The strong total rainbow connection of graphs

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Abstract

A graph is called *total colored* if both all its edges and all its vertices are colored. A path in a total colored graph is called *total rainbow* if all the edges and the internal vertices on the path have distinct colors. A graph is called *total rainbow connected* if any two vertices of the graph are connected by a total rainbow path. In this paper we introduce the concept of strong total rainbow connection of graphs. A graph is called *strongly total rainbow connected* if any two vertices of the graph are connected by a total rainbow geodesic, i.e., a path of length equals to the distance between the two vertices. For a connected graph G, the *strong total rainbow connection number*, denoted by $strc(G)$, is the minimum number of colors that are needed to make G strongly total rainbow connected. Among our results we state some simple observations about $strc(G)$ for a connected graph G. We also investigate the strong total rainbow connection numbers of some special graphs. Finally, for any pair of integers a and b with $a = 5$ and $b \geq 6$ or $a \geq 6$ and $b \geq a + 4$, we construct a connected graph G such that $trc(G) = a$ meanwhile $strc(G) = b$.

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1 Introduction

In this paper, all graphs under our consideration are finite, undirected and simple. For more notation and terminology that will be used in the sequel, we refer to [1], unless otherwise stated.

In 2008, Chartrand et al. [4] introduced the concept of rainbow connection. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \ldots, k\} (k \in \mathbb{N})$ is defined, where adjacent edges may be colored the same. For any two vertices u and v of G, a path in G connecting u and v is abbreviated as a uv-path. A uv-path P is a *rainbow uv-path* if no two edges of P are colored the same. The graph G is *rainbow connected* (with respect to c) if G contains a rainbow uv-path for every two vertices u and v in it, and the coloring c is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring*. The minimum k for which there exists a rainbow

k-coloring of the edges of G is the *rainbow connection number* of G, denoted by $rc(G)$. The topic of rainbow connection is fairly interesting and numerous relevant papers have been written. In addition, the concept of strong rainbow connection was introduced by the same authors. For two vertices u and v of G, a *rainbow uv-geodesic* in G is a rainbow uv-path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v (the length of a shortest uv-path in G). The graph G is *strongly rainbow connected* if G contains a rainbow uv-geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum k for which there exists a coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G such that G is strongly rainbow connected is the *strong rainbow connection number* of G, denoted by $src(G)$. The investigation of $src(G)$ is more challenging than that of $rc(G)$, and few papers have been obtained on it, for details see [4, 7, 11, 17].

As a natural counterpart of the rainbow connection, Krivelevich and Yuster proposed the concept of rainbow vertex-connection in [10]. A vertex colored graph G is *rainbow vertex-connected* if any two vertices of G are connected by a path whose internal vertices have different colors, and such a path is called a *vertex-rainbow path.* The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors needed to make G rainbow vertex-connected. Corresponding to the strong rainbow connection, Li et al. [13] introduced the notion of strong rainbow vertex-connection. A vertex colored graph G is *strongly rainbow vertex-connected*, if for every pair of distinct vertices u and v, there exists a vertex rainbow uv-geodesic. The minimum number k for which there exists a k-coloring of the vertices of G that results in a strongly rainbow vertex-connected graph is called the *strong rainbow vertex-connection number* of G, denoted by $srvc(G)$. For more results on rainbow vertex-connection, we refer to [14].

It was also shown that computing the rainbow connection number and rainbow vertex-connection number of an arbitrary graph is NP-hard [2, 3, 5, 6, 8, 12]. For more results on rainbow connections, we refer to the survey [15] and the book [16].

Subsequently, Liu et al. [18] proposed the concept of total rainbow connection. A *total colored graph* is a graph G such that both all edges and all vertices of G are colored. A total colored path is *total rainbow* if its edges and internal vertices have distinct colors. A total colored graph G is *total rainbow connected* if any two vertices of G are connected by a total rainbow path. The *total rainbow connection number* of G, denoted by $trc(G)$, is defined as the minimum number of colors required to make G total rainbow connected. Note the trivial fact that $trc(G) = 1$ if and only if G is a complete graph. Moreover, $2diam(G) - 1 \leq trc(G) \leq 2n - 3$ for any nontrivial connected graph G, where $diam(G)$ denotes the diameter of G. For more results on the total rainbow connection number, see [9, 18, 19].

Inspired by the concepts of total rainbow connection number $trc(G)$, strong rainbow connection number $src(G)$ and strong rainbow vertex-connection number $srvc(G)$, a natural idea is to introduce the concept of strong total rainbow connection number.

Let G be a total colored graph. For any two vertices u and v of G, a *total rainbow uv-geodesic* in G is a total rainbow uv-path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v. The graph G is *strongly total rainbow connected* if any two vertices of G are connected by a total rainbow geodesic. The *strong total rainbow connection number* of G, denoted by $strc(G)$, is defined to be the smallest number of colors needed to make G strongly total rainbow connected. Clearly, the trivial lower bound $strc(G) \geq 2diam(G) - 1$ is immediate from the definition, where G is any nontrivial connected graph and $diam(G)$ is the diameter of G. Further, this bound is tight as we will show $strc(P_n) = 2n - 3 = 2diam(P_n) - 1$ for an n-vertices path P_n. Besides, by definition, $trc(G) \leq strc(G)$ holds for every connected graph G.

In this paper, we firstly state some simple observations about $strc(G)$ for a nontrivial connected graph G. Then we investigate the values of the strong total rainbow connection number of some special graphs. Finally, for any pair of integers a and b with $a = 5$ and $b \geq 6$ or $a \geq 6$ and $b \geq a + 4$, we construct a connected graph G such that $trc(G) = a$ meanwhile $strc(G) = b$.

2 Preliminaries

We in this section state some observations about $strc(G)$ for a nontrivial connected graph G, certain necessary lemmas are also listed.

For a nontrivial connected graph G, since $diam(G) = 1$ if and only if G is a complete graph, then the following proposition is immediate.

Proposition 1. *Let G be a nontrivial connected graph. Then $strc(G) = 1$ if and only if G is a complete graph; and $strc(G) \geq 3$ if G is not complete.*

Let G be a connected graph. For any strong total rainbow coloring of G, the colors of bridges and non-pendant vertices incident with bridges must be pairwise distinct. Hence, we have the following result.

Proposition 2. *Let G be a connected graph of order $n \geq 3$. Suppose B is the set of all bridges in G, and C is the set of all non-pendant vertices that incident with bridges. Denote by b and c the cardinalities of B and C, respectively. Then $strc(G) \geq b + c$.*

Apparently, for a nontrivial connected graph G, according to the definition of $strc(G)$, $strc(G) \geq \max\{srvc(G), src(G)\}$ holds trivially. Additionally, the following associative upper bound is obvious and we omit the proof.

Proposition 3. *Let G be a connected graph with n vertices and p pendant vertices. Then $strc(G) \leq src(G) + n - p$.*

In [18], the authors derived an upper bound on the total rainbow connection number, which will be helpful in the next section.

Lemma 1 ([18]). *Let G be a connected graph on n vertices, with q vertices having degree at least 2. Then, $trc(G) \leq n-1+q$, with equality if and only if G is a tree.*

Let C_n denote the cycle of order n. The total rainbow connection number of C_n for $n \geq 3$ is established in [18].

Lemma 2 ([18]). *For $3 \leq n \leq 12$, the values of $trc(C_n)$ are given in the following table.*

n	3	4	5	6	7	8	9	10	11	12
$trc(C_n)$	1	3	3	5	6	7	8	9	11	11

For $n \geq 13$, we have $trc(C_n) = n$.

The *wheel* W_n is the graph obtained from the cycle C_n by joining a new vertex v to every vertex of C_n. The vertex v is the *centre* of W_n. $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. In [18], the authors determined $trc(W_n)$ for $n \geq 3$ and $trc(K_{m,n})$ for $2 \leq m \leq n$.

Lemma 3 ([18]). *$trc(W_3) = 1$, $trc(W_n) = 3$ for $4 \leq n \leq 6$, $trc(W_n) = 4$ for $7 \leq n \leq 9$, and $trc(W_n) = 5$ for $n \geq 10$.*

Lemma 4 ([18]). *For $2 \leq m \leq n$, we have $trc(K_{m,n}) = \min(\lceil \sqrt[m]{n} \rceil + 1, 7)$.*

3 Strong total rainbow connection numbers of some graphs

In this section, we study the strong total rainbow connection numbers for some specific graphs. We begin with trees of order n.

Theorem 1. *Let T be a nontrivial tree of order n with p pendant vertices. Then $strc(T) = 2n-p-1$.*

Proof. From Lemma 1 we know that $trc(T) = 2n-p-1$. Thus $strc(T) \geq trc(T) = 2n-p-1$.

On the other hand, for any total rainbow coloring c of T with $trc(T) = 2n-p-1$ colors and arbitrary two vertices u and v of T, the total rainbow path between u and v is also the total rainbow geodesic between u and v since u and v are connected by exactly one path. Therefore, c is also a strong total rainbow coloring of T. That is, $strc(T) \leq 2n-p-1$. Hence $strc(T) = 2n-p-1$. ∎

It is well known that every nontrivial tree has at least two pendant vertices and every nontrivial tree with $n \geq 3$ has at most $n-1$ pendant vertices. Thus for a nontrivial tree T of order $n \geq 3$, we have $n \leq strc(T) \leq 2n-3$. In addition, both the lower bound and the upper bound can reached by the n-vertices star S_n and the n-vertices path P_n, respectively. Particularly, $strc(P_n) = 2n-3 = 2diam(P_n) - 1$, as mentioned in the section of Introduction.

By taking full advantages of the fact that $strc(G) \geq trc(G)$ and the proof of Lemma 2 (see the proof of Theorem 2 in [18]), one can easily check that $strc(C_3) = 1$, $strc(C_4) = 3$, $strc(C_5) = 3$, $strc(C_6) = 5$, $strc(C_7) = 6$, $strc(C_8) = 7$, $strc(C_9) = 8$, $strc(C_{10}) = 9$, $strc(C_{11}) = 11$, $strc(C_{12}) = 11$. We aim at determining $strc(C_n)$ for $n \geq 13$ in the following theorem.

Theorem 2. *For $n \geq 13$, we have $strc(C_n) = n$.*

Proof. First, since $strc(G) \geq trc(G)$ for any connected graph G. The lower bound $strc(C_n) \geq n$ is easy to get by Lemma 2.

Then we prove the upper bound. By the definition of strong total rainbow connection number, it suffices to find a strong total rainbow coloring using n colors for C_n. Let $C_n = v_0v_1 \cdots v_{n-1}v_0$. We discuss the following two cases.

Case 1 If n is even.

Then color the edges v_0v_1, v_1v_2, $\ldots$, $v_{\frac{n}{2}-1}v_{\frac{n}{2}}$, $v_{\frac{n}{2}}v_{\frac{n}{2}+1}$, $v_{\frac{n}{2}+1}v_{\frac{n}{2}+2}$, $\ldots$, $v_{n-1}v_0$ with colors $0, 1, \ldots, \frac{n}{2}-1, 0, 1, \ldots, \frac{n}{2}-1$, respectively. Meanwhile, gradually color the vertices v_0, v_1, $\ldots$, $v_{\frac{n}{2}-1}$, $v_{\frac{n}{2}}$, $\ldots$, v_{n-1} with colors $\frac{n}{2}$, $\frac{n}{2}+1$, $\ldots$, $n-1$, $\frac{n}{2}$, $\ldots$, $n-1$. Whereupon a total coloring using n colors is assigned to C_n. Under this coloring, any path with length no more than $\frac{n}{2}$ is a total rainbow path. Since $diam(C_n) = \frac{n}{2}$, then for any two vertices v_i, v_j of C_n, there is a total rainbow v_iv_j-geodesic. Hence C_n is strongly total rainbow connected. That is, $strc(C_n) \leq n$.

Case 2 If n is odd.

Then color the edges v_0v_1, v_1v_2, $\ldots$, $v_{\frac{n-3}{2}}v_{\frac{n-1}{2}}$, $v_{\frac{n+1}{2}}v_{\frac{n+3}{2}}$, $v_{\frac{n+3}{2}}v_{\frac{n+5}{2}}$, $\ldots$, $v_{n-1}v_0$ with colors $0, 1, \ldots, \frac{n-3}{2}, 0, 1, \ldots, \frac{n-3}{2}$ stepwise. The remaining edge $v_{\frac{n-1}{2}}v_{\frac{n+1}{2}}$ is assigned the color $\frac{n-1}{2}$. Next, color the vertices v_1, v_2, $\ldots$, $v_{\frac{n-1}{2}}$, $v_{\frac{n+1}{2}}$, $\ldots$, v_{n-1} with colors $n-1$, $n-2$, $\ldots$, $\frac{n+1}{2}$, $n-1$, $\ldots$, $\frac{n+1}{2}$. Further, assign the color $\frac{n-1}{2}$ to the vertex v_0. Now we have a total coloring of C_n with n colors. Similarly, under this coloring, one can check that for any pair of vertices v_i and v_j ($i \neq j$), there exists a total rainbow v_iv_j-geodesic in C_n. Accordingly, $strc(C_n) \leq n$.

In summary, we finally arrive at $strc(C_n) = n$ for $n \geq 13$. ∎

We give the following theorem to study the strong total rainbow connection number of W_n, whose proof is partially based on the fact that $strc(W_n) \geq trc(W_n)$.

Theorem 3. *Let W_n be a wheel with $n \geq 3$. Then*

$$\begin{cases} strc(W_3) = 1, \\ strc(W_n) = \lceil \frac{n}{3} \rceil + 1 \quad \text{for } n \geq 4. \end{cases}$$

Proof. Let v be the centre of W_n, and v_1, v_2, …, v_n be the vertices of W_n in the cycle C_n.

(i) Since W_3 is precisely the complete graph K_4, then obviously $strc(W_3) = 1$.

(ii) It follows from Lemma 3 that $strc(W_n) \geq trc(W_n) = 3$ when $4 \leq n \leq 6$. On the other hand, when $4 \leq n \leq 6$, we define a total coloring f_n with 3 colours for W_n as the same as Liu et al. [18]. Namely, let $f_n(vv_i) = \lceil \frac{i}{3} \rceil$, $f_n(v_iv_{i+1}) = 1$ if $i \equiv 1 \pmod 3$, $f_n(v_iv_{i+1}) = 2$ if $i \equiv 2 \pmod 3$, $f_n(v_iv_{i+1}) = 3$ if $i \equiv 0 \pmod 3$ and $f_n(v) = f_n(v_i) = 3$ for $i = 1, 2, \ldots, n$. Note that $v_{n+1} = v_1$. Then under this coloring, W_n is strongly total rainbow connected. Hence $strc(W_n) \leq 3$. Therefore, we get $strc(W_n) = 3 = \lceil \frac{n}{3} \rceil + 1$ for $4 \leq n \leq 6$.

(iii) What remains to be shown is $strc(W_n) = \lceil \frac{n}{3} \rceil + 1$ for $n \geq 7$. Firstly we prove the lower bound, that is, $strc(W_n) \geq \lceil \frac{n}{3} \rceil + 1$. Assume to the contrary, $strc(W_n) \leq \lceil \frac{n}{3} \rceil$. Let f'_n be a strong total rainbow coloring of W_n with $strc(W_n)$ colors. Since $n \geq 7$, then for each vertex $v_i (1 \leq i \leq n)$, there exists at least a vertex v_j with $j \in \{1, 2, \ldots, n\}, j \neq i$ such that the unique v_iv_j-geodesic of length 2 passes the centre v. Thus $f'_n(v) \neq f'_n(vv_i)$ for $i = 1, 2, \ldots, n$. Therefore, the n edges $vv_i (i = 1, 2, \ldots, n)$ use at most $strc(W_n) - 1 \leq \lceil \frac{n}{3} \rceil - 1 < \frac{n}{3}$ different colors. Which deduces that there exist at least four different edges, say vv_i, vv_j, vv_k, vv_l, such that $f'_n(vv_i) = f'_n(vv_j) = f'_n(vv_k) = f'_n(vv_l)$. Again since $n \geq 7$, there exist at least two vertices, without loss of generality, say v_i and v_j, such that the unique v_iv_j-geodesic is precisely the path v_ivv_j. So, there are no total rainbow v_iv_j-geodesics as $f'_n(vv_i) = f'_n(vv_j)$, a contradiction. Consequently, $strc(W_n) \geq \lceil \frac{n}{3} \rceil + 1$.

Now we prove the upper bound by giving W_n a strong total rainbow coloring g_n with colors from $\{1, 2, \ldots, \lceil \frac{n}{3} \rceil + 1\}$, as follows. Let $g_n(vv_i) = \lceil \frac{i}{3} \rceil$ for $i = 1, 2, \ldots, n$; $g_n(v) = \lceil \frac{n}{3} \rceil + 1$; $g_n(v_iv_{i+1}) = 1$ if $i \equiv 1 \pmod 3$; $g_n(v_iv_{i+1}) = 2$ if $i \equiv 2 \pmod 3$; $g_n(v_iv_{i+1}) = 3$ if $i \equiv 0 \pmod 3$; $g_n(v_{3j+2}) = 4$ for $j = 0, 1, \ldots, \lfloor \frac{n-2}{3} \rfloor$. Lastly, color the remaining vertices arbitrarily with colors in $\{1, 2, \ldots, \lceil \frac{n}{3} \rceil + 1\}$. Notice that here we stipulate $v_{n+1} = v_1$ and since $n \geq 7$, we have at least $\lceil \frac{7}{3} \rceil + 1 = 4$ different colors available. One can check that this total coloring g_n is indeed a strong total rainbow coloring of W_n. Accordingly, by definition, $strc(W_n) \leq \lceil \frac{n}{3} \rceil + 1$.

In conclusion, we have established that $strc(W_n) = \lceil \frac{n}{3} \rceil + 1$ for $n \geq 7$. And then the theorem is proved. ∎

By Proposition 1 and Theorem 1, we know that $strc(K_{1,1}) = 1$ and $strc(K_{1,n}) = n+1$ for $n \geq 2$. In the sequel, we will determine $strc(K_{m,n})$ for $2 \leq m \leq n$.

Theorem 4. *For $2 \leq m \leq n$, we have $\lceil \sqrt[m]{n} \rceil \leq strc(K_{m,n}) \leq \lceil \sqrt[m]{n} \rceil + 1$.*

Proof. Let the classes of $K_{m,n}$ be U and V, where $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. We first show the lower bound by contradiction. Assume that $strc(K_{m,n}) < \lceil \sqrt[m]{n} \rceil$, that is, $strc(K_{m,n}) \leq \lceil \sqrt[m]{n} \rceil - 1$. Then there exists a strong total rainbow coloring, say g, using colors from $\{1, 2, \ldots, \lceil \sqrt[m]{n} \rceil - 1\}$ such that $K_{m,n}$ is strongly total rainbow connected. Assign a vector $z_i = (z_{i1}, z_{i2}, \ldots, z_{im})$ of length m to every vertex $v_i (i = 1, 2, \ldots, n)$ satisfying $z_{ij} = g(v_i u_j) (j = 1, 2, \ldots, m)$. Since $g(v_i u_j) \in \{1, 2, \ldots, \lceil \sqrt[m]{n} \rceil - 1\}$, there are at most $(\lceil \sqrt[m]{n} \rceil - 1)^m$ distinct vectors. Clearly, $(\lceil \sqrt[m]{n} \rceil - 1)^m < n$. Therefore, there exist at least two vertices v_{k_1} and v_{k_2} such that $z_{k_1} = z_{k_2}$. Namely, $g(v_{k_1} u_j) = g(v_{k_2} u_j)$ for each $j \in \{1, 2, \ldots, m\}$. Since any $v_{k_1}v_{k_2}$-geodesic in $K_{m,n}$ must contain some vertex of U, then there exist no total rainbow geodesics between v_{k_1} and v_{k_2}, which contradicts to g being a strong total rainbow coloring. Hence $strc(K_{m,n}) \geq \lceil \sqrt[m]{n} \rceil$.

Now we begin to show the upper bound. We construct a strong total rainbow coloring f for $K_{m,n}$ by using colors from $\{1, 2, \ldots, \lceil \sqrt[m]{n} \rceil + 1\}$ as follows. Assign to the vertices of V distinct vectors of length m with entries from $\{1, 2, \ldots, \lceil \sqrt[m]{n} \rceil\}$. Since $(\lceil \sqrt[m]{n} \rceil)^m \geq n = |V|$, we can make sure that any pair of vertices of V are assigned with different vectors. Simultaneously, since $m \leq n$, we can stipulate that the m vectors $(2, 1, 1, \ldots, 1)$, $(1, 2, 1, \ldots, 1)$, $\ldots$, $(1, 1, \ldots, 2, 1)$, $(1, 1, \ldots, 1, 2)$ are all present. For $v_i \in V$, let $z'_i = (z'_{i1}, z'_{i2}, \ldots, z'_{im})$ denote the vector assigned to v_i. Then for $u_j \in U$ and $v_i \in V$, let $f(u_j v_i) = z'_{ij}$ $(j = 1, 2, \ldots, m; i = 1, 2, \ldots, n)$. Further, let $f(w) = \lceil \sqrt[m]{n} \rceil + 1$ for all $w \in V(K_{m,n})$. In such a way, we define a total coloring for $K_{m,n}$. Under this coloring, for $u_{j_1}, u_{j_2} \in U$, the path $u_{j_1} z u_{j_2}$ is a total rainbow geodesic between u_{j_1} and u_{j_2}, where $z \in V$ is the vertex assigned the vector $z' = (1, \ldots, 1, 2, 1, \ldots, 1)$, with 2 in the j_1-th position. For $v_{k_1}, v_{k_2} \in V$, since $z'_{k_1} \neq z'_{k_2}$, there exists at least an index $j \in \{1, 2, \ldots, m\}$ such that $z'_{k_1 j} \neq z'_{k_2 j}$. It follows that there exists at least a vertex $u_j \in U$ such that $f(u_j v_{k_1}) \neq f(u_j v_{k_2})$. Hence the path $v_{k_1} u_j v_{k_2}$ is a total rainbow geodesic between v_{k_1} and v_{k_2}. While for any $u \in U$ and any $v \in V$, there obviously exists a total rainbow uv-geodesic. Therefore, for any two vertices of $K_{m,n}$, there exists at least a total rainbow geodesic connecting them. Namely, $K_{m,n}$ is strongly total rainbow connected under the coloring f. Hence $strc(K_{m,n}) \leq \lceil \sqrt[m]{n} \rceil + 1$.

To sum up, we arrive at $\lceil \sqrt[m]{n} \rceil \leq strc(K_{m,n}) \leq \lceil \sqrt[m]{n} \rceil + 1$. ∎

Actually, in Theorem 4, if $strc(K_{m,n}) = \lceil \sqrt[m]{n} \rceil$, then for any strong total rainbow coloring g of $K_{m,n}$ with colors from $\{1, 2, \ldots, \lceil \sqrt[m]{n} \rceil\}$, we conjecture that there exists at least a vertex $w \in V(K_{m,n})$ such that $g(w)$ is different from all the colors of the edges in $K_{m,n}$. That is, the edges of $K_{m,n}$ use at most $\lceil \sqrt[m]{n} \rceil - 1$ colors, which will deduce a contradiction. Hence we give the following conjecture.

Conjecture 1. *If* $2 \leq m \leq n$, *then* $strc(K_{m,n}) = \lceil \sqrt[m]{n} \rceil + 1$.

It's worth mentioning that according to the proof of Theorem 4 in [18], we know that $trc(K_{m,n}) = \lceil \sqrt[m]{n} \rceil + 1$ for $m \leq n \leq 6^m$. Hence $strc(K_{m,n}) \geq trc(K_{m,n}) = \lceil \sqrt[m]{n} \rceil + 1$ if $m \leq n \leq 6^m$, which means that the conjecture above holds for $2 \leq m \leq n \leq 6^m$.

Lastly we discuss the case when G is a complete multipartite graph $K_{n_1,\ldots,n_t}$. Let M be a matching. The two ends of each edge in M are said to be matched under M, and each vertex incident with an edge in M is said to be covered by M. Moreover, a matching M is called a k-matching if the size of M is k.

Theorem 5. *Let $t \geq 3$, $1 \leq n_1 \leq \cdots \leq n_t$, $m = \sum\limits_{i=1}^{t-1} n_i$ and $n = n_t$. Then*

$$\begin{cases} strc(K_{n_1,\ldots,n_t}) = 1 & \text{if } n = 1; \\ strc(K_{n_1,\ldots,n_t}) = 3 & \text{if } n \geq 2 \text{ and } m > n; \\ \lceil \sqrt[m]{n} \rceil \leq strc(K_{n_1,\ldots,n_t}) \leq \lceil \sqrt[m]{n} \rceil + 1 & \text{if } n \geq 2 \text{ and } m \leq n. \end{cases}$$

Proof. For convenience, we write G instead of $K_{n_1,\ldots,n_t}$. Let V_i be the i-th class with n_i vertices for $1 \leq i \leq t$. Set $U = \bigcup\limits_{i=1}^{t-1} V_i$.

(i) If $n = 1$, then $G = K_{n_1,\ldots,n_t} = K_{1,\ldots,1} = K_t$ is a complete graph. Hence clearly $strc(G) = strc(K_t) = 1$.

(ii) If $n \geq 2$ and $m > n$. Then G is not complete, which implies that $strc(G) \geq 3$. Since $|U| = \sum\limits_{i=1}^{t-1} |V_i| = \sum\limits_{i=1}^{t-1} n_i = m > n$, there exists a n-matching E_0 between U and V_t. Let $U_0 = \{u \in U : u \text{ is covered by } E_0\}$, $U_i = V_i \cap U_0$ and $E_0|_{U_i} = \{e = uv \in E_0 : u \in U_i \text{ and } v \in V_t\}$. For each i with $1 \leq i \leq t-1$, since $n_i \leq n_t$, there is a $|V_i \setminus U_i|$-matching E_i between $V_i \setminus U_i$ and V_t such that E_i and $E_0|_{U_i}$ are vertex-disjoint. In order to prove $strc(G) \leq 3$ in this case, we give G such a total coloring c using 3 colors, as defined as follows. Let $c(e) = 1$ for $e \in E_0 \cup E_1 \cdots \cup E_{t-1}$; $c(v) = 2$ for all $v \in G$; color the remaining edges with color 3. We will show that c is a strong total rainbow coloring of G. Let u and v be arbitrary two different vertices of G. If u and v are in different classes, then there exists a total rainbow geodesic connecting u and v obviously. If u and v are in the same class, then according to the positions of u and v, the discussion is divided into four aspects regardless of the symmetric cases.

Case 1 $u, v \in V_t$.
Then there exists some vertex $w \in U$ such that $wv \in E_0$ while $wu \notin E_0 \cup E_1 \cdots \cup E_{t-1}$. By the definition of the coloring c, we know $c(wv) = 1$, $c(w) = 2$, $c(wu) = 3$. Hence uwv is a total rainbow geodesic between u and v.

Case 2 $\{u, v\} \subseteq U_i$ for some $i \in \{1, 2, \ldots, t-1\}$.
Then there exists some vertex $x \in V_t$ such that $vx \in E_0$ while $ux \notin E_0 \cup E_1 \cdots \cup E_{t-1}$. Hence $c(vx) = 1$, $c(x) = 2$, $c(ux) = 3$. Accordingly, uxv is a total rainbow geodesic connecting u and v.

Case 3 $u \in U_i$, $v \in V_i \setminus U_i$ for some $i \in \{1, 2, \ldots, t-1\}$.

Case 4 $\{u, v\} \subseteq V_i \setminus U_i$ for some $i \in \{1, 2, \ldots, t-1\}$.

The arguments of Case 3 and Case 4 are similar to that of Case 2, we will not elaborate.

Based on the above analysis, we have illustrated the existence of a total rainbow uv-geodesic for any two vertices u and v in G. Thus c is a strong total rainbow coloring of G with 3 colors. It follows that $strc(G) \leq 3$. Therefore, $strc(G) = 3$ if $n \geq 2$ and $m > n$.

(iii) If $n \geq 2$ and $m \leq n$. Firstly, we prove the lower bound. Consider the edges between U and V_t. For any two vertices u, $v \in V_t$, any uv-geodesic must contain a

vertex of U. Being similar to the argument of the lower bound of strong total rainbow connection number for complete bipartite graphs, we can get $strc(G) \geq \lceil \sqrt[m]{n} \rceil$.
Then we prove the upper bound by defining a strong total rainbow coloring c for G with $\lceil \sqrt[m]{n} \rceil + 1$ colors. Let $U = \{v_1, v_2, \ldots, v_m\}$. Assign to the vertices of V_t distinct vectors of length m with entries from $\{1, 2, \ldots, \lceil \sqrt[m]{n} \rceil\}$. This is possible since $(\lceil \sqrt[m]{n} \rceil)^m \geq n = |V_t|$. In addition, since $m \leq n$, the vectors $(2, 1, 1, \ldots, 1)$, $(1, 2, 1, \ldots, 1)$, $\ldots$, $(1, 1, \ldots, 2, 1)$, $(1, 1, \ldots, 1, 2)$ can all be present. For $v \in V_t$, let $v' = (v'_1, v'_2, \ldots, v'_m)$ denote the vector assigned to v. Then for $v_i \in U$ and $v \in V_t$, let $c(vv_i) = v'_i$. For the remaining edges, we color them with color $\lceil \sqrt[m]{n} \rceil + 1$. Finally let $c(v) = \lceil \sqrt[m]{n} \rceil + 1$ for all $v \in G$. One can check that such a coloring c is actually a strong total rainbow coloring of G with $\lceil \sqrt[m]{n} \rceil + 1$ colors (the proof is as the same as Theorem 4). Hence $strc(G) \leq \lceil \sqrt[m]{n} \rceil + 1$. Consequently, $\lceil \sqrt[m]{n} \rceil \leq strc(G) \leq \lceil \sqrt[m]{n} \rceil + 1$ if $n \geq 2$ and $m \leq n$.

The proof is thus completed. ∎

4 On the (strong) total rainbow connection numbers with prescribed values

We have seen that $strc(G) \geq trc(G)$ holds for every connected graph G. In addition, we also find graphs for which the equality holds. For example, $strc(C_n) = trc(C_n) = n$ if $n \geq 13$. However, we want to know what difference is there between these two parameters. In this section, we aim to find a graph G for every pair of integers a, b with $a = 5$ and $b \geq 6$ or $a \geq 6$ and $b \geq a + 4$ such that $trc(G) = a$ while $strc(G) = b$.

In order to prove the main result, we need some assistant lemmas.

Lemma 5. *Let G be a connected graph and $e = uv$ be a pendant edge of G with pendant vertex u. Then $trc(G - u) \leq trc(G)$.*

Proof. By definition, for a given total rainbow coloring of G with $trc(G)$ colors, there is a total rainbow path between each pair of vertices of G. Since e is a pendant edge of G, every total rainbow path connecting two vertices of $G - u$ misses e. Therefore, there is still a total rainbow path between any two vertices of $G - u$, which implies that $trc(G - u) \leq trc(G)$. ∎

Remark: Similarly, we can show the result of Lemma 5 is also true for the version of strong total rainbow connection number, that is, we also have $strc(G-u) \leq strc(G)$.

Before we give the following lemmas, a new graph was constructed. Denote by $W_n^r (r \geq 1)$ the graph obtained from a wheel W_n by attaching r pendant edges at the center v of W_n. The r pendant vertices are denoted by $u_1, u_2, \ldots, u_r$ and the n vertices in the cycle C_n of W_n are denoted by $v_1, v_2, \ldots, v_n$. We will investigate $trc(W_n^r)$ and $strc(W_n^r)$ respectively.

Lemma 6. *For $n \geq 10$, we have $trc(W_n^r) = 5$ for $1 \leq r \leq 4$, and $trc(W_n^r) = r+1$ for $r \geq 5$.*

Proof. According to Lemma 3 and Lemma 5, we know that $trc(W_n^r) \geq trc(W_n) = 5$ when $n \geq 10$. For W_n^4, we give it a total coloring g as follows. Let $g(vv_i) = 1$ if $i \equiv 1(\text{mod } 2)$, $g(vv_i) = 2$ if $i \equiv 0(\text{mod } 2)$, $g(v_i) = 3$ for $i = 1, 2, \ldots, n$, $g(v_iv_{i+1}) = 4$ for $i = 1, 2, \ldots, n$, where $v_{n+1} = v_1$, $g(v) = g(u_i) = 5$ and $g(vu_i) = i$ for $i = 1, 2, 3, 4$. Then one can check that g is indeed a total rainbow coloring of W_n^4 using 5 colors. Hence $trc(W_n^4) \leq 5$. It follows that $trc(W_n^4) = 5$. In addition, for $1 \leq r \leq 3$, we have $trc(W_n^r) \leq trc(W_n^4) = 5$ by Lemma 5. Thus $trc(W_n^r) = 5$ for $r = 1, 2, 3$.

Now we discuss the case of $r \geq 5$. Let u_i and u_j ($i \neq j$, $i, j = 1, 2, \ldots, r$) be any two pendant vertices in W_n^r. In order to get a total rainbow path connecting u_i and u_j, vu_i and vu_j must have different colors, which means that the r pendant edges must be assigned r different colors. Furthermore, any total rainbow path connecting two pendant vertices passes the vertex v. Hence the color of v is different from the color of vu_i for each $i \in \{1, 2, \ldots, r\}$. In summary, to get a total rainbow coloring of W_n^r, at least $r+1$ colors are required. Namely, $trc(W_n^r) \geq r+1$. We in the following construct a total coloring f for W_n^r by using colors from $\{1, 2, \ldots, r+1\}$.

Let $f(vv_i) = 1$ if $i \equiv 1(\text{mod } 2)$, $f(vv_i) = 2$ if $i \equiv 0(\text{mod } 2)$, $f(v_i) = 3$ and $f(v_iv_{i+1}) = 4$ for $i = 1, 2, \ldots, n$, similarly, $v_{n+1} = v_1$, $f(v) = 5$, $f(u_i) = 5$ for $i = 1, 2, \ldots, r$, $f(vu_i) = i$ for $i = 1, 2, 3, 4$ and $f(vu_i) = i+1$ for $i = 5, 6, \ldots, r$. Under this coloring, we can find a total rainbow path for any two vertices of W_n^r. Hence $trc(W_n^r) \leq r+1$. Accordingly, $trc(W_n^r) = r+1$ for $r \geq 5$. And the lemma is thus proved. ∎

Lemma 7. *For $n \geq 10$, we have $strc(W_n^r) = \lceil \frac{n}{3} \rceil + r + 1$.*

Proof. Firstly we prove the lower bound. For any two pendant vertices u_i and u_j in W_n^r, where $i \neq j$, $i, j = 1, 2, \ldots, r$, in order to obtain a total rainbow geodesic connecting them, the colors of vu_i, v and vu_j are all distinct. Which deduces that there should be $r+1$ distinct colors for $v, vu_1, vu_2, \ldots, vu_r$. Without loss of generality, these $r+1$ colors are set as $1, 2, \ldots, r+1$ respectively. Moreover, let $v_k (k = 1, 2, \ldots, n)$ and $u_i (i = 1, 2, \ldots, r)$ be arbitrary two vertices in W_n^r. Since $d(v_k, u_i) = 2$, to make sure that there exists a total rainbow geodesic between v_k and u_i, the colors of vv_k, v and vu_i are all distinct. Hence the color of vv_k is different from the colors in $\{1, 2, \ldots, r+1\}$, i.e., new colors should be used for the edges $vv_k (k = 1, 2, \ldots, n)$. We claim that among these edges, there are at most three edges assigned the same color.

Assume, to the contrary, that there exist at least four edges having the same color, say vv_{k_1}, vv_{k_2}, vv_{k_3}, vv_{k_4}. Since $n \geq 10$, then among the four vertices v_{k_1}, v_{k_2}, v_{k_3} and v_{k_4}, there are at least two of them with distance greater than 2 in C_n. Hence there exists no total rainbow geodesic between such two vertices, a contradiction. Therefore, the n edges $vv_k (k = 1, 2, \ldots, n)$ use at least $\lceil \frac{n}{3} \rceil$ distinct colors. As a consequence, $strc(W_n^r) \geq \lceil \frac{n}{3} \rceil + r + 1$.

We now prove the upper bound by defining a strong total rainbow coloring c for W_n^r as follows. $c(vu_i) = i$ for $i = 1, 2, \ldots, r$; $c(v) = r + 1$; $c(vv_k) = c(vv_{k+1}) = c(vv_{k+2}) = r+1+\lceil \frac{k}{3} \rceil$ for $k = 1, 4, 7, \ldots, 3(\lceil \frac{n}{3} \rceil - 1) + 1$; $c(v_k v_{k+1}) = r+1$ if $k \equiv 1 (\text{mod } 3)$, $c(v_k v_{k+1}) = r + 2$ if $k \equiv 2 (\text{mod } 3)$, $c(v_k v_{k+1}) = r + 3$ if $k \equiv 0 (\text{mod } 3)$, where $k = 1, 2, \ldots, n$ and $v_{n+1} = v_1$; $c(v_{3j+2}) = r + 4$ for $j = 0, 1, \ldots, \lfloor \frac{n-2}{3} \rfloor$; color the remaining vertices with arbitrary colors from $\{1, 2, \ldots, \lceil \frac{n}{3} \rceil + r + 1\}$. Under this coloring, one can check that any two vertices of W_n^r have a total rainbow geodesic connecting them. Hence this coloring is really a strong total rainbow coloring of W_n^r with $\lceil \frac{n}{3} \rceil + r + 1$ colors. Consequently, $strc(W_n^r) \leq \lceil \frac{n}{3} \rceil + r + 1$.

Combining with the lower and the upper bound, we finally arrive at $strc(W_n^r) = \lceil \frac{n}{3} \rceil + r + 1$. ∎

With the aids of Lemma 6 and Lemma 7, we are now able to state our main result of this section.

Theorem 6. *Let a and b be two integers with $a = 5$ and $b \geq 6$ or $a \geq 6$ and $b \geq a + 4$. Then there exists a connected graph G such that $trc(G) = a$ and $strc(G) = b$.*

Proof. In the following, we shall apply Lemmas 6 and 7 repeatedly, which we will no longer state.

If $a = 5$ and $6 \leq b \leq 9$, let $G = W_{12}^{b-5}$. Since $6 \leq b \leq 9$, then $1 \leq b - 5 \leq 4$. Hence $trc(G) = trc(W_{12}^{b-5}) = 5 = a$, and $strc(G) = strc(W_{12}^{b-5}) = \lceil \frac{12}{3} \rceil + b - 5 + 1 = b$.

If $a = 5$ and $b \geq 10$, let $G = W_{3b-15}^4$, where $3b - 15 > 10$. Then $trc(G) = trc(W_{3b-15}^4) = 5 = a$ and $strc(G) = strc(W_{3b-15}^4) = \lceil \frac{3b-15}{3} \rceil + 4 + 1 = b$.

If $a \geq 6$ and $b \geq a + 4$. Let $G = W_{3b-3a}^{a-1}$, where $a - 1 \geq 5$ and $3b - 3a > 10$. Then $trc(G) = trc(W_{3b-3a}^{a-1}) = a - 1 + 1 = a$, meanwhile $strc(G) = strc(W_{3b-3a}^{a-1}) = \lceil \frac{3b-3a}{3} \rceil + a - 1 + 1 = b$.

The proof of this theorem is thus completed. ∎

5 Conclusion

As mentioned earlier, $strc(G) \geq \max\{srvc(G), src(G)\}$ holds trivially for a nontrivial connected graph G. Moreover, if G is complete, then $strc(G) = 1 = src(G) = \max\{srvc(G), src(G)\}$, that is, the bound is sharp. On the other hand, since $strc(W_n) = \lceil \frac{n}{3} \rceil + 1$ for $n \geq 4$, while $srvc(W_n) = 1$ (see [13]), $src(W_n) = \lceil \frac{n}{3} \rceil$ (see [4]). Then there exists a graph G of order n such that $strc(G) > srvc(G)$ and $strc(G) > src(G)$ for arbitrary $n \geq 4$. In conclusion, for $srvc(G)$, we conjecture that $strc(G) > srvc(G)$. For $src(G)$, we at present cannot be sure that how tight is the inequality $strc(G) \geq src(G)$. Does there exist a graph G with $strc(G) = src(G) = s$ for sufficiently large integer s? Or could we possibly expect that $strc(G) > src(G)$ for any non-complete graph G? These problems will be the objects of our future study.

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