Non-conforming finite element methods for transmission eigenvalue problem

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Abstract : The transmission eigenvalue problem is an important and challenging topic arising in the inverse scattering theory. In this paper, for the Helmholtz transmission eigenvalue problem, we give a weak formulation which is a nonselfadjoint linear eigenvalue problem. Based on the weak formulation, we first discuss the non-conforming finite element approximation, and prove the error estimates of the discrete eigenvalues obtained by the Adini element, Morley-Zienkiewicz element, modified-Zienkiewicz element et. al. And we report some numerical examples to validate the efficiency of our approach for solving transmission eigenvalue problem.

Keywords : transmission eigenvalue, the weak formulation, non-conforming finite elements, error estimates.

1 Introduction

The transmission eigenvalue problems have important physical background, for example, they can be used to obtain estimates for the material properties of the scattering object [7, 8, 24]. In addition, transmission eigenvalues have theoretical importance in the uniqueness and reconstruction in inverse scattering theory [14]. Before 2010, significant progresses of the existence of transmission eigenvalues and applications have been made (see [8] and the survey paper [9]).

In recent years, the computation of transmission eigenvalues has attracted the attention of many researchers. The first numerical treatment of the transmission eigenvalue problem appears in [15] where three finite element methods, including the Argyris, continuous and mixed finite element methods, are proposed for the Helmholtz transmission eigenvalues, and has been further developed by [2, 11, 15, 16, 17, 19, 25, 26, 29]. In particular, [11] studied the mixed method using the Argyris conforming elements and [29] the H^2 conforming finite element method, and made rigorous error analysis. Moreover, based on H^2 conforming finite element approximations, the iterative methods in [25] and the multigrid method in [17] were proposed for computing real transmission eigenvalues, and two-grid method in [29] for computing real and complex transmission eigenvalues. And the spectral-element method was studied in [2]. However, to the best of our knowledge, there has no research on the non-conforming finite element methods for the transmission eigenvalues even for arbitrary nonselfadjoint elliptic eigenvalue problem.

Inspired by the works mentioned above, we transform the fourth order equation of transmission eigenvalue problem into a weak formulation, which is suitable to nonconforming elements. This formulation is a nonselfadjoint linear eigenvalue problem (see (2.11)) with a selfadjoint, continuous and coercive sesquilinear form $A(\cdot,\cdot)$. Based on the weak formulation we build a type of nonconforming finite element discretizations with good algebraic structure, including the Adini element [1], modified-Zienkiewicz element [27], Morley-Zienkiewicz element [23], 12-parameter triangle plate element, 15-parameter triangle plate element et. al. (see [23]). And we prove the error estimates of the numerical eigenvalues. The proof difficulty lies in the non-symmetry of right-hand sides of eigenvalue problem that involves derivatives. To overcome this difficulty, based on Babuska-Osborn spectral approximation theory [3], the new proof method employed in this paper is to establish a fundamental relationship (4.18) and use it to prove the optimal error estimates of non-conforming element eigenvalues.

For fourth order equation in $\mathbb{R}^3$, it is difficult to implement conforming elements in H^2, whereas many non-conforming elements such as the Morley-Zienkiewicz element have had their three dimensional versions at present (e.g., see [23]). Hence it is an essential and significant work to study the non-conforming element approximation for transmission eigenvalues.

Our non-conforming finite element discretization is easy to realize under the package of iFEM [12] with Matlab. We use the sparse matrix eigenvalue solver *eigs* to compute the numerical eigenvalues, and numerical results indicate that our methods are efficient for computing real and complex transmission eigenvalues as expected.

In this paper, regarding the basic theory of finite element methods, we refer to [3, 6, 13, 20, 23].

Throughout this paper, C denotes a positive constant independent of h, which may not be the same constant in different places. For simplicity, we use the symbol $a \lesssim b$ to mean that $a \leq Cb$.

2 The weak formulation and non-conforming element method

Consider the Helmholtz transmission eigenvalue problem: Find $k \in \mathbb{C}$, $w, \sigma \in L^2(\Omega)$, $w - \sigma \in H^2(\Omega)$ such that

$$\Delta w + k^2 n w = 0, \quad in\ \Omega, \tag{2.1}$$

$$\Delta \sigma + k^2 \sigma = 0, \quad in\ \Omega, \tag{2.2}$$

$$w - \sigma = 0, \quad on\ \partial\Omega, \tag{2.3}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial \sigma}{\partial \nu} = 0, \quad on\ \partial\Omega, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^d$ (d=2,3) is a bounded simply connected inhomogeneous medium, ν is the unit outward normal to $\partial\Omega$ and the index of refraction $n(x)$ is positive.

Let $W^{s,p}(\Omega)$ denote the usual Sobolev space with norm $\|\cdot\|_{s,p}$, $H^s(\Omega) = W^{s,2}(\Omega)$, and $\|\cdot\|_{s,2} = \|\cdot\|_s$, $H^0(\Omega) = L^2(\Omega)$ with the inner product $(u,v)_0 = \int\limits_\Omega u\overline{v}dx$. Denote $H_0^2(\Omega) = \{v \in H^2(\Omega) : v|_{\partial\Omega} = \frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0\}$. Let $H^{-s}(\Omega)$ be the "negative space", with norm given by

$$\|v\|_{-s} = \sup_{0 \neq f \in H_0^s(\Omega)} \frac{|(v,f)_0|}{\|f\|_s}.$$

It is clear that for any real functions v_1 and v_2, norms $\|v_1+iv_2\|_{s,p}$ and $\|v_1\|_{s,p} + \|v_2\|_{s,p}$ are equivalent in $W^{s,p}(\Omega)$, and norms $\|v_1+iv_2\|_{-s}$ and $\|v_1\|_{-s}+\|v_2\|_{-s}$ are equivalent in $H^{-s}(\Omega)$.

Define Hilbert space $\mathbf{H} = H_0^2(\Omega)\times L^2(\Omega)$ with norm $\|(v,z)\|_{\mathbf{H}} = \|v\|_2+\|z\|_0$, and define $\mathbf{H}_1 = H_0^1(\Omega) \times H^{-1}(\Omega)$ with norm $\|(v,z)\|_{\mathbf{H}_1} = \|v\|_1 + \|z\|_{-1}$. Since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ compactly (see pp.31-39 in [4]) and $H^2(\Omega) \hookrightarrow H^1(\Omega)$ compactly, $\mathbf{H} \hookrightarrow \mathbf{H}_1$ compactly.

In this paper, we suppose that $n = n(x) \in L^\infty(\Omega)$ satisfying either one of the following assumptions

$$(C1) \quad 1+\delta \leq \inf_\Omega n(x) \leq n(x) \leq \sup_\Omega n(x) < \infty,$$
$$(C2) \quad 0 < \inf_\Omega n(x) \leq n(x) \leq \sup_\Omega n(x) < 1-\beta,$$

for some constant $\delta > 0$ or $\beta > 0$.

From [9, 22] we know that the problem (2.1)-(2.4) can be written as an equivalent fourth order equation for $u = w - \sigma \in H_0^2(\Omega)$:

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0,$$

i.e.,

$$\Delta(\frac{1}{n-1}\Delta u) = -k^2\frac{n}{n-1}\Delta u - k^2\Delta(\frac{1}{n-1}u) - k^4\frac{n}{n-1}u. \tag{2.5}$$

Then the weak formulation for the transmission eigenvalue problem (2.1)-(2.4) can be stated as follows: Find $k \in \mathbb{C}$, $u \in H_0^2(\Omega)$ such that

$$\begin{aligned}(\frac{1}{n-1}\Delta u, \Delta v)_0 = k^2(\nabla u, \nabla(\frac{n}{n-1}v)_0 + k^2(\nabla(\frac{1}{n-1}u), \nabla v)_0 \\ - k^4(\frac{n}{n-1}u, v)_0, \quad \forall v \in H_0^2(\Omega).\end{aligned} \tag{2.6}$$

Introduce an auxiliary variable

$$\omega = k^2 u, \tag{2.7}$$

then

$$(\omega, z)_0 = k^2(u,z)_0, \quad \forall z \in L^2(\Omega). \tag{2.8}$$

Thus, combining (2.6) and (2.8), we arrive at a linear weak formulation: Find $(k^2, u, \omega) \in \mathbb{C} \times H_0^2(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned}(\frac{1}{n-1}\Delta u, \Delta v)_0 = k^2(\nabla(\frac{1}{n-1}u), \nabla v)_0 \\ + k^2(\nabla u, \nabla(\frac{n}{n-1}v))_0 - k^2(\frac{n}{n-1}\omega, v)_0, \quad \forall v \in H_0^2(\Omega),\end{aligned} \tag{2.9}$$

$$(\omega, z)_0 = k^2(u,z)_0, \quad \forall z \in L^2(\Omega). \tag{2.10}$$

With this weak formulation, we have discussed the conforming finite element approximations (see [29]). However, for the non-conforming element approximations, the weak formulation can not guarantee that the discrete bilinear form satisfies the uniform $\mathbf{H}_h$-ellipticity (see Remark 49.1 in [13]). To study the non-conforming element approximations, next we will give a new weak formulation referring to the weak formulation of the plate problem (see (49.3) in [13]).

If $(C1)$ holds, let

$$A((u,\omega),(v,z)) = ((\frac{1}{n-1}-\mu_1)\Delta u,\Delta v)_0 + (\mu_1\Delta u,\Delta v)_0 + (\omega,z)_0$$
$$= ((\frac{1}{n-1}-\mu_1)\Delta u,\Delta v)_0 + \mu_1\int_\Omega \sum_{1\le i,j\le d}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2\overline{v}}{\partial x_i\partial x_j}dx + (\omega,z)_0,$$
$$B((u,\omega),(v,z)) = (\nabla(\frac{1}{n-1}u),\nabla v)_0 + (\nabla u,\nabla(\frac{n}{n-1}v))_0 - (\frac{n}{n-1}\omega,v)_0 + (u,z)_0,$$

and if $(C2)$ holds, let

$$A((u,\omega),(v,z)) = ((\frac{1}{1-n}-\mu_2)\Delta u,\Delta v)_0 + (\mu_2\Delta u,\Delta v)_0 + (\omega,z)_0$$
$$= ((\frac{1}{1-n}-\mu_2)\Delta u,\Delta v)_0 + \mu_2\int_\Omega \sum_{1\le i,j\le d}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2\overline{v}}{\partial x_i\partial x_j}dx + (\omega,z)_0,$$
$$B((u,\omega),(v,z)) = (\nabla(\frac{1}{1-n}u),\nabla v)_0 + (\nabla u,\nabla(\frac{n}{1-n}v))_0 - (\frac{n}{1-n}\omega,v)_0 + (u,z)_0,$$

where $\mu_1>0$ and $\mu_2>0$ are chosen as good approximations of $\min(\frac{1}{n-1})$ and $\min(\frac{1}{1-n})$ respectively such that $\frac{1}{n-1}-\mu_1\ge 0$ and $\frac{1}{1-n}-\mu_2\ge 0$.
Let $\lambda=k^2$, then (2.9)-(2.10) can be rewritten as: Find $\lambda\in\mathbb{C}$, $(u,\omega)\in\mathbf{H}\setminus\{0\}$ such that

$$A((u,\omega),(v,z)) = \lambda B((u,\omega),(v,z)),\quad \forall (v,z)\in\mathbf{H}. \tag{2.11}$$

Next we shall see that the discrete bilinear form of (2.11) satisfies the uniform $\mathbf{H}_h$-ellipticity automatically for many non-conforming elements (see (2.22)).

Thus we get the following.

Theorem 2.1. The weak formulations (2.11) and (2.6) are equivalent.

Proof. If (k^2,u) is an eigenpair of (2.6), then together with (2.8) we get that (k^2,u,ω) is an eigenpair of (2.9)-(2.10), thus it is an eigenpair of (2.11). Conversely, if (k^2,u,ω) satisfies (2.11), then (k^2,u,ω) also satisfies (2.9)-(2.10); from (2.10) we get $\omega=k^2u$, and substituting it into (2.9) we get (2.6). The above argument indicates that (2.11) and (2.6) are equivalent. □

For simplicity, in the next discussion we assume that $(C1)$ holds. And the argument is the same if $(C2)$ holds.

It is obvious that $A(\cdot,\cdot)$ is a selfadjoint, continuous sesquilinear form on $\mathbf{H}\times\mathbf{H}$, and

$$A((v,z),(v,z)) \ge \mu_1|v|_2^2 + \|z\|_0^2 \gtrsim \|(v,z)\|_{\mathbf{H}}^2, \tag{2.12}$$

i.e., $A(\cdot,\cdot)$ is coercive.

We use $A(\cdot,\cdot)$ and $\|\cdot\|_A = A(\cdot,\cdot)^{\frac{1}{2}}$ as an inner product and norm on $\mathbf{H}$, respectively.

Obviously, $k = 0$ is not an eigenvalue since $A((u,\omega),(u,\omega)) = 0$ implies $(u,\omega) = 0$.

When $n \in W^{1,\infty}(\Omega)$, a simple calculation shows that

$$\begin{aligned}
&|B((f,g),(v,z))| \\
&\quad = |(\nabla(\frac{1}{n-1}f),\nabla v)_0 + (\nabla f,\nabla(\frac{n}{n-1}v))_0 - (\frac{n}{n-1}g,v)_0 + (f,z)_0| \\
&\quad \lesssim \|f\|_1\|v\|_1 + \|f\|_1\|v\|_1 + \|g\|_{-1}\|v\|_1 + \|f\|_1\|z\|_{-1} \\
&\quad \lesssim (\|f\|_1 + \|g\|_{-1})(\|v\|_1 + \|z\|_{-1}) \\
&\quad \lesssim \|(f,g)\|_{\mathbf{H}_1}\|(v,z)\|_{\mathbf{H}_1}, \quad \forall (f,g),(v,z) \in \mathbf{H}_1.
\end{aligned} \tag{2.13}$$

We can see from (2.13) that for any given $(f,g) \in \mathbf{H}_1$, $B((f,g),(v,z))$ is a continuous linear form on $\mathbf{H}$.

The source problem associated with (2.11) is as follows: Find $(\psi,\varphi) \in \mathbf{H}$ such that

$$A((\psi,\varphi),(v,z)) = B((f,g),(v,z)), \quad \forall (v,z) \in \mathbf{H}. \tag{2.14}$$

From Lax-Milgram theorem we know that (2.14) has one and only one solution. Therefore, we define the corresponding solution operators $T : \mathbf{H}_1 \to \mathbf{H}$ by

$$A(T(f,g),(v,z)) = B((f,g),(v,z)), \quad \forall (v,z) \in \mathbf{H}. \tag{2.15}$$

Then (2.11) has the equivalent operator form:

$$T(u,\omega) = \frac{1}{\lambda}(u,\omega). \tag{2.16}$$

Theorem 2.2. Suppose $n \in W^{1,\infty}(\Omega)$, then $T : \mathbf{H} \to \mathbf{H}$ is compact, and $T : \mathbf{H}_1 \to \mathbf{H}_1$ is compact.

Proof. Let $(v,z) = T(f,g)$ in (2.14), then from (2.12) and (2.13) we have

$$\|T(f,g)\|^2_{\mathbf{H}} \lesssim A(T(f,g),T(f,g)) = B((f,g),T(f,g)) \lesssim \|(f,g)\|_{\mathbf{H}_1}\|T(f,g)\|_{\mathbf{H}_1},$$

thus

$$\|T(f,g)\|_{\mathbf{H}} \lesssim \|(f,g)\|_{\mathbf{H}_1}, \tag{2.17}$$

which implies that $T : \mathbf{H}_1 \to \mathbf{H}$ is continuous. Because of the compact embedding $\mathbf{H} \hookrightarrow \mathbf{H}_1$, $T : \mathbf{H} \to \mathbf{H}$ is compact and $T : \mathbf{H}_1 \to \mathbf{H}_1$ is compact. $\square$

Consider the dual problem of (2.11): Find $\lambda^* \in \mathbb{C}$, $(u^*,\omega^*) \in \mathbf{H} \setminus \{0\}$ such that

$$A((v,z),(u^*,\omega^*)) = \overline{\lambda^*}B((v,z),(u^*,\omega^*)), \quad \forall (v,z) \in \mathbf{H}. \tag{2.18}$$

The source problem associated with (2.18) is as follows: Find $(\psi^*,\varphi^*) \in \mathbf{H}$ such that

$$A((v,z),(\psi^*,\varphi^*)) = B((v,z),(f,g)), \quad \forall (v,z) \in \mathbf{H}. \tag{2.19}$$

Define the corresponding solution operators $T^* : \mathbf{H}_1 \to \mathbf{H}$ by

$$A((v,z),T^*(f,g)) = B((v,z),(f,g)), \quad \forall (v,z) \in \mathbf{H}. \tag{2.20}$$

Then (2.18) has the equivalent operator form:

$$T^*(u^*, \omega^*) = \lambda^{*-1}(u^*, \omega^*). \tag{2.21}$$

It can be proved that T^* is the adjoint operator of T in the sense of inner product $A(\cdot, \cdot)$. In fact, from (2.15) and (2.20) we have

$$A(T(f,g),(v,z)) = B((f,g),(v,z)) = A((f,g),T^*(v,z)), \quad \forall (f,g),(v,z) \in \mathbf{H}.$$

Note that since T^* is the adjoint operator of T, the primal and dual eigenvalues are connected via $\lambda = \overline{\lambda^*}$.

Let π_h be a shape-regular mesh with size h. Let $\mathbf{H}_h = S^h \times S^h \subset \mathbf{H}_1$ and $\mathbf{H}_h \not\subset \mathbf{H}$ be a non-conforming finite element space; for example, $S^h \subset H_0^1(\Omega)$ is the finite element space associated with one of the Adini element, Morley-Zienkiewicz element, modified Zienkiewicz element, 12-parameter triangle plate element and 15-parameter triangle plate element et. al.

Let

$$A_h((u_h,\omega_h),(v,z)) = \sum_{\kappa\in\pi_h}\int_\kappa \{(\frac{1}{n-1} - \mu_1)\Delta u_h \Delta\overline{v} + \mu_1 \sum_{1\le i,j\le d} \frac{\partial^2 u_h}{\partial x_i \partial x_j}\frac{\partial^2 \overline{v}}{\partial x_i \partial x_j}\}dx + (\omega_h, z)_0.$$

Denote

$$A_h((v,z),(v,z)) \equiv \|v\|_h^2 + \|z\|_0^2 \equiv \|(v,z)\|_h^2.$$

For the finite element spaces mentioned above, from [13] and Lemma 5.4.3 of [23], we know that $A_h(\cdot,\cdot)$ satisfies the uniform $\mathbf{H}_h$-ellipticity.

$$A_h((v,z),(v,z)) \gtrsim \sum_{\kappa\in\pi_h} |v|_{2,\kappa}^2 + \|z\|_0^2, \quad \forall (v,z) \in \mathbf{H}_h. \tag{2.22}$$

Thus $\|(v,z)\|_h$ is a norm in $\mathbf{H}_h$, and the generalized Poincare-Friedrichs inequality holds:

$$\|(v,z)\|_{\mathbf{H}_1} \lesssim \|(v,z)\|_h, \quad \forall (v,z) \in \mathbf{H}_h. \tag{2.23}$$

The non-conforming finite element approximation of (2.11) is given by the following: Find $\lambda_h \in \mathbb{C}$, $(u_h, \omega_h) \in \mathbf{H}_h \setminus \{0\}$ such that

$$A_h((u_h,\omega_h),(v,z)) = \lambda_h B((u_h,\omega_h),(v,z)), \quad \forall (v,z) \in \mathbf{H}_h. \tag{2.24}$$

Consider the approximate source problem: Find $(\psi_h, \varphi_h) \in \mathbf{H}_h$ such that

$$A_h((\psi_h,\varphi_h),(v,z)) = B((f,g),(v,z)), \quad \forall (v,z) \in \mathbf{H}_h. \tag{2.25}$$

We introduce the corresponding solution operator: $T_h : \mathbf{H}_1 \to \mathbf{H}_h$:

$$A_h(T_h(f,g),(v,z)) = B((f,g),(v,z)), \quad \forall (v,z) \in \mathbf{H}_h. \tag{2.26}$$

Then (2.24) has the operator form:

$$T_h(u_h,\omega_h) = \frac{1}{\lambda_h}(u_h,\omega_h). \tag{2.27}$$

The non-conforming finite element approximation of (2.18) is given by: Find $\lambda_h^* \in \mathbb{C}$, $(u_h^*, \omega_h^*) \in \mathbf{H}_h \setminus \{0\}$ such that

$$A_h((v,z),(u_h^*,\omega_h^*)) = \overline{\lambda_h^*} B((v,z),(u_h^*,\omega_h^*)), \quad \forall (v,z) \in \mathbf{H}_h. \tag{2.28}$$

Define the solution operator $T_h^* : \mathbf{H}_1 \to \mathbf{H}_h$ satisfying

$$A_h((v,z), T_h^*(f,g)) = B((v,z),(f,g)), \quad \forall\ (v,z) \in \mathbf{H}_h. \tag{2.29}$$

And (2.28) has the following equivalent operator form

$$T_h^*(u_h^*,\omega_h^*) = \lambda_h^{*-1}(u_h^*,\omega_h^*). \tag{2.30}$$

It can be proved that T_h^* is the adjoint operator of T_h in the sense of inner product $A_h(\cdot,\cdot)$. In fact, from (2.26) and (2.29) we have

$$A_h(T_h(u,\omega),(v,z)) = B((u,\omega),(v,z)) = A_h((u,\omega), T_h^*(v,z)), \quad \forall (u,\omega),(v,z) \in \mathbf{H}_h.$$

Hence, the primal and dual eigenvalues are connected via $\lambda_h = \overline{\lambda_h^*}$.

Denote

$$\mathbb{S} = (\frac{d}{2}, 2].$$

Define interpolation operator $I_h^1 : H_0^2(\Omega) \cap W^{3,p}(\Omega) \to S^h$ ($p \in \mathbb{S}$), and define $I_h^2 : L^2(\Omega) \to S^h$ by

$$(\varphi - I_h^2\varphi, z)_0 = 0, \quad \forall z \in S^h.$$

And let $I_h(\psi,\varphi) = (I_h^1\psi, I_h^2\varphi)$.

For the finite element spaces mentioned above, when $\psi \in W^{3,p}(\Omega)$ with $p \in \mathbb{S}$, the following estimates are valid:

$$\|I_h^1\psi - \psi\|_h \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}, \tag{2.31}$$

$$\|I_h^1\psi - \psi\|_s \lesssim h^{3-s+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}, \quad s = 0,1, \tag{2.32}$$

and when $\varphi \in H_0^1(\Omega)$

$$\|I_h^2\varphi - \varphi\|_0 = \inf_{v \in S^h}\|\varphi - v\|_0 \lesssim h\|\varphi\|_1, \tag{2.33}$$

$$\|I_h^2\varphi - \varphi\|_{-1} = \sup_{v \in H_0^1(\Omega)} \frac{(I_h^2\varphi - \varphi, v - I_h^2 v)_0}{\|v\|_1} \lesssim h^2\|\varphi\|_1. \tag{2.34}$$

3 The consistency term and Strang lemma

Let (ψ,φ) and (ψ^*,φ^*) be the solutions of (2.14) and (2.19), respectively. Define the consistency terms: For any $(v,z) \in \mathbf{H}_h + \mathbf{H}$,

$$D_h((\psi,\varphi),(v,z)) = B((f,g),(v,z)) - A_h((\psi,\varphi),(v,z)), \tag{3.1}$$

$$D_h^*((v,z),(\psi^*,\varphi^*)) = B((v,z),(f,g)) - A_h((v,z),(\psi^*,\varphi^*)). \tag{3.2}$$

The following estimations of the consistency term play an crucial role in our analysis.

$$|D_h((\psi,\varphi),(v,z))| \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}\|v\|_h, \tag{3.3}$$

$$|D_h^*((v,z),(\psi^*,\varphi^*))| \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi^*\|_{3,p}\|v\|_h. \tag{3.4}$$

Next, we will prove the estimates (3.3) and (3.4) of the consistency term.

It is well known that the following (C3) is valid for the non-conforming finite elements mentioned in Section 2 except Adini element (see Section 2.6 in [23]).

(C3) If F is the common face of element κ and κ', then

$$\int_F \nabla(v|_\kappa)ds = \int_F \nabla(v|_{\kappa'})ds, \quad \forall v \in S^h; \tag{3.5}$$

if F is a face of element κ and $F \in \partial\Omega$, then

$$\int_F \nabla(v|_\kappa)ds = 0, \quad \forall v \in S^h. \tag{3.6}$$

Define the face and element average interpolation operators

$$P_F^0 f = \frac{1}{meas(F)} \int_F f ds, \quad R_F^0 f = f - P_F^0 f,$$
$$P_\kappa^0 f = \frac{1}{meas(\kappa)} \int_\kappa f dx, \quad R_\kappa^0 f = f - P_\kappa^0 f,$$

where element $\kappa \in \pi_h$ and F is an arbitrary element face of π_h.
A simple calculation shows that for arbitrary constant C_0,

$$\frac{1}{meas(F)} \int_F (f - P_F^0 f) C_0 ds = 0. \tag{3.7}$$

Theorem 3.1. Suppose that $\psi, \psi^* \in W^{3,p}(\Omega)$ ($p \in \mathbb{S}$), and (C3) is valid. Then for any $(v, z) \in \mathbf{H}_h + \mathbf{H}$, (3.3) and (3.4) hold.

Proof. For any $(v, z) \in C_0^\infty(\Omega) \times L^2(\Omega)$, by the Green's formula we deduce

$$\begin{aligned} B((f,g),(v,z)) &= A((\psi,\varphi),(v,z)) \\ &= \sum_{\kappa \in \pi_h} \int_\kappa -\nabla(\frac{1}{n-1}\Delta\psi) \cdot \nabla\overline{v} + \varphi\overline{z} dx. \end{aligned} \tag{3.8}$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, for any $(v, z) \in \mathbf{H}_h + \mathbf{H}$ the above (3.8) holds.

Thus

$$
\begin{aligned}
&D_h((\psi,\varphi),(v,z)) = B((f,g),(v,z)) - A_h((\psi,\varphi),(v,z)) \\
&= \sum_{\kappa\in\pi_h}\int_\kappa -\nabla(\frac{1}{n-1}\Delta\psi)\cdot\nabla\overline{v} + \varphi\overline{z}dx - \sum_{\kappa\in\pi_h}\int_\kappa \{(\frac{1}{n-1}-\mu_1)\Delta\psi\Delta\overline{v} \\
&\quad + \mu_1 \sum_{1\le i,j\le d}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2 \overline{v}}{\partial x_i\partial x_j}\}dx - (\varphi,z)_0 \\
&= -\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\frac{1}{n-1}\Delta\psi\nabla\overline{v}\cdot\gamma ds \\
&\quad + \mu_1\sum_{\kappa\in\pi_h}\int_\kappa\{\Delta\psi\Delta\overline{v} - \sum_{1\le i,j\le d}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2 \overline{v}}{\partial x_i\partial x_j}\}dx \\
&= -\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\frac{1}{n-1}\Delta\psi\nabla\overline{v}\cdot\gamma ds \\
&\quad + \mu_1\sum_{\kappa\in\pi_h}\int_\kappa(\sum_{1\le i\ne j\le d}\frac{\partial^2\psi}{\partial x_i^2}\frac{\partial^2\overline{v}}{\partial x_j^2} - \sum_{1\le i\ne j\le d}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial^2\overline{v}}{\partial x_i\partial x_j})dx \\
&= -\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\frac{1}{n-1}\Delta\psi\nabla\overline{v}\cdot\gamma ds + \mu_1\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\sum_{1\le i\ne j\le d}\frac{\partial^2\psi}{\partial x_i^2}\frac{\partial\overline{v}}{\partial x_j}\gamma_j ds \\
&\quad - \mu_1\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\sum_{1\le i\ne j\le d}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial\overline{v}}{\partial x_j}\gamma_i ds \\
&\equiv I_1 + I_2 + I_3. \qquad (3.9)
\end{aligned}
$$

Note that (C3) and (3.7) are valid, and for all $v\in H_0^2(\Omega)$ (3.5)-(3.6) also hold, we deduce that $\forall(v,z)\in\mathbf{H}_h+\mathbf{H}$,

$$
I_1 = -\sum_{\kappa\in\pi_h}\sum_{F\in\partial\kappa}\int_F R_F^0(\frac{1}{n-1}\Delta\psi)R_F^0(\nabla\overline{v}\cdot\gamma)ds. \qquad (3.10)
$$

Let $\hat\kappa$ is a reference element, κ and $\hat\kappa$ be affine-equivalent. When $\hat w\in W_{1,\iota}(\hat\kappa)$ and $1\le\rho<\frac{(d-1)\iota}{d-\iota}$, by the trace theorem we get $W_{1,\iota}(\hat\kappa)\hookrightarrow L_\rho(\partial\hat\kappa)$, thus we deduce the following trace inequality:

$$
\begin{aligned}
&\int_{\partial\kappa}|w|^\rho ds = \int_{\partial\hat\kappa}|\hat w|^\rho\frac{|\partial\kappa|}{\partial\hat\kappa}d\hat s \lesssim h_\kappa^{d-1}\|\hat w\|_{0,\rho,\partial\hat\kappa}^\rho \lesssim h_\kappa^{d-1}\|\hat w\|_{1,\iota,\hat\kappa}^\rho \\
&\quad\lesssim h_\kappa^{d-1}(\|\hat w\|_{0,\iota,\hat\kappa}^\rho + |\hat w|_{1,\iota,\hat\kappa}^\rho) \lesssim h_\kappa^{d-1}(h_\kappa^{-\frac{\rho d}{\iota}}\|w\|_{0,\iota,\kappa}^\rho + h_\kappa^{\rho-\frac{\rho d}{\iota}}|w|_{1,\iota,\kappa}^\rho) \\
&\quad\lesssim h_\kappa^{d-\frac{\rho d}{\iota}-1}\|w\|_{0,\iota,\kappa}^\rho + h_\kappa^{\rho+d-\frac{\rho d}{\iota}-1}|w|_{1,\iota,\kappa}^\rho, \quad \forall\kappa\in\pi_h. \qquad (3.11)
\end{aligned}
$$

Since $p\in\mathbb{S}$, $W_{1,p}(\hat\kappa)\hookrightarrow L_\rho(\partial\hat\kappa)$ with $\rho\in(d-1,\frac{(d-1)p}{d-p})$. Choose $\frac{1}{\rho'}=1-\frac{1}{\rho}$, then $\rho'<\frac{d-1}{d-2}$ and $W_{1,2}(\hat\kappa)\hookrightarrow L_{\rho'}(\partial\hat\kappa)$. And thus, by the Hölder inequality, the trace inequality (3.11) and the interpolation error estimate we deduce that

$$
\begin{aligned}
|I_1| &\lesssim \sum_{\kappa\in\pi_h}\sum_{F\in\partial\kappa}\|R_F^0(\frac{1}{n-1}\Delta\psi)\|_{0,\rho,F}\|R_F^0(\nabla\overline{v}\cdot\gamma)\|_{0,\rho',F} \\
&\lesssim \sum_{\kappa\in\pi_h}\sum_{F\in\partial\kappa}\|R_\kappa^0(\frac{1}{n-1}\Delta\psi)\|_{0,\rho,F}\|R_\kappa^0(\nabla\overline{v}\cdot\gamma)\|_{0,\rho',F} \\
&\lesssim \sum_{\kappa\in\pi_h}(h_\kappa^{d-\frac{\rho d}{p}-1}\|R_\kappa^0(\frac{1}{n-1}\Delta\psi)\|_{0,p,\kappa}^{\rho}+h_\kappa^{\rho+d-\frac{\rho d}{p}-1}|R_\kappa^0(\frac{1}{n-1}\Delta\psi)|_{1,p,\kappa}^{\rho})^{\frac{1}{\rho}} \\
&\quad\times(h_\kappa^{d-\frac{\rho' d}{2}-1}\|R_\kappa^0(\nabla v\cdot\gamma)\|_{0,\kappa}^{\rho'}+h_\kappa^{\rho'+d-\frac{\rho' d}{2}-1}|R_\kappa^0(\nabla\overline{v}\cdot\gamma)|_{1,\kappa}^{\rho'})^{\frac{1}{\rho'}} \\
&\lesssim \sum_{\kappa\in\pi_h}(h_\kappa^{\rho+d-\frac{\rho d}{p}-1})^{\frac{1}{\rho}}\|\psi\|_{3,p,\kappa}\times(h_\kappa^{\rho'+d-\frac{\rho' d}{2}-1})^{\frac{1}{\rho'}}\|v\|_{2,\kappa} \\
&\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}\|v\|_h, \quad \forall (v,z)\in\mathbf{H}_h+\mathbf{H}. \qquad (3.12)
\end{aligned}
$$

Similarly we deduce

$$|I_2|\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}\|v\|_h, \quad \forall (v,z)\in\mathbf{H}_h+\mathbf{H}, \qquad (3.13)$$

$$|I_3|\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}\|v\|_h, \quad \forall (v,z)\in\mathbf{H}_h+\mathbf{H}. \qquad (3.14)$$

Substituting (3.12), (3.13) and (3.14) into (3.9) we get (3.3).

Using the same argument as above, we can prove (3.4). □

Next, we shall analyze Adini rectangle element approximation. We suppose that $\Omega\subset\mathbb{R}^2$, and the boundary of Ω and the edges of elements are parallel to the coordinate axis. Although (C3) is not valid, thanks to [13], we can prove (3.3) and (3.4) still hold.

Theorem 3.2. Suppose that S^h is Adini element space, then for any $(v,z)\in\mathbf{H}_h$ (3.3) and (3.4) are valid.

Proof. We shall analyze the terms I_1, I_2 and I_3 on the right-hand side of (3.9). Noticing that the edges of elements are parallel to the coordinate axis, using the proof method of Theorem 50.1 in [13], we can deduce that for any $(v,z)\in\mathbf{H}_h$,

$$|I_1|+|I_2|\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p}\|v\|_h.$$

And from line 11 on page 304 in [13], we see

$$I_3=0.$$

Substituting the above estimates into (3.9) we get (3.3). Similarly we can prove (3.4). □

The following lemma is a generalization of Strang Lemma (1972).

Lemma 3.1. Let (ψ,φ) be the solution of (2.14) and (ψ_h,φ_h) be the solution of (2.25), then

$$\inf_{(v,z)\in \mathbf{H}_h} \|(\psi,\varphi)-(v,z)\|_h + \sup_{(v,z)\in \mathbf{H}_h\setminus\{0\}} \frac{D_h((\psi,\varphi),(v,z))}{\|(v,z)\|_h}$$
$$\lesssim \|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h$$
$$\lesssim \inf_{(v,z)\in \mathbf{H}_h} \|(\psi,\varphi)-(v,z)\|_h + \sup_{(v,z)\in \mathbf{H}_h\setminus\{0\}} \frac{D_h((\psi,\varphi),(v,z))}{\|(v,z)\|_h}. \quad (3.15)$$

Let (ψ^*,φ^*) be the solution of (2.19) and (ψ_h^*,φ_h^*) be its finite element solution, then

$$\inf_{(v,z)\in \mathbf{H}_h} \|(\psi^*,\varphi^*)-(v,z)\|_h + \sup_{(v,z)\in \mathbf{H}_h\setminus\{0\}} \frac{D_h((v,z),(\psi^*,\varphi^*))}{\|(v,z)\|_h}$$
$$\lesssim \|(\psi^*,\varphi^*)-(\psi_h^*,\varphi_h^*)\|_h$$
$$\lesssim \inf_{(v,z)\in \mathbf{H}_h} \|(\psi^*,\varphi^*)-(v,z)\|_h + \sup_{(v,z)\in \mathbf{H}_h\setminus\{0\}} \frac{D_h((v,z),(\psi^*,\varphi^*))}{\|(v,z)\|_h}. \quad (3.16)$$

Proof. For any $(v,z)\in \mathbf{H}_h$,

$$\|(\psi_h,\varphi_h)-(v,z)\|_h^2 = A_h((\psi_h,\varphi_h)-(v,z),(\psi_h,\varphi_h)-(v,z))$$
$$= A_h((\psi,\varphi)-(v,z),(\psi_h,\varphi_h)-(v,z)) + B((f,g),(\psi_h,\varphi_h)-(v,z))$$
$$- A_h((\psi,\varphi),(\psi_h,\varphi_h)-(v,z)).$$

When $\|(\psi_h,\varphi_h)-(v,z)\|_h\neq 0$, dividing it in both sides of the above we obtain

$$\|(\psi_h,\varphi_h)-(v,z)\|_h \le \|(\psi,\varphi)-(v,z)\|_h$$
$$-\frac{A_h((\psi,\varphi),(\psi_h,\varphi_h)-(v,z)) - B((f,g),(\psi_h,\varphi_h)-(v,z))}{\|(\psi_h,\varphi_h)-(v,z)\|_h}$$
$$\lesssim \|(\psi,\varphi)-(v,z)\|_h + \sup_{(v,z)\in \mathbf{H}_h\setminus\{0\}} \frac{D_h((\psi,\varphi),(v,z))}{\|(v,z)\|_h}.$$

This together with the triangular inequality

$$\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h \le \|(\psi,\varphi)-(v,z)\|_h + \|(v,z)-(\psi_h,\varphi_h)\|_h$$

yields the second inequality of (3.15). From

$$A_h((\psi,\varphi)-(\psi_h,\varphi_h),(v,z)) \le \|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h\|(v,z)\|_h, \quad \forall\ (v,z)\in S^h,$$

we get

$$\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h \ge \frac{A_h((\psi,\varphi),(v,z)) - A_h((\psi_h,\varphi_h),(v,z))}{\|(v,z)\|_h} = -\frac{D_h((\psi,\varphi),(v,z))}{\|(v,z)\|_h},$$

which together with $\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h \ge \inf_{(v,z)\in S^h}\|(\psi,\varphi)-(v,z)\|_h$ we obtain the first inequality of (3.15).

Similarly we can prove (3.16). The proof is completed. $\square$

By Lemma 3.1, we get:

Theorem 3.3. Suppose that $\psi,\psi^*\in W^{3,p}(\Omega)$ $(p\in\mathbb{S})$, for any $(v,z)\in\mathbf{H}_h$ (3.3) and (3.4) hold. Then

$$\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}(\|\psi\|_{3,p}+\|\varphi\|_1), \quad (3.17)$$
$$\|(\psi^*,\varphi^*)-(\psi_h^*,\varphi_h^*)\|_h \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}(\|\psi^*\|_{3,p}+\|\varphi^*\|_1). \quad (3.18)$$

Proof. By the interpolation error estimates (2.31) and (2.33), we get

$$\begin{aligned}
\inf_{(v,z)\in\mathbf{H}_h} \|(\psi,\varphi)-(v,z)\|_h &\lesssim \|(\psi,\varphi)-I_h(\psi,\varphi)\|_h \\
&= \|\psi - I_h^1\psi\|_h + \|\varphi - I_h^2\varphi\|_0 \\
&\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|\psi\|_{3,p} + h\|\varphi\|_1 \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}(\|\psi\|_{3,p}+\|\varphi\|_1). \quad (3.19)
\end{aligned}$$

Substituting (3.19) and (3.3) into (3.15) we get (3.17). By the same argument we can prove (3.18). The proof is completed. □

Remark 3.1. *We tried to use the Nitsche technique to prove the error estimate in $\|\cdot\|_{\mathbf{H}_1}$ is of higher order than that in $\|\cdot\|_h$, but failed because of the non-symmetry of right-hand sides that involves derivatives, of (2.14) and (2.19).*

4 The error analysis of the non-conforming element eigenvalues

Let (λ, u, ω) and $(\lambda^*, u^*, \omega^*)$ be the eigenpair of (2.11) and (2.18), respectively. Then from (3.1) and (3.2) we get that for any $(v,z)\in\mathbf{H}_h+\mathbf{H}$,

$$D_h((u,\omega),(v,z)) = B(\lambda(u,\omega),(v,z)) - A_h((u,\omega),(v,z)), \quad (4.1)$$
$$D_h^*((v,z),(u^*,\omega^*)) = B((v,z),\lambda^*(u^*,\omega^*)) - A_h((v,z),(u^*,\omega^*)). \quad (4.2)$$

We need the following regularity assumption:
$R(\Omega)$**.** *For any $\xi\in H^{-1}(\Omega)$, there exists $\psi\in W_{3,p_0}(\Omega)$ satisfying*

$$\Delta(\frac{1}{n-1}\Delta\psi)=\xi, \quad \text{in } \Omega, \quad \psi=\frac{\partial\psi}{\partial\nu}=0 \quad \text{on } \partial\Omega,$$

and

$$\|\psi\|_{3,p_0}\le C_R\|\xi\|_{-1}, \quad (4.3)$$

where $p_0\in\mathbb{S}$, C_R denotes the prior constant dependent on the $n(x)$ and Ω but independent of the right-hand side ξ of the equation.

It is well known that (4.3) is valid when n and $\partial\Omega$ are appropriately smooth. For example, when $\Omega\subset\mathbb{R}^2$ is a convex polygon, from Theorem 2 in [5], we can get that $p_0=2$.

Consider the source problem associated with (2.5) and (2.7):

$$\Delta(\frac{1}{n-1}\Delta\psi) = -\frac{n}{n-1}\Delta f - \Delta(\frac{1}{n-1}f) - \frac{n}{n-1}g, \quad (4.4)$$
$$\varphi = f. \quad (4.5)$$

When $n\in W^{1,\infty}(\Omega)\cap H^2(\Omega)$ and f is appropriately smooth, from $R(\Omega)$ we can deduce that $\psi\in W^{3,p_0}(\Omega)$ and

$$\begin{aligned}
\|\psi\|_{3,p_0} &\le C_R\|-\frac{n}{n-1}\Delta f - \Delta(\frac{1}{n-1}f) - \frac{n}{n-1}g\|_{-1} \\
&\lesssim \|f\|_1 + \|g\|_{-1} \lesssim \|(f,g)\|_{\mathbf{H}_1} \quad (4.6)
\end{aligned}$$
$$\|\varphi\|_1 = \|f\|_1. \quad (4.7)$$

In this paper, for simplicity, we assume that the dual and primal problems have the same regularity.

Theorem 4.1. Assume $n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$, (2.2) and $R(\Omega)$ hold, and for any $(v,z) \in \mathbf{H}_h$ (3.3)-(3.4) are valid. Then

$$\|T - T_h\|_{\mathbf{H}_1} \lesssim h^{1+(\frac{1}{2}-\frac{1}{p_0})d}, \tag{4.8}$$

$$\|T^* - T_h^*\|_{\mathbf{H}_1} \lesssim h^{1+(\frac{1}{2}-\frac{1}{p_0})d}. \tag{4.9}$$

Proof. For any $(f,g) \in \mathbf{H}_1$, with $\|(f,g)\|_{\mathbf{H}_1} = 1$, there is $(f_j,g) \in C_0^\infty(\Omega) \times H^{-1}(\Omega)$, such that

$$\|(f,g) - (f_j,g)\|_{\mathbf{H}_1} = \|f - f_j\|_1 \leq h.$$

By (2.23), (2.31)-(2.34), (3.17) and (4.6) we deduce

$$\begin{aligned}
&\|(T-T_h)(f_j,g)\|_{\mathbf{H}_1} \leq \|T(f_j,g) - I_hT(f_j,g)\|_{\mathbf{H}_1} + \|I_hT(f_j,g) - T_h(f_j,g)\|_{\mathbf{H}_1} \\
&\quad \leq \|T(f_j,g) - I_hT(f_j,g)\|_{\mathbf{H}_1} + \|I_hT(f_j,g) - T_h(f_j,g)\|_h \\
&\quad \leq Ch^{1+(\frac{1}{2}-\frac{1}{p_0})d}\|T(f_j,g)\|_{W^{3,p_0}(\Omega)\times H^1(\Omega)} + \|T(f_j,g) - T_h(f_j,g)\|_h \\
&\quad \leq h^{1+(\frac{1}{2}-\frac{1}{p_0})d}\|T(f_j,g)\|_{W^{3,p_0}(\Omega)\times H^1(\Omega)} \leq Ch^{1+(\frac{1}{2}-\frac{1}{p_0})d}\|(f_j,g)\|_{\mathbf{H}_1}.
\end{aligned}$$

From (2.26) we know that T_h has a upper bound uniformly with respect to h. Thus we have

$$\begin{aligned}
&\|(T-T_h)(f,g)\|_{\mathbf{H}_1} \leq \|(T-T_h)((f,g)-(f_j,g))\|_{\mathbf{H}_1} + \|(T-T_h)(f_j,g)\|_{\mathbf{H}_1} \\
&\quad \leq (\|T\|_{\mathbf{H}_1} + \|T_h\|_{\mathbf{H}_1})\|(f,g)-(f_j,g)\|_{\mathbf{H}_1} + Ch^{1+(\frac{1}{2}-\frac{1}{p_0})d}\|(f_j,g)\|_{\mathbf{H}_1} \\
&\quad \leq (\|T\|_{\mathbf{H}_1} + \|T_h\|_{\mathbf{H}_1})h + Ch^{1+(\frac{1}{2}-\frac{1}{p_0})d}(\|(f_j,g)-(f,g)\|_{\mathbf{H}_1} + \|(f,g)\|_{\mathbf{H}_1}) \\
&\quad \leq (\|T\|_{\mathbf{H}_1} + \|T_h\|_{\mathbf{H}_1} + C)h^{1+(\frac{1}{2}-\frac{1}{p_0})d}.
\end{aligned}$$

And by the definition of operator norm we have

$$\|T - T_h\|_{\mathbf{H}_1} = \sup_{(f,g)\in\mathbf{H}_1, \|(f,g)\|_{\mathbf{H}_1}=1} \|(T-T_h)(f,g)\|_{\mathbf{H}_1} \lesssim h^{1+(\frac{1}{2}-\frac{1}{p_0})d}.$$

Hence, (4.8) is valid. Similarly we can deduce (4.9). The proof is completed. □

In this paper, we suppose that λ be an eigenvalue of (2.11) with the algebraic multiplicity q and the ascent α. Then $\lambda^* = \overline{\lambda}$ is an eigenvalue of (2.18). Since $\|T_h - T\|_{\mathbf{H}_1} \to 0$, q eigenvalues $\lambda_{1,h}, \cdots, \lambda_{q,h}$ of (2.24) will converge to λ.

Let E be the spectral projection associated with T and λ, then $R(E) = N((\lambda^{-1} - T))$ is the space of generalized eigenfunctions associated with λ and T, where R denotes the range and N denotes the null space. Let E_h be the spectral projection associated with T_h and the eigenvalues $\lambda_{1,h}, \cdots, \lambda_{q,h}$, then $R(E_h)$ is the space spanned by all generalized eigenfunctions corresponding to all eigenvalues $\lambda_{1,h}, \cdots, \lambda_{q,h}$. In view of the adjoint problem (2.18) and (2.28), the definitions of E^*, $R(E^*)$, E_h^* and $R(E_h^*)$ are analogous to E, $R(E)$, E_h and $R(E_h)$ (see [3]).

Let $\lambda_h \in \{\lambda_{1,h}, \cdots, \lambda_{q,h}\}$. From [3] we get the following results.

Theorem 4.2. Assume that the conditions of Theorem 4.1 are valid. Let (u_h, ω_h) be eigenfunction corresponding to λ_h and $\|(u_h, \omega_h)\|_h = 1$. Then there exists eigenfunction (u,ω) corresponding to λ, such that

$$\|(u_h,\omega_h)-(u,\omega)\|_{\mathbf{H}_1} \lesssim \|(T-T_h)|_{R(E)}\|_{\mathbf{H}_1}^{\frac{1}{\alpha}}, \tag{4.10}$$

$$|\lambda_h-\lambda| \lesssim \|(T-T_h)|_{R(E)}\|_{\mathbf{H}_1}^{\frac{1}{\alpha}}. \tag{4.11}$$

$$|(\frac{1}{q}\sum_{i=1}^{q}\lambda_{i,h}^{-1})^{-1}-\lambda| \lesssim \|(T-T_h)|_{R(E)}\|_{\mathbf{H}_1}. \tag{4.12}$$

Furthermore assume $R(E)\subset W^{3,p}(\Omega)$ $(p\in\mathbb{S})$, then

$$\|(T-T_h)|_{R(E)}\|_{\mathbf{H}_1} \lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}, \tag{4.13}$$

and

$$\|(u_h,\omega_h)-(u,\omega)\|_{\mathbf{H}_1} \lesssim h^{\frac{1}{\alpha}+(\frac{1}{2}-\frac{1}{p})\frac{d}{\alpha}}, \tag{4.14}$$

$$\|(u_h,\omega_h)-(u,\omega)\|_{h} \lesssim h^{\frac{1}{\alpha}+(\frac{1}{2}-\frac{1}{p})\frac{d}{\alpha}}, \tag{4.15}$$

with $\|(u,\omega)\|_h=1$.

Proof. From Theorem 4.1 we know $\|T-T_h\|_{\mathbf{H}_1}\to 0$ $(h\to 0)$, thus from Theorem 7.4, Theorem 7.3 and Theorem 7.2 of [3] we get (4.10), (4.11) and (4.12), respectively. By the way to show (4.8), we get (4.13). Substituting (4.13) into (4.10), we get (4.14). By calculation we get

$$\begin{aligned}\|(u_h,\omega_h)-(u,\omega)\|_h &= \|\lambda_h T_h(u_h,\omega_h)-\lambda T(u,\omega)\|_h\\ &\le \|\lambda_h T_h(u_h,\omega_h)-\lambda T_h(u,\omega)\|_h+\|\lambda T_h(u,\omega)-\lambda T(u,\omega)\|_h\\ &\lesssim \|\lambda T_h(u,\omega)-\lambda T(u,\omega)\|_h+\|\lambda_h(u_h,\omega_h)-\lambda(u,\omega)\|_{\mathbf{H}_1}\\ &\lesssim |\lambda|\|T(u,\omega)-T_h(u,\omega)\|_h+h^{\frac{1}{\alpha}+(\frac{1}{2}-\frac{1}{p})\frac{d}{\alpha}}.\end{aligned} \tag{4.16}$$

Combining (3.17) with the above relation we get (4.15). By calculation we have

$$\begin{aligned}\|(u_h,\omega_h)-\frac{(u,\omega)}{\|(u,\omega)\|_h}\|_{\mathbf{s}} &\lesssim \|(u_h,\omega_h)-(u,\omega)\|_h\\ &+\|(u_h,\omega_h)-(u,\omega)\|_{\mathbf{s}},\quad \mathbf{s}=\mathbf{H}_1,h,\end{aligned} \tag{4.17}$$

thus, when replacing (u,ω) by $\frac{(u,\omega)}{\|(u,\omega)\|_h}$, (4.14) and (4.15) also hold. □

Starting from (4.11), if we use $\|(T-T_h)|_{R(E)}\|_{\mathbf{H}_1}$ we can not derive the optimal estimates for the eigenvalue when the eigenfunction is smooth on concave domain because the error estimate in $\|\cdot\|_{\mathbf{H}_1}$ depends on the Nitsche technique and the regularity. To avoid this problem, we employ a new method and give an identity in the following lemma, and use it to prove the optimal error estimates of non-conforming element eigenvalues. The identity and proof method are also valid for general nonselfadjoint eigenvalue problems.

Lemma 4.1. Let (λ,u,ω) and (λ^*,u^*,ω^*) be the eigenpairs of (2.11) and (2.18) respectively. Then for any $(v,z),(v^*,z^*)\in\mathbf{H}_h$, when $B((v,z),(v^*,z^*))\neq 0$ it is valid that

$$\begin{aligned}\frac{A_h((v,z),(v^*,z^*))}{B((v,z),(v^*,z^*))}-\lambda &= \frac{A_h((u,\omega)-(v,z),(u^*,\omega^*)-(v^*,z^*))}{B((v,z),(v^*,z^*))}\\ &-\lambda\frac{B((u,\omega)-(v,z),(u^*,\omega^*)-(v^*,z^*))}{B((v,z),(v^*,z^*))}\\ &+\frac{D_h((u,\omega),(v^*,z^*))}{B((v,z),(v^*,z^*))}+\frac{D_h((v,z),(u^*,\omega^*))}{B((v,z),(v^*,z^*))}.\end{aligned} \tag{4.18}$$

Proof. From (2.11), (2.18), (4.1) and (4.2) we have

$$\begin{aligned}
&A_h((u,\omega)-(v,z),(u^*,\omega^*)-(v^*,z^*))-\lambda B((u,\omega)-(v,z),(u^*,\omega^*)-(v^*,z^*))\\
&\quad= A_h((u,\omega),(u^*,\omega^*))+A_h((v,z),(v^*,z^*))-A_h((u,\omega),(v^*,z^*))\\
&\qquad-A_h((v,z),(u^*,\omega^*))-\lambda(B((u,\omega),(u^*,\omega^*))+B((v,z),(v^*,z^*))\\
&\qquad-B((u,\omega),(v^*,z^*))-B((v,z),(u^*,\omega^*)))\\
&\quad=\lambda B((u,\omega),(u^*,\omega^*))+A_h((v,z),(v^*,z^*))-B(\lambda(u,\omega),(v^*,z^*))\\
&\qquad-D_h((u,\omega),(v^*,z^*))-B((v,z),\lambda^*(u^*,\omega^*))-D_h((v,z),(u^*,\omega^*))\\
&\qquad-\lambda B((u,\omega),(u^*,\omega^*))-\lambda B((v,z),(v^*,z^*))\\
&\qquad+\lambda B((u,\omega),(v^*,z^*))+\lambda B((v,z),(u^*,\omega^*))\\
&\quad=A_h((v,z),(v^*,z^*))-\lambda B((v,z),(v^*,z^*))\\
&\qquad+D_h((u,\omega),(v^*,z^*))+D_h((v,z),(u^*,\omega^*)),
\end{aligned}$$

dividing $B((v,z),(v^*,z^*))$ in both side of the above we obtain the desired conclusion. □

Theorem 4.3. Assume that the conditions of Theorem 4.2 are valid, and $R(E),R(E^*)\subset W^{3,p}(\Omega)$ $(p\in\mathbb{S})$, the ascent α of λ is equal to 1, for any $(v,z)\in\mathbf{H}_h+\mathbf{H}$, (3.3) and (3.4) hold, then

$$|\lambda_h-\lambda|\lesssim h^{2+2(\frac{1}{2}-\frac{1}{p})d}. \tag{4.19}$$

Proof. From $\alpha=1$, we know $R(E^*)$ is the space of eigenfunctions associated with λ^*. Let (u,ω) and (u_h,ω_h) satisfy (4.10) and (4.15), since $(u,\omega)\in R(E)$, $\|(u,\omega)\|_A=1$, Define

$$f((v,z))=A(E(v,z),(u,\omega)),\quad \forall (v,z)\in\mathbf{H}.$$

Since for all $(v,z)\in\mathbf{H}$ one has

$$\begin{aligned}
|f((v,z))|&=|A(E(v,z),(u,\omega))|\le\|E(v,z)\|_A\|(u,\omega)\|_A\\
&\lesssim\sqrt{\lambda}\|E(v,z)\|_{\mathbf{H_1}}\lesssim\|E\|_{\mathbf{H_1}}\|(v,z)\|_A,
\end{aligned}$$

f is a linear and bounded functional on $\mathbf{H}$ and $\|f\|_A\lesssim\|E\|_{\mathbf{H}_1}$. Using Riesz Theorem, we know there exists $(u^*,\omega^*)\in\mathbf{H}$ satisfying $\|(u^*,\omega^*)\|_A=\|f\|_A$ and

$$A((v,z),(u^*,\omega^*))=A(E(v,z),(u,\omega)). \tag{4.20}$$

For any $(v,z)\in\mathbf{H}$, notice $E(I-E)(v,z)=0$,

$$\begin{aligned}
&A((v,z),(\lambda^{*-1}-T^*)(u^*,\omega^*))=A((\lambda^{-1}-T)^{\alpha}(v,z),(u^*,\omega^*))\\
&\quad=A((\lambda^{-1}-T)E(v,z),(u^*,\omega^*))+A((\lambda^{-1}-T)(I-E)(v,z),(u^*,\omega^*))=0,
\end{aligned}$$

i.e., $(\lambda^{*-1}-T^*)(u^*,\omega^*)=0$, hence $(u^*,\omega^*)\in R(E^*)$. From (2.31)-(2.34) we have

$$\|(u^*,\omega^*)-I_h(u^*,\omega^*)\|_h\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}, \tag{4.21}$$
$$\|(u^*,\omega^*)-I_h(u^*,\omega^*)\|_{\mathbf{H_1}}\lesssim h^{2+(\frac{1}{2}-\frac{1}{p})d}, \tag{4.22}$$

By (4.20) we have

$$A((u,\omega),(u^*,\omega^*))=A(E(u,\omega),(u,\omega))==A(E(u,\omega),(u,\omega))=1. \tag{4.23}$$

Then, from (4.15), (4.21) and (4.23), when h is small enough, $|A_h((u_h,\omega_h),I_h(u^*,\omega^*))|$ has a positive lower bound uniformly with respect to h, thus there is a positive constant C_0 independent of h such that

$$|B((u_h,\omega_h),I_h(u^*,\omega^*))| = |\lambda_h^{-1}A_h((u_h,\omega_h),I_h(u^*,\omega^*))| \geq C_0. \tag{4.24}$$

In (4.18), let $(v,z)=(u_h,\omega_h),(v^*,z^*)=I_h(u^*,\omega^*)$, noting that

$$\lambda_h = A((u_h,\omega_h),I_h(u^*,\omega^*))/B((u_h,\omega_h),I_h(u^*,\omega^*)),$$

then

$$\begin{aligned}|\lambda_h-\lambda| \lesssim\ & \|(u,\omega)-(u_h,\omega_h)\|_h\|(u^*,\omega^*)-I_h(u^*,\omega^*)\|_h\\ &+\|(u,\omega)-(u_h,\omega_h)\|_{\mathbf{H}_1}\|(u^*,\omega^*)-I_h(u^*,\omega^*)\|_{\mathbf{H}_1}\\ &+|D_h((u,\omega),I_h(u^*,\omega^*))|+|D_h((u_h,\omega_h),(u^*,\omega^*))|.\end{aligned} \tag{4.25}$$

From (3.3) and (4.21),

$$\begin{aligned}&|D_h((u,\omega),I_h(u^*,\omega^*))| = |D_h((u,\omega),I_h(u^*,\omega^*)-(u^*,\omega^*))|\\ &\quad\lesssim h^{1+(\frac{1}{2}-\frac{1}{p})d}\|(u,\omega)\|_{3,p}\|I_h(u^*,\omega^*)-(u^*,\omega^*)\|_h\\ &\quad\lesssim h^{2+2(\frac{1}{2}-\frac{1}{p})d}\|(u,\omega)\|_{3,p}\|(u^*,\omega^*)\|_{3,p},\end{aligned} \tag{4.26}$$

and from (3.4) and (4.15),

$$\begin{aligned}&|D_h((u_h,\omega_h),(u^*,\omega^*))| = |D_h((u_h,\omega_h)-(u,\omega),(u^*,\omega^*))|\\ &\quad\lesssim h^{2+2(\frac{1}{2}-\frac{1}{p})d}\|(u,\omega)\|_{3,p}\|(u^*,\omega^*)\|_{3,p}.\end{aligned} \tag{4.27}$$

Substituting (4.15), (4.21), (4.22), (4.26) and (4.27) into (4.25), we get (4.19). □

Remark 4.1. Using the same argument as in this section we can prove the error estimates of finite element approximation for the dual problem (2.18): Let $R(E^*)\subset W^{3,p}(\Omega)$ $(p\in\mathbb{S})$, then

$$\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_h \lesssim h^{\frac{1}{\alpha}+(\frac{1}{2}-\frac{1}{p})\frac{d}{\alpha}}, \tag{4.28}$$

$$\|(u_h^*,\omega_h^*)-(u^*,\omega^*)\|_{\mathbf{H}_1} \lesssim h^{\frac{1}{\alpha}+(\frac{1}{2}-\frac{1}{p})\frac{d}{\alpha}}. \tag{4.29}$$

Theorem 4.4. Assume that the conditions of Theorem 4.2 are valid, and $D\subset\mathbb{R}^2$, $R(E)\subset H^4(\Omega)$, n is a constant. Let S^h be the Adini element space defined on the uniform rectangle mesh. Then

$$\|(u_h,\omega_h)-(u,\omega)\|_h \lesssim h^{\frac{2}{\alpha}}, \tag{4.30}$$

$$|\lambda_h-\lambda| \lesssim h^{\frac{2}{\alpha}}, \tag{4.31}$$

$$|(\frac{1}{q}\sum_{i=1}^{q}\lambda_{i,h}^{-1})^{-1}-\lambda| \lesssim h^2. \tag{4.32}$$

Proof. From line 11 on page 304 in [13], we know for Adini element, the third term on the right-hand of (3.9) is equal to 0, thus we obtain

$$\begin{aligned}D_h((\psi,\varphi),(v,z)) = &-\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\frac{1}{n-1}\Delta\psi\nabla\overline{v}\cdot\gamma ds\\ &+\mu_1\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\{\frac{\partial^2\psi}{\partial x_1^2}\frac{\partial\overline{v}}{\partial x_2}\gamma_2+\frac{\partial^2\psi}{\partial x_2^2}\frac{\partial\overline{v}}{\partial x_1}\gamma_1\}ds+0.\end{aligned} \tag{4.33}$$

Comparing it with the consistency term of the clamped plate bending problem (see (50.7) of [13]), from [18] we can deduce

$$|D_h((\psi,\varphi),(v,z))| \lesssim h^2\|\psi\|_4\|v\|_h, \quad \forall (v,z)\in \mathbf{H}_h, \tag{4.34}$$
$$\|(\psi,\varphi)-(\psi_h,\varphi_h)\|_h \lesssim h^2(\|\psi\|_4+\|\varphi\|_2). \tag{4.35}$$

Thus we have

$$\begin{aligned}\|(T-T_h)|_{R(E)}\|_{\mathbf{H}_1} &= \sup_{(u,\omega)\in R(E),\|(u,\omega)\|_{\mathbf{H}_1}=1}\|(T-T_h)(u,\omega)\|_{\mathbf{H}_1}\\ &\lesssim \sup_{(u,\omega)\in R(E),\|(u,\omega)\|_{\mathbf{H}_1}=1} h^2\|T(u,\omega)\|_{H^4(\Omega)\times H^2(\Omega)} \lesssim h^2. \end{aligned}\tag{4.36}$$

Substituting (4.36) into (4.11) and (4.12) we get (4.31) and (4.32), respectively.
By the way to show (4.15), we can prove (4.30). □

The literature [28] proved that the order of convergence is just 2 for the Adini finite element eigenvalues for the clamped plate vibration problem. Based on [28], we can prove the estimate (4.31) and (4.32) are optimal and cannot be improved further.

5 Numerical Experiment

In this section, we will report some numerical experiments for non-conforming finite element discretizations to validate our theoretical results.

We use Matlab 2012a to solve (2.1)-(2.4) on a Lenovo G480 PC with 4G memory. Our program is compiled under the package of iFEM [12].

Let $\{\xi_j\}_{j=1}^{N_h}$ be a basis of S^h and $u_h=\sum_{j=1}^{N_h}u_j\xi_j, \omega_h=\sum_{j=1}^{N_h}\omega_j\xi_j$. Denote $\overrightarrow{u}=(u_1,\cdots,u_{N_h})^T$ and $\overrightarrow{\omega}=(\omega_1,\cdots,\omega_{N_h})^T$. To describe our algorithm, we specify the following $N_h\times N_h$ matrices in the discrete case.

Matrix	Definition
A_h	$a_{lj}=\sum\int_\kappa\{(\frac{1}{n-1}-\mu_1)\Delta\xi_j\Delta\xi_l+\mu_1(\frac{\partial^2\xi_j}{\partial x_1^2}\frac{\partial^2\xi_l}{\partial x_1^2}+2\frac{\partial^2\xi_j}{\partial x_1\partial x_2}\frac{\partial^2\xi_l}{\partial x_1\partial x_2}+\frac{\partial^2\xi_j}{\partial x_2^2}\frac{\partial^2\xi_l}{\partial x_2^2})\}$
B_h	$b_{lj}=\int_D\{\nabla(\frac{1}{n-1}\xi_j)\cdot\nabla\xi_l+\nabla\xi_j\cdot\nabla(\frac{n}{n-1}\xi_l)\}dx$
C_h	$c_{lj}=-\int_D\frac{n}{n-1}\xi_j\xi_l dx$
G_h	$g_{lj}=\int_D\xi_j\xi_l dx$

where $N_h=\dim(S^h)$. Then (2.23) can be written as a generalized eigenvalue problem

$$\begin{pmatrix} A_h & 0\\ 0 & G_h\end{pmatrix}\begin{pmatrix}\overrightarrow{u}\\ \overrightarrow{\omega}\end{pmatrix}=\lambda_h\begin{pmatrix} B_h & C_h\\ G_h & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{u}\\ \overrightarrow{\omega}\end{pmatrix}. \tag{5.1}$$

Note that in (5.1) A_h is a positive definite Hermitian matrix, and G_h can be equivalently replaced by the identity matrix I_h, which will lead to two sparser coefficient matrices with good structure. Based on this fact, we use the sparse matrix eigenvalue solver *eigs* to compute the numerical eigenvalues and the resulting numerical eigenvalues are ideal.

We consider the model problems (2.1)-(2.4) with the refraction index $n=8+x_1-x_2$ and $n=16$ on the unit square $(0,1)^2$, L-shaped $(-1,1)^2\backslash\big([0,1)\times(-1,0]\big)$, triangle whose vertices are given by $(-\frac{\sqrt{3}}{2},-\frac{1}{2})$, $(\frac{\sqrt{3}}{2},-\frac{1}{2})$ and $(0,1)$, and disk

with radius $\frac{1}{2}$ and center $(0,0)$. We adopt the Morley-Zienkiewicz(MZ) element and Adini element to compute the transmission eigenvalues on quasi-uniform meshes. The Morley-Zienkiewicz element was put forward in [23], and its finite element space is defined as:
$S^h = \{v \in V_h | v\ and\ \nabla v\ vanish\ at\ all\ vertices\ on\ \partial\Omega,\ and\ over\ any\ face\ F \subset \partial\Omega,\ the\ mean\ value\ of\ \frac{\partial v}{\partial \nu_F}\ vanishes\}$,
where
$V_h = \{v \in L^2(\Omega) : v|_\kappa \in P_\kappa,\ \forall \kappa \in \pi_h;\ v\ and\ \nabla v\ are\ continuous\ at\ all\ vertices\ of\ \pi_h, and\ over\ each\ interelement\ face\ F\ of\ \pi_h,\ the\ jump\ of\ the\ mean\ value\ of\ \frac{\partial v}{\partial \nu_F}\ is\ zero\}$,
$P_\kappa = P_3''(\kappa) + span\{l_1^2 l_2 \cdots l_{d+1}, l_1 l_2^2 \cdots l_{d+1}, \cdots, l_1 l_2 \cdots l_{d+1}^2\}$ with $P_3''(\kappa)$ being the Zienkiewicz element shape function space and $l_i (i = 1, \cdots, d+1)$ being the barycentric coordinates.

In our computation, for all the domains mentioned above we set $\mu_1 = \frac{1}{9}$ when the refraction index $n = 8 + x_1 - x_2$ and $\mu_1 = \frac{1}{15}$ when the refraction index $n = 16$. The associated numerical eigenvalues computed by the MZ element and Adini element are listed partially in Tables 1-2 and Table 3, while the error curves of these numerical eigenvalues whose slopes are computed by procedure of curve fitting are depicted in Figures 1-3.

For reading conveniently, in our tables and figures we use the notation $k_{j,h}^{\Omega} = \sqrt{\lambda_{j,h}^{\Omega}}$ to denote the jth eigenvalue on the domain $\Omega = S, L, T, D$ obtained by (2.23) on π_h, where the symbols S, L, T, D denote the domains square, L-shaped, triangle and disk, respectively.

It is seen from Figures 1-3 that the convergence orders of the numerical eigenvalues on the unit square, triangle and disk computed by the two elements are around 2, which coincides with the theoretical result. Nevertheless, the convergence orders on the L-shaped domain of the numerical eigenvalues $k_{1,h}, k_{2,h}, k_{5,h}, k_{6,h}$ with $n = 8 + x_1 - x_2$ and $k_{1,h}, k_{3,h}$ with $n = 16$ are less than 2 (see Figures 1-2). This fact suggests that the eigenfunctions corresponding to these eigenvalues on the L-shaped domain do have singularities to different degrees.

Numerical results indicates our discretizations by the MZ element and the Adini element are efficient and consistent with theoretical analysis.

References

[1] A. Adini, R. Clough, *Analysis of plate bending by the finite element method.* NSF Rept. G. 7337, 1961.

[2] J. An, J. Shen, *A spectral-element method for transmission eigenvalue problems.* J. Sci. Comput., *57* (2013) 670–688.

[3] I. Babuska, J.E. Osborn, *Eigenvalue Problems.* in: P.G. Ciarlet, J.L. Lions,(Ed.), Finite Element Methods (Part 1), Handbook of Numerical Analysis, vol.2, Elsevier Science Publishers, North-Holand, 1991, pp. 640-787.

[4] Ju.M. Berezanskiǐ, *Expansion in Eigenfunctions of Selfadjoint Operators.* Transl. Math. Monos., Vol.17, Amer. Math. Soc., Providence, R.I., 1968

Table 1: The eigenvalues obtained by MZ element, $n = 8 + x_1 - x_2$.

j	h	$k^S_{j,h}$	$k^L_{j,h}$	$k^T_{j,h}$	h	$k^D_{j,h}$
1	$\frac{\sqrt{2}}{32}$	2.8218574	2.3035843	2.7388174	0.025	2.9775769
1	$\frac{\sqrt{2}}{64}$	2.8220628	2.3028188	2.7389418	0.012	2.9771919
1	$\frac{\sqrt{2}}{128}$	2.8221545	2.3024576	2.7389765	0.006	2.9771000
2	$\frac{\sqrt{2}}{32}$	3.5381161	2.3953577	3.2915472	0.025	3.7774560
2	$\frac{\sqrt{2}}{64}$	3.5384282	2.3955964	3.2917188	0.012	3.7770363
2	$\frac{\sqrt{2}}{128}$	3.5386203	2.3956673	3.2917696	0.006	3.7769414
5,6	$\frac{\sqrt{2}}{32}$	4.4959659	2.9255876	4.1666454	0.025	4.8741035
		±0.8714721i	±0.5654338i	±0.7836432i		±0.8760355i
5,6	$\frac{\sqrt{2}}{64}$	4.4963441	2.9248145	4.1666973	0.012	4.8733986
		±0.8714728i	±0.5650876i	±0.7836699i		±0.8758772i
5,6	$\frac{\sqrt{2}}{128}$	4.4964963	2.9244878	4.1667103	0.006	4.8732345
		±0.8714802i	±0.5648487i	±0.7836780i		±0.8758363i

Table 2: The eigenvalues obtained by MZ element, $n = 16$.

j	h	$k^S_{j,h}$	$k^L_{j,h}$	$k^T_{j,h}$	j	h	$k^D_{j,h}$
1	$\frac{\sqrt{2}}{32}$	1.8795675	1.4775023	1.8184414	1	0.025	1.9883914
1	$\frac{\sqrt{2}}{64}$	1.8795717	1.4767526	1.8184573	1	0.012	1.9880919
1	$\frac{\sqrt{2}}{128}$	1.8795854	1.4764066	1.8184622	1	0.006	1.9880191
2	$\frac{\sqrt{2}}{32}$	2.4440863	1.5696996	2.2870296	2,3	0.025	2.6134315
2	$\frac{\sqrt{2}}{64}$	2.4441734	1.5697172	2.2870557	2,3	0.012	2.6130503
2	$\frac{\sqrt{2}}{128}$	2.4442186	1.5697237	2.2870651	2,3	0.006	2.6129596
3	$\frac{\sqrt{2}}{32}$	2.4442285	1.7053198	2.2870296	13,14	0.049	4.9056584
3	$\frac{\sqrt{2}}{64}$	2.4441893	1.7051917	2.2870557			±0.5787253i
3	$\frac{\sqrt{2}}{128}$	2.4442212	1.7051196	2.2870651	13,14	0.025	4.9018623
4	$\frac{\sqrt{2}}{32}$	2.8667518	1.7830953	2.8375736			±0.5781361i
4	$\frac{\sqrt{2}}{64}$	2.8664156	1.7831002	2.8376056	13,14	0.006	4.9009219
4	$\frac{\sqrt{2}}{128}$	2.8664256	1.7831114	2.8376222			±0.5781031i

Table 3: The eigenvalues obtained by Adini element on the unit square.

h	j	$k^S_{j,h}(n = 8 + x_1 - x_2)$	j	$k^S_{j,h}(n = 16)$
$\frac{\sqrt{2}}{32}$	1	2.8178682	1	1.8778418
$\frac{\sqrt{2}}{64}$	1	2.8211011	1	1.8791512
$\frac{\sqrt{2}}{128}$	1	2.8219168	1	1.8794810
$\frac{\sqrt{2}}{32}$	2	3.532859351	2,3	2.4413924
$\frac{\sqrt{2}}{64}$	2	3.537222143	2,3	2.4435179
$\frac{\sqrt{2}}{128}$	2	3.538327097	2,3	2.4440561
$\frac{\sqrt{2}}{32}$	5,6	4.4949831±0.8710067i	4	2.8588866
$\frac{\sqrt{2}}{64}$	5,6	4.4961529±0.8713583i	4	2.8645286
$\frac{\sqrt{2}}{128}$	5,6	4.4964517±0.8714506i	4	2.8659601

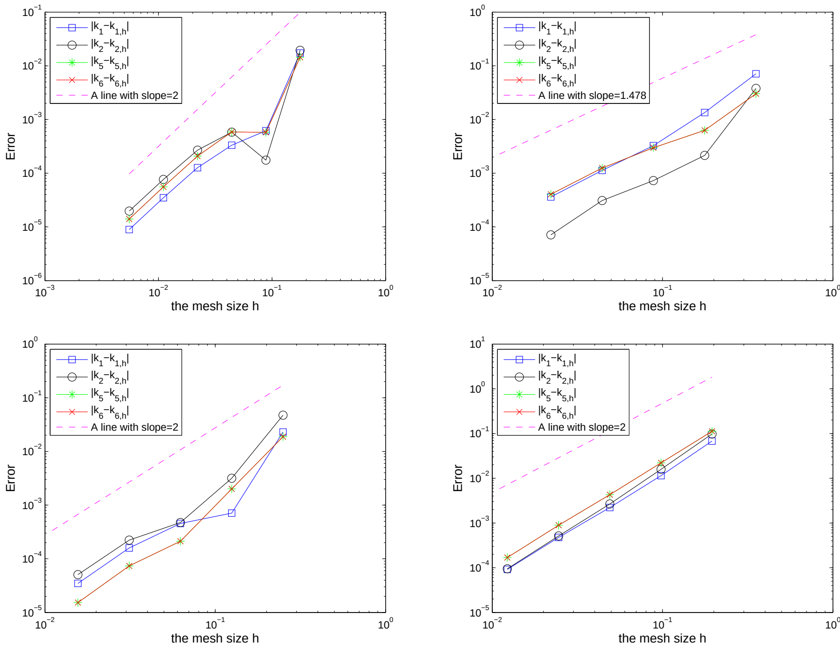

Figure 1: Error curves computed by MZ element with $n = 8 + x_1 - x_2$ on the unit square (left top), on the L-shaped (right top), on the triangle (left bottom), on the disk (right bottom).

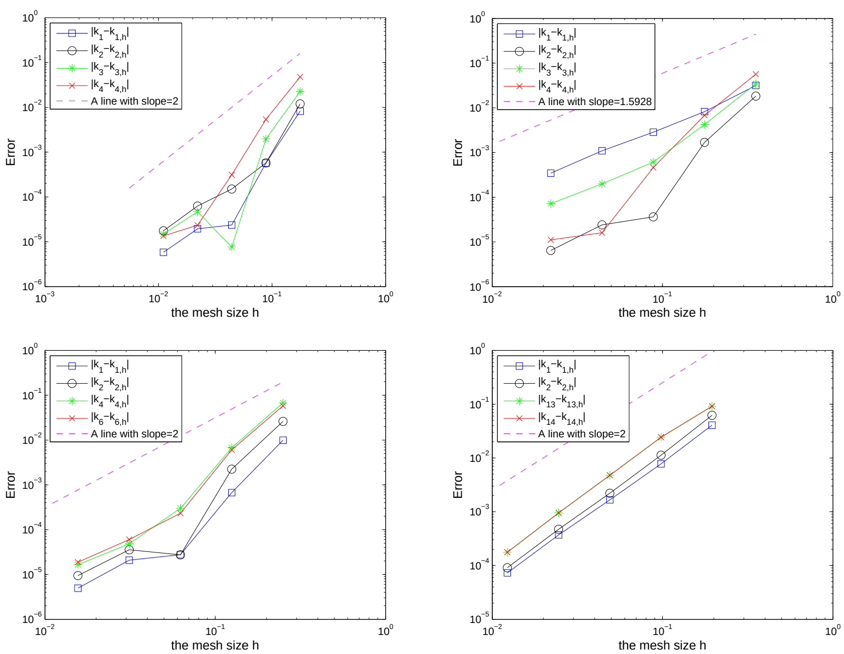

Figure 2: Error curves computed by MZ element with $n = 16$ on the unit square (left top), on the L-shaped (right top), on the triangle (left bottom), on the disk (right bottom).

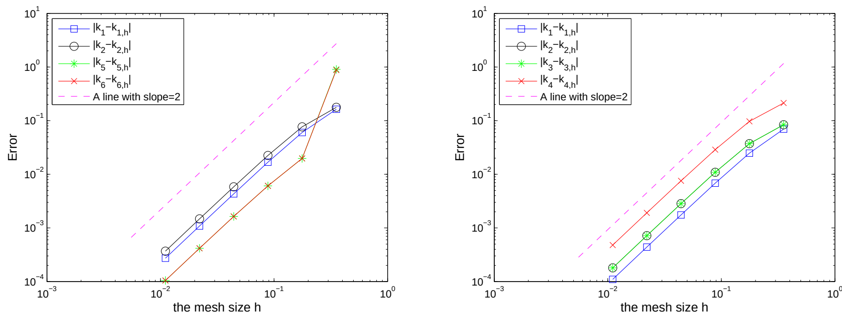

Figure 3: Error curves computed by Adini element on the unit square with $n = 8 + x_1 - x_2$ (left) and with $n = 16$ (right).

[5] H. Blum, R. Rannacher, *On the boundary value problem of the biharmonic operator on domains with angular corners.* Math. Method Appl. Sci., *2* (1980) 556–581.

[6] S.C. Brenner, L.R. Scott, *The Mathematical Theory of Finite Element Methods.* 2nd ed., Springer-Verlag, New york, 2002.

[7] F. Cakoni, M. Cayoren, D. Colton, *Transmission eigenvalues and the nondestructive testing of dielectrics.* Inverse Problems, *24* (2008) 065016.

[8] F. Cakoni, D. Gintides, H. Haddar, *The existence of an infinite discrete set of transmission eigenvalues.* SIAM J. Math. Anal., *42* (2010) 237–255.

[9] F. Cakoni, H. Haddar, *On the existence of transmission eigenvalues in an inhomogeneous medium.* Appl. Anal., *88* (2009) 475–493.

[10] F. Cakoni, H. Haddar, *The computation of lower bounds for the norm of the index of refraction in an anisotropic media from far field data.* J. Int. Equ. Appl., *21* (2009) 203–227.

[11] F. Cakoni, P. Monk, P. Sun, *Error analysis for the finite element approximation of transmission eigenvalues.* Comput. Meth. Appl. Math., *14* (2014) 419–427.

[12] L. Chen, *iFEM: an integrated finite element method package in MATLAB.* Technical Report, University of California at Irvine, 2009.

[13] P.G. Ciarlet, *Basic error estimates for elliptic proplems.* in: P.G. Ciarlet, J.L. Lions, (Ed.), Finite Element Methods (Part1), Handbook of Numerical Analysis, vol.2, Elsevier Science Publishers, North-Holand, 1991, pp.21–343.

[14] D. Colton, R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory.* 2nd ed., Vol. 93 in Applied Mathematical Sciences, Springer, New York, 1998.

[15] D. Colton, P. Monk, J. Sun, *Analytical and computational methods for transmission eigenvalues.* Inverse Problems, *26* (2010) 045011.

[16] X. Ji, J. Sun, T. Turner, *Algorithm 922: a mixed finite element method for Helmholtz transmission eigenvalues.* ACM Transaction on Math. Soft., *38* (2012) 29:1–8.

[17] X. Ji, J. Sun, H. Xie, *A multigrid method for Helmholtz transmission eigenvalue problems.* J. Sci. Comput., *60* (2014) 276–294.

[18] P. Lascaux and P. Lesaint, *Some nonconforming finite elements for the plate bending problem.* RAIRO Anal. Numer., *9* (1975) 9–53.

[19] P. Monk, J. Sun, *Finite element methods of Maxwell transmission eigenvalues.* SIAM J. Sci. Comput., *34* (2012) B247–264.

[20] J. T. Oden, J. N. Reddy, *An Introduction to the Mathematical Theory of Finite Elements.* Courier Dover Publications, New York, 2012.

[21] L. Päivärinta, J. Sylvester, *Transmission eigenvalues.* SIAM J. Math. Anal., *40* (2008) 738–753.

[22] B. P. Rynne, B.D. Sleeman, *The interior transmission problem and inverse scattering from inhomogeneous media.* SIAM J. Math. Anal., *22* (1991) 1755–1762.

[23] Z. Shi, M. Wang, *Finite Element Methods.* Beijing, Scientific Publishers, 2013.

[24] J. Sun, *Estimation of transmission eigenvalues and the index of refraction from Cauchy data.* Inverse Problems, *27* (2011) 015009.

[25] J. Sun, *Iterative methods for transmission eigenvalues.* SIAM J. Numer. Anal., *49* (2011) 1860–1874.

[26] J. Sun, L. Xu, *Computation of Maxwells transmission eigenvalues and its applications in inverse medium problems.* Inverse Problems, *29* (2013) 104013 (18pp)

[27] M. Wang, Z. Shi, J. Xu, *A new class of Zienkiewicz-type nonconforming element in any dimensions.* Numer. Math., *106* (2007) 335–347.

[28] Y. Yang, *A posteriori error estimates in Adini finite element for eigenvalue problems.* J. Comput. Math., *18* (2000) 413–418.

[29] Y. Yang, J. Han, H. Bi, *A new weak formulation and finite element approximation for transmission eigenvalues.* arXiv: 1506.06486v1 [math. NA] 22 Jun 2015.