ENTROPY IN THE CATEGORY OF PERFECT COMPLEXES WITH COHOMOLOGY OF FINITE LENGTH

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ABSTRACT. Let R be a Noetherian local ring. For an endomorphism $\varphi\colon R\to R$ with zero-dimensional closed fiber, the total derived inverse image functor $\mathbb{L}\varphi^\star$ induces an exact (triangulated) functor from the category of perfect complexes over R with cohomology of finite length, to itself. The *triangulated* entropy of this endofunctor, as defined in [3], exists. We show that when R is regular, the triangulated entropy of the total derived inverse image functor $\mathbb{L}\varphi^\star$ is equal to the *local* entropy of φ, as defined in [7]. We also compute the triangulated entropy of the the total derived inverse image functor of the Frobenius endomorphism over a complete local ring of equal characteristic p, and show that this entropy is equal to $d\cdot\log(p)$, where d is the dimension of the ring.

1. Introduction

Recently in [3] a notion of entropy was defined and studied for an exact (triangulated) endofunctor of a triangulated category with generator. In this note we will specialize to a specific triangulated category, namely the category of perfect complexes with cohomology of finite length over a Noetherian local ring R, and will study the entropy of the total derived inverse image functor $\mathbb{L}\varphi^\star$, where $\varphi\colon R\to R$ is an endomorphism with zero-dimensional closed fiber. Our goal is to compute the *triangulated* entropy of the functor $\mathbb{L}\varphi^\star$ in terms of the *local* entropy (cf. [7]) of the endomorphism φ. We find that local entropy is always less than or equal to triangulated entropy. When R is regular we are able to show that the two entropies are equal. We use this result to compute the triangulated entropy of the the total derived inverse image functor of the Frobenius endomorphism over a complete local ring of equal characteristic p, and show that this entropy is equal to $d\cdot\log(p)$, where d is the dimension of the ring.

The organization of the content of this paper is as follows: In Sections 2 and 3 we recall the definitions of triangulated and local entropies. In Section 4 we review the definition of the category of perfect complexes with cohomology of finite length over a Noetherian local ring R and show that in this category every nonzero object is a generator. We also explain why the total derived inverse image functor $\mathbb{L}\varphi^\star\colon D(R)\to D(R)$, where $\varphi\colon R\to R$ is an endomorphism with zero-dimensional closed fiber, induces an exact (triangulated) functor from the category of perfect complexes with cohomology of finite length over R, to itself. The aim of Section 4 is to show that it makes sense to speak of the triangulated entropy of the functor $\mathbb{L}\varphi^\star$, viewed as an endofunctor of the category of perfect complexes with cohomology of finite length over R. Finally, Section 5 contains our main results, as described above.

2. Entropy of exact (triangulated) endofunctors

Let $\mathcal{D}$ be a triangulated category. Recall that a subcategory of $\mathcal{D}$ is called *thick* if it is triangulated, contains every object isomorphic to any of its objects, and contains all direct summands of its objects ([11, Definition 2.1.6, p. 74]). An object G of $\mathcal{D}$ is called a (classical) *generator* if the smallest thick subcategory of $\mathcal{D}$ containing G is equal to $\mathcal{D}$ itself (cf. [2, Section 2.1]). Let $\mathcal{D}$ be a triangulated category with a generator G. To say that G is a generator is equivalent to saying that for every object E of $\mathcal{D}$ there is an object E'

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and a tower of (distinguished) triangles

$$\begin{array}{ccccccccc} 0 = F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 \to \cdots \to & F_{k-1} & \longrightarrow & F_k \cong E \bigoplus E' \\ & \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow & & & \nwarrow \quad \swarrow & \\ & G[n_1] & & G[n_2] & \cdots & & G[n_k] & \end{array}$$

with $k \geqslant 0$ and $n_i \in \mathbb{Z}$.

Definition 2.1 ([3, Definition 2.1])**.** *Let E_1 and E_2 be objects in a triangulated category $\mathcal{D}$. The* complexity *of E_2 with respect to E_1 is the function $\delta_t(E_1, E_2)\colon \mathbb{R} \to [0, \infty]$ given by*

$$\delta_t(E_1, E_2) = \inf \left\{ \sum_{i=1}^{k} e^{n_i t} \;\middle|\; \begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \cdots & A_{k-1} & \longrightarrow & E_2 \bigoplus E_2' \\ & \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & E_1[n_1] & & E_1[n_2] & & \cdots & & E_1[n_k] & \end{array} \right\}.$$

Note that $\delta_t(E_1, E_2) = +\infty$ if and only if E_2 does not lie in the thick subcategory generated by E_1. The complexity function $\delta_t(-, -)$ has the following properties:

Proposition 2.2 ([3, Proposition 2.3])**.** *Let E_1, E_2 and E_3 be objects in a triangulated category $\mathcal{D}$. Then*

(a) *(Triangle Inequality) $\delta_t(E_1, E_3) \leqslant \delta_t(E_1, E_2) \cdot \delta_t(E_2, E_3)$.*
(b) *(Retraction) $\delta_t(F(E_1), F(E_2)) \leqslant \delta_t(E_1, E_2)$ for any exact (triangulated) functor $F\colon \mathcal{D} \to \mathcal{D}'$.*

Definition 2.3 ([3, Definition 2.5])**.** *Let $F\colon \mathcal{D} \to \mathcal{D}$ be an exact (triangulated) endofunctor of a triangulated category $\mathcal{D}$ with a generator G. The* entropy *of F is the function $h_t(F)\colon \mathbb{R} \to [-\infty, +\infty)$ of t given by*

$$h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t\left(G, F^n(G)\right).$$

It is shown in [3, Lemma 2.5] that $h_t(F)$ is well-defined, i.e., the limit exists and is independent of the choice of generator G.

3. Entropy of local endomorphisms

The notion of local entropy associated with an endomorphism of finite length of a Noetherian local ring was introduced in [7]. We recall its definition.

Definition 3.1 ([7, Definition 1])**.** *A local homomorphism $f\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ of Noetherian local rings is said to be* of finite length *if the ideal $f(\mathfrak{m})S$ is $\mathfrak{n}$-primary, or equivalently, if the closed fiber of f is of dimension zero. These conditions are also equivalent to the statement that if $\mathfrak{p}$ is a prime ideal of S such that $f^{-1}(\mathfrak{p}) = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{n}$.*

Definition 3.2 ([7, Theorem 1])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\varphi\colon R \to R$ an endomorphism of finite length. Let* $\text{length}_R(-)$ *denote the length of an R-module. The* local entropy *of φ is given by*

$$h_{\text{loc}}(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log\left(\text{length}_R(R/\varphi^n(\mathfrak{m})R)\right).$$

It is shown in [7, Theorem 1] that $h_{\text{loc}}(\varphi)$ is well-defined, i.e., the limit exists. Furthermore, it is shown that local entropy is non-negative and can be calculated using any module of finite length ([7, Proposition 18]). In particular, any ideal of definition of R can be used to calculate local entropy:

Lemma 3.3 (cf. [7, Proposition 18])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\varphi\colon R \to R$ an endomorphism of finite length. If $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R, then*

$$h_{\text{loc}}(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log\left(\text{length}_R(R/\varphi^n(\mathfrak{q})R)\right).$$

Remark 3.4. In a Cohen-Macaulay Noetherian local ring of dimension d, a sequence of d elements in the maximal ideal form a system of parameters if and only if they form a (maximal) regular sequence. The reader can find a proof of this fact in [9, Theorem 17.4].

4. Perfect complexes with cohomology of finite length

In Section 5 we will work in the category of perfect complexes with finite length cohomology over a Noetherian local ring R and for an endomorphism of finite length $\varphi\colon R \to R$ we will study relations between $h_{\mathrm{loc}}(\varphi)$ and $h_t(\mathbb{L}\varphi^\star)$, where $\mathbb{L}\varphi^\star$ is the total derived inverse image functor. Our goal in this section is to show that it makes sense to speak of $h_t(\mathbb{L}\varphi^\star)$. We have also collected a number of definitions and standard facts about the category of perfect complexes. The main references for this section are [1, 14, 15].

Definition 4.1. *Let R be a commutative ring and let $D(R)$ denote the derived category of the category of R-modules. A* strictly perfect *complex on R is a bounded complex of projective R-modules of finite type. The category of* perfect *complexes over R, which we will denote by $D(R)_{\mathrm{perf}}$, is the full subcategory of $D(R)$ consisting of all complexes that are quasi-isomorphic to a strictly perfect complex on R.*

The category $D(R)_{\mathrm{perf}}$ is a thick subcategory of $D(R)$, by [1, Exposé I, Propositions 4.10, 4.17] or [15, Proposition 2.2.13].

Remark 4.2. For definitions of strictly perfect complexes and perfect complexes on schemes see [1, Exposé I, Definitions 2.1 and 4.7] or [15, Definitions 2.2.2 and 2.2.10].

Let $f\colon S \to R$ be a homomorphism of commutative rings. Denote the homotopy category of complexes of S-modules (respectively R-modules) by $\mathbf{K}(S)$ (respectively $\mathbf{K}(R)$). Let $f^\star\colon \mathbf{K}(S) \to \mathbf{K}(R)$ be the inverse image functor, that is, the functor that sends a complex of S-modules $E^\bullet$ to the complex of R-modules $E^\bullet \otimes_S R$. Let $\mathbb{L}f^\star\colon D(S) \to D(R)$ denote the total derived inverse image functor. Recall that $\mathbb{L}f^\star$ is an exact (triangulated) functor.

Remark 4.3. For a morphism $f\colon Y \to X$ of schemes, the total inverse image functor $\mathbb{L}f^\star$ was generally only defined as a functor $D^-(X) \to D^-(Y)$ in [1] and [4, p. 99]. Spaltenstein extended the definition of $\mathbb{L}f^\star$ to a functor $D(X) \to D(Y)$ in [13, Proposition 6.7]. Over an affine scheme, however, this will not make any difference because the category of perfect complexes over an affine scheme is equivalent to the triangulated category obtained from the category of bounded chain complexes of finitely generated projective modules by inverting the quasi-isomorphisms.

Proposition 4.4 (cf. [1, Exposé I, Corollaire 4.19.1] or [15, p. 303, 2.5.1])**.** *Let $f\colon S \to R$ be a homomorphism of commutative rings and $\mathbb{L}f^\star\colon D(S) \to D(R)$ the total derived inverse image functor. Then $\mathbb{L}f^\star$ induces a functor $D(S)_{\mathrm{perf}} \to D(R)_{\mathrm{perf}}$.*

Proof. For $E^\bullet$ a strictly perfect complex of S-modules, $f^\star E^\bullet$ is clearly a strictly perfect complex of R-modules and this complex represents $\mathbb{L}f^\star E^\bullet$. Since (over an affine scheme) any perfect complex is (globally) quasi-isomorphic to a strictly perfect complex, the result follows immediately. □

Definition 4.5 (cf. [14, Definition 3.2])**.** *Let R be a commutative ring and $E^\bullet$ a complex of R-modules. The* cohomological support *of $E^\bullet$ is the subspace $\mathrm{Supph}(E^\bullet) \subseteq \operatorname{Spec} R$ of those prime ideals $\mathfrak{p} \in \operatorname{Spec} R$ at which the complex $E^\bullet_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$-modules is not acyclic.*

Thus $\mathrm{Supph}(E^\bullet) = \bigcup_{n\in\mathbb{Z}} \operatorname{Supp} H^n(E^\bullet)$ is the union of the supports in the classic sense of the cohomology modules of $E^\bullet$.

Suppose now that $(R, \mathfrak{m})$ is a Noetherian *local* ring and denote by $D_{\mathfrak{m}}(R)_{\mathrm{perf}}$ the full subcategory of $D(R)_{\mathrm{perf}}$ consisting of perfect complexes $E^\bullet$ with $\mathrm{Supph}\, E^\bullet \subseteq \{\mathfrak{m}\}$. Clearly a perfect complex $E^\bullet$ is in $D_{\mathfrak{m}}(R)_{\mathrm{perf}}$ if and only if $H^n(E^\bullet)$ is an R-module of finite length for every $n \in \mathbb{Z}$. The proof of the next proposition is straightforward.

Proposition 4.6 (cf. [14, Example 3.9.1])**.** *Let $(R, \mathfrak{m})$ be a Noetherian* local *ring. Then $D_{\mathfrak{m}}(R)_{\mathrm{perf}}$ is a thick subcategory of $D(R)_{\mathrm{perf}}$.*

Proposition 4.7. *Let $f\colon (S,\mathfrak{n}) \to (R,\mathfrak{m})$ be a homomorphism* of finite length *of Noetherian local rings. Then the total derived inverse image functor $\mathbb{L}f^\star\colon D(S)_{\text{perf}} \to D(R)_{\text{perf}}$ (cf. Proposition 4.4) induces an exact (triangulated) functor $D_{\mathfrak{n}}(S)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$.*

Proof. This statement quickly follows from the following more general fact proved in [14, Lemma 3.3]: "Let X be a quasi-compact and quasi-separated scheme (e.g., an affine scheme). Let $E^\bullet$ be a perfect complex on X. If Y is a quasi-compact and quasi-separated scheme and $f\colon Y \to X$ is a morphism of schemes, then $\text{Supph}(\mathbb{L}f^\star E^\bullet) = f^{-1}(\text{Supph}(E^\bullet))$." Note that denoting by ${}^a f$ the morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ corresponding to the given homomorphism $f\colon S \to R$, we have $({}^a f)^{-1}(\{\mathfrak{n}\}) = \{\mathfrak{m}\}$, as f is assumed to be of finite length. □

Finally, we show that the category $D_{\mathfrak{m}}(R)_{\text{perf}}$ has generators:

Lemma 4.8. *Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then every nonzero object in $D_{\mathfrak{m}}(R)_{\text{perf}}$ is a generator, in the sense defined in Section 2.*

Proof. This statement quickly follows from Proposition 4.6 and the following more general fact that was first proved in [5, Proof of Theorem 11]. Also see [10, Lemma 1.2] or [14, Lemma 3.14]: "Let R be a commutative Noetherian ring and let $E^\bullet, F^\bullet \in D(R)_{\text{perf}}$ be two perfect complexes on R. Suppose that $\text{Supph}(E^\bullet) \subseteq \text{Supph}(F^\bullet)$. Then $E^\bullet$ is in the smallest thick subcategory of $D(R)_{\text{perf}}$ containing $F^\bullet$." □

In this section we have shown that the category $D_{\mathfrak{m}}(R)_{\text{perf}}$ of perfect complexes with cohomology of finite length on a Noetherian local ring $(R,\mathfrak{m})$ is a triangulated category in which every nonzero object is a generator. Moreover, given an endomorphism of finite length $\varphi\colon R \to R$, the exact (triangulated) functor $\mathbb{L}\varphi^\star\colon D(R) \to D(R)$ induces an exact (triangulated) functor $\mathbb{L}\varphi^\star\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$. Therefore, it makes sense to speak of $h_t(\mathbb{L}\varphi^\star)$ as defined in Definition 2.3.

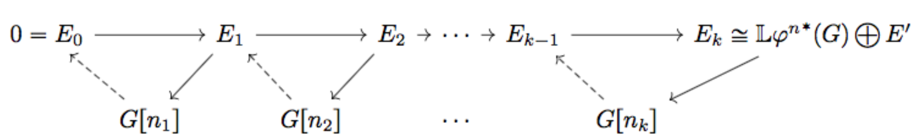
5. Connections between triangulated entropy and local entropy

Let $D_{\mathfrak{m}}(R)_{\text{perf}}$ be the category of perfect complexes with cohomology of finite length over a Noetherian local ring $(R,\mathfrak{m})$.

Lemma 5.1. *Let $\varphi\colon R \to R$ be an endomorphism of finite length of a Noetherian local ring $(R,\mathfrak{m})$ and let $\mathbb{L}\varphi^\star\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$ be the total derived inverse image functor. Let $G \in D_{\mathfrak{m}}(R)_{\text{perf}}$ be a generator. Assume that $H^k(G) = 0$ for $\mid k \mid > N$ and let $B := \max\{\text{length}_R(H^k(G)) \mid -N \leqslant k \leqslant N\}$. Then for any integer $n \geqslant 1$ and any real number t:*

$$\text{length}_R\left(H^0\left(\mathbb{L}\varphi^{n\star}(G)\right)\right) \leqslant Be^{N|t|} \cdot \delta_t\left(G, \mathbb{L}\varphi^{n\star}(G)\right). \tag{5.1}$$

Proof. Fix an integer $n \geqslant 1$ and consider a tower of distinguished triangles in $D_{\mathfrak{m}}(R)_{\text{perf}}$ of the form:



Let $S := \{i \mid -N \leqslant n_i \leqslant N\}$. Since $H^0(-)$ is a *cohomological* functor ([6, Definition 1.5.2, p. 39]), it quickly follows that for any distinguished triangle $X \to Y \to Z \to X[1]$ in $D_{\mathfrak{m}}(R)_{\text{perf}}$:

$$\text{length}_R\left(H^k(Y)\right) \leqslant \text{length}_R\left(H^k(X)\right) + \text{length}_R\left(H^k(Z)\right).$$

Using this inequality one can immediately check that in the above tower of distinguished triangles:

$$\text{length}_R\left(H^k(\mathbb{L}\varphi^{n\star}(G))\right) \leqslant \text{length}_R\left(H^k(G[n_k])\right) + \ldots + \text{length}_R\left(H^k(G[n_2])\right) + \text{length}_R\left(H^k(G[n_1])\right).$$

We will use this inequality, below.

First assume that $t \leqslant 0$. Then

$$\begin{aligned}\sum_{i=1}^{k} e^{n_i t} &\geqslant \sum_{i\in S} e^{n_i t} \geqslant e^{Nt} \mid S \mid \\ &\geqslant \frac{e^{Nt}}{B}\sum_{i\in S}\operatorname{length}_R(H^{n_i}(G)) \\ &= \frac{e^{Nt}}{B}\sum_{i\in S}\operatorname{length}_R(H^0(G[n_i])) \\ &\geqslant \frac{e^{Nt}}{B}\cdot\operatorname{length}_R(H^0(\mathbb{L}\varphi^{n\star}(G))).\end{aligned}$$

Hence, when $t \leqslant 0$ we see that $\operatorname{length}(H^0(\mathbb{L}\varphi^{n\star}(G))) \leqslant Be^{-Nt}\cdot\delta_t(G, \mathbb{L}\varphi^{n\star}(G))$.

Next, assume that $t > 0$. Then

$$\begin{aligned}\sum_{i=1}^{k} e^{n_i t} \geqslant \sum_{i\in S} e^{n_i t} &\geqslant e^{-Nt} \mid S \mid \\ &\geqslant \frac{e^{-Nt}}{B}\sum_{i\in S}\operatorname{length}_R(H^{n_i}(G)) \\ &= \frac{e^{-Nt}}{B}\sum_{i\in S}\operatorname{length}_R(H^0(G[n_i])) \\ &\geqslant \frac{e^{-Nt}}{B}\cdot\operatorname{length}_R(H^0(\mathbb{L}\varphi^{n\star}(G))).\end{aligned}$$

Hence, when $t > 0$ we see again that $\operatorname{length}(H^0(\mathbb{L}\varphi^{n\star}(G))) \leqslant Be^{Nt}\cdot\delta_t(G, \mathbb{L}\varphi^{n\star}(G))$. □

Corollary 5.2. *Let $\varphi\colon R \to R$ be an endomorphism of finite length of a Noetherian local ring $(R, \mathfrak{m})$ and let $\mathbb{L}\varphi^\star\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$ be the total derived inverse image functor. Then $h_{\text{loc}}(\varphi) \leqslant h_t(\mathbb{L}\varphi^\star)$ for any real number t.*

Proof. Let $\{x_1, \ldots, x_d\}$ be a system of parameters of R and let $\mathfrak{q}$ be the ideal of R that they generate. Let $G^\bullet(\underline{x})$ be the Koszul complex over R constructed from $x_1, \ldots, x_d$. Then $G^\bullet(\underline{x})$ is a generator for the triangulated category $D_{\mathfrak{m}}(R)_{\text{perf}}$. Since $H^0(G^\bullet(\underline{x})) = R/\mathfrak{q}$ and tensor product is a right-exact functor, it quickly follows that $H^0(\mathbb{L}\varphi^{n\star}(G^\bullet(\underline{x}))) = R/\varphi^n(\mathfrak{q})R$. The desired inequality $h_{\text{loc}}(\varphi) \leqslant h_t(\mathbb{L}\varphi^\star)$ follows by taking the logarithm, dividing by n, and passing to the limit as $n \to \infty$ on both sides of (5.1), and using Lemma 3.3. □

Lemma 5.3. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. Suppose $\varphi\colon R \to R$ is an endomorphism of finite length, and let $\mathbb{L}\varphi^\star\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$ be the total derived inverse image functor. Let $\{x_1, \ldots, x_d\}$ be a regular system of parameters of R and let $G^\bullet(\underline{x})$ be the Koszul complex over R constructed from $x_1, \ldots, x_d$. The following inequality holds for any integer $n \geqslant 1$ and any real number t:*

$$\delta_t\left(G^\bullet(\underline{x}), \mathbb{L}\varphi^{n\star}\left(G^\bullet(\underline{x})\right)\right) \leqslant \operatorname{length}_R\left(R/\varphi^n(\mathfrak{m})R\right).$$

Proof. The idea behind the proof is that since R is regular, every finitely generated R-module has finite projective dimension and is therefore an object of the category $D(R)_{\text{perf}}$. Hence, one can work with modules instead of complexes. To be more precise, consider a composition series of $R/\varphi^n(\mathfrak{m})R$ over R:

$$0 = N_\ell \subsetneq N_{\ell-1} \subsetneq \ldots \subsetneq N_1 \subsetneq N_0 = R/\varphi^n(\mathfrak{m})R, \tag{5.2}$$

with successive quotients N_{i-1}/N_i isomorphic to the residue field $k = R/\mathfrak{m}$, for $1 \leqslant i \leqslant \ell$. Now, beginning from the left end of the composition series (5.2) we will inductively build a tower of triangles as follows: the

Koszul complex $G^\bullet(\underline{\mathbf{x}})$ provides a bounded free resolution of $k \cong N_{\ell-1}$. There exists (cf. [12, Theorem 16, p. 82]) a bounded free resolution $L^\bullet_{\ell-2}$ of $N_{\ell-2}$ that completes the diagram

$$\begin{array}{ccccccccc} & & G^\bullet(\underline{\mathbf{x}}) & & & & G^\bullet(\underline{\mathbf{x}}) & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & N_{\ell-1} & \longrightarrow & N_{\ell-2} & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

into a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^\bullet(\underline{\mathbf{x}}) & \longrightarrow & L^\bullet_{\ell-2} & \longrightarrow & G^\bullet(\underline{\mathbf{x}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{\ell-1} & \longrightarrow & N_{\ell-2} & \longrightarrow & k & \longrightarrow & 0. \end{array} \tag{5.3}$$

We use the distinguished triangle $G^\bullet(\underline{\mathbf{x}}) \to L^\bullet_{\ell-2} \to G^\bullet(\underline{\mathbf{x}}) \to G^\bullet(\underline{\mathbf{x}})[1]$ in $D_{\mathfrak{m}}(R)_{\text{perf}}$ associated to the first row in (5.3) (cf. [6, Proposition 1.7.5, p. 46]) to build the beginning of our tower of triangles, as follows:

$$\begin{array}{ccccc} 0 & \longrightarrow & G^\bullet(\underline{\mathbf{x}}) & \longrightarrow & L^\bullet_{\ell-2} \\ & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & \\ & G^\bullet(\underline{\mathbf{x}}) & & G^\bullet(\underline{\mathbf{x}}). & \end{array}$$

Next, there exists (cf. [12, Theorem 16, p. 82]) a bounded free resolution $L^\bullet_{\ell-3}$ of $N_{\ell-3}$ completing the diagram

$$\begin{array}{ccccccccc} & & L^\bullet_{\ell-2} & & & & G^\bullet(\underline{\mathbf{x}}) & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & N_{\ell-2} & \longrightarrow & N_{\ell-3} & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

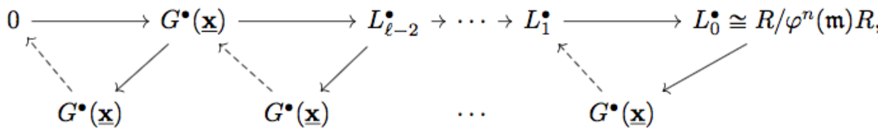
into a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\bullet_{\ell-2} & \longrightarrow & L^\bullet_{\ell-3} & \longrightarrow & G^\bullet(\underline{\mathbf{x}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{\ell-2} & \longrightarrow & N_{\ell-3} & \longrightarrow & k & \longrightarrow & 0. \end{array} \tag{5.4}$$

We use the distinguished triangle $L^\bullet_{\ell-2} \to L^\bullet_{\ell-3} \to G^\bullet(\underline{\mathbf{x}}) \to L^\bullet_{\ell-2}[1]$ in $D_{\mathfrak{m}}(R)_{\text{perf}}$ associated to the first row in (5.4) to build the next triangle in our tower of triangles, as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G^\bullet(\underline{\mathbf{x}}) & \longrightarrow & L^\bullet_{\ell-2} & \longrightarrow & L^\bullet_{\ell-3} \\ & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & \\ & G^\bullet(\underline{\mathbf{x}}) & & G^\bullet(\underline{\mathbf{x}}) & & G^\bullet(\underline{\mathbf{x}}). & \end{array}$$

Continuing this process, we obtain a tower of triangles in $D_{\mathfrak{m}}(R)_{\text{perf}}$:



in which the quasi-isomorphism $L_0^\bullet \cong R/\varphi^n(\mathfrak{m})R$ exists because $L_0^\bullet$ provides a bounded free resolution $L_0^\bullet \to N_0 \cong R/\varphi^n(\mathfrak{m})R \to 0$. We conclude:

$$\delta_t\left(G^\bullet(\underline{\mathbf{x}}), \mathbb{L}\varphi^{n\star}\left(G^\bullet(\underline{\mathbf{x}})\right)\right) \leqslant \sum_{1}^{\ell} e^{0\cdot t} = \ell = \operatorname{length}_R\left(R/\varphi^n(\mathfrak{m})R\right).$$

□

Theorem 5.4. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension d. Suppose $\varphi\colon R \to R$ is an endomorphism of finite length, and let $\mathbb{L}\varphi^\star\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$ be the total derived inverse image functor. Then $h_t(\mathbb{L}\varphi^\star)$ is constant and equal to $h_{\text{loc}}(\varphi)$.*

Proof. Let $\{x_1,\dots,x_d\}$ be a regular system of parameters of R. Let $G^\bullet(\underline{\mathbf{x}})$ be the Koszul complex over R constructed from $x_1,\dots,x_d$. Then $G^\bullet(\underline{\mathbf{x}})$ is a generator for the triangulated category $D_{\mathfrak{m}}(R)_{\text{perf}}$. By Corollary 5.2 we have $h_{\text{loc}}(\varphi) \leqslant h_t(\mathbb{L}\varphi^\star)$. On the other hand by Lemma 5.3 for any integer $n \geqslant 1$ and any real number t:

$$\delta_t\left(G^\bullet(\underline{\mathbf{x}}), \mathbb{L}\varphi^{n\star}\left(G^\bullet(\underline{\mathbf{x}})\right)\right) \leqslant \operatorname{length}_R\left(R/\varphi^n(\mathfrak{m})R\right). \tag{5.4}$$

Taking the logarithm, dividing by n, and passing to the limit as $n \to \infty$ in (5.4), we obtain $h_t(\mathbb{L}\varphi^\star) \leqslant h_{\text{loc}}(\varphi)$. This concludes the proof. □

Corollary 5.5. *Let k be a field and $R = k[\![X_1,\dots X_d]\!]$. Suppose $\xi_1,\dots,\xi_d$ are positive integers and let $\varphi\colon R \to R$ be the endomorphism that maps $X_i \mapsto X_i^{\xi_i}$ for $1 \leqslant i \leqslant d$. Then $h_t(\mathbb{L}\varphi^\star)$ is constant and equal to $\sum_{i=1}^d \log(\xi_i)$.*

Proof. By Theorem 5.4 the entropy $h_t(\mathbb{L}\varphi^\star)$ is constant and equal to $h_{\text{loc}}(\varphi)$. Thus, it suffices to show that $h_{\text{loc}}(\varphi) = \sum_{i=1}^d \log(\xi_i)$. This is done by induction on $\dim R$. If $\dim R = 1$ then $R = k[\![X_1]\!]$, $\varphi^n(X_1) = X_1^{\xi_1^n}$ and $R/\varphi^n(X_1)$ has a composition series of length ξ^n

$$0 \subsetneq (X_1^{\xi_1^{n}-1})/(X_1^{\xi_1^n}) \subsetneq \ldots \subsetneq (X_1^2)/(X_1^{\xi_1^n}) \subsetneq (X_1)/(X_1^{\xi_1^n}) \subsetneq R/(X_1^{\xi_1^n}).$$

We see by Definition 3.2 that $h_{\text{loc}}(\varphi) = \log(\xi_1)$. Next, assume the result holds for the ring $k[\![X_1,\dots X_{d-1}]\!]$. We want to prove it for the ring $k[\![X_1,\dots X_d]\!]$. We define a homomorphism $\alpha\colon k[\![Y]\!] \to k[\![X_1,\dots X_d]\!]$ by mapping Y to X_d. This homomorphism is flat (cf. [9, Theorem 23.1, p. 179]). If we equip $k[\![Y]\!]$ with the endomorphism $\psi\colon k[\![Y]\!] \to k[\![Y]\!]$ that maps Y to Y^{ξ_d}, then $\alpha\circ\psi = \varphi\circ\alpha$. By [8, Theorem 1] and using the induction hypothesis we see that $h_{\text{loc}}(\varphi) = \sum_{i=1}^d \log(\xi_i)$. □

Proposition 5.6. *Let $(R,\mathfrak{m})$ be an arbitrary Noetherian local ring with an endomorphism $\varphi\colon R \to R$ of finite length. Assume there exists a homomorphism of finite length $\xi\colon S \to R$, where $(S,\mathfrak{n})$ is a regular local ring, and suppose there is an endomorphism $\psi\colon S \to S$ of finite length, such that $\xi\circ\psi = \varphi\circ\xi$. Let $\mathbb{L}\varphi^\star\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$ and $\mathbb{L}\psi^\star\colon D_{\mathfrak{n}}(S)_{\text{perf}} \to D_{\mathfrak{n}}(S)_{\text{perf}}$ be the corresponding total derived inverse image functors. If $h_{\text{loc}}(\varphi) = h_{\text{loc}}(\psi)$, then $h_t(\mathbb{L}\varphi^\star)$ is constant and equal to $h_{\text{loc}}(\varphi)$.*

Proof. Let $\{x_1,\dots,x_d\}$ be a regular system of parameters of S, $d = \dim S$, and let $y_i = \xi(x_i)$ for $1 \leqslant i \leqslant d$. Let $G_S^\bullet(\underline{\mathbf{x}})$ and $G_R^\bullet(\underline{\mathbf{y}})$ be the Koszul complexs over S and R, respectively, constructed from $x_1,\dots,x_d$ and $y_1,\dots,y_d$. The complexes $G_S^\bullet(\underline{\mathbf{x}})$ and $G_R^\bullet(\underline{\mathbf{y}})$ are generators for the triangulated categories $D_{\mathfrak{n}}(S)_{\text{perf}}$ and $D_{\mathfrak{m}}(R)_{\text{perf}}$, respectively. Since ξ is a homomorphism of finite length, by Proposition 4.7 the restriction of the total derived inverse image functor $\mathbb{L}\xi^\star\colon D(S) \to D(R)$ to $D_{\mathfrak{n}}(S)_{\text{perf}}$ provides an exact (triangulated) functor $D_{\mathfrak{n}}(S)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$. It is clear that $\mathbb{L}\xi^\star(G^\bullet_S(\underline{\mathbf{x}})) = G^\bullet_R(\underline{\mathbf{y}})$. Since $\xi \circ \psi = \varphi \circ \xi$, it follows that $\mathbb{L}\xi^\star \circ \mathbb{L}\psi^\star = \mathbb{L}\varphi^\star \circ \mathbb{L}\xi^\star$. Using Proposition 2.2, for any integer $n \geqslant 1$ and any real number t we obtain:

$$\begin{aligned}
\delta_t\big(G^\bullet_R(\underline{\mathbf{y}}), \mathbb{L}\varphi^{n\star}\left(G^\bullet_R(\underline{\mathbf{y}})\right)\big) &= \delta_t\left(\mathbb{L}\xi^\star\left(G^\bullet_S(\underline{\mathbf{x}})\right), \mathbb{L}\varphi^{n\star}\left(\mathbb{L}\xi^\star\left(G^\bullet_S(\underline{\mathbf{x}})\right)\right)\right)\\
&= \delta_t\left(\mathbb{L}\xi^\star\left(G^\bullet_S(\underline{\mathbf{x}})\right), \mathbb{L}\xi^\star\left(\mathbb{L}\psi^{n\star}\left(G^\bullet_S(\underline{\mathbf{x}})\right)\right)\right)\\
&\leqslant \delta_t\left(G^\bullet_S(\underline{\mathbf{x}}), \mathbb{L}\psi^{n\star}\left(G^\bullet_S(\underline{\mathbf{x}})\right)\right).
\end{aligned}$$

By taking the logarithm, dividing by n, and passing to the limit as $n \to \infty$ we obtain $h_t(\mathbb{L}\varphi^\star) \leqslant h_t(\mathbb{L}\psi^\star)$. From Corollary 5.2 and Theorem 5.4 we get:

$$h_{\text{loc}}(\varphi) \leqslant h_t(\mathbb{L}\varphi^\star) \leqslant h_t(\mathbb{L}\psi^\star) = h_{\text{loc}}(\psi).$$

Hence, if $h_{\text{loc}}(\varphi) = h_{\text{loc}}(\psi)$, then $h_t(\mathbb{L}\varphi^\star)$ is constant and equal to $h_{\text{loc}}(\varphi)$. □

Corollary 5.7. *Let $(R, \mathfrak{m})$ be an arbitrary complete Noetherian local ring of prime characteristic p and of dimension d. Suppose $f_R\colon R \to R$ is the Frobenius endomorphism and let $\mathbb{L}f^\star_R\colon D_{\mathfrak{m}}(R)_{\text{perf}} \to D_{\mathfrak{m}}(R)_{\text{perf}}$ be the total derived inverse image functor. Then $h_t(\mathbb{L}f^\star_R)$ is constant and equal to $d \cdot \log(p)$.*

Proof. Let $\{x_1, \ldots, x_d\}$ be a system of parameters of R. We recall that R is a module-finite extension of the regular ring $S := k[\![X_1, \ldots, X_d]\!]$ via the injective ring homomorphism $\xi\colon S \to R$ that maps X_i onto x_i, for $1 \leqslant i \leqslant d$ (cf. [9, Theorem 29.4, p. 225]). Let f_S be the Frobenius endomorphism of S. By [7, Theorem 1] the local entropy of the Frobenius endomorphism of a Noetherian local ring of prime characteristic p and of dimension d is equal to $d \cdot \log(p)$. That is, $h_{\text{loc}}(f_R) = h_{\text{loc}}(f_S) = d \cdot \log p$. Since $\xi \circ f_S = f_R \circ \xi$, the result follows from Proposition 5.6. □

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