac{\dot{v} R_E}{3v}\right) \tag{27}$$

and thus the second time derivative of the total entropy is given by

$$\begin{aligned} \ddot{S}_T \;=\; & \frac{2\pi}{G}(R_E f_E + v_E^2) - \frac{8\pi}{(v_A H^3)^2}\left[\left\{(v_A H^3)\left(f_E(\rho+p) + v_E \frac{\partial}{\partial t}(\rho + p)\right)\right.\right. \\ & \left.\left. - \; v_E(\rho+p)(f_A H^3 + 3 v_A H^2 \dot{H})\right\}\left(1 + \frac{\dot{v} R_E}{3v}\right) + \frac{v_A v_E H^3 (\rho + p)}{3v^2}\left\{v(\ddot{v} R_E + \dot{v} v_E) - \dot{v}^2 R_E\right\}\right] \end{aligned} \tag{28}$$

where $v_E = \dot{R}_E$ and $f_E = \dot{v}_E$.

Case-2

In this case the horizon entropy is modified as [27]

$$S_h = \beta \frac{\pi R_E^2}{G} \tag{29}$$

where,

$$\beta = \frac{2}{R_E^2} \int \frac{R_E^2 dR_A}{R_A}$$

and Hawking temperature $(= T^h)$ is taken to be [25]

$$T^h = \frac{R_E}{2\pi R_A^2}. \tag{30}$$

Here we can write β as

$$\beta = \frac{2}{R_E^2} \int \frac{R_E^2 dR_A}{R_A} = \frac{2}{R_E^2} \int \frac{R_E^2 v_A}{R_A} dt$$

and from Gibb's relation (5) we have

$$\dot{S}_f = -\frac{8\pi^2 R_E(\rho + p)}{H^2} \left(1 + \frac{\dot{v} R_E}{3v}\right) \tag{31}$$

Hence the first order time variation of the total entropy is

$$\dot{S}_T = \frac{2\pi R_E^2 v_A}{G R_A} - \frac{8\pi^2 R_E(\rho + p)}{H^2} \left(1 + \frac{\dot{v} R_E}{3v}\right), \tag{32}$$

and the second time derivative of the total entropy is

$$\begin{aligned} \ddot{S}_T &= \frac{2\pi}{G R_A^2} \left[R_A R_E^2 f_A + 2 R_A R_E v_A v_E - R_E^2 v_A^2\right] - \frac{8\pi^2}{H^4} \left[\left\{H^2 \left(v_E(\rho + p) + R_E \frac{\partial(\rho + p)}{\partial t}\right)\right.\right. \\ &- \left.\left. 2R_E(\rho + p) H \dot{H}\right\} \left(1 + \frac{\dot{v} R_E}{3v}\right) + \frac{H^2 R_E(\rho + p)}{3v^2} \left\{v(\ddot{v} R_E + \dot{v} v_E) - v^2 R_E\right\}\right]. \end{aligned} \tag{33}$$

Case-3

Here we consider horizon entropy as the Bekenstien entropy and temperature as Hawking temperature [26] *i.e.*,

$$S_h = \frac{\pi R_E^2}{G} \tag{34}$$

and

$$T^h = \frac{R_E}{2\pi R_A^2}. \tag{35}$$

In this case the first derivative of the total entropy is

$$\dot{S}_T = \frac{2\pi R_E v_E}{G} - \frac{8\pi^2 R_E(\rho+p)}{H^2}\left(1+\frac{\dot{v}R_E}{3v}\right). \tag{36}$$

The second derivative of the total entropy is given by

$$\begin{aligned}\ddot{S}_T \;&=\; \frac{2\pi}{G}\left(R_E f_E + v_E^2\right) - \frac{8\pi^2}{H^4}\left[\left\{H^2\left(v_E(\rho+p) + R_E\frac{\partial(\rho+p)}{\partial t}\right)\right.\right.\\ &-\; \left.\left. 2R_E(\rho+p)H\dot{H}\right\}\left(1+\frac{\dot{v}R_E}{3v}\right) + \frac{H^2 R_E(\rho+p)}{3v^2}\left\{v(\ddot{v}R_E+\dot{v}v_E) - v^2 R_E\right\}\right]. \end{aligned} \tag{37}$$

Case-4

Finally, in this case we take the horizon entropy as [27]

$$S_h = \beta\frac{\pi R_E^2}{G} \tag{38}$$

and the modified Hawking temperature as [27]

$$T^h = \frac{\alpha R_E}{2\pi R_A^2}. \tag{39}$$

Then the first time derivative of the total entropy is

$$\dot{S}_T = \frac{2\pi R_E^2 v_A}{G R_A} - \frac{8\pi^2 v_E(\rho+p)}{v_A H^3}\left(1+\frac{\dot{v}R_E}{3v}\right). \tag{40}$$

The second time derivative of the total entropy is given by

$$\begin{aligned}\ddot{S}_T \;&=\; \frac{2\pi}{G R_A^2}\left[R_A R_E^2 f_A + 2R_A R_E v_A v_E - R_E^2 v_A^2\right] - \frac{8\pi}{(v_A H^3)^2}\left[\left\{(v_A H^3)\left(f_E(\rho+p) + v_E\frac{\partial}{\partial t}(\rho+p)\right)\right.\right.\\ &-\; \left.\left. v_E(\rho+p)(f_A H^3 + 3v_A H^2\dot{H})\right\}\left(1+\frac{\dot{v}R_E}{3v}\right) + \frac{v_A v_E H^3(\rho+p)}{3v^2}\left\{v(\ddot{v}R_E+\dot{v}v_E) - \dot{v}^2\dot{R}_E\right\}\right] \end{aligned} \tag{41}$$

As in all four cases the time variation of the total entropy are very complicated, so we cannot definitely conclude about their sign analytically. Hence we plot these time variation of entropies $\dot{S}_T$ and $\ddot{S}_T$. For simplicity we consider the universe filled with perfect fluid having a constant equation of state *i.e.* $p = \omega\rho$.

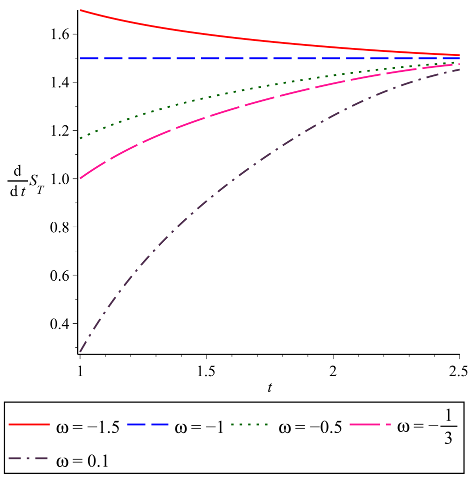

Fig.1 : The time derivative of the total entropy is plotted against t for Type-I in Case-1, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

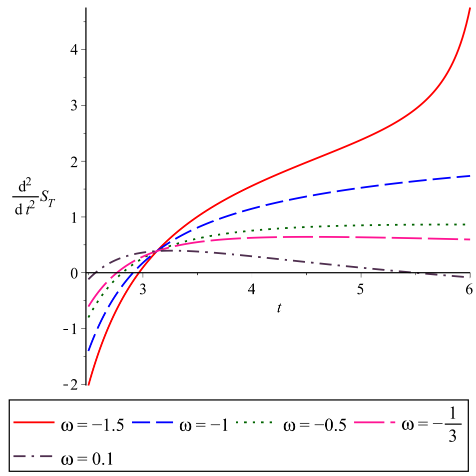

Fig.2 : The second order time derivative of the total entropy is plotted against t for Type-I in Case-1, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

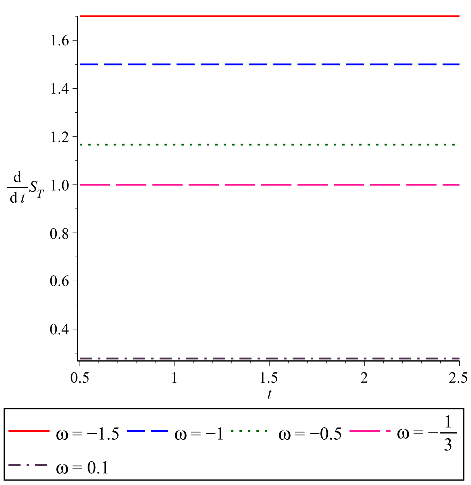

Fig.3 : The time derivative of the total entropy is plotted against t for Type-II in Case-1, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

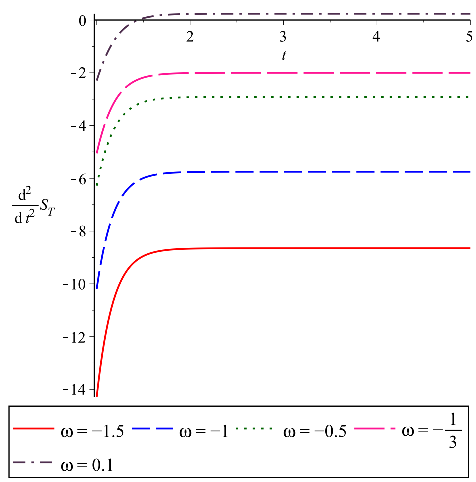

Fig.4 : The second order time derivative of the total entropy is plotted against t for Type-II in Case-1, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

V. DISCUSSION

In the present work we have considered two different types of fractional function v and we have examined GSLT and TE for both cases taking four different combinations of (modified) Hawking temperature and (modified) Bekenstein entropy. Due to complicated expressions of the time variation of the total entopy we cannot definitely conclude about validity of GSLT

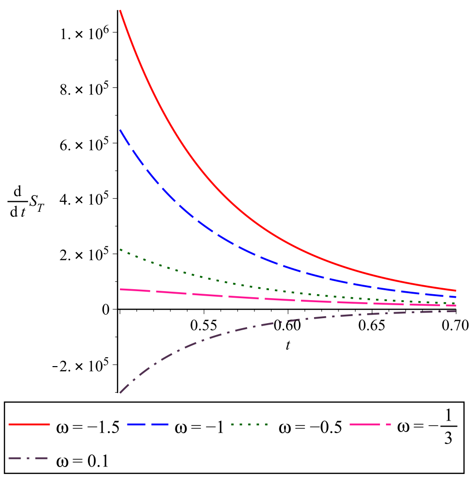

Fig.5 : The time derivative of the total entropy is plotted against t for Type-I in Case-2, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

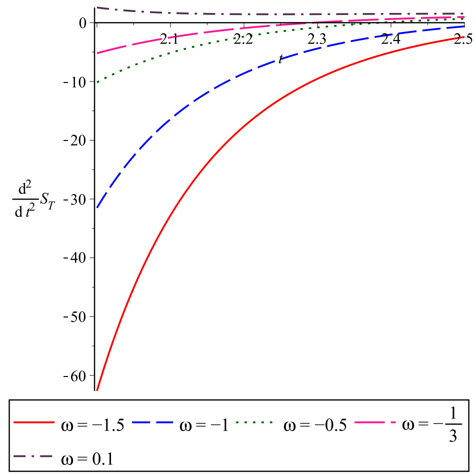

Fig.6 : The second order time derivative of the total entropy is plotted against t for Type-I in Case-2, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

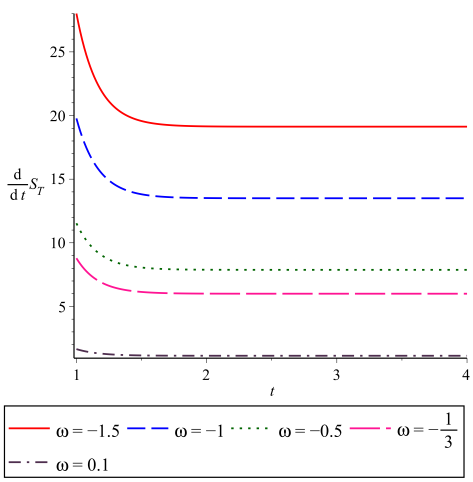

Fig.7 : The time derivative of the total entropy is plotted against t for Type-II in Case-2, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

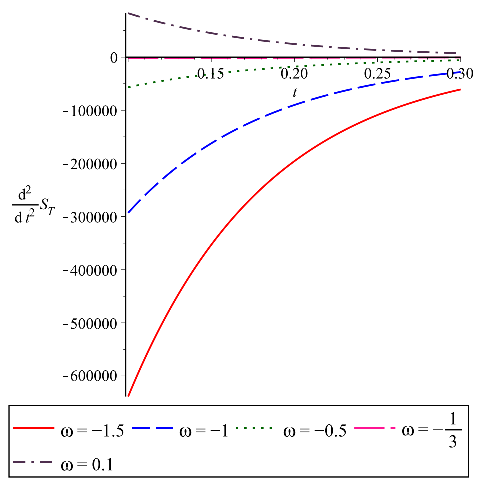

Fig.8 : The second order time derivative of the total entropy is plotted against t for Type-II in Case-2, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

and TE. However, we have drawn some inferences only from graphical analysis considering $8\pi = 1 = G$, $H = 1$ and $R_E = 3$.

From the above figures (1−16), we have the following observations:

1) Both GSLT and TE hold for a wider range when the fractional function is an exponential function of time (*i.e.* Type-I) rather than power law form (*i.e.* Type-II).

2) In most of the cases, GSLT and TE hold good when the values of equation of state

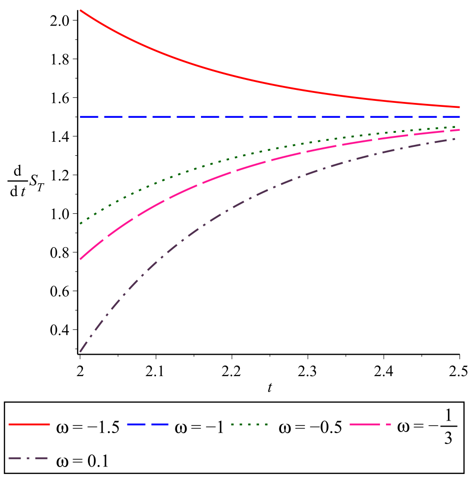

Fig.9 : The time derivative of the total entropy is plotted against t for Type-I in Case-3, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

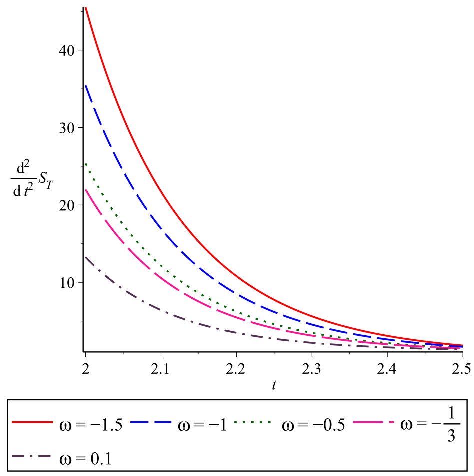

Fig.10 : The second order time derivative of the total entropy is plotted against t for Type-I in Case-3, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

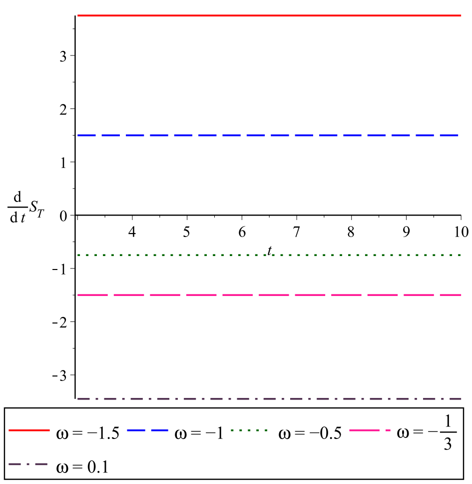

Fig.11 : The time derivative of the total entropy is plotted against t for Type-II in Case-3, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

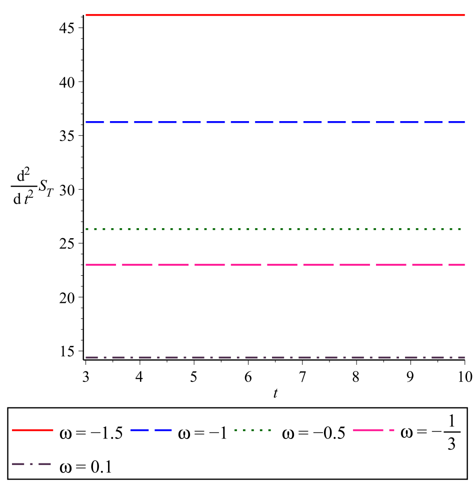

Fig.12 : The second order time derivative of the total entropy is plotted against t for Type-II in Case-3, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

parameter is taken to be negative, *i.e.* $\omega < 0$.

3) Among all the four cases presented above, we have observed that case 2 is better than all other cases.

Therefore from the above comparative study we may conclude that the exponential form for the fractional function v is better then power law form. Finally, we may conclude that

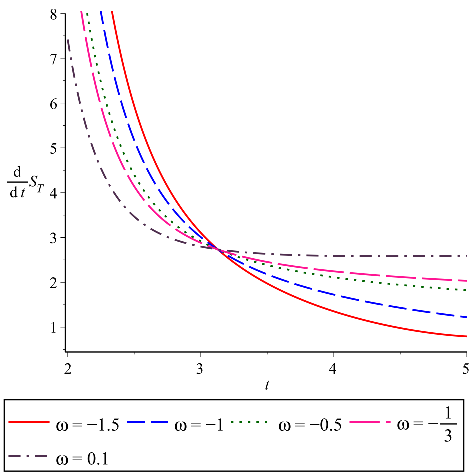

Fig.13 : The time derivative of the total entropy is plotted against t for Type-I in Case-4, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

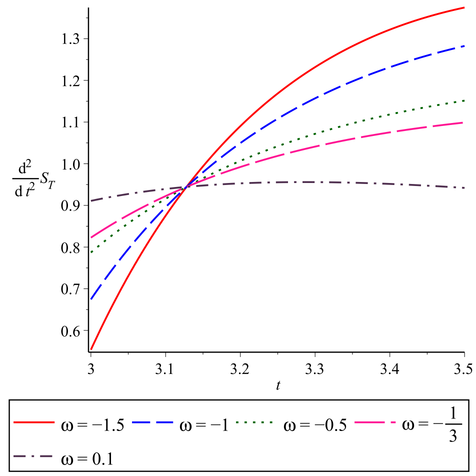

Fig.14 : The second order time derivative of the total entropy is plotted against t for Type-I in Case-4, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

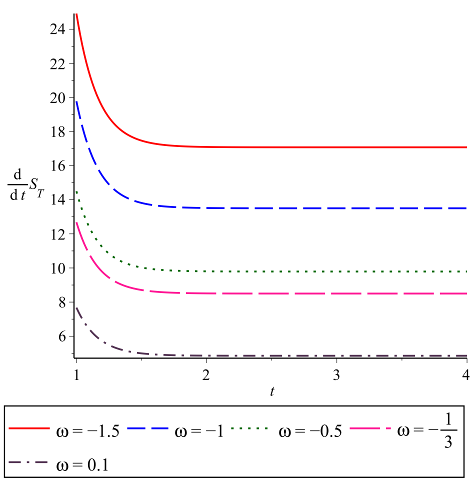

Fig.15 : The time derivative of the total entropy is plotted against t for Type-II in Case-4, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

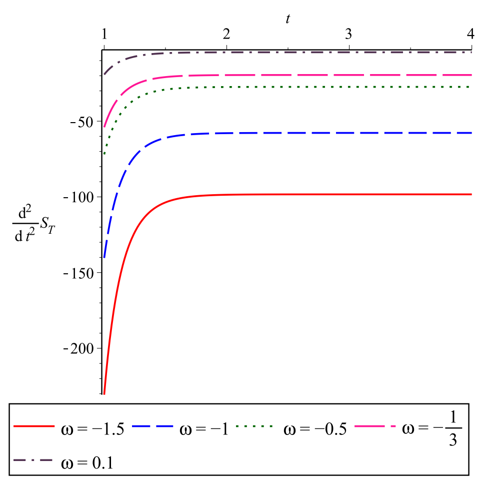

Fig.16 : The second order time derivative of the total entropy is plotted against t for Type-II in Case-4, considering $v_0 = 5$, $\beta = 3$ and $w = 10$.

modified bekenstien entropy and modified Hawking temperature can be considered as realistic thermodynamical parameters on the event horizon.

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