Expansion shock solutions of the BBM and Boussinesq equations

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We identify a stationary expansion shock solution of a class of dispersive nonlinear wave equations that include the BBM equation $u_t + uu_x = u_{xxt}$ [1] and the Boussinesq equations of shallow water wave theory. The persistence of the expansive shock in initial value problems is analyzed and justified using matched asymptotic expansions and numerical simulations. We establish the algebraic decay of the shock as it is gradually eroded by a simple wave on either side. More generally, we observe a robustness of the expansion shock in the presence of weak dissipation and in simulations of asymmetric initial conditions where a train of solitary waves is shed from one side of the shock.

In this paper, we consider a class of nonlinear, dispersive equations most concisely exemplified by a version of the Benjamin-Bona-Mahoney (BBM) equation

$$u_t + uu_x = u_{xxt}. \tag{1}$$

The original BBM equation, which contains an additional u_x term, is an important model for the description of unidirectional propagation of weakly nonlinear, long waves in the presence of dispersion. It first appeared in a numerical study of shallow water undular bores [2] and later was proposed in [1] as an analytically advantageous alternative to the KdV equation

$$u_t + uu_x = -u_{xxx}. \tag{2}$$

In the context of shallow water waves, the BBM and KdV equations (1) and (2) are reduced, normalized versions of corresponding asymptotic models derived from the general Euler equations of fluid mechanics using small amplitude, long wave expansions. If $\delta \ll 1$ is the ratio of the undisturbed depth to a typical wave length and $\epsilon \ll 1$ is the ratio of a typical wave amplitude to the undisturbed depth, then the asymptotic KdV and BBM equations occur under the same balance $\epsilon \sim \delta^2$ [3] and so can be used interchangeably within their common domain of asymptotic validity.

Despite asymptotic equivalence, the mathematical properties of the BBM and KdV equations are very different, which is acutely captured by their normalized versions (1) and (2). The KdV equation (2) is known to be integrable via the inverse scattering transform and to possess an infinite number of conservation laws. The BBM equation (1), on the contrary, does not enjoy full integrability and has only three independent conservation laws. But as a mathematical model, it yields more satisfactory short-wave behavior, regularizing the unbounded growth of frequency, phase and group velocity values present in the KdV equation.

Equations (1) and (2) describe two different dispersive mechanisms to regularize the scalar conservation law, the inviscid Burger's equation

$$u_t + (\tfrac{1}{2}u^2)_x = 0. \tag{3}$$

Dispersive regularization of hyperbolic conservation laws is known to give rise to dispersive shock waves (DSWs), which are in many respects very different from their diffusive or diffusive-dispersive counterparts [4]. These DSWs have a distinct oscillatory structure and expand with time so that the Rankine-Hugoniot relations are not applicable to them. Instead, DSW closure is achieved via an appropriate solution of the Whitham modulation equations obtained by a nonlinear wave averaging procedure applied to the full dispersive equation [3, 5, 6]. Regularizing DSWs are evolutionary if they satisfy causality conditions [5] and thus represent dispersive counterparts of classical, Lax shocks [7]. All shock solutions of the KdV equation are evolutionary DSWs [4]. In contrast, we show in this paper that the BBM equation (1) admits a family of stationary (non-propagating), non-oscillatory *expansion shocks* that (i) satisfy the Rankine-Hugoniot jump condition, and (ii) violate causality. BBM expansion shocks are very different from both classical shocks of the inviscid Burger's equation (3) and KdV (2) DSWs.

Nonlinear partial differential equations of hyperbolic type, such as those modeling inviscid gas dynamics, e.g., (3), can have discontinuous solutions. These weak solutions may or may not be physical, depending on whether they are stable, or persist under small changes to initial conditions or the governing equations. Shock waves are physical, discontinuous solutions that typically satisfy side conditions associated with either a physical or mathematical notion of entropy. In gas dynamics, these conditions force shock waves to be compressive in that they compress the gas as they pass a fixed location. This kind of condition was expressed by Lax in the 1950s in terms of characteristics, requiring that shock waves are evolutionary, meaning that they are uniquely determined from initial conditions.

In this paper, we show that a non-evolutionary stationary shock wave of the BBM equation (1) persists but decays algebraically in time. This example is surprising because hyperbolic theory would suggest that the stationary shock would immediately give way to a continuous solution, namely a rarefaction wave. The persistence is explained through the interaction of the particular nature of dispersion in the BBM equation and a length scale associated with the stationary shock, that sets the time scale for decay. Expansion shocks are not unique to the BBM equation. We show that they also persist in one of the versions of the classical bi-directional Boussinesq equations for dispersive shallow-water waves [3, 8]. Similar to the BBM equation, these Boussinesq equations have the term u_{xxt} in the momentum equation. More broadly, we identify a large class of non-evolutionary partial differential equations — i.e., equations not explicitly resolvable with respect to the first time derivative [9] — that exhibit decaying expansion shock solutions, indicating the ubiquity of these new solutions.

I. SHOCKS AND RAREFACTIONS

If the dispersive right hand side of the BBM (1) or KdV (2) equation is deleted, we are left with the inviscid Burger's equation (3), a scalar conservation law that admits shock wave weak solutions

$$u(x,t) = \begin{cases} u_-, & x < st \\ u_+, & x > st, \end{cases} \tag{4}$$

provided the speed s is the average of the characteristic speeds $u_\pm$ on either side of the shock: $s = \frac{1}{2}(u_+ + u_-)$. Such shocks are stable provided that characteristics enter the shock from each side, $u_+ < u_-$, a condition known as the Lax entropy condition [7]. In this case, the shock is called an entropy shock, or by analogy with gas dynamics, a *compressive* shock.

By contrast, a shock wave (4) solution of (3) is called *expansive* if $u_- < u_+$. Expansive shocks are thought to be unstable and to violate causality, because characteristics leave rather than approach the shock. Instead of an expansive shock, a self-similar rarefaction wave resolves the discontinuity between u_- and u_+:

$$u(x,t) = \begin{cases} u_-, & x < u_- t \\ x/t, & u_- t < x < u_+ t \\ u_+, & x > u_+ t. \end{cases} \tag{5}$$

When $u_+ = -u_-$, the shock wave (4) is stationary, and hence is also a weak solution of the BBM equation (1). With $u_- > 0$, the stable case, the stationary shock persists. However, for $u_- < 0$, the unstable case, hyperbolic theory would suggest that the jump is immediately replaced by the self-similar rarefaction wave (5) or some approximation to it. However, we find that the BBM equation sustains solutions in which a stationary shock persists but decays algebraically, as shown in

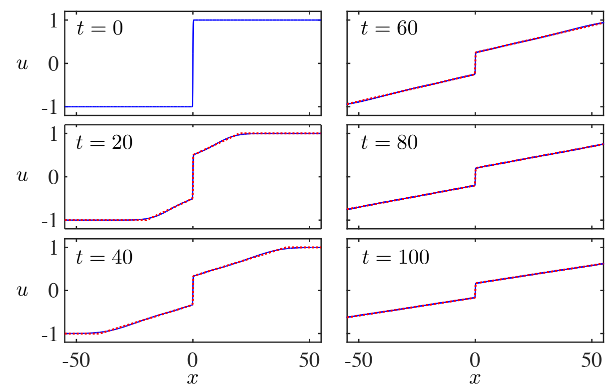

FIG. 1: Numerical (solid, blue) and asymptotic (dashed, red) solutions of the initial value problem for eq. (1) with initial data eq. (6) where $\varepsilon = 0.1$ and $A = 1$.

Fig. 1. More precisely, we study the initial value problem with initial data being a smoothed stationary shock, with width $\varepsilon > 0$.

II. THE EXPANSION SHOCK.

To see the effect of dispersion on a stationary shock, we pose initial data

$$u(x,0) = A \tanh \frac{x}{\varepsilon}, \quad -\infty < x < \infty, \tag{6}$$

with amplitude $A > 0$ for the BBM equation (1). Thus, as $\varepsilon \to 0$, the initial data converge to a jump from $u = -A$ to $u = A$, representing a stationary expansive shock solution to the inviscid Burger's equation (3). The numerical solution of (1),(6) is shown in Fig. 1. We observe the development of a rarefaction wave on either side of a stationary but decaying shock. We analyze the solution by matched asymptotics. First note that the initial function $u(x,0)$ is an odd function, and the solution should therefore be an odd function.

A. The inner solution

To capture the inner solution, we introduce the short space $\xi = x/\varepsilon$ and long time $T = \varepsilon t$ scalings of the independent variables x and t into eq. (1)

$$\varepsilon u_T + \frac{1}{\varepsilon} u u_\xi = \frac{1}{\varepsilon} u_{\xi\xi T}. \tag{7}$$

Expanding the dependent variable

$$u = u_0(\xi, T) + \varepsilon u_1(\xi, T) + \ldots, \tag{8}$$

and substituting this ansatz into (7) yields the leading order equation

$$u_0 \partial_\xi u_0 = \partial_{\xi\xi T} u_0.$$

This equation admits separated solutions $u_0(\xi, T) = f(\xi)a(T)$, leading to

$$ff_\xi a^2 = f_{\xi\xi} a_T,$$

so that

$$\frac{a_T}{a^2} = \frac{ff_\xi}{f_{\xi\xi}} = -K.$$

Thus,

$$a(T) = \frac{a_0}{1 + a_0 KT},$$

and

$$f(\xi) = \sqrt{c}\tanh\left(\frac{\sqrt{c}}{2K}(\xi - \xi_0)\right).$$

In these formulas, K, a_0, c and ξ_0 are arbitrary constants. To agree with the initial data (6), we set $c = 1, K = \frac{1}{2}, a_0 = A$, and $\xi_0 = 0$. Thus, the leading order inner solution is

$$u \sim u_{in}(\xi, T) = \frac{A}{1 + \frac{1}{2}AT}\tanh\xi. \qquad (9)$$

B. The outer solution

The outer solution has a different, long space and time scaling

$$X = \varepsilon x, \quad T = \varepsilon t.$$

This leads to the scaled equation

$$\varepsilon u_T + \varepsilon u u_X = \varepsilon^3 u_{XXT}. \qquad (10)$$

With the expansion $u(X,T) = \tilde{u}_0(X,T) + \varepsilon\tilde{u}_1(X,T) + \cdots$, we have the leading order conservation law

$$\partial_T\tilde{u}_0 + \tilde{u}_0\partial_X\tilde{u}_0 = 0.$$

We write the general, implicit solution by characteristics in the form

$$\tilde{u}_0(X,T) = f(T - \frac{X}{\tilde{u}_0}).$$

Matching to the inner solution, we have, for $x > 0$,

$$\lim_{X\to 0+}\tilde{u}_0(X,T) = f(T) = \lim_{\xi\to\infty} u_0(\xi, T) = \frac{A}{1 + \frac{1}{2}AT}.$$

The matching for $x < 0$ is similar, giving an odd function for the outer solution.

$$\tilde{u}_0 = \frac{A\,\mathrm{sgn}(X)}{1 + \frac{1}{2}A(T - \frac{X}{\tilde{u}_0})}.$$

Solving for $\tilde{u}_0$, we find the leading order outer solution

$$u \sim u_{out}(X,T) = \frac{A(\frac{1}{2}X + \mathrm{sgn}(X))}{1 + \frac{1}{2}AT}, \quad |X| < AT. \qquad (11)$$

Matching to the constant, far field conditions we obtain

$$u_{out}(X,T) = \mathrm{sgn}(X)A, \quad |X| \geq AT. \qquad (12)$$

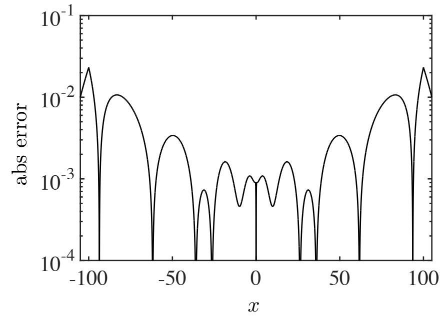

FIG. 2: Pointwise error between the uniform asymptotic expansion and numerical solution of Fig. 1 at $t = 100$ with $\varepsilon = 0.1$, and $A = 1$.

C. Uniformly valid asymptotic solution

Using the standard technique from asymptotics, we can formulate a composite solution that is asymptotically valid over the entire range of x. Based on the outer solution (11), (12), we define

$$F(x,t,\varepsilon) = \begin{cases} -A, & x < -At \\ \dfrac{A(\frac{1}{2}\varepsilon x + \mathrm{sgn}(\varepsilon x))}{1 + \frac{1}{2}A\varepsilon t}, & |x| < At \\ A, & x > At \end{cases}$$

Then the uniformly valid asymptotic solution is

$$u(x,t) = \frac{A}{1 + \frac{1}{2}A\varepsilon t}\left(\tanh\frac{x}{\varepsilon} - \mathrm{sgn}(x)\right) + F(x,t,\varepsilon). \qquad (13)$$

A comparison of the uniform asymptotic expansion to the numerical solution is shown in Fig. 1. The two solutions are hardly distinguishable. Figure 2 displays the absolute error. Note that the largest error occurs at the outermost edges of the rarefaction wave where the asymptotic solution has a weak discontinuity. The error in the inner solution is approximately $\varepsilon^3 = 10^{-3}$, which can be formally identified by going to higher order terms in eq. (7). In Fig. 3(b) we show characteristics calculated from the outer solution (11), (12) with $\varepsilon = 0.1$ and $A = 1$.

D. Boussinesq equations

The Boussinesq equations, formulated in the 1870s [8], can take a variety of asymptotically equivalent forms [3]. While having the same level of accuracy as the KdV and BBM equations in reproducing dispersive shallow water dynamics, the Boussinesq equations have the advantage of bi-directionality. The system considered here

$$\begin{aligned} h_t + (uh)_x &= 0 \\ u_t + uu_x + h_x - \tfrac{1}{3}u_{xxt} &= 0, \end{aligned} \qquad (14)$$

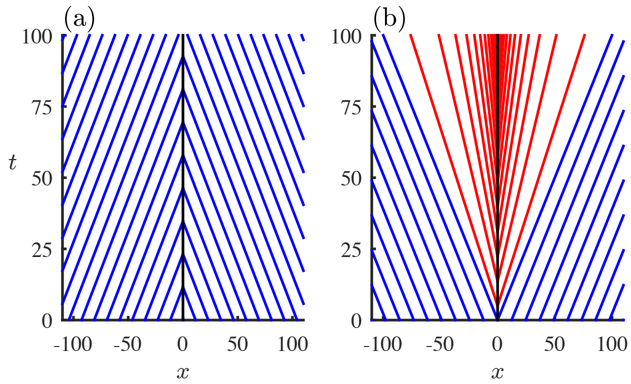

FIG. 3: Characteristics for: (a) the compressive shock solution of equation (3);
(b) the expansion shock solution of equation (1).

is a reduced, normalized version of the equation analyzed by Bona et al [10], which includes a u_x term in the dynamical equation for h. The nondimensional variables h, u represent the height of the water free surface above a flat horizontal bottom, and the depth-averaged horizontal component of the water velocity, respectively. A stationary shock solution of this system

$$h(x,t) = \begin{cases} h_-, & x<0 \\ h_+, & x>0, \end{cases} \quad u(x,t) = \begin{cases} u_-, & x<0 \\ u_+, & x>0, \end{cases} \tag{15}$$

will satisfy Rankine-Hugoniot (RH) jump conditions derived from the time-independent equations,

$$h_+u_+ = h_-u_-; \quad h_+ + \tfrac{1}{2}u_+^2 = h_- + \tfrac{1}{2}u_-^2. \tag{16}$$

The RH conditions (16) are attained for the two-parameter loci of states

$$u_\pm = h_\mp \left(\frac{2}{h_- + h_+}\right)^{1/2}, \tag{17}$$

with arbitrary, positive total water depths $h_\pm$. In Fig. 4, we show the result of a numerical simulation demonstrating the persistence of a stationary shock wave for the Boussinesq system (14).

The characteristic speeds for the dispersionless system ((14) with $u_{xxt} \to 0$) are $\lambda_\pm(h,u) = u \pm \sqrt{h}$. Therefore, if $u_\pm \geq 0$ as in the loci (17), then the characteristics with speed λ_+ pass through the stationary shock from left to right. However, the characteristics with speed λ_- leave the shock on both sides if $u_- < \sqrt{h_-}$ and $u_+ > \sqrt{h_+}$. This is the case for the choices of $h_\pm, u_\pm$ in Fig. 4. These choices also satisfy the Rankine-Hugoniot conditions for a stationary shock (16). To see that similar data with $u < 0$ can make a stationary shock expansive in the λ_+ characteristic family, note that the system (14) is unchanged under the transformation $x \to -x$, $u \to -u$.

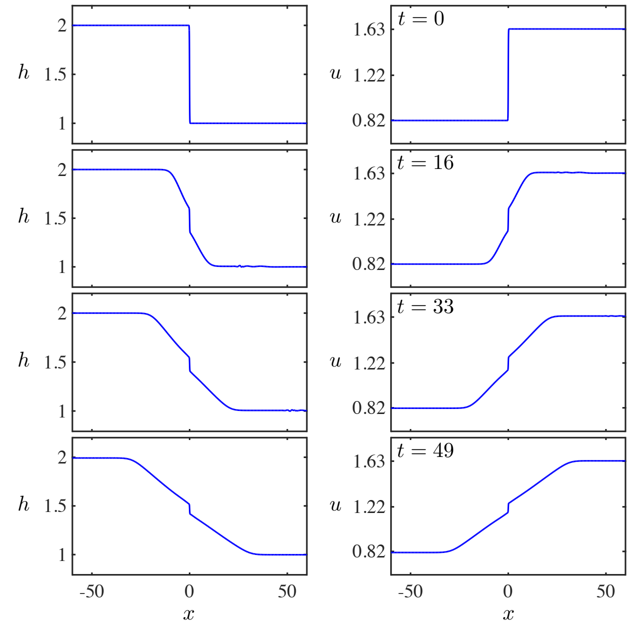

FIG. 4: Evolution of h (left) and u (right) in an expansion shock for the Boussinesq system with $h_- = 2$, $h_+ = 1$ and u_-, u_+ given by eq. (17). Jump initial data is smoothed by $\tanh(x/\varepsilon)$, as in eq. (6), with $\varepsilon = 0.1$.

E. Discussion

The expansion shock solutions we have discovered here do decay slowly in time, but their persistence in the face of the usual rules of causality is a surprise. For the BBM expansion shock, we can identify further robustness to perturbation by considering the asymmetric initial condition passing through zero

$$u(x,0) = \tfrac{1}{2}\left((u_+ - u_-)\tanh(\frac{x}{\varepsilon}) + u_+ + u_-\right),$$

where $u_- < 0 < u_+$. The numerical simulation of eq. (1) with this asymmetric data is shown in Fig. 5. As t increases, the solution quickly develops a stationary, expansion shock with initial amplitude $A = \min\{u_+, |u_-|\}$, that decays according to the inner solution (9). However, the solution also sheds a train of rank ordered solitons.

The expansion shock also persists in the presence of weak dissipation in the BBM-Burger's equation

$$u_t + uu_x = u_{xxt} + \nu u_{xx}, \tag{18}$$

where $\nu > 0$ is the dissipation coefficient. If we consider the initial data (6) for (18), then the inner solution exhibits exponential temporal decay

$$u_{in}(\xi, T) = \frac{Ae^{-\nu T/\varepsilon}}{1 + \frac{A\varepsilon}{2\nu}(1 - e^{-\nu T/\varepsilon})}\tanh\xi. \tag{19}$$

Thus, for $\nu = \mathcal{O}(\varepsilon)$, the expansion shock decays exponentially as $t \to \infty$, rather than the algebraic decay in

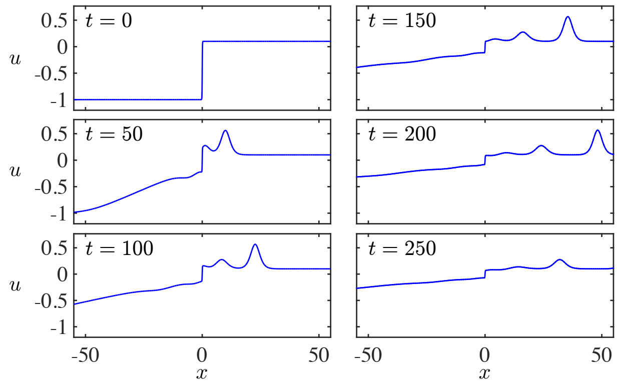

FIG. 5: Expansion shock and solitons as components of the initial value problem $u(x, 0) = 0.55\tanh(x/\varepsilon) - 0.45$ for eq. (1) where $\varepsilon = 0.1$.

the absence of diffusion. In fact, if $\nu \ll \varepsilon$, then (19) is asymptotically equivalent to (9).

The construction presented here can be generalized to higher order nonlinearity $f(u)$ and higher order, positive differential operators $\mathcal{L}$ in the form

$$\mathcal{L}[u_t] + f(u)_x = \nu u_{xx}, \tag{20}$$

so long as the non-evolutionary, dispersive character is maintained. For example, $\mathcal{L} = 1 - \partial_{xx}$ and $f(u) = u^4$ or $\mathcal{L} = 1 + \partial_{xxxx}$ and $f(u) = u^2$ admit expansion shock solutions that can be approximated with matched asymptotic methods.

F. Conclusions

We have identified decaying expansion shocks as robust solutions to conservation laws of non-evolutionary type, even in the presence of weak dissipation. These models include versions of the well-known BBM and Boussinesq equations, which are weakly nonlinear models for uni-directional and bi-directional long wave propagation, respectively. Expansion shocks represent a new class of purely dispersive and diffusive-dispersive shock waves. We stress, however, that the presented expansion shock solutions constitute a formal mathematical result inspired by, but not directly applicable to, shallow water wave theory because the BBM (1) and Boussinesq (14) equations as written are not asymptotically resolved with respect to the small dispersion parameter. The solutions obtained here are therefore outside the asymptotic validity of the equations as shallow water models. An important open question is whether expansion shock solutions can be physically realized in other systems exhibiting a similar dispersive regularization mechanism.

Appendix

The numerical methods utilized here for both the BBM (1) and Boussinesq (14) equations incorporate a standard fourth order Runge-Kutta timestepper (RK4) and a pseudospectral Fourier spatial discretization, similar to the method described in [4]. We briefly describe the method for BBM here.

We are interested in solutions $u(x, t)$ to (1) that rapidly decay to the far field boundary conditions $u(\pm\infty, t) = u_\pm$. The derivative $v = u_x$ therefore rapidly decays to zero and satisfies

$$(1 - \partial_{xx})v_t + (uv)_x = 0, \tag{21}$$

where $u(x, t) = \int_{-L}^{x} v(y, t)\, \mathrm{d}y + u_-$. The Fourier transform (written $\widehat{f}(k)$ with wavenumber k for a function $f(x)$) of eq. (21) can therefore be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{v} = -\frac{ik}{1 + k^2}\widehat{uv}. \tag{22}$$

The term $\widehat{uv}$ is well-defined because the function uv is rapidly decaying. Suitable truncation of the spatial and Fourier domains turn eq. (22) into a nonlinear system of ordinary differential equations, which we temporally evolve according to RK4. The computation of the nonlinear term in (22) is efficiently implemented using the fast Fourier transform (see [4] for further details). The numerical computations were performed on the domain $x \in [-L, L]$ with N Fourier modes and the timestep Δt. For BBM, $(L, N, \Delta t) = (200, 2^{15}, 0.01)$ (Figs. 1, 2), $(300, 2^{15}, 0.01)$ (Fig. 5). For Boussinesq, $(L, N, \Delta t) = (100, 2^{14}, 0.005)$.

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