STRATIFYING ENDOMORPHISM ALGEBRAS USING EXACT CATEGORIES

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We dedicate this paper to the memory of J.A. Green

1. INTRODUCTION

This paper is the second in a series aimed at proving versions of a conjecture made by the authors in 1996. The conjecture concerns the enlargement, in a framework involving Kazhdan-Lusztig cell theory, of those Hecke endomorphism algebras which occur naturally in the cross characteristic representation theory of finite groups of Lie type. See [DPS98a] for the original version of the conjecture, and [DPS15a] for a reformulation.

The [DPS98a] conjecture is set in the context of a Hecke algebra H for a finite Weyl group, using the dual left cell modules S_ω, $\omega \in \Omega$, in the sense of [Lu03]. (Thus, each S_ω is a right H-module.) The base ring (in [DPS15a]) is $\mathbb{Z}[v, v^{-1}]$, where v is an indeterminate. One of the conjecture's implications is that there is a faithful right H-module $T^\dagger$, filtered by various S_ω, such that the modules $\Delta(\omega) := \mathrm{Hom}_H(S_\omega, T^\dagger)$, with $\omega \in \Omega$, form a stratifying system (in the sense of [DPS98a]) for the endomorphism algebra $A^\dagger := \mathrm{End}_H(T^\dagger)$. Using exact category methods, we are able to prove this statement. See Theorem 4.8 below.

A strength of the "stratifying system" construction is that it is well-behaved under base change, so that the resulting algebra $A^\dagger \otimes k$ inherits a stratification from that of $A^\dagger$ over any commutative ring or field k in which v is specialized to an invertible element.

The endomorphism algebras $A^\dagger$ constructed here have other good properties. In particular, based changed versions $\widetilde{A}^\dagger, \widetilde{T}^\dagger$ can be shown to satisfy the particular "cyclotomic" local versions of the conjecture which were treated in [DPS15a, Theorem 5.6], using results of [GGOR03] on the module categories $\mathscr{O}$ for rational Cherednik algebras. The present paper raises the possibility that the [DPS98a] conjecture can be proved directly within the global framework of $\mathbb{Z}[v, v^{-1}]$-algebras and modules, perhaps even with the present versions of $A^\dagger$.

The authors began developing a general theory in [DPS98a] for constructing the required enlarged algebras, centered around a set of requirements contained in what we call the "stratification hypothesis." The most difficult condition to verify in this hypothesis is an Ext^1-vanishing requirement for some of the modules involved. The present paper takes a novel approach to this problem by building new exact categories containing the relevant modules, effectively making the Ext^1-groups involved smaller and better behaved. While there are Specht modules and analogues for all finite Weyl groups, there are no troublesome self-extensions, or extensions in the "wrong order," because of the exact structure we construct. As a result, many issues of "bad characteristic" do not arise.

The present paper also contains new results on exact category constructions. In particular, the main Lemma 3.1 gives a new, very general construction in an abstract setting. It very quickly

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leads to new exact module categories $(\mathscr{A},\mathscr{E})$ for algebras B over domains $\mathscr{K}$ such that the K-algebra B_K obtained by base change from $\mathscr{K}$ to its quotient field K is semisimple. The underlying additive category $\mathscr{A}$ is the full subcategory of B-mod consisting of all modules which are finitely generated and torsion-free over $\mathscr{K}$. The "exact sequences" in $\mathscr{E}$ are required to be exact on certain filtrations. Both this construction and that of Lemma 3.1 apply to all standard axiom systems for exact categories. However, a further construction, especially useful for the cell module setting, focuses on the Quillen axioms [Q73], [K90].[1]

2. Stratified and highest weight module categories.

Throughout this section, let $\mathscr{K}$ be a fixed Noetherian commutative ring. Often $\mathscr{K}$ will be the ring $\mathscr{Z} := \mathbb{Z}[t,t^{-1}]$ of Laurent polynomials in a variable t, or the ring $\mathscr{Z}^{\#}$ obtained by localizing $\mathscr{Z}$ at a multiplicative set generated by a finite number of rational prime integers in $\mathscr{Z}$. (The primes will be the set of bad primes for a fixed root system.) A $\mathscr{K}$-module V is called finite if it is finitely generated as a $\mathscr{K}$-module. In particular, A will often be a finite $\mathscr{K}$-algebra.

By a quasi-poset, we mean a (usually finite) set Λ with a transitive and reflexive relation $\leq$. An equivalence relation $\sim$ is defined on Λ by putting $\lambda\sim\mu$ if and only if $\lambda\leq\mu$ and $\mu\leq\lambda$. Let $\bar\lambda$ be the equivalence class containing $\lambda\in\Lambda$. Of course, $\bar\Lambda$ inherits a poset structure itself.

2.1 Stratifying systems. We will briefly review the notion of a *stratifying system*[2] for a finite $\mathscr{K}$-algebra A and a quasi-poset Λ. Assume that A is projective over $\mathscr{K}$. For $\lambda\in\Lambda$, we require a finitely generated A-module $\Delta(\lambda)$, and a finitely generated, projective A-module $P(\lambda)$, together with an epimorphism $P(\lambda)\twoheadrightarrow\Delta(\lambda)$. The following conditions are assumed to hold:

(SS1) For $\lambda,\mu\in\Lambda$,
$$\mathrm{Hom}_A(P(\lambda),\Delta(\mu))\neq 0 \implies \lambda\leq\mu.$$
(SS2) Every irreducible A-module L is a homomorphic image of some $\Delta(\lambda)$.
(SS3) For $\lambda\in\Lambda$, the A-module $P(\lambda)$ has a finite filtration by A-submodules with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\bar\mu>\bar\lambda$.

When these conditions all hold, the data consisting of the $\Delta(\lambda)$, $P(\lambda)$, etc. forms (by definition) a *stratifying system* for the category A–mod of finitely generated A-modules. It is also clear that $\Delta(\lambda)_{\mathscr{K}'}, P(\lambda)_{\mathscr{K}'},\dots$ is a stratifying system for $A_{\mathscr{K}'}$-mod *for any base change* $\mathscr{K}\to\mathscr{K}'$, provided $\mathscr{K}'$ is a Noetherian commutative ring. (Notice that condition (SS2) is redundant, if it is known that the direct sum of the projective modules in (SS3) is a progenerator—a property preserved by base change.)

[1] Contrary to popular beliefs, the notion of an "exact category" is not exactly well-defined. There are at least three axiom systems, all quite useful. The weakest set of axioms is that of Quillen [Q73], as reduced to a smaller set by Keller [K90]. See our Appendix 1, §5. Then there is the axiom system of Gabriel-Roiter [GR97]. Keller shows in the appendix to [DRSS99] that this set is equivalent to that of Quillen after adding the additional condition that retractions have kernels. This axiom set is generally easier to use for producing new exact sequences from others, but the retraction axiom may be hard to verify in integral settings, or simply is not true. It is implied by the stronger, yet simpler requirement, that all idempotents split. The latter has several applications, including a six term "long exact sequence" for Hom and Ext^1 in [DRSS99], and it is used by Neeman [Ne90] to build derived categories. But in the context of the proof of Theorem 4.8 below, our main result, idempotents do not split. For a discussion of derived categories in the Quillen framework, see [K96].

[2] In [DPS98a], these systems were called *strict* stratifying systems. In this paper, we drop the word "strict" and do not consider more general systems. (The more general stratifying systems in [DPS98a] allowed $\bar\mu\geq\bar\lambda$ in condition (SS3).)

An ideal J in the $\mathscr{K}$-algebra A above is called a *stratifying ideal* provided that the inclusion $J \hookrightarrow A$ is $\mathscr{K}$-split (or, equivalently, A/J is $\mathscr{K}$-projective), and, for $M, N \in A/J$-mod, inflation from A/J to A defines an isomorphism

$$\text{(2.1.1)} \qquad \operatorname{Ext}^n_{A/J}(M,N) \xrightarrow{\sim} \operatorname{Ext}^n_A(M,N), \quad \forall n \geq 0$$

of Ext-groups. (In particular, the $n = 1$ case implies that $J^2 = J$, see [CPS90].) A *standard stratification* of length n of A is a sequence $0 = J_0 \subsetneq J_1 \subseteq \cdots \subsetneq J_n = A$ of stratifying ideals[3] of A such that each J_i/J_{i-1} is a projective A/J_{i-1}-module. If A–mod has a stratifying system with quasi-poset Λ, then it has a standard stratification of length $n = |\bar{\Lambda}|$; see [DPS98a, Thm. 1.2.8].

Lemma 2.1. *Suppose that A has a stratifying system. Then $\operatorname{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda < \mu$.*

Proof. Assume that $\lambda \not< \mu$, and let $Q(\lambda)$ be the kernel of the given epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$. Then $\operatorname{Ext}^1_A(\Delta(\lambda), \Delta(\mu))$ is homomorphic image of $\operatorname{Hom}_A(Q(\lambda), \Delta(\mu))$. But $Q(\lambda)$ has a filtration with sections of the form $\Delta(\tau)$ for $\bar{\tau} > \bar{\lambda}$, so that $\operatorname{Hom}_A(\Delta(\tau), \Delta(\mu))) = 0$ since $\tau \not\leq \mu$. □

Given a finite quasi-poset Λ, a *height function* on Λ is a mapping $\text{ht} : \Lambda \to \mathbb{Z}$ with the property that $\lambda < \mu \implies \text{ht}(\lambda) < \text{ht}(\mu)$. Given $\lambda \in \Lambda$, a sequence $\lambda = \lambda_n > \lambda_1 > \cdots > \lambda_0$ is called a chain of length n starting at $\lambda = \lambda_n$. Then the standard height function $\text{ht} : \Lambda \to \mathbb{N}$ is defined by setting $\text{ht}(\lambda)$ to be the maximal length of a chain beginning at λ.

Given A-modules X, Y, recall that the trace module $\text{trace}_X(Y)$ of Y in X is the submodule of X generated by the images of all morphisms $Y \to X$.

Proposition 2.2. *Suppose that A has a stratifying system. Then the Δ-sections arising from the filtration (SS3) of $P(\lambda)$ can be reordered with respect to any height function. Moreover, if we set*

$$P(\lambda)_j = \text{trace}_{P(\lambda)}\Big(\bigoplus_{\text{ht}(\mu)\geq j} P(\mu)\Big),$$

then $P(\lambda)_{j+1} \subseteq P(\lambda)_j$, for $j \in \mathbb{Z}$, and

$$P(\lambda)_j / P(\lambda)_{j+1}$$

is a direct sum of modules $\Delta(\mu)$ satisfying $\text{ht}(\mu) = j$.

Proof. First, fix j maximal with a section $\Delta(\mu)$ appearing in $P(\lambda)$ such that $\text{ht}(\mu) = j$. Apply Lemma 2.1 to construct a submodule $P(\lambda)(j)$ which is filtered by modules $\Delta(\nu)$ with $\text{ht}(\nu) \geq j$, and $P(\lambda)/P(\lambda)(j)$ filtered by modules $\Delta(\nu)$ with $\text{ht}(\nu) < j$. Axiom (SS1) clearly gives $P(\lambda)(j) = P(\lambda)_j$, and $P(\lambda)_{j+1} = 0$. Clearly, $P(\lambda)_j/P(\lambda)_{j+1}$ is the direct sum as equipped by the proposition. We have not used projectivity of $P(\lambda)$, only its filtration properties. Induction applied to the quotient module $P(\lambda)/P(\lambda)_j$ completes the proof. □

Remark 2.3. The proposition above shows that the projective modules have a canonically described filtration, given any height function ht. This suggests that, if A is to be realized as an endomorphism algebra of a given module, that module might also reflect that filtration in a canonical way. In §§3,4, this is successfully approached using semisimple base change and exact categories. The latter also builds in a height function version of the vanishing condition in Lemma 2.1.

[3] If (2.1.1) is not assumed, but projectivity of each J_i/J_{i-1} is assumed, then (2.1.1) for all $J = J_i$ follows for each n from the $n = 1$ case (idempotence of the ideals J_i). See Appendix II. This is the more usual definition of a standard stratification [DPS98a], but not our focus here.

The proposition can also be used, in conjunction with Lemma 2.4 below, to build stratifying ideals in an algebra Morita equivalent to A, and then in A. See [DPS98a, Lem. 1.2.7, Thm. 1.2.8]. We will not need to return to this in this paper.

Lemma 2.4. *A has a stratifying system. Then*

$$P := \bigoplus_{\lambda\in\Lambda} P(\lambda)$$

is a projective generator for A-mod.

Proof. Obvious from (SS2) and (SS3). □

2.2 Exact categories and the stratification hypothesis. This section provides a way to constructing stratifying systems in an endomorphism ring setting. Previously, the construction was based on assuming a "stratification hypothesis" in [DPS98a, Hyp. 1.2.9, Thm. 1.2.10]. The method required a difficult Ext^1-vanishing condition (see [DPS98a, Thm. 2.3.9, 2.4.4]). This subsection gives an important generalization using exact categories. The advantage of this approach is that the relevant Ext^1-vanishing conditions involve smaller spaces (and so are hopefully easier to make vanish).

Let $(\mathscr{A},\mathscr{E})$ be exact category in the sense of Quillen [Q73], as discussed in Appendix 1, §6 using axioms of Keller [K90]. In particular, $\mathscr{A}$ is an additive category and $\mathscr{E}$ is a class of sequences $X \to Y \to Z$ satisfying certain properties. In the hypotheses below we will assume the more explicit setup in which $\mathcal{A}$ is an additive full subcategory of mod–B where B is a finite and projective $\mathscr{K}$-algebra. The sequences $X \to Y \to Z \in \mathscr{E}$ are among the short exact sequences $0 \to X \to Y \to Z \to 0$ in mod–B. Thus, $\mathscr{A}$ is an "exact subcategory" of mod–B. Note, however, we do *not* assume that all exact sequences in mod–B whose object terms lie in $\mathscr{A}$ necessarily belong to $\mathscr{E}$.

Next, we discuss the variation of the stratification hypothesis based on the notion of an exact category. First, thee are several preliminary assumptions.

Assume there is given a collection of objects $S_\lambda \in \mathscr{A}$ indexed by the elements λ of a finite quasi-poset Λ. For each $\lambda \in \Lambda$, S_λ is a subobject of $T_\lambda \in \mathscr{A}$. Write

$$T := \bigoplus_{\lambda\in\Lambda} T_\lambda \in \mathscr{A}.$$

With this notation, the following statements make up the most straightforward version of the "stratification hypothesis":

Hypothesis 2.5. *The stratification hypothesis holds provided the following statements hold.*

(1) *For $\lambda \in \Lambda$, there is a fixed sequence $\nu_{\lambda,0}, \cdots, \nu_{\lambda,l(\lambda)}$ where $l(\lambda) \geq 0$, $\nu_{\lambda,0} = \lambda$, and $\nu_{\lambda,i} > \lambda$ for each $i > 0$. Also, there is an increasing filtration*

$$0 = F_\lambda^{-1} \subseteq F_\lambda^0 \subseteq \cdots \subseteq F_\lambda^{l(\lambda)} = T_\lambda$$

such that each inclusion $F_\lambda^{i-1} \subseteq F_\lambda^i$ is an inflation,[4] *such that*

$$F_\lambda^i / F_\lambda^{i-1} \cong S_{\nu_{\lambda,i}}$$

for $0 \leq i \leq l(\lambda)$.

[4]It an abstract exact category setting, $F_\lambda^{i-1} \subseteq F_\lambda^i$ may be taken as a notation for a monomorphism $F_\lambda^{i-1} \to F_\lambda^i$. In this case, it has a cokernel isomorphic to $F_\lambda^i / F_\lambda^{i-1}$.

(2) *For* $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_{\mathscr{A}}(S_\mu, T_\lambda) \neq 0 \implies \lambda \leq \mu$.
(3) *For all* $\lambda \in \Lambda$, $\mathrm{Ext}^1_{\mathscr{E}}(T_\lambda/F^i_\lambda, T) = 0, \quad \forall i \geq 0$. *(See Appendix I for a definition of* $\mathrm{Ext}^1_{\mathscr{E}}$*.)*

The proof of the following result parallels the analogous result in [DPS98a, Thm. 1.2.10].

Theorem 2.6. *Let* $\mathscr{A}, B, T$ *be as above. Assume that Hypothesis 2.5 holds. Put*

$$A = \mathrm{End}_B(T)$$

and, for $\lambda \in \Lambda$, *define* $\Delta(\lambda) := \mathrm{Hom}_B(S_\lambda, T) \in A$*-mod. Assume that each* $\Delta(\lambda)$ *is* $\mathscr{K}$*-projective. Then* $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ *is a stratifying system for* A*–mod.*

Remark 2.7. The main function of condition (3) in Hypothesis 2.5 in proving Theorem 2.6 is to ensure the existence of various exact sequences when $\mathrm{Hom}_A(-, T)$ is applied. This exactness still works and Theorem 2.6 still holds if S_λ is used in place of T_λ/F^i_λ, at least for the exact categories we use. For one precise formulation, see Lemma 3.10. This discussion is necessary when using the Quillen axiom system. In the idempotent split context studied in [DRSS99], the functor $\mathrm{Ext}^1_{\mathscr{E}}$ is half-exact in each variable; see [DRSS99, Thm. 1.3], who quote arguments of [BH61, Thm. 1.1]. In this case, the original version of condition (3) holds as written when all the $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S_\lambda, T)$ vanish. Finally, another useful modification of Hypothesis 2.5 (1) is obtained by replacing $S_{\nu_{\lambda,i}}$, $i \geq 1$, by the direct sum of such objects, all with $i \geq 1$. Again, Theorem 2.6 holds with essentially the same arguments.

3. A CONSTRUCTION OF EXACT CATEGORIES.

Let $(\mathscr{A}, \mathscr{E})$ be an exact category in the sense of Quillen [Q73], see Appendix 1, §5. Suppose that $\mathscr{C}$ is a given abelian category, and let $F : \mathscr{A} \to \mathscr{C}$ be an additive functor. Then F is called $\mathscr{E}$-exact (resp., left $\mathscr{E}$-exact) if given any $(X \to Y \to Z) \in \mathscr{E}$, the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ (resp., $0 \to F(X) \to F(Y) \to F(Z)$) is exact in $\mathscr{C}$.

Lemma 3.1. *Let* $\mathscr{C}$ *be an abelian category. Also, let* $(\mathscr{A}, \mathscr{E}')$ *be an exact category and let* $F : \mathscr{A} \to \mathscr{C}$ *be an left* $\mathscr{E}'$*-exact, additive functor. Define* $\mathscr{E}$ *to be the class of those* $(X \to Y \to Z) \in \mathscr{E}'$ *such that* $0 \to F(X) \to F(Y) \to F(Z) \to 0$ *is exact in* $\mathscr{C}$*. Then* $(\mathscr{A}, \mathscr{E})$ *is an exact category.*

Proof. First, since F is left $\mathscr{E}'$-exact, $\mathscr{E}$ can also be described as the class of all $(X \to Y \to Z \in \mathscr{E}'$ such that $F(X) \to F(Z)$ an epimorphism in $\mathscr{C}$. Axioms 0, 1 in Appendix §5 are immediate. Consider Axiom 2 and the following commutative diagram in $\mathscr{A}$

$$\begin{array}{ccccc} X & \longrightarrow & Y' & \xrightarrow{d'} & Z' \\ \Vert & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y & \xrightarrow{d} & Z \end{array}$$

in which the bottom row belongs to $\mathscr{E}$ (so that the sequence is $\mathscr{E}'$-exact and $F(d) : F(Y) \to F(Z)$ is an epimorphism), and the top row is the pullback of the bottom row (through the map f). The object Y' is identified as the kernel of the epimorphism $(-f, d) : Z' \oplus Y \to Z$ in the bottom row of the commutative diagram

$$\begin{array}{ccccc} & & Y & \xrightarrow{d} & Z \\ & & {\scriptstyle \binom{0}{1_Y}}\big\downarrow & & \big\| \\ Y' & \longrightarrow & Z' \oplus Y & \xrightarrow{(-f,d)} & Z \end{array}$$

The bottom row $Y' \longrightarrow Z' \oplus Y \stackrel{(-f,d)}{\longrightarrow} Z$ is isomorphic to $Y' \stackrel{\binom{-d'}{f'}}{\longrightarrow} Z' \oplus Y \stackrel{(f,d)}{\longrightarrow} Z$, which is shown in [K90, p. 406] to belong to $\mathscr{E}'$. (See also Remark 5.1(d) in Appendix I below for an alternate argument.) Now apply the functor F, and use the natural isomorphism $F(Z' \oplus Y) \cong F(Z') \oplus F(Y)$ to obtain the following commutative diagram

$$\begin{array}{ccccccccc} & & & & F(Y) & \xrightarrow{F(d)} & F(Z) & \longrightarrow & 0 \\ & & & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & F(Y') & \longrightarrow & F(Z') \oplus F(Y) & \xrightarrow{(-F(f),F(d))} & F(Z) & \longrightarrow & 0. \end{array}$$

As noted above, the morphism $F(d)$ is an epimorphism. Thus, since F is left exact, the bottom row is exact, and it identifies $F(Y')$ as the the pullback in the abelian category $\mathscr{C}$ of $F(f)$ and $F(d)$. Since $F(d)$ is an epimorphism, so is its pullback $F(d')$. This verifies Axiom 2.

Finally, we must check that Axiom 2° holds. Consider a commutative pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & Y & \xrightarrow{d} & Z & \longrightarrow & 0 \\ & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & X' & \xrightarrow{i'} & Y' & \xrightarrow{d'} & Z & \longrightarrow & 0 \end{array} \tag{3.0.1}$$

in which the top row belongs to $\mathscr{E}$. We must prove that $X' \to Y' \to Z$ also belongs to $\mathscr{E}$. But the diagram (3.0.1) gives the following commutative diagram

$$\begin{array}{ccccccccc} & & & & Y & \xrightarrow{=} & Y & & \\ & & & & {\scriptstyle h}\big\downarrow & & {\scriptstyle d}\big\downarrow & & \\ 0 & \longrightarrow & X' & \xrightarrow{i'} & Y' & \xrightarrow{d'} & Z & \longrightarrow & 0 \end{array} . \tag{3.0.2}$$

After applying F, we get the following commutative diagram

$$\begin{array}{ccccccccc} & & & & F(Y) & \xrightarrow{=} & F(Y) & & \\ & & & & {\scriptstyle F(h)}\big\downarrow & & {\scriptstyle F(d)}\big\downarrow & & \\ 0 & \longrightarrow & F(X') & \xrightarrow{F(i')} & F(Y') & \xrightarrow{F(d')} & F(Z) & \longrightarrow & 0 \end{array}$$

in which $F(d)$ is an epimorphism, since the top row of (3.0.1) belongs to $\mathscr{E}$. This implies that $F(d')$ is an epimorphism, and, hence, the bottom row of (3.0.2) is exact in $\mathscr{C}$. Thus, the bottom row of (3.0.1) belongs to $\mathscr{E}$, and Axiom 2° holds, completing the proof of the lemma. □

We now make some assumptions which will often be in force for the rest of this paper.

Assumptions 3.2. *Let $\mathscr{K}$ be a fixed Noetherian integral domain with fraction field K. Let H be $\mathscr{K}$-algebra which is finite and torsion-free as a $\mathscr{K}$-module. Assume that H_K is semisimple. The isomorphism classes of irreducible right H_K-modules are indexed by a finite set Λ. Given $\lambda \in \Lambda$, let E_λ denote a representative from the corresponding irreducible class. Fix a function* $\mathrm{ht} : \Lambda \to \mathbb{Z}$, *taking, for convenience, non-negative values. (We call* ht *a height function, though there is no immediate assumption that Λ is a quasi-poset.)*

Let mod–H be the category of $\mathscr{K}$-finite right H-modules, and let mod–H_K be category of finite dimensional right H_K-modules. Let $\mathscr{A}$ be the full subcategory of mod–H which consists of $\mathscr{K}$-torsion-free $\mathscr{H}$-modules.

For $N \in$ mod–H_K, the height function ht induces a natural increasing (finite) filtration

$$0 = N^{-1} \subseteq N^0 \subseteq \cdots \subseteq N^i \subseteq N^{i+1} \subseteq \cdots \subseteq N,$$

defining N^i to be the sum of all irreducible right H_K-submodules isomorphic to E_λ with $\mathrm{ht}(\lambda) \leq i$.

Then if $M \in \mathscr{A}$, there is an induced filtration

$$0 = M^{-1} \subseteq M^0 \subseteq \cdots \subseteq M^i \subseteq M^{i+1} \subseteq \cdots \subseteq M$$

on M defined by setting

$$M^i = M \cap (M_K)^i, \quad i \geq 0.$$

Observe that each $M^i \in \mathscr{A}$, as are the modules M/M^i and M^i/M^{i-1}. Also, $(M^i/M^{i-1})_K$ is a direct sum of H_K-modules E_λ with $\mathrm{ht}(\lambda) = i$.

Our goal is to show that the above data define the structure of an exact category on the additive category $\mathscr{A}$ of $\mathscr{K}$-torsion-free right H-modules, once an appropriate family $\mathscr{E}$ of conflations $X \to Y \to Z$ has been defined.

First, we require more preliminaries, including the proposition below. Note that if $X \xrightarrow{f} Y$ is a map in $\mathscr{A}$, then f induces a map $f_i : X^i \to Y^i$ and a map $\overline{f_i} : X^i/X^{i-1} \to Y^i/Y^{i-1}$ for each integer i. In addition, if $g : Y \to Z$ is another morphism in $\mathscr{A}$, then $(gf)_i = g_i f_i$ and $\overline{g_i}\,\overline{f_i} = \overline{g_i f_i}$ for each i. Finally, if $f : X \to Y$ is an inclusion $X \subseteq Y$, then

$$X \cap (Y_K)^i = X \cap (X_K)^i = X^i, \quad \forall i. \tag{3.0.3}$$

In the following proposition, we continue to assume that Assumptions 3.2 are in force.

Proposition 3.3. *Suppose $X, Y, Z \in \mathscr{A}$ and $0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$ is an exact sequence in mod–H. Then for each $i \in \mathbb{Z}$, the following statements hold.*

(a) The sequence $0 \to X^i \to Y^i \to Z^i$ is exact in mod–H.

(b) The sequence $0 \to X^h \to Y^h \to Z^h \to 0$ is a short exact sequence in mod–H for each $h \leq i$ if and only if

$$0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$$

is exact for each $j \leq i$.

(c) The sequence $0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$ is a short exact sequence for all $j \leq i$ if and only if $Y^g/Y^{g-1} \to Z^g/Z^{g-1}$ is an epimorphism for all $g \leq i$.

Proof. Throughout this proof, the word "exact" means exact in the usual sense in the category of right H– (or possibly H_K–) modules.

Without loss, we can assume that the map $f : X \to Y$ is an inclusion of a submodule. Clearly, each f_i is an inclusion. Also, $g_i f_i = (gf)_i = 0$, so that the image of f_i is contained in the kernel of g_i. To prove the reverse inclusion, let $y \in \ker g_i$. Thus, $y \in \ker g$, so $y \in X$. But also $y \in Y^i \subseteq (Y_K)^i$. So $y \in X \cap (Y_K)^i = X^i$, as per (3.0.3). This proves (a).

We next prove (b). For every integer j, we have a 3×3 diagram

$$\begin{array}{ccccc} X^{j-1} & \longrightarrow & Y^{j-1} & \longrightarrow & Z^{j-1} \\ \downarrow & & \downarrow & & \downarrow \\ X^j & \longrightarrow & Y^j & \longrightarrow & Z^j \\ \downarrow & & \downarrow & & \downarrow \\ X^j/X^{j-1} & \longrightarrow & Y^j/Y^{j-1} & \longrightarrow & Z^j/Z^{j-1} \end{array} \tag{3.0.4}$$

in which the columns are short exact sequences. Then assume that each $0 \to X^h \to Y^h \to Z^h \to 0$ is exact for each $h \leq j$. Then the 3×3 Lemma [Mac94, p. 49] implies that $0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$ is exact for all $j \leq i$.

Conversely, assume that, for any $j \leq i$, the sequence $0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$ is exact. By induction, we can assume that $0 \to X^{i-1} \to Y^{i-1} \to Z^{i-1} \to 0$ is exact. In addition, the composition map $X^i \to Y^i \to Z^i$ is zero. Since the top and bottom rows of (3.0.4) are short exact sequences, [Mac94, Ex. 2, p. 51] implies the middle horizonal line is a short exact sequence, as required.

As for (c), the $\Longrightarrow$ direction is obvious. Conversely, it is easy to see that if the maps $Y^g/Y^{g-1} \to Z^g/Z^{g-1}$ are epimorphisms for all $g \leq i$, then each map $Y^h \to Z^h$, $h \leq i$, is an epimorphism. Now apply (a). □

In the context of Proposition 3.3(b), it is easy to give examples where $0 \to X^h \to Y^h \to Z^h \to 0$ is not a short exact sequence.

Example 3.4. Let $\mathscr{K} = \mathbb{Z}$, and let $H = \mathbb{Z}C_2$, where $C_2 = \{1, s\}$ is the cyclic group of order 2. Let S_2 be the trivial module for H. It is free of rank 1 over $\mathbb{Z}$ with basis vector 1. Let S_1 be the sign module for H, also free of rank 1 with basis vector denoted ϵ (so that $s \cdot \epsilon := -\epsilon$). Consider the short exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ of torsion-free H-modules where $X = S_2$, $Y = H$, and $Z = S_1$. Here $\alpha(1) = 1 + s$, and $\beta(1) = -\beta(s) = -\epsilon$. Assign $S_{2,\mathbb{Q}}$ height 1 and $S_{1,\mathbb{Q}}$ height 2, then

$$\begin{cases} X^1 = S_2^1 = 0; \\ Y^1 = H^1 = \mathbb{Z}(1-s); \\ Z^1 = Z. \end{cases}$$

Then $\beta(Y^1) = 2\mathbb{Z}\epsilon$, so that $Y^1 \to Z^1$ is not surjective. Thus, taking $h = 1$, the short exact sequence $0 \to X^h \to Y^h \to Z^h \to 0$ is not exact. However, assigning S_2 to have height 2 and S_1 to have height 1, and interchanging the roles of X and Y, the short exact sequence $0 \to Z \xrightarrow{\phi} H \xrightarrow{\psi} X \to 0$ (where $\phi(\epsilon) = 1 - s$, and $\psi(1) = \psi(s) = 1$) has the property that $0 \to Z^h \to Y^h \to X^h \to 0$ is short exact for all h.

Construction 3.5. *Keep Assumptions 3.2 with $\mathscr{A}$ as described there.* An exact pair (ι, d), $X \xrightarrow{\iota} Y$ and $Y \xrightarrow{d} Z$ belongs, by definition, to $\mathscr{E}$ provided that, for each integer i, the sequence $0 \to$

$X^i/X^{i-1} \to Y^i/Y^{i-1} \to Z^i/Z^{i-1} \to 0$ is a short exact sequence of right H-modules. By Proposition 3.3, each sequence $0 \to X^i \to Y^i \to Z^i \to 0$ is also short exact.

Theorem 3.6. *The pair $(\mathscr{A}, \mathscr{E})$ is an exact category.*

Proof. First observe that there is the standard exact category $(\mathscr{A}, \mathscr{E}')$. Here $\mathscr{E}'$ consists of all exact triples $X \to Y \to Z$ in mod–H with X, Y, Z objects in $\mathscr{A}$ (i.e., X, Y, Z are $\mathscr{K}$-torsion-free). Let $\mathscr{C}$ be the abelian category of right H-modules (not necessarily finitely generated), and $F : \mathscr{A} \to \mathscr{C}$ the functor $FX = \bigoplus_{i\geq 0} X^i$. Then F is left $\mathscr{E}'$-exact, and $\mathscr{E}$ (as defined in Construction 3.5) consists of precisely those $(X \to Y \to Z) \in \mathscr{E}'$ for which $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is a short exact sequence in $\mathscr{C}$. (Apply Proposition 3.3(d).) Thus, $(\mathscr{A}, \mathscr{E})$ is an exact category by Lemma 3.1. □

Remark 3.7. Though the construction of $(\mathscr{A}, \mathscr{E})$ requires the tools of exact category theory, they can all be interpreted here in the larger (and more familiar) category mod–H. Similar remarks apply to the second construction below.

Construction 3.8. *Keep Assumptions 3.2. For each integer i, let $\mathscr{S}_i$ be a full, additive subcategory of $\mathscr{A}$ such that if $S \in \mathscr{S}_i$, then S_K is a direct sum of irreducible irreducible right H_K-modules having height i.*[5] *(If i is not in the image of the height function, then put $\mathscr{S}_i := [0]$.) Let $\mathscr{S}$ be the set-theoretic union of the $\mathscr{S}_i$. Let $\mathscr{A}(\mathscr{S})$ be the full subcategory of $\mathscr{A}$ above having objects M satisfying $M^j/M^{j-1} \in \mathscr{S}_j$ for all j (or, equivalently, M^j/M^{j-1} is in $\mathscr{S}$ for all integers j).*

Define $\mathscr{E}(\mathscr{S})$ to be the class of those conflations $X \to Y \to Z$ in $\mathscr{E}$ such that $X, Y, Z \in \mathscr{A}(\mathscr{S})$ and with the additional property that, for each integer i,

$$0 \to X^i/X^{i-1} \to Y^k/Y^{i-1} \to Z^i/Z^{i-1} \to 0$$

is a split short exact sequence in mod–H. (Thus, by definition, $\mathscr{E}(\mathscr{S}) \subseteq \mathscr{E}$.)

Theorem 3.9. *$(\mathscr{A}(\mathscr{S}), \mathscr{E}(\mathscr{S}))$ is an exact category.*

Proof. The first two axioms are easily verified. (Note again that $\mathscr{E}(\mathscr{S}) \subseteq \mathscr{E}$.) To check Axiom 2, consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y' & \longrightarrow & Z' \\ \big\downarrow & & \big\downarrow & & {\scriptstyle f}\big\downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where the bottom row is in $\mathscr{E}(\mathscr{S})$ and the top row is a pullback (in mod–H) with $Z' \in \mathscr{A}(\mathscr{S})$. However, since the bottom row lies in $\mathscr{E}$, we have that $X' \to Y' \to Z'$ also belongs to $\mathscr{E}$. The issue is whether it splits section by section (which, in particular, would imply that $Y' \in \mathscr{A}(\mathscr{S})$). This splitting at the section level follows easily from the fact that the pullback of a split short exact sequence is split. A similar argument gives Axiom 2°. □

The following lemma shows a common vanishing condition leads to expected exact sequences.

[5] We think of $\mathscr{S}_i$ as a special class of objects in $\mathscr{A}$; the stated condition on S_K is necessary, but not always sufficient for membership in $\mathscr{S}_i$.

Lemma 3.10. *Suppose that $X \in \mathscr{A}(\mathscr{S})$ satisfies $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S,X)=0$ for all $S\in\mathscr{S}$. Let $E\to F\to G$ belong to $\mathscr{E}(\mathscr{S})$. For any $S\in\mathscr{S}$,*

$$0\to \mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(G,X)\to \mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(F,X)\to \mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(E,X)\to 0.$$

is a short exact sequence of $\mathscr{K}$-modules.

Proof. The lemma is obvious when $F=F^h$ for some $h\in\mathbb{Z}$ and $E=F^{h-1}$, since $F^h/F^{h-1}\in\mathscr{S}_h$.

This special case applies to all columns of the commutative diagram, upon applying the functor $\mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(-,X)$ to the diagram

$$\begin{array}{ccccc}
E^{h-1} & \longrightarrow & F^{h-1} & \longrightarrow & G^{h-1}\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
E & \longrightarrow & F & \longrightarrow & G\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
E^h/E^{h-1} & \longrightarrow & F^h/F^{h-1} & \longrightarrow & G^h/G^{h-1}.
\end{array}$$

Here, h is chosen so that $F=F^h$, and it follows that $E=E^h$ and $G=G^h$. Moreover, we can assume the top row of the resulting diagram is exact by induction (on, say, the number of indices j for which $F^j/F^{j-1}\neq 0$). Finally, the bottom split row, of course, remains split exact in the new 3×3 diagram. Since the middle row of the latter satisfies the hypothesis of [Mac94, Ex. 2, p. 51], it defines a short exact sequence. This proves the lemma. □

4. Some further results for $(\mathscr{A}(\mathscr{S}),\mathscr{E}(\mathscr{S}))$ and construction of $T^\dagger$

In this section, we consider further the exact category $(\mathscr{A}(\mathscr{S}),\mathscr{E}(\mathscr{S}))$ introduced in Construction 3.8.

Proposition 4.1. *Let $M,N\in\mathscr{A}(\mathscr{S})$, and let h be any integer.*

(a) There is a natural isomorphism $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(N^h,M)\cong \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(N^h,M^h)$.

(b) In particular, if $S\in\mathscr{S}_h$, we have

$$\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S,M)=\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S,M^h).$$

(c) Assume that $S\in\mathscr{S}_h$. Suppose that $M=M^h$ and $M^{h-1}=0$. Then $\mathrm{Ext}^1_{\mathscr{A}(\mathscr{S})}(S,M)=0$.

Proof. Without loss, take $N=N^h$ in (a). Obviously, there is a natural transformation

$$\eta(N,M):\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(N,M)\to \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(N^h,M^h)$$

which sends $(M\to Y\to N)\in\mathscr{E}(\mathscr{S})$ to $(M^h\to Y^h\to N^h)\in\mathscr{E}(\mathscr{S})$. The inverse is obtained by pushout. This proves (a), and (b) follows. Finally, (c) follows immediately from the definition of $\mathscr{E}(\mathscr{S})$. □

We also have the following result. It is immediate from the definitions.

Lemma 4.2. *Let $M\in\mathscr{A}(\mathscr{S})$. If $S\in\mathscr{S}_h$, then $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S,M^{h-1})\cong \mathrm{Ext}^1_H(S,M^{h-1})$.*

Proposition 4.3. *Let $S \in \mathscr{S}_h$, let $M \in \mathscr{A}(\mathscr{S})$, and let j be a non-negative integer. There is a 6-term exact sequence*

$$0 \to \mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(S, M^j) \to \mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(S, M) \to \mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(S, M/M^j)$$
$$\xrightarrow{f} \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M^j) \xrightarrow{g} \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M) \to \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M/M^j).$$

It is compatible with first 6 terms of the long exact sequence for the functor $\mathrm{Hom}_{\mathscr{A}}(S, -) = \mathrm{Hom}_H(S, -)$ *applied to the short exact sequence* $0 \to M^j \to M \to M/M^j \to 0$.

Proof. All the maps are standard: the connecting map uses pullbacks, and the other $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}$-maps arise from functorality (and use pushouts). The composition of any two consecutive maps is zero. All $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}$–groups are contained in (and are compatible with) their Ext^1_H counterparts. Now an object in the kernel of g is also in the kernel of is classical counterpart, so lies in the image of f, since the first three terms of the long exact sequence are identical to those in the classical case.

Now consider exactness at the 5th node. If $M = M^j$, then g is clearly an isomorphism and exactness at the 5th node follows. If g is not an isomorphism, then j is smaller than h, so that $(M^j)^{h-1} = M^j$, so Lemma 4.2 can be applied. □

Remark 4.4. Observe that exactness of the first 5 terms of the proposition holds for any $S \in \mathscr{A}(\mathscr{S})$, not just in $\mathscr{S}$. Also, as noted in Proposition 5.2, the $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}$-groups above are all naturally $\mathscr{K}$-modules. The proof of that proposition shows they are $\mathscr{K}$-submodules of the corresponding $\mathscr{K}$-modules Ext^1_H. All maps in the above proposition are $\mathscr{K}$-module maps.

Corollary 4.5. *Let $S \in \mathscr{S}$ have height h, and let $M \in \mathscr{A}(\mathscr{S})$.*

(a) The map $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M^{h-1}) \to \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M^h) = \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M)$ *is surjective.*

(b) We have $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M) = 0$ *if and only if the map*

$$\mathrm{Hom}_{\mathscr{A}(\mathscr{S})}(S, M^h/M^{h-1}) \to \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M^{h-1})$$

is surjective.

(c) Suppose that $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M^{h-1})$ *is generated as a $\mathscr{K}$-module by $\epsilon_1, \cdots, \epsilon_n$. Let $M^{h-1} \to N \to S^{\oplus n}$ represent the element of*

$$\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S^{\oplus n}, M^{h-1}) \cong \mathrm{Ext}^1_H(S^{\oplus n}, M^{h-1}) \quad \text{(see Lemma 4.2)}$$

corresponding to $\chi := \epsilon_1 \oplus \cdots \oplus \epsilon_n$. Finally, suppose there is a commutative diagram

$$\begin{array}{ccccc} M^{h-1} & \longrightarrow & N & \longrightarrow & S^{\oplus n} \\ \| & & \downarrow & & \downarrow f \\ M^{h-1} & \longrightarrow & M^h & \longrightarrow & M^h/M^{h-1} \end{array}$$

where f is a morphism in $\mathscr{A}(\mathscr{S})$. Then $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M) = 0$.

Proof. (a) follows from the 6-term exact sequence and the fact that $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, S) = 0$. (The equality follows from Lemma 4.2.) The proof of (b) is similar, using Proposition 4.1. Next, if $M^{h-1} \to N_i$ corresponds to ϵ_i, there is a pullback with the top row ϵ_i. Thus, ϵ_i is a pullback of $M^{h-1} \to M^h \to M^h/M^{h-1}$ under the evident composite $g_i : S \to S^{\oplus n} \xrightarrow{f} M^h/M^{h-1}$. Consequently, the image of $g_i \in \mathrm{Hom}_{\mathscr{A}}(S, M^h/M^{h-1})$ under the connecting homomorphism to

$\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M^{h-1})$ is ϵ_i. Since i was arbitrary, the connecting homomorphism f in (b) is surjective. Hence, $\mathrm{Ext}^1_{\mathscr{A}(\mathscr{S})}(S, M) = 0$. □

Remark 4.6. The argument above has already appeared in a module theoretic form in [DPS15a]. However, the argument given there required stronger hypotheses, e.g., that $\mathrm{Ext}^1_H(S, S) = 0$.

Theorem 4.7. *Assume that each $\mathscr{S}_i$ is strictly generated as an additive category by finitely many objects, i.e., every object in $\mathscr{S}_i$ is isomorphic to a finite direct sum of a given finite set of objects in $\mathscr{S}_i$. Let $M \in \mathscr{A}(\mathscr{S})$. Then there exists an object X in $\mathscr{A}(\mathscr{S})$ and an inflation $M \xrightarrow{i} X$ such that $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, X) = 0$ for all $S \in \mathscr{S}$. In addition if h is chosen minimal such that $M^{h-1} \neq 0$, it may be assumed that the inflation induces an isomorphism $M^{h-1} \cong X^{h-1}$.*

Proof. Without loss, we can assume that $M \neq 0$, and also that $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, M) \neq 0$ for some $S \in \mathscr{S}$. Choose an integer h minimal with such a non-vanishing occurring for some $S \in \mathscr{S}_h$. Note that $M^{h-1} \neq 0$ by Proposition 4.1(c). We will next enlarge M to an object X, closer to the X required in the theorem.

Let $S_1, \cdots, S_m$ be generators for $\mathscr{S}_h$. For each index i, let $\epsilon_{i,1}, \cdots, \epsilon_{i,n_i}$ be a finite set of generators for $\mathrm{Ext}^1_H(S_i, M^{h-1}) \cong \mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S_i, M^{h-1})$. Form an extension $0 \to M^{h-1} \to Y_i \to S_i^{\oplus n_i} \to 0$ corresponding to $\chi_i := \epsilon_{i,1} \oplus \cdots \oplus \epsilon_{i,n_i} \in \mathrm{Ext}^1_H(S_i^{\oplus n_i}, M^{h-1})$. Put $\chi := \chi_1 \oplus \cdots \oplus \chi_m$, and let $\chi' \in \mathrm{Ext}^1_H(M^h/M^{h-1}, M^{h-1})$ correspond to the extension $0 \to M^{h-1} \to M^h \to M^h/M^{h-1} \to 0$. Put $Z := \oplus_i S_i^{\oplus n_i} \oplus M^h/M^{h-1}$, and let $M^{h-1} \to X^h \to Z$ correspond to $\chi \oplus \chi'$. Observe there is a commutative diagram

$$\begin{array}{ccccc} M^{h-1} & \longrightarrow & Y_i & \longrightarrow & S_i^{\oplus n_i} \\ \| & & \downarrow & & \downarrow \\ M^{h-1} & \longrightarrow & X^h & \longrightarrow & Z, \end{array}$$

in which the top row corresponds to χ_i and the bottom row to $\chi \oplus \chi'$. Comparison with Corollary 4.5(c), allowing for the differences in notation, shows $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S_i, X^h) = 0$ for all i. Thus, $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, X^h) = 0$, for all S in $\mathscr{S}_h$. Note we have the same vanishing for $S \in \mathscr{S}_j$ with $j < h$, by our choice of h. In all cases, we can replace X^h with any X' containing it with $(X')^h = X^h$.

So far, we have not constructed an object X, only X^h. However, the latter may be viewed as the middle term of an exact sequence of right H-modules $0 \to M^h \to X^h \to S' \to 0$, where $S' := \bigoplus S_i^{\oplus n_i} \in \mathscr{S}_h$. This sequence clearly corresponds to a conflation in $\mathscr{E}(\mathscr{S})$. (Note how Z above is split.) Applying a pushout construction using $M^h \to M$ (see Propostion 4.1(b) and its proof), we obtain an object X in $\mathscr{A}(\mathscr{S})$ which contains a copy of M under an inflation, and has our constructed X^h as its image under the functor $(-)^h$. In addition $X^j = M^j$ for $j \leq h-1$.

Applying Proposition 4.1(b) again, we find that $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(-, X)$ vanishes on all objects in $\mathscr{S}_j$ with $j \leq h-1$ (and, thus, $j \leq h$). Now repeat the argument with X in the role of M. This requires a bigger h, unless $\mathrm{Ext}^1_{\mathscr{E}(\mathscr{S})}(S, X)$ already vanishes for all $S \in \mathscr{S}$. Eventually the process stops, at which point X satisfies all requirements of the theorem. □

For the main result, we let H be the Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ associated to a finite Coxeter system (W, S). See [Lu03, Ch. 8] for a very general "unequal parameter" version of H, and a corresponding Kazhdan-Lusztig cell theory. We use dual left cell modules S_ω as generators for the various additive categories $\mathscr{S}_i$. Here ω is a left cell in W. There are also right cells, and two-sided cells. These are all defined as equivalence classes associated to certain quasi-posets in W. We shall make use of the opposite $\leq_{LR}^{\mathrm{op}}$ of the quasi-poset order $\leq_{LR}$, defined in [Lu03, Ch. 8]. However, we view it as an order on the set Ω of left cells (rather than on W). Using $\leq_{LR}^{\mathrm{op}}$ for $\leq$ in the discussion above Proposition 2.2 earlier in this paper, choose a height function $\mathrm{ht} : \Omega \to \mathbb{Z}$. Then, for each integer i, define $\mathscr{S}_i$ as the additive category generated by all dual left cell modules S_ω for which $\mathrm{ht}(\omega) = i$.

For each $\omega \in \Omega$, construct $X = X_\omega$ as in the above theorem, with $M = S_\omega$. Choose positive integers m_ω, $\omega \in \Omega$, and let

$$T^\dagger = \bigoplus X_\omega^{\oplus m_\omega}.$$

The use of chosen positive integers m_ω is a useful flexibility—all choices of $m_\omega > 0$ lead to Morita equivalent endomorphism algebras $A^\dagger$ in the statement below.

Theorem 4.8. *The $\mathbb{Z}[t, t^{-1}]$-algebra $A^\dagger := \mathrm{End}_H(T^\dagger)$ is standardly stratified. In fact, it has a stratification system consisting of all $\Delta(\omega) := \mathrm{Hom}_H(S_\omega, T^\dagger)$, with S_ω ranging over the dual left cell modules.*

Proof. The result follows by applying Theorem 2.6, as modified by Remark 2.7, using Lemma 3.10. The projective $A^\dagger$-modules for (SS1) and (SS3) in (2.1) may be taken as the various $\mathrm{Hom}_H(X_\omega, T^\dagger)$. We leave the straightforward details to the reader. □

We mention without proof that $T^\dagger$ can be choosen with the regular module H as a direct summand. We do not yet know if it is possible to do the same with other permutation module analogs.

5. Appendix I: A summary of exact categories

This brief appendix summarizes, for the convenience of the reader, some basic material concerning exact categories. We closely follow Keller's treatment in the appendix to [DRSS99]. (See also Keller's paper [K90].)

Let $\mathscr{A}$ be an additive category. We do not repeat the standard definition, but refer to [Mac94, Chp. 9, §1] for a precise discussion. A pair (i, d) of composable morphisms $i : X \to Y$ and $d : Y \to Z$ in $\mathscr{A}$ is called exact if $i : X \to Y$ is the kernel of $d : Y \to Z$ and d is the cokernel of i. Let $\mathscr{E}$ be a class of exact pairs, which is closed under isomorphisms. If $(i, d) \in \mathscr{E}$, then i (resp., d) is called an inflation (resp., deflation), and the pair (i, d) itself can be called a conflation. We often just write $X \overset{i}{\to} Y \overset{j}{\to} Z$ or merely $X \to Y \to Z$ to denote elements (i.e., conflations) in $\mathscr{E}$.

The pair $(\mathscr{A}, \mathscr{E})$ is called an exact category provided the following axioms hold:

0. $1_0 \in \mathrm{Hom}(0, 0)$ is a deflation, where 0 is the zero object in $\mathscr{A}$.
1. The composition of two deflations is a deflation.
2. Morphisms $Y \xrightarrow{d} Z \xleftarrow{f} Z'$ in $\mathscr{A}$ in which d is a deflation can be completed to a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z \end{array}$$

in which d' is a deflation.

2°. Morphisms $X' \xleftarrow{f} X \xrightarrow{i} Y$ in $\mathscr{A}$ in which i is an inflation can be completed to a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ X' & \xrightarrow{i'} & Y' \end{array}$$

in $\mathcal{A}$ in which i' is an inflation.

Remarks 5.1. (a) The axioms above are part of Quillen's axioms [Q73] for an exact category, and they are shown in [K90] to be equivalent to the full set of axioms. Since the Quillen axioms are self-dual, it follows that any exact category in the sense of the above conditions also satisfies each corresponding dual condition. For example, the composition of any two inflations is an inflation.

(b) Continuing the above remark, note that the opposite category $\mathscr{A}^{\rm op}$ inherits an exact category structure from that of $\mathscr{A}$. Now assume that $\mathscr{A}$ is small. (If one believes in the set-theoretic philosophy of universes, every $\mathscr{A}$ can be regarded as small in an appropriate set-theoretic universe.) Applying [K90, Prop. A2] to the opposite category $\mathscr{A}^{\rm op}$, we find that there is an abelian category $\mathscr{B}$ and faithful full embedding $G : \mathscr{A} \to \mathscr{B}$, such that an exact pair (i, d) belongs to $\mathscr{E}$ if and only if $0 \to G(X) \stackrel{G(i)}{\to} G(Y) \stackrel{G(d)}{\to} G(Z) \to 0$ is a short exact sequence in $\mathscr{B}$. Moreover, we can assume that the strict image $\mathscr{M}$ of F (which is equivalent to $\mathscr{A}$) is closed under extensions in $\mathscr{B}$.

(c) It is an exercise to show that if $i' : X' \to Y'$ is an inflation with $(i', d') \in \mathscr{E}$, then the morphism $X' \to X$ induced by the zero composition $X' \to Y' \to Y \to Z$ is an isomorphism, with inverse given by the map $X \to X'$ induced from the evident zero morphism $X \to Y' \to Z$. This is all in $(\mathscr{A}, \mathscr{E})$, but it follows that the diagram in Axiom 2 is a pullback in $\mathscr{B}$ or in any abelian category in which $(\mathscr{A}, \mathscr{E})$ is fully and exactly embedded. Similar remarks hold for Axiom 2°.

(d) The embedding in (b) can be used to prove "with elements" that useful exact sequences belong to $\mathscr{E}$. For example, we can use Axiom 2, to obtain in $Y' \stackrel{i}{\to} Z' \oplus Y \stackrel{e}{\to} Z$ in $\mathscr{B}$ where the left map sends $y' \in Y'$ to $d'(y') \oplus f'(y')$ and the right map sends $z' \oplus y$ to $f(z') - d(y) \in Z$ for $z' \in Z', y \in Y$. This sequence is short exact in $\mathscr{B}$ if and only if the diagram in Axiom 2 is a pullback in $\mathscr{B}$, which is the case if and only if it is a pullback in $\mathscr{A}$. Moreover, it is short exact in $\mathscr{B}$ if and only if $(i, e) \in \mathscr{E}$.

(e) The abelian category $\mathscr{B}$ can also be used to extend the exact sequence in Proposition 5.2 below to the right by one term, as in the argument for Proposition 4.3. As previously mentioned, [DRSS99] effectively give a general 6 term version, using the "split idempotent" hypothesis, which we cannot assume.[6]

Let $(\mathscr{A}, \mathscr{E})$ be an exact category. For $X, Z \in \mathscr{A}$, let $\mathscr{E}(Z, X)$ be the set of sequences $X \to Y \to Z$ in $\mathscr{E}$. Define the usual equivalence relation $\sim$ on $\mathscr{E}(Z, X)$ by putting $X \to Y \to Z \sim X \to Y' \to Z$ provided there is a morphism $Y \to Y'$ giving a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \| & & \downarrow & & \| \\ X & \longrightarrow & Y' & \longrightarrow & Z \end{array}$$

[6]An idempotent $e : A \to A$ in an (additive) category is called split, if e can be factored as $e = \alpha\beta$, $\alpha : B \to A$ and $\beta : A \to B$, where $\beta\alpha = 1_Z$, i.e., β is a retraction.

The morphism $Y \to Y'$ is necessarily an isomorphism (as follows from Remark 5.1(b) and a diagram chase, for example). Let $\mathrm{Ext}^1_{\mathscr{E}}(Z, X) = \mathscr{E}(Z, X)/\sim$.

Proposition 5.2. *(a)* $\mathrm{Ext}^1_{\mathscr{E}}(Z, X)$ *has a natural abelian group structure such that given any* $A \to B \to C$ *in* $\mathscr{E}$ *and object* $Z \in \mathscr{A}$*, there are exact sequences*

$$0 \to \mathrm{Hom}_{\mathscr{A}}(Z, A) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Z, B) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Z, C) \xrightarrow{f} \mathrm{Ext}^1_{\mathscr{E}}(Z, A)$$

where f *is defined by pullback as in Axiom 2. A dual contravariant version holds, using the contravariant functor* $\mathrm{Hom}_{\mathscr{A}}(-, Z)$ *and pullback as in Axiom 2°.*

(b) Let $\mathscr{K}$ *be a fixed commutative, Noetheran ring. If* $\mathscr{A}$ *is a* $\mathscr{K}$*-category, then* $\mathrm{Ext}^1_{\mathscr{E}}(Z, A)$ *is naturally a* $\mathscr{K}$*-module.*

Proof. The usual argument involving the Baer sum ($\alpha + \beta = \nabla_X(\alpha \oplus \beta)\Delta_Z$) proves (a); see [Mac94, p. 85, (5.4)]. We next prove (b). Using standard embedding theorems, we can reduce to the case where $\mathscr{A}$ is a $\mathscr{K}$-category of $\mathscr{K}$-modules (We remark that this is the only case to which we make applications in this paper.). Assuming this, we have what appears to be two actions of $\mathscr{K}$ on $\mathrm{Ext}^1_{\mathscr{E}}(Z, X)$, one through the action of $\mathscr{K}$ on Z, and one through its action on X. The first of the two actions uses a pullback of multiplication by any given element b in $\mathscr{K}$ on Z, and the second uses a pushout of the $X \to Y \to Z$ action of b on X. We take part (b) as asserting, in this context, that the actions are the same, and that is (all of) what we will prove.

Suppose we are given an element of $\mathrm{Ext}^1_{\mathscr{E}}(Z, X)$ represented by $X \xrightarrow{i} Y \xrightarrow{d} Z$, and let $b \in \mathscr{K}$. Form the pullback and pushout objects as above, denoting the pullback by Y' and the pushout by $Y^{\#}$. The pullback object is formed by all pairs (y, z) with $dy = bz (y \in Y, z \in Z)$. It is an object in $\mathscr{A}$ which is a subobject of $Y \oplus Z$. There is an evident sequence $X \to Y' \to Z$, which we also call a pullback. The pushout object $Y^{\#}$ is formed as a quotient of $X \oplus Y$ by the subobject W consisting of all pairs $(-bx, ix)$, with $x \in X$. We represent an element of this quotient as a bracketed pair $[x, y]$, with the representative pair (x, y) well-defined only up to addition of an element of W. There is a corresponding pushout sequence $X \to Y^{\#} \to Z$. We claim this sequence represents the same element of $\mathrm{Ext}^1_{\mathscr{E}}(Z, X)$ as the pullback sequence with Y'. To prove this, all we have to do is exhibit a map $Y^{\#} \to Y'$ in the $\mathscr{K}$-category $\mathscr{A}$ giving the expected commutative diagram. Such a map may be defined by sending a pair $x, y \in X \oplus Y$ to $(by + ix, dy) \in Y \oplus Z$, a pair which is actually in Y', since $d(by + ix) = b(dy)$. Moreover, the map has W in its kernel since, if $x \in X$, $(b(ix) + (-bx), d(ix)) = (0, 0)$. Thus, induces a map to $Y^{\#} \to Y'$. We leave it to the reader to check the required commutativites. This proves the claim and completes the proof of part (b). □

For a relatively recent survey of exact categories, starting from the Quillen axioms (though without any explicit discussion of $\mathrm{Ext}^1_{\mathscr{E}}$), see [Bü10].

6. Appendix II: Idempotent ideals

The following result is proved in [CPS90]. For convenience, we indicate a short proof.

Proposition 6.1. *Let* J *be an idempotent ideal in a ring* A*. Assume that* ${}_AJ$ *is projective. Let* M, N *be* A/J*-modules. For any integer* $n \geq 0$*, inflation provides an isomorphism*

$$\mathrm{Ext}^n_{A/J}(M, N) \xrightarrow{\sim} \mathrm{Ext}^n_A(M, N)$$

of abelian groups. (On the right hand side, M, N are regarded as A-modules through the morphism $A \to A/J$.)

Proof. Using the short exact sequence $0 \to J \to A \to A/J \to 0$ of left A-modules, the projectivity of ${}_AJ$ implies that $\mathrm{Ext}^n_A(A/J, N) = 0$ for $n > 1$. Since $J^2 = J$, $\mathrm{Hom}_A(J, N) = 0$. Thus, any projective A/J-module is acyclic for the functor $\mathrm{Hom}_A(-, N)$. The proposition follows. □

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