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A Unified Analysis of Nonconforming Virtual Element Methods for Convection Diffusion Reaction Problem

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Abstract

We discuss nonconforming virtual element method for convection dominated (diffusive coefficient is very small compared to convective coefficient and reaction coefficient) convection-diffusion-reaction equation using L^2 projection operator.In this paper we stabilize the stabilization terms $\int_T(\vec{b}.\nabla u)(\vec{b}.\nabla v)$ using same technique which is used for stabilization of symmetric part in VEM, where T is an arbitrary element, and assume $H^2(T)$ regularity of $v_h|_T$ on each element to prove polynomial consistency where v_h is approximate solution.We have shown that linear nonconforming VE is not convergent for convection dominated convection-diffusion reaction problem and higher regularity of f,source term is also needed for convergence analysis.The novelty of this paper is we introduce a new SDFEM type nonconforming virtual element method for convection-dominated convection diffusion reaction equation, and discuss the computability issue using degrees of freedom of element without explicit knowledge of basis functions of virtual element methods.The present framework is stable in the limit of vanishing diffusion.

Keywords: Virtual element, SDFEM, L^2-projection

1. Introduction

In recent times the virtual element method has been successfully analysed and applied to a great variety of problems in 2D as well as 3D. Virtual element method has a noticeable similarity with mimetic finite difference method [4, 10, 9, 12]. The advantage of virtual element method is that we can generalize analysis framework for any order of desired accuracy and we can compute solution on each element T using only degree of freedoms without having explicit knowledge of basis functions. The test and trial functions contain polynomials of degree $\leq k$ with some additional continuous functions which are solution of model problem.

Conforming virtual element has been studied for elliptic equation in [2], linear elasticity problem in [8], biharmonic plate bending problem [5]. Recently nonconforming virtual element method has been analysed for diffusion problem by Ayuso et al [10]. The variational or weak formulation of all these above said model problem are symmetric hence stability analysis and polynomial consistency can be easily carried out using elliptic projector operator[1], for any order of accuracy k. But if the variational or weak formulation is non symmetric then the stability analysis does not follow easily and requires further analysis.

In this paragraph we discuss about the computational aspects of the methods to be proposed. We compute moments of test function using degrees of freedom of finite element. It has been shown in [2, 1] that elliptic projector operator is computable for polynomial of degree upto k where k is accuracy of the element, but L^2 projection operator is partially computable when degree of polynomial $\leq k-2$. But after little modification of virtual element space we can compute L^2-projection using elliptic projector operator which is explained in [1]. The main advantage of using L^2 projection is we can use this operator even for non-symmetric bilinear form like convection-diffusion-reaction equation, and it has been observed that polynomial consistency, stability analysis and convergence analysis can be easily carried out without any difficulty.

In this paper we consider the convection-diffusion-reaction problem

$$\begin{aligned} -\epsilon\,\Delta u + \vec{b}\cdot\nabla u + c\,u &= f \text{ in } \Omega \\ u &= u_b \text{ on } \partial\Omega \end{aligned} \tag{1}$$

under the assumptions that $\vec{b}, c, f$ are sufficiently smooth functions and ϵ is a small perturbation ($\epsilon \ll 1$). The above said problem is not stable in standard Galerkin method hence numerical solution produce non-physical oscillation at boundary layer, which contaminates the global computed numerical solution. There are several remedy to overcome this oscillation. Streamline-diffusion finite element is famous among them. Discrete bilinear form $a_h(u_h, v_h)$ is coercive i.e. $a_h(v_h, v_h) \geq C\|v_h\|_1^2$ in $H^1(\Omega)$ norm which is unable to capture large growth of $\|\vec{b}\cdot\nabla u\|_0$ at the boundary layer hence produce non physical oscillation. In streamline diffusion method we add additional term $\int_T |\vec{b} \cdot \nabla u|^2 \, dT$ and introduce new convective term dependent norm $|||.|||$ to capture this oscillation. Using the domain dependent norm a new nonconforming analysis of streamline diffusion method for convection dominated convection-diffusion equation has been studied by Tobiska et al [16, 17, 15].

In this paper we analyse non conforming virtual element method for convection-diffusion-reaction equation using external L^2 projection operator. To prove polynomial consistency we have assumed higher regularity $u_h|_T \in H^3(T)$ on each element T. In nonconforming VEM approximate solutions are discontinuous along interior edges except at Gauss-Lobatto points on the edges. In convergence analysis we see that two additional jump term which is known as consistency error [16, 3, 7], arise because of this nonconformity along interior edges. We shall use patch test [14, 13] which is a crucial property of nonconforming method to bound these jump terms. The layout of the paper is as follows. In section 2 we discuss continuous weak formulation and basic setting of the model problem (1). Since construction of non conforming virtual element and degrees of freedom are already defined in [10] we introduce those discussion with local and global settings of nonconforming VEM briefly in section 3. Discrete stability and polynomial consistency have been studied in section 4. Using mesh dependent norm [15] discrete coercivity has been discussed in section 5. Boundedness of discrete and continuous bilinear form have been studied in section 6. Consistency error estimation is discussed in section 6.2 and computational issue is discussed in section 6.3. In the section 7 we discuss convergence analysis and a priori error estimation. Finally some conclusions are drawn in section 8.

2. Continuous Problem

Let the domain Ω in $\mathbb{R}^d$ with $d = 2, 3$ be a bounded open polygonal domain with straight boundary edges for $d = 2$ and polyhedral domain with flat boundary faces for $d = 3$. We consider the model problem described in (1) with sufficient regularity of $\vec{b}, c, f$. The variational formulation of the model problem (1) reads as, find $u \in V$ such that

$$A(u, v) =< f, v > \quad \forall\, v \in V \tag{2}$$

where the bilinear form $A(\cdot, \cdot) : V \times V \to \mathbb{R}$ is given by

$$A(u,v) = \int_\Omega \epsilon \, \nabla u \cdot \nabla v \; dT + \int_\Omega \vec{b} \cdot \nabla u \; v + \int_\Omega c \, u \, v \quad \forall \; u, v \in V$$

and

$$< f, v > = \int_\Omega f v$$

$< \cdot, \cdot >$ denotes the duality product between the function space V' and V, where V is defined by $V := H_0^1(\Omega)$.

We consider the assumption

$$\left(c - \frac{1}{2} \nabla \cdot \vec{b} \right) \geq c_0 > 0 \tag{3}$$

We define the elemental contributions of the bilinear form $A(\cdot, \cdot)$ by

$$\begin{aligned} a(u,v) &:= \int_\Omega \epsilon \, \nabla u \cdot \nabla v \; d\Omega \\ b(u,v) &:= \int_\Omega (\vec{b} \cdot \nabla u) \, v \; d\Omega \\ c(u,v) &:= \int_\Omega c \, u \, v \, d\Omega \end{aligned}$$

$$\begin{aligned} A(v,v) &= \int_\Omega \epsilon \, |\nabla v|^2 + \int_\Omega (\vec{b} \cdot \nabla v) \, v + \int_\Omega c \, v^2 \\ &= \int_\Omega \epsilon \, |\nabla v|^2 + \int_\Omega \left(c - \frac{1}{2} \nabla \cdot \vec{b} \right) v^2 \\ &\geq \int_\Omega \epsilon \, |\nabla v|^2 + \int_\Omega c_0 \, v^2 \\ &\geq C \, \|v\|^2_{H^1(\Omega)} \end{aligned} \tag{4}$$

Inequality (4) implies that $A(\cdot, \cdot)$ is coercive in $H^1(\Omega)$ norm. Together with coercivity and boundedness of $A(\cdot, \cdot)$ Lax-milgram theorem ensures that there exists a unique solution to the variational form (2).

2.1. Basic Setting

Let $\{\tau_h\}_h$ be a family of triangulation of the domain Ω. Each triangulation τ_h consists of a finite number of elements T such that $\bar{\Omega} = \bigcup_{T\in\tau_h} T$ and any two different elements $T_1, T_2 \in \tau_h$ are either disjoint or possess either a common vertex or a common edge. We assume following on the family of partitions.

There exists a positive $\rho > 0$ such that

(Z1) for every element T and for every edge/face $e \subset \partial T$, we have $h_e \geq \rho h_T$.

(Z2) every element T is star-shaped with respect to all the points of a sphere of radius $\geq \rho h_T$.

(Z3) for $d = 3$, every face $e \in \varepsilon_h$ is star-shaped w.r.t. all the points of a disk having radius $\geq \rho h_e$.

ε_h^0 and ε_h^∂ denote the set of interior and boundary edges respectively. The maximum of the diameters of element $T \in \tau_h$ will be denoted by h,

$$\text{i.e.} \quad h = \max_{T\in\tau_h}\{h_T\}$$

We introduce the broken sobolev space for any integer $m > 0$

$$H^m(\tau_h) := \prod_{T\in\tau_h} H^m(T) = \{v \in L^2(\Omega) : v|_T \in H^m(T)\}$$

and define the broken H^m norm

$$\|v\|_{m,h}^2 = \sum_{T\in\tau_h} \|v\|_{m,T}^2 \quad \forall v \in H^m(\tau_h)$$

For $m = 1$ the broken H^1-seminorm

$$|v|_{1,h}^2 = \sum_{T\in\tau_h} \|\nabla v\|_{0,T}^2 \quad \forall v \in H^1(\tau_h)$$

Let $e \in \varepsilon_0^h$ be an interior edge and let T^+ and T^- be two triangle that shares the common edge e and let $n_e^\pm$ denote the unit normal on e in the outward direction w.r.t. $T^\pm$. We then define the jump operator as:

$$[|v|] := v^+ n_e^+ + v^- n_e^- \quad \text{on} \quad e \in \varepsilon_h^0$$

and

$$[|v|] := v n_e \quad \text{on} \quad e \in \varepsilon_h^\partial$$

Now we introduce the space that satisfies the continuity at Gauss-Lobatto points on the edges. For an integer $k \geq 1$, we define

$$H^{1,nc}(\tau_h; k) = \left\{ v \in H^1(\tau_h) : \int_e [|v|] \cdot n_e \, q \, ds = 0 \quad q \in \mathbb{P}^{k-1}(e), \forall e \in \varepsilon_h \right\}$$

$H^{1,nc}(\tau_h; k)$ satisfy patch-test of order k [14]. $H^{1,nc}(\tau_h, 1)$ is the space with minimal required order of patch test to ensure convergence analysis of diffusion dominated convection-diffusion-reaction equation. But if the problem is convection dominated then we need higher order of patch test atleast $k = 2$ which is explained in section 6.3.

3. Nonconforming Virual Element Method

We have already discussed the continuous setting of variational form of model problem (1) in section 2. We shall present the variational form in a different way which will help us to prove the convergence analysis without disturbing the weak solution and its corresponding weak formulation (2).

Applying integration by parts to the convective term $(\vec{b} \cdot \nabla u, v)$ we obtain

$$\int_\Omega (\vec{b} \cdot \nabla u) \, v = \frac{1}{2} \left[\int_\Omega (\vec{b} \cdot \nabla u) v - \int_\Omega (\vec{b} \cdot \nabla v) u + \int_\Omega \nabla \cdot \vec{b} \, u \, v \right] \quad \forall \, u, v \in H_0^1(\Omega)$$

Bilinear form can be written as

$$A(u, v) = \sum_{T \in \tau_h} A^T(u, v)$$

where $A^T(\cdot, \cdot)$ is restriction of $A(\cdot, \cdot)$ on each triangle $T \in \tau_h$.

$$A^T(u, v) = \int_T \epsilon \, \nabla u \cdot \nabla v + b^T(u, v) + \int_T c \; u \, v + b^{T,stab}(u, v)$$

where

$$b^T(u, v) = \frac{1}{2} \left[\int_T (\vec{b} \cdot \nabla u) v - \int_T (\vec{b} \cdot \nabla v) u - \int_T (\nabla \cdot \vec{b}) \, u \, v \right] \tag{5}$$

$b^T(u,v)$ can be split into two parts $b^{T,sym}(u,v)$ and $b^{T,skew}(u,v)$ where

$$b^{T,sym}(u,v) = \frac{1}{2}\int_T (\nabla \cdot \vec{b})uv$$

$$b^{T,skew}(u,v) = \frac{1}{2}\left[\int_T (\vec{b}\cdot\nabla u)v - \int_T (\vec{b}\cdot\nabla v)u\right] \tag{6}$$

and

$$b^{T,stab}(u,v) = \int_T (-\epsilon\,\Delta u + \vec{b}\cdot\nabla u + c\,u)\ \delta_T\ \vec{b}\cdot\nabla v$$

and right hand side function $F(\cdot)$ can be written as

$$F(v) = \int_\Omega fv + \sum_{T\in\tau_h}\int_T f\ \delta_T\,\vec{b}\cdot\nabla v$$

We introduce the nonconforming virtual finite element method for the model problem (1) which read as, find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) = F_h(v_h) \quad \forall\ v_h \in V_h^k \tag{7}$$

where $V_h^k \subset H^{1,nc}(\tau_h, k)$ is a global nonconforming virtual finite element space.

We will refer local virtual element space by $V_h^k(T)$. For $k \geq 1$ the finite dimensional space $V_h^k(T)$ associated to the polygon T is given by

$$V_h^k(T) := \left\{v \in H^1(T) : \frac{\partial v}{\partial \mathbf{n}} \in \mathbb{P}^{k-1}(e)\ \forall\ e \subset \partial T,\ \Delta v \in \mathbb{P}^{k-2}(T)\right\} \tag{8}$$

For each polygon T, the dimension of $V_h^k(T)$ is given by

$$nk + (k-1)k/2 \ \text{ where } \ T \subset \Omega \subset \mathbb{R}^2$$

$$nk(k+1)/2 + (k-1)k(k+1)/6 \ \text{ where } \ T \subset \Omega \subset \mathbb{R}^3$$

Now we shall introduce degrees of freedom

For $l \geq 0$, $M^l(e)$ and $M^l(T)$ respectively denote the set of scaled monomials on e and T

$$M^l(e) = \left\{\left(\frac{x - x_e}{h_e}\right)^s, |s| \leq l\right\}$$

and

$$M^l(T) = \left\{ \left(\frac{x - x_T}{h_T} \right)^s, |s| \le l \right\}$$

Using scaled monomial as basis function we define degrees of freedom

(L1) all the moments of v_h of order upto $k-1$ on each edge/face $e \subset \partial T$

$$\mu_e^{k-1}(v_h) = \left\{ \frac{1}{|e|} \int_e v_h \, m \, ds, \ \forall m \in M^{k-1}(e) \right\} \tag{9}$$

(L2) all the moments of v_h of order upto $(k-2)$ on T

$$\mu_T^{k-2}(v_h) = \left\{ \frac{1}{|T|} \int_T v_h \, m \, dT, \quad \forall \ m \in M^{k-2}(T) \right\} \tag{10}$$

The degrees of freedom(9)-(10) are unisolvent for $V_h^k(T)$, which is proved in detail in [10].

3.1. Global nonconforming virtual element space V_h^k

We have defined local nonconforming virtual element space for each element $T \in \tau_h$.The global nonconforming virtual element space V_h^k of order k is given by

$$n_{edg} \, k + n_{ele} \, (k-1) \, k/2 \ \text{ for } \ d = 2$$

$$n_{faces} \, k \, (k+1)/2 + n_{ele} \, (k-1) \, k \, (k+1)/6 \ \text{ for } \ d = 3$$

where n_{edg} and n_{faces} denote total no of edges $(d = 2)$ and faces$(d = 3)$ respectively. n_{ele} denotes total no of elements in τ_h

$$V_h^k = \{ v_h \in H^{1,nc}(\tau_h; k) : v_h|_T \in V_h^k(T) \ \forall \ T \in \tau_h \}$$

3.2. Interpolation operator

Assuming assumption (Z1)-(Z3), there exists a local polynomial approximation $u_\Pi \in \mathbf{P}^k(T)$, which satisfies the following approximation property

$$\|u - u_\Pi\|_{0,T} + h_T \, |u - u_\Pi|_{1,T} \le C \, h_T^s \, |u|_{s,T} \tag{11}$$

where $u \in H^s(T)$, $2 \le s \le k+1$, and C is a positive constant independent of h_T and depends on regularity constant ρ.

In standard finite element method literature we have seen interpolation operator depending on polynomial basis function rigorously. But in the case of virtual element theory we do not have explicit knowledge of the basis function. In this paragraph we define an interpolation operator on V_h^k having optimal approximation properties using degrees of freedom only without explicit knowledge of basis function. Since detail construction of interpolation operator has been shown in [10, 6], we state here only the result.

For every $v \in H^s(T)$, there exists an unique interpolant $v_I \in V_h^k$ satisfies

$$\|v - v_I\|_{0,T} + h_T \, \|v - v_I\|_{1,T} \leq C \, h_T^s \, \|v\|_{s,T} \tag{12}$$

where $2 \leq s \leq k+1$ and $T \in \tau_h$ be an arbitrary element.

3.3. Construction of A_h

In this section we explicitly describe the discrete bilinear form $A_h(u_h, v_h)$ and right hand side function F_h. As we mentioned earlier if the model problem (1) is convection dominated then we need to add additional diffusion in streamline direction. In the virtual element formulation we rewrite the discrete bilinear form into two parts one is polynomial part constructed by various type of projection operators like elliptic projection operator $\Pi_k^\nabla : V_h^k \to \mathbb{P}^k(T)$ or L^2 projection $\Pi_k : V_h^k \to \mathbb{P}^k(T)$ and another is stabilization part which is responsible to stabilize the bilinear form. We have added additional term $\int_T (\vec{b} \cdot \nabla u)(\vec{b} \cdot \nabla v) \, dT$ to the bilinear form $A_h^T(u, v)$ to reduce the oscillation. In virtual element formulation we will reveal the stabilization part into two parts one is polynomial part and another is stabilization part, same as what we do for diffusion or reaction part. The present framework is stable for very small value of ϵ. Convergence analysis independent of ϵ will be shown in section 7.

Let us write the discrete bilinear form $A_h(u_h, v_h)$ as

$$A_h(u_h, v_h) = \sum_{T \in \tau_h} A_h^T(u_h, v_h) \; \forall \; u_h, v_h \in V_h^k \tag{13}$$

$A_h^T : V_h^k \times V_h^k \to \mathbb{R}$ denoting the restriction of $A_h(u_h, v_h)$ to the local space $V_h^k(T)$.

The bilinear form A_h^T can be decomposed as

$$\begin{aligned} A_h^T(u_h, v_h) \;\; &:= \;\; a_h^T(u_h, v_h) + b_h^T(u_h, v_h) \\ &+ \;\; c_h^T(u_h, v_h) + b_h^{T,stab}(u_h, v_h) \end{aligned} \tag{14}$$

where

$$a_h^T(u_h, v_h) := \int_T \epsilon\, \Pi_{k-1}(\nabla u_h) \cdot \Pi_{k-1}(\nabla v_h) + s_a^T((I - \Pi_k)u_h, (I - \Pi_k)v_h) \quad (15)$$

$$b_h^T(u_h, v_h) := -b_h^{T,sym}(u_h, v_h) + b_h^{T,skew}(u_h, v_h)$$

symmetric part is defined as

$$b_h^{T,sym}(u_h, v_h) := \frac{1}{2} \int_T (\nabla \cdot \vec{b})\, \Pi_k(u_h)\, \Pi_k(v_h)\, dT + s^{T,sym}((I - \Pi_k)u_h, (I - \Pi_k)v_h)\, dT$$

skew-symmetric part of convection term is defined as

$$b_h^{T,skew}(u_h, v_h) := \frac{1}{2} \left[\int_T \vec{b} \cdot \Pi_{k-1}(\nabla u_h)\, \Pi_k(v_h)\, dT - \int_T \vec{b} \cdot \Pi_{k-1}(\nabla v_h) \Pi_k(u_h)\, dT \right] \quad (16)$$

$$c_h^T(u_h, v_h) := \int_T c\, \Pi_k(u_h)\, \Pi_k(v_h)\, dT + s_c^T((I - \Pi_k)u_h, (I - \Pi_k)v_h)$$

stabilization term

$$\begin{aligned} b_h^{T,stab}(u_h, v_h) \quad &:= \quad \int_T (-\epsilon\, \Pi_{k-2}(\Delta u) + c\, \Pi_k(u))\ \delta_T(\vec{b} \cdot \Pi_{k-1}(\nabla v))\ dT \\ &+ \quad \int_T (\vec{b} \cdot \Pi_{k-1}(\nabla u))(\vec{b} \cdot \Pi_{k-1}(\nabla v))\ dT \\ &+ \quad s_b^{T,stab}((I - \Pi_k)u_h, (I - \Pi_k)v_h) \end{aligned} \quad (17)$$

$s_a^T, s_c^T, s_b^{T,stab}, s^{T,sym}$ are the stabilization terms. These terms are symmetric and vanish on the polynomial space $\mathbb{P}_k(T)$.

4. Discrete stability

Our model problem (1) contains three parts diffusion, reaction, convection. Weak formulation of diffusion and reaction parts are symmetric hence stability analysis of these two parts follows same as discussed in the literature of VEM [2, 5, 10], convection part is not symmetric which does not follow same stability analysis as diffusion and reaction parts. We need some extra effort to stabilize convection part. In discrete formulation $A_h^T(u_h, v_h)$, we add additional term $\int_T (\vec{b}\cdot\nabla u_h)\,(\vec{b}\cdot\nabla v_h)$ to capture non physical oscillation produced by convection term $\vec{b}\cdot\nabla u$ in convection dominated region ($\epsilon << 1$). Fortunately $\int_T (\vec{b}\cdot\nabla u_h)\,(\vec{b}\cdot\nabla v_h)$ is symmetric and we can stabilize the term using same technique as diffusion and reaction parts, which prevent drastic change of $\vec{b}.\nabla u_h$ in convection dominated region. Finally we can conclude that there exist positive constants $\alpha_*, \alpha^*, \gamma_*, \gamma^*, s_*, s^*, \Gamma_*, \Gamma^*$ such that

$$\alpha_*\, a^T(v_h, v_h) \le a_h^T(v_h, v_h) \le \alpha^*\, a^T(v_h, v_h) \tag{18}$$

$$\gamma_*\, c^T(v_h, v_h) \le c_h^T(v_h, v_h) \le \gamma^*\, c^T(v_h, v_h) \tag{19}$$

$$\Gamma_*\, G^T(v_h, v_h) \le G_h^T(v_h, v_h) \le \Gamma^*\, G^T(v_h, v_h) \tag{20}$$

$$s_*\, b^{T,sym}(v_h, v_h) \le b_h^{T,sym}(v_h, v_h) \le s^* b^{T,sym}(v_h, v_h) \tag{21}$$

for all $v_h \in V_h^k(T)$ where T is an arbitrary element and $G^T(\cdot,\cdot), G_h^T(\cdot,\cdot)$ are defined by

$$\begin{aligned} G^T(u_h, v_h) &:= \int_T \delta_T\,(\vec{b}\cdot\nabla u_h)\,(\vec{b}\cdot\nabla v_h) \\ G_h^T(u_h, v_h) &:= \int_T \delta_T\,\vec{b}\cdot\Pi_{k-1}(\nabla u_h)\;\vec{b}\cdot\Pi_{k-1}(\nabla v_h) \\ &\quad + \; s_b^{T,stab}((I-\Pi_k)u_h, (I-\Pi_k)v_h) \end{aligned} \tag{22}$$

4.1. Polynomial consistency

Lemma 4.1. *Let $u_h|_T \in \mathbb{P}^k(T)$ and $v_h|_T \in H^2(T)$. Then the bilinear form $A_h^T(u_h, v_h)$ defined in (14) satisfies polynomial consistency property, i.e. $A_h^T(u_h, v_h) = A^T(u_h, v_h)$ for all $h > 0$ and for all $T \in \tau_h$.*

Proof. If u_h or v_h, or both are polynomial of degrees k, then the stabilization terms $s_a^T, s_c^T, s_b^{T,stab}$ vanish.Now we will prove the following

$$\begin{aligned} a_h^T(u_h, v_h) &= a^T(u_h, v_h) \\ b_h^{T,skew}(u_h, v_h) &= b^{T,skew}(u_h, v_h) \\ b_h^{T,sym}(u_h, v_h) &= b^{T,sym}(u_h, v_h) \\ c_h^T(u_h, v_h) &= c^T(u_h, v_h) \\ b_h^{T,stab}(u_h, v_h) &= b^{T,stab}(u_h, v_h) \end{aligned}$$

L^2 projection Π_k is invariant on polynomial space $\mathbb{P}^k(T)$, i.e., $\Pi_k(p) = p$, where $p \in \mathbb{P}^k$. Using orthogonality property of L^2 projection we can estimate required polynomial consistency property

$$\begin{aligned}
a_h^T(p, v_h) &= \int_T \epsilon \, \nabla p \cdot \Pi_{k-1}(\nabla v_h) \, dT \\
&= \int_T (\Pi_{k-1}(\nabla v_h) - \nabla v_h) \, \epsilon \, \nabla p \, dT \\
&+ \int_T \epsilon \, \nabla p \cdot \nabla v_h \, dT \\
&= \int_T \epsilon \, \nabla p \, \nabla v_h \, dT \\
&= a^T(p, v_h)
\end{aligned} \tag{23}$$

$b_h^T(p, v_h)$ contains three parts. Since ∇p is a polynomial of degrees $k - 1$ less than $\deg(\Pi_{k-1}(v_h))$, using orthogonality property of L^2 operator we estimate polynomial consistency of first term

$$b_h^{T,skew}(p, v_h) = \frac{1}{2} \left[\int_T (\vec{b} \cdot \nabla p) \, \Pi_k(v_h) \, dT - \int_T \vec{b} \cdot \Pi_{k-1}(\nabla v_h) \, p \, dT \right] \tag{24}$$

$$\begin{aligned}
\int_T \vec{b} \cdot \nabla p \; \Pi_k(v_h) \, dT &= \int_T \vec{b} \cdot \nabla p \, (\Pi_k(v_h) - v_h) \, dT \\
&+ \int_T \vec{b} \cdot \nabla p \, v_h dT \\
&= \int_T \vec{b} \cdot \nabla p \, v_h \; dT
\end{aligned} \tag{25}$$

$\Pi_{k-1}(\nabla v_h)$ is a polynomial of degree $k - 1$, hence we cannot apply orthogonality property of L^2 projection. We use polynomial approximation property of ∇v_h and $H^2(T)$ regularity of v_h to establish the following result,

$$
\begin{aligned}
\int_T \vec{b}\,\Pi_{k-1}(\nabla v_h)\,p &= \int (\Pi_{k-1}(\nabla v_h) - \nabla v_h)\,\vec{b}\,p \\
&+ \int_T \vec{b}\cdot\nabla v_h\,p \\
&\leq \|\vec{b}\|_{\infty,T}\,\|p\|_{0,T}\,\|\nabla v_h - \Pi_{k-1}(\nabla v_h)\| + \int_T \vec{b}\,\nabla v_h\,p \\
&\leq C\|\vec{b}\|_{\infty,T}\,h_T\,|\nabla v_h|_{1,T}\|p\|_{0,T} \\
&+ \int_T \vec{b}\cdot\nabla v_h\,p \\
&\approx \int_T \vec{b}\cdot\nabla v_h\,p
\end{aligned} \tag{26}
$$

for very small values of h_T.

Again using orthogonality property of L^2 projection we establish

$$
\begin{aligned}
b_h^{T,sym}(p, v_h) &= \frac{1}{2}\int_T (\nabla\cdot\vec{b})\,p\,\Pi_k(v_h)\;dT \\
&= \frac{1}{2}\int_T (\nabla\cdot\vec{b})(\Pi_k(v_h) - v_h)p + \frac{1}{2}\int_T (\nabla\cdot\vec{b})\,p\,v_h \\
&= \frac{1}{2}\int_T (\nabla\cdot\vec{b})\,p\,v_h
\end{aligned} \tag{27}
$$

and

$$
c_h^T(p, v_h) = c^T(p, v_h) \tag{28}
$$

Now we will discuss polynomial consistency of additional terms. Δp is a polynomial of degree $k-2$ less than deg $(\Pi_{k-1}(\nabla v))$, hence we use orthogonality property of L^2 function to establish the following results.

$$
\begin{aligned}
b_h^{T,stab}(u_h, v_h) &= \int_T (-\epsilon\,\Delta p + c\,p)\,\delta_T\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h))\,dT \\
&+ \int_T \delta_T\,(\vec{b}\cdot\nabla p)\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h))\,dT
\end{aligned} \tag{29}
$$

$$\begin{aligned}
\int_T -\epsilon\,\Delta p\,\delta_T(\vec{b}\cdot\Pi_{k-1}(\nabla v_h))\,dT &= \delta_T\int_T -\epsilon\,\Delta p\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h))\,dT \\
&= \delta_T\int_T -\epsilon\,\Delta p\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h)-\vec{b}\cdot\nabla v_h)\,dT \\
&+ \delta_T\int_T -\epsilon\,\Delta p\,(\vec{b}\cdot\nabla v_h)\,dT \\
&= \delta_T\int_T -\epsilon\,\Delta p\,(\vec{b}\cdot\nabla v_h)\,dT \qquad (30)
\end{aligned}$$

Using same technique as (26) we establish the following result

$$\begin{aligned}
\int_T c\,p\,\delta_T\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h)) &= \delta_T\int_T c\,p\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h)-\vec{b}\cdot\nabla v_h) \\
&+ \delta_T\int_T c\,p\,\vec{b}\cdot\nabla v_h \\
&\leq \delta_T\,c_{\max}\,C\,\|p\|_{0,T}\,\|\vec{b}\|_{0,\infty}\,h_T\,|\nabla v_h|_{1,T} \\
&+ \delta_T\int_T c\,p\,\vec{b}\cdot\nabla v_h \\
&\approx \delta_T\int_T c\,p\,\vec{b}\cdot\nabla v_h \qquad (31)
\end{aligned}$$

for very small values of h_T.

Again using orthogonality property of L^2 function we derive the following result.

$$\begin{aligned}
\int_T(\vec{b}\cdot\nabla p)\,\delta_T\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h))\,dT &= \delta_T\int_T(\vec{b}\cdot\nabla p)(\vec{b}\cdot\Pi_{k-1}(\nabla v_h)-\vec{b}\cdot\nabla v_h)\,dT \\
&+ \delta_T\int_T(\vec{b}\cdot\nabla p)(\vec{b}\cdot\nabla v_h)\,dT \\
&= \delta_T\int_T(\vec{b}\cdot\nabla p)(\vec{b}\cdot\nabla v_h)\,dT \qquad (32)
\end{aligned}$$

□

4.2. Remark:

In estimation (26) we have approximated the term $\int_T \vec{b}\, \Pi_{k-1}(\nabla v_h)\, p$ by $\int_T \vec{b}\cdot\nabla v_h\, p$. Hence corresponding error is $\left| \int_T \vec{b}\, \Pi_{k-1}(\nabla v_h)\, p - \int_T \vec{b}\cdot\nabla v_h\, p \right| \leq C\, \|\vec{b}\|_{\infty,T}\, h_T\, |\nabla v_h|_{1,T}\ \ \|p\|_{0,T}$. Since we have assumed that $v_h|_T \in H^2(T)$, $|\nabla v_h|_{1,T}$ and other terms are well defined. Therefore making mesh diameter h_T sufficiently small we can reduce the error less than any positive quantity ϵ_1 which is different from the diffusion coefficient ϵ. Hence for sufficiently small size of diameter of mesh element T the error is negligible. In estimation (31) we have followed same methodology.

5. Discrete coercivity

Now we discuss the coerciveness of $A_h(u_h, v_h)$ on V_h^k. We assume the assumptions (Z1)-(Z3) and (3) hold then there exist constants μ_1 and μ_2 independent of $T \in \tau_h$ such that the following local inverse inequality hold

$$\|\Delta v_h\|_{0,T} \leq \mu_1\, h_T^{-1}\, |v_h|_{1,T} \quad \forall\ v_h \in V_h^k, T \in \tau_h \tag{33}$$

$$|v_h|_{1,T} \leq \mu_2\, h_T^{-1}\, \|v_h\|_{0,T} \quad \forall v_h \in V_h^k, T \in \tau_h \tag{34}$$

Let us introduce domain dependent norm

$$|||v_h||| := \sum_T \left\{ \epsilon\, |v_h|_{1,T}^2 + c_0\, \|v_h\|_{0,T}^2 + \delta_T\, \|\vec{b}\cdot\nabla v_h\|_{0,T}^2 \right\} \tag{35}$$

We assume the following assumption on control parameter δ_T

$$0 < \delta_T \leq \min\left\{ \frac{c_0\, \min\{s^*, \gamma_*\}}{4\, c_{\max}^2}, \frac{h_T^2\, \alpha_*}{2\, \epsilon\, \mu_1^2}, \min\left\{1, \frac{1}{c_I}\right\} \frac{c_0 \min\{s^*, \gamma_*\}\, h_T^2}{4\, \|\vec{b}\|_{0,\infty}^2\, \mu_2^2} \right\} \tag{36}$$

Lemma 5.1. *Let the virtual element space V_h^k satisfies the assumptions $(Z1), (Z2), (Z3)\&(3)$. Let the virtual element space satisfies the condition (12). Then the discrete bilinear form $A_h(u_h, v_h)$ is coercive on V_h^k, i.e.*

$$A_h(v_h, v_h) \geq \alpha |||v_h|||^2$$

where α is a positive constant.

Proof. Let us consider the bilinear form $A_h(\cdot,\cdot)$.

$$A_h(u_h, v_h) = \sum_T A_h^T(u_h, v_h)$$

$$\begin{aligned} A_h^T(v_h, v_h) &= a_h^T(v_h, v_h) + b_h^T(v_h, v_h) + c_h^T \\ &+ \int_T [-\epsilon\, \Pi_{k-2}(\Delta v_h) + c\, \Pi_k(v_h)]\, \delta_T\, (\vec{b} \cdot \Pi_{k-1}(\nabla v_h)) \\ &+ \int_T (\vec{b} \cdot \Pi_{k-1}(\nabla v_h))\, \delta_T\, (\vec{b} \cdot \Pi_{k-1}(\nabla v_h)) \\ &+ s_h^{T,stab}((I - \Pi_k)v_h, (I - \Pi_k)v_h) \end{aligned} \tag{37}$$

Using stability property of $a_h^T(v_h, v_h), G_h^T(v_h, v_h),\ c_h^T(v_h, v_h)$ and $b_h^{T,sym}$ and considering the assumptions mentioned in the lemma we can write

$$\begin{aligned} A_h^T(v_h, v_h) &\geq \alpha_*\, a^T(v_h, v_h) + \min\{s^*, \gamma_*\}\, c_0\, \|v_h\|_{0,T}^2 + \Gamma_*\, \delta_T\, \|\vec{b} \cdot \nabla v_h\|_{0,T}^2 \\ &+ \int_T [-\epsilon\, \Pi_{k-2}(\Delta v_h) + c\, \Pi_k(v_h)]\, \delta_T(\vec{b} \cdot \Pi_{k-1}(\nabla v_h)) \end{aligned} \tag{38}$$

Using boundedness property of projection operator and satisfying the condition (36) we estimate the additional term.

$$\begin{aligned} \left| \int_T -\epsilon\, \Pi_{k-2}(\Delta v_h)\, \delta_T\, (\vec{b} \cdot \Pi_{k-1}(\nabla v_h)) \right| &\leq \delta_T\, \epsilon\, \|\Delta v_h\|_{0,T}\ \|\vec{b} \cdot \nabla v_h\|_{0,T} \\ &\leq \delta_T \big[\frac{\epsilon^2}{2} \|\Delta v_h\|_{0,T}^2 \\ &+ \frac{1}{2} \|\vec{b}.\nabla v_h\|_{0,T}^2 \big] \end{aligned} \tag{39}$$

using inverse inequality property of VE space (33) we get,

$$\begin{aligned} \frac{\delta_T}{2}\, \epsilon^2\, \|\Delta v_h\|_{0,T}^2 &\leq \frac{\delta_T}{2}\, \epsilon^2\, \mu_1^2\, h_T^{-2} |v_h|_{1,T}^2 \\ &\leq \frac{\epsilon}{4}\, \alpha_*\, |v_h|_{1,T}^2 \end{aligned} \tag{40}$$

using polynomial approximation property (11) of virtual element space and assuming control parameter δ_T satisfying the condition (36) we can estimate

$$
\begin{aligned}
\int_T c\,\Pi_k(v_h)\,\delta_T\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h)) &= \int_T c\,v_h\,\delta_T\,(\vec{b}\cdot\Pi_{k-1}(\nabla v_h)) \\
&= \int_T c\,v_h\,\delta_T\,\vec{b}\cdot[\Pi_{k-1}(\nabla v_h)-(\nabla v_h)] \\
&+ \int_T c\,v_h\,\delta_T\,(\vec{b}\cdot\nabla v_h) \qquad (41)
\end{aligned}
$$

Since c is smooth enough we can bound it by $c_{\max}$

$$
\begin{aligned}
\left|\int_T c\,v_h\,\delta_T\,(\vec{b}\cdot\nabla v_h)\right| &\leq \delta_T\,\|c\,v_h\|_{0,T}\,\|\vec{b}\cdot\nabla v_h\|_{0,T} \\
&\leq \delta_T\left[\frac{1}{2}\|c\,v_h\|_{0,T}^2+\frac{1}{2}\|\vec{b}\cdot\nabla v_h\|_{0,T}^2\right] \\
&= \frac{\delta_T}{2}\|c\,v_h\|_{0,T}^2+\frac{\delta_T}{2}\|\vec{b}\cdot\nabla v_h\|_{0,T}^2 \qquad (42)
\end{aligned}
$$

If the control parameter δ_T satisfies the condition (36) we can estimate

$$
\begin{aligned}
\frac{\delta_T}{2}\|c\,v_h\|_{0,T}^2 &\leq \frac{\delta_T}{2}c_{\max}^2\|v_h\|^2 \\
&\leq \frac{c_0}{8}\min\{s^*,\gamma_*\}\|v_h\|_{0,T}^2 \qquad (43)
\end{aligned}
$$

We consider vector valued convection coefficient $\vec{b}$ is regular enough. Taking it outside of integration and bounding it by L_∞ norm $|\vec{b}|_{0,\infty}$ and using (34), we can write

$$
\begin{aligned}
\frac{1}{2}\,\delta_T\,\|\vec{b}\cdot\nabla v_h\|_{0,T}^2 &\leq \frac{1}{2}\,\delta_T\,|\vec{b}|_{0,\infty}^2\|\nabla v_h\|_{0,T}^2 \\
&\leq \frac{1}{2}\,\delta_T\,|\vec{b}|_{0,\infty}^2\,\mu_2^2\,h_T^{-2}\,\|v_h\|_{0,T}^2 \\
&\leq \frac{c_0}{8}\,\min\{s^*,\gamma_*\}\|v_h\|_{0,T}^2 \qquad (44)
\end{aligned}
$$

First term of (41) can be estimated as

$$
\begin{aligned}
\left|\int_T cv_h\,\delta_T\,\vec{b}\cdot(\Pi_{k-1}(\nabla v_h)-(\nabla v_h))\right| &\leq \delta_T\,\|c\,v_h\|_{0,T}\|\vec{b}\cdot(\Pi_{k-1}(\nabla v_h)-\nabla v_h)\|_{0,T} \\
&\leq \delta_T\,[\frac{1}{2}\|c\,v_h\|_{0,T}^2 \\
&+ \frac{1}{2}\|\vec{b}\cdot(\Pi_{k-1}(\nabla v_h)-\nabla v_h)\|_{0,T}^2] \qquad (45)
\end{aligned}
$$

$$\delta_T \, \frac{1}{2} \, \|c \, v_h\|_{0,T}^2 \quad \leq \quad \frac{c_0}{8} \min\{s^*, \gamma_*\} \|v_h\|_{0.T}^2 \tag{46}$$

Using polynomial approximation property (11) and local inverse inequality (34) of VE space we can write

$$\begin{aligned}
\delta_T \, \frac{1}{2} \|\vec{b} \cdot (\Pi_{k-1}(\nabla v_h) - \nabla v_h)\|_{0,T}^2 \quad &\leq \quad \frac{1}{2} \, \delta_T \, \|\vec{b}\|_{0,\infty}^2 \, c_I \, \|\nabla v_h\|_{0,T}^2 \\
&\leq \quad \frac{1}{2} \, \delta_T \, \|\vec{b}\|_{0,\infty}^2 \, c_I \, \mu_2^2 \, h_T^{-2} \|v_h\|_{0,T}^2 \\
&\leq \quad \frac{1}{8} \, c_0 \min\{s^*, \gamma_*\} \|v_h\|_{0,T}^2
\end{aligned} \tag{47}$$

Finally we estimate local bilinear form $A_h^T(v_h, v_h)$

$$\begin{aligned}
A_h^T(v_h, v_h) \quad &\geq \quad \frac{3}{4} \, \alpha_* \, \epsilon \, |v_h|_{1,T}^2 \\
&+ \quad \frac{3}{8} \min\{s^*, \gamma_*\} \, c_0 \, \|v_h\|_{0,T}^2 \\
&+ \quad \Gamma_* \, \delta_T \, \|\vec{b} \cdot \nabla v_h\|_{0,T}^2
\end{aligned} \tag{48}$$

Using local estimation(48) we can write

$$\begin{aligned}
A_h(v_h, v_h) \quad &\geq \quad \sum_T A_h^T(v_h, v_h) \\
&\geq \quad \alpha \sum_T \left\{ \epsilon \, |v_h|_{1,T}^2 + c_0 \, \|v_h\|_{0,T}^2 + \delta_T \, \|\vec{b} \cdot \nabla v_h\|_{0,T}^2 \right\} \\
&\geq \quad \alpha \, |||v_h|||^2
\end{aligned} \tag{49}$$

Positive constant α is defined by $\alpha :=$ min $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = \frac{3}{4} \, \alpha_*$, $\alpha_2 = \frac{3}{8} \min\{s^*, \gamma_*\}$ and $\alpha_3 = \Gamma_*$. □

6. Boundedness

In this section we will bound $\sum_T A^T(u_\Pi - u, \delta)$. Let $\tilde{u} = u_\Pi - u$, and $\delta = u_h - u_I$.

Lemma 6.1. *Let the virtual element spaces satisfy* $(Z1) - (Z3)$ *and the control parameter* δ_T *fulfill the assumption(36) .Then the following estimation holds*

$$\big|\sum_T A^T(\tilde{u},\delta)\big| \le C\,h^k \left(\sum_T \eta\,|u|^2_{k+1,T}\right)^{1/2} |||\delta|||$$

where η *is a positive constant.*

Proof. We will bound $\sum_T A^T(\tilde{u},\delta)$ term-wise. Using Cauchy-Schwarz inequality and polynomial approximation property (11) of VE space we can bound,

$$\begin{aligned}
\sum_T \epsilon \int_T \nabla\tilde{u}\,\nabla\delta \;&\le\; \sum_T \epsilon\,\|\nabla\tilde{u}\,\|_{0,T}\|\nabla\delta\|_{0,T} \\
&\le\; C\,\epsilon^{1/2}\left(\sum_T h_T^{2k}\,|u|^2_{k+1}\right)^{1/2}\left(\sum_T \epsilon^{1/2}\,|\delta|^2_{1,T}\right)^{1/2} \\
&\le\; C\,\epsilon^{1/2}\,h^k\left(\sum_T |u|^2_{k+1}\right)^{1/2} |||\delta||| \qquad (50)
\end{aligned}$$

Using Greens theorem on each element T we rewrite the bilinear form (5) as,

$$\begin{aligned}
&\frac{1}{2}\Big[< (\vec{b}\cdot\nabla\tilde{u}),\delta > - < (\vec{b}\cdot\nabla\delta),\tilde{u} > - < \operatorname{div} b,\tilde{u}\,\delta >\Big] \\
&=< (\vec{b}\cdot\nabla\tilde{u}),\delta > -\frac{1}{2}\int_{\partial T}(\vec{b}\cdot\mathbf{n})\,\tilde{u}\,\delta \\
&= - < (\vec{b}.\nabla\delta),\tilde{u} > - < \operatorname{div} b,\tilde{u}\,\delta > +\frac{1}{2}\int_{\partial T}(\vec{b}\cdot\mathbf{n})\,\tilde{u}\,\delta \qquad (51)
\end{aligned}$$

$\int_T c\,\tilde{u}\,\delta$ along with (51) can be written as,

$$< (c - \operatorname{div}\vec{b}),\tilde{u}\,\delta > - < (\vec{b}\cdot\nabla\delta),\tilde{u} > +\frac{1}{2}\int_T(\vec{b}\cdot\mathbf{n})\,\tilde{u}\,\delta \qquad (52)$$

Using Cauchy-Schwarz inequality and the assumption (3) we can estimate

$$
\begin{aligned}
\left| \sum_T < (c - \operatorname{div} \vec{b}), \tilde{u}\, \delta > \right| &\leq C \sum_T \|\tilde{u}\|_{0,T} \|\, \delta\|_{0,T} \\
&\leq C \left(\sum_T h_T^{2k+2} \, |u|^2_{k+1,T} \right)^{1/2} |||\delta||| \qquad (53)
\end{aligned}
$$

using Cauchy-Schwarz inequality and local polynomial approximation (11) and interpolation approximation (12) of VE space we can estimate second term of (52),

$$
\begin{aligned}
\sum_T < (\vec{b} \cdot \nabla \delta), \tilde{u} > &\leq \sum_T \delta_T^{1/2} \, \|\vec{b} \cdot \nabla \delta\| \ \delta_T^{-1/2} \|\tilde{u}\|_{0,T} \\
&\leq C \left(\sum_T \delta_T^{-1} \, h_T^{2k+2} \, |u|^2_{k+1,T} \right)^{1/2} \left(\sum_T \delta_T \, \|\vec{b} \cdot \nabla \delta\|^2 \right)^{1/2} \\
&= C \left(\sum_T \left(\frac{h_T^2}{\delta_T} \right) \ h_T^{2k} \, |u|^2_{k+1,T} \right)^{1/2} |||\delta||| \qquad (54)
\end{aligned}
$$

Again using Cauchy-Schwarz inequality, patch test [14] of $[|\delta|]$ and polyomial approximation property of VE space we bound last term of (52),

$$
\begin{aligned}
\sum_T \int_{\partial T} (\vec{b}.\mathbf{n}) \ \tilde{u}\delta &= \sum_e \int_e (\vec{b}.\mathbf{n}) \tilde{u} [|\delta|] \\
&\leq C \sum_T h_T^{k+1} |u|_{k+1,T} \ |\delta|_{1,T} \\
&\leq C (\sum_T (\frac{h_T^2}{\epsilon}) h_T^{2k} |u|^2_{k+1})^{1/2} (\sum_T \epsilon |\delta|^2_{1,T})^{1/2} \\
&= C h^k (\sum_T (\frac{h_T^2}{\epsilon}) |u|^2_{k+1})^{1/2} |||\delta||| \qquad (55)
\end{aligned}
$$

Now we bound additional term of bilinear form $A^T(\tilde{u}, \delta)$.Assuming (11) and considering control parameter δ_T satisfies the condition (36) we can write

$$\sum_T < -\epsilon\Delta\tilde{u} + \vec{b}.\nabla\tilde{u} + c\tilde{u}, \delta_T\vec{b}.\nabla\delta >_T$$
$$\leq \sum_T \delta_T^{1/2}\| - \epsilon\Delta\tilde{u} + \vec{b}.\nabla\tilde{u} + c\tilde{u}\| \ \delta_T^{1/2}\|\vec{b}.\nabla\delta\|$$
$$\leq \sum_T \delta_T^{1/2}\{\epsilon\|\Delta\tilde{u}\| + \|\vec{b}.\nabla\tilde{u}\| + \|c\tilde{u}\|\}\delta_T^{1/2}\|\vec{b}.\nabla\delta\|$$
$$\leq (\sum_T \delta_T\{\epsilon\|\Delta\tilde{u}\|_{0,T} + Ch_T^k|u|_{k+1,T}\}^2)^{1/2}$$
$$(\sum_T \delta_T\|\vec{b}.\nabla\delta\|_{0,T}^2)^{1/2}$$
$$C(\sum_T(\epsilon + \delta_T)h^{2k}|u|_{k+1,T}^2)^{1/2}|||\delta||| \tag{56}$$

In last inequality of (56) we have used the assumption $\epsilon\delta_T < Ch_T^2$, where C is a constant.

Therefore

$$|\sum_T A^T(u_\Pi - u, \delta)| \leq Ch^k(\sum_T \eta|u|_{k+1,T}^2)^{1/2}|||\delta|||$$

where $\eta = \epsilon + h_T^2 + (\frac{h_T^2}{\delta}) + \delta_T + (\frac{h_T^2}{\epsilon})$ □

6.1. boundedness of discrete bilinear form

We will bound $\sum_T A_h^T(u', \delta)$, where $u' = u_I - u_\Pi$ and $\delta = u_h - u_I$

Lemma 6.2. *Let the virtual element spaces satisfy* $(Z1)-(Z3)$ *and the control parameter* δ_T *fulfill the assumption(36) and the bilinear form (14) satisfies stability condition defined in section(4).Then the following estimation holds*

$$|\sum_T A_h^T(u', \delta)| \leq Ch^k(\sum_T \zeta|u|_{k+1,T}^2)^{1/2}|||\delta|||$$

where ζ *is a positive constant.*

Proof. Bilinear form $a_h^T(.,.)$ is symmetric hence it defines an inner-product on $V_h^k \times V_h^k$.We can bound inner product by energy norm . Using stability property of $a_h^T(.,.)$ and considering the property (12) we can bound diffusion tern

$$
\begin{aligned}
a_h^T(u^{'},\delta) &\leq (a_h^T(u^{'},u^{'}))^{1/2}(a_h^T(\delta,\delta))^{1/2} \\
&\leq \alpha^*(a^T(u^{'},u^{'}))^{1/2}(a^T(\delta,\delta))^{1/2} \\
&\leq \alpha^*\epsilon^{1/2}\|\nabla u^{'}\|_{0,T}\epsilon^{1/2}\|\nabla\delta\|_{0,T} \\
&\leq C\alpha^*\epsilon^{1/2}h_T^k|u|_{k+1,T}\epsilon^{1/2}\|\nabla\delta\|_{0,T}
\end{aligned}
$$

$$
\sum_T a_h^T(u^{'},\delta) \leq Ch^k(\sum_T \epsilon|u|_{k+1,T}^2)^{1/2}|||\delta||| \tag{57}
$$

similarly

$$
c_h^T(u^{'},\delta) \leq \gamma^* c_{\max} h_T^{k+1}|u|_{k+1}\|\delta\|_{0,T}
$$

$$
\sum_T c_h^T(u^{'},\delta) \leq Ch^k(\sum_T h_T^2|u|_{k+1}^2)^{1/2}|||\delta||| \tag{58}
$$

Symmetric part of convection part can be bounded using same idea as earlier

$$
b_h^{T,sym}(u^{'},\delta) \leq s^*\frac{1}{2}|\nabla.\vec{b}|h_T^{k+1}|u|_{k+1}\|\delta\|_{0,T}
$$

summing over $T \in \tau_h$ and using Cauchy-Schwarz inequality and approximation property (11, 12) we get

$$
\sum_T b_h^{T,sym}(u^{'},\delta) \leq Ch^k(\sum_T h_T^2|u|_{k+1}^2)^{1/2}|||\delta||| \tag{59}
$$

Using same technique as diffusion and reaction part we bound additional symmetric stabilization part

$$
\begin{aligned}
&\int_T(\vec{b}.\Pi_{k-1}(\nabla u^{'}))\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) + s_b^{T,stab}((I-\Pi_k)u^{'},(I-\Pi_k)\delta) \\
&\leq \Gamma^*\delta_T^{1/2}C\|u^{'}\|_{0,T}\delta_T^{1/2}\|\vec{b}.\nabla\delta\|_{0,T}
\end{aligned}
$$

Summing over all element T

$$
\begin{aligned}
&\sum_T\{\int_T(\vec{b}.\Pi_{k-1}(\nabla u^{'}))\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) + s_b^{T,stab}((I-\Pi_k)u^{'},(I-\Pi_k)\delta)\} \\
&\leq Ch^k(\sum_T \delta_T|u|_{k+1,T}^2)^{1/2}|||\delta||| \qquad (60)
\end{aligned}
$$

Using Cauchy-Schwarz inequality and boundedness property of projection operator Π_k we can estimate

$$\begin{aligned}\int_T [\vec{b}.\Pi_{k-1}(\nabla u^{'})]\Pi_k(\delta) &= \int_T [\vec{b}.\Pi_{k-1}(\nabla u^{'})]\delta \\ &\leq C\|\vec{b}\|_{0,\infty}\|\nabla u^{'}\|_{0,T}\|\delta\|_{0,T} \\ &\leq C\|\vec{b}\|_{0,\infty} h_T^k |u|_{k+1,T}\|\delta\|_{0,T}\end{aligned}$$

Summing over all element $T \in \tau_h$ we get

$$\sum_T \int_T [\vec{b}.\Pi_{k-1}(\nabla u^{'})]\Pi_k(\delta) \leq C\|\vec{b}\|_{0,\infty} h^k (\sum_T |u|_{k+1}^2)^{1/2} |||\delta||| \tag{61}$$

Using Cauchy-Schwarz inequality and boundedness and orthogonality property of projection operator we can estimate

$$\begin{aligned}\int_T [\vec{b}.\Pi_{k-1}(\nabla \delta)]\Pi_k(u^{'}) &= \int_T [\vec{b}.\Pi_{k-1}(\nabla \delta)]u^{'} \\ &\leq C h_T^{k+1}\|\vec{b}\|_{0,\infty}|u|_{k+1,T}\|\nabla\delta\|_{0,T} \\ &\leq C\|\vec{b}\|\epsilon^{-1/2} h_T^{k+1}|u|_{k+1,T}\epsilon^{1/2}|\delta|_{1,T}\end{aligned}$$

summing over all $T \in \tau_h$

$$\begin{aligned}\sum_T \int_T [\vec{b}.\Pi_{k-1}(\nabla \delta)]\Pi_k(u^{'}) &\leq \sum_T C\|\vec{b}\|\epsilon^{-1/2} h_T^{k+1}|u|_{k+1,T}\epsilon^{1/2}|\delta|_{1,T} \\ &\leq C\|\vec{b}\|_{0,\infty}(\sum_T (\frac{h_T^2}{\epsilon}) h_T^{2k}|u|_{k+1}^2)^{1/2}(\sum_T \epsilon|\delta|_{1,T}^2)^{1/2} \\ &\leq C h^k\|\vec{b}\|_{0,\infty}(\sum_T (\frac{h_T^2}{\epsilon})|u|_{k+1}^2)^{1/2}|||\delta||| \qquad (62)\end{aligned}$$

Now we shall bound additional term .We consider the control parameter δ_T satisfies the condition (36). Using boundedness property of projection operator Π_{k-2} we estimate

$$\begin{aligned}\int_T -\epsilon\Pi_{k-2}(\Delta u^{'})\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) &\leq \epsilon\delta_T^{1/2}\|\Pi_{k-2}\ (\Delta u^{'})\|_{0,T}\ \delta_T^{1/2}\|\vec{b}.\nabla\delta\|_{0,T} \\ &\leq C\epsilon\delta_T^{1/2}\|\Delta u^{'}\|_{0,T}\ \delta_T^{1/2}\|\vec{b}.\nabla\delta\|_{0,T}\end{aligned}$$

Using Cauchy-Schwarz inequality and using property of the control parameter δ_T, $\epsilon\delta_T < Ch_T^2$ we estimate

$$\begin{aligned}
&\sum_T \int_T -\epsilon \Pi_{k-2}(\Delta u^{'})\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) \\
&\leq C(\sum_T \epsilon^2 \delta_T \|\Delta u^{'}\|^2)^{1/2}(\sum_T \delta_T \|\vec{b}.\nabla\delta\|_{0,T}^2)^{1/2} \\
&\leq C(\sum_T \epsilon h_T^{2k}|u|_{k+1}^2)^{1/2}|||\delta||| \\
&\leq Ch^k(\sum_T \epsilon |u|_{k+1}^2)^{1/2}|||\delta||| \qquad (63)
\end{aligned}$$

Using boundedness property of projection operator Π_{k-1} we can write

$$\begin{aligned}
\int_T c\Pi_k(u^{'})\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) &= \int_T cu^{'}\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) \\
&\leq c_{max}\|u^{'}\|_{0,T}\delta_T\|\vec{b}.\Pi_{k-1}(\nabla\delta)\|_{0,T} \\
&\leq C\|\vec{b}\|_{0,\infty}\|\nabla\delta\|_{0,T}\delta_T h_T^{k+1}|u|_{k+1}
\end{aligned}$$

Taking sum over all element T we establish the following inequality

$$\begin{aligned}
&\sum_T c\Pi_k(u^{'})\delta_T(\vec{b}.\Pi_{k-1}(\nabla\delta)) \\
&\leq C(\sum_T \delta_T^2 \epsilon^{-1} h_T^{2k+2}|u|_{k+1,T}^2)^{1/2}(\sum_T \epsilon|\delta|_{1,T}^2)^{1/2} \\
&\leq Ch^k(\sum_T \delta_T^2(\frac{h_T^2}{\epsilon})|u|_{k+1,T}^2)^{1/2}|||\delta||| \qquad (64)
\end{aligned}$$

Using all above established inequalities we finally obtain

$$\left|\sum_T A_h^T(u_I - u_\Pi, \delta)\right| \leq Ch^k(\sum_T \zeta|u|_{k+1,T}^2)^{1/2}|||\delta||| \qquad (65)$$

where $\zeta = 1 + \epsilon + \delta_T + h_T^2 + (\frac{h_T^2}{\epsilon}) + \delta_T^2(\frac{h_T^2}{\epsilon})$

□

6.2. consistency error estimates

Nonconforming VE space $V_h^k \nsubseteq H^1(\Omega)$. Function v_h on virtual element space V_h^k is not continuous along interior edges except gauss lobatto points which introduce additional consistency error term$[|v_h|]$. We will see that two consistency error terms will arise in the proof of convergence analysis in next section.Diffusion part introduce $\sum_T \int_{\partial T}(\epsilon \nabla u.\mathbf{n})\delta$ and convection part introduce $\sum_T \int_{\partial T}(\vec{b}.\mathbf{n})u\delta$.We use patch test to bound these error terms.Before going into detail proof we introduce some basic result which will help us bound the consistency error terms.

Let $\mathbf{P}_s^e : L^2(e) \rightarrow \mathbb{P}^s(e)$ is the L^2- orthogonal projection operator onto the space $\mathbb{P}^s(e)$ for $s \geq 0$. Let $e \in \varepsilon_h^0$ be an interior edge and e is shared by two elements T^+ and T^- as a common edge.Standard approximation results [6, 3, 11] say that

$$\begin{aligned} \|\nabla u - \mathbf{P}_{k-1}^e(\nabla u)\|_{0,e} &\leq Ch^{k-1/2}\|u\|_{k+1,T^+\cup T^-} \\ \|[|v_h|] - \mathbf{P}_0^e([|v_h|])\|_{0,e} &\leq Ch^{1/2}|v_h|_{0,T^+\cup T^-} \end{aligned} \tag{66}$$

Let $u \in H^m(\Omega), m \geq 3/2$.$H^{3/2}$ is the space with minimum regularity to ensure that the analysis can be carried out. Using patch test of $v_h \in V_h^k$ and stated result(66) we can estimate

$$\begin{aligned} \sum_T \int_{\partial T}(\epsilon \nabla u.\mathbf{n})\delta &= \sum_{e\in\varepsilon_h} \int_e \epsilon \nabla u.[|\delta|] \\ &\leq \sum_{e\in\varepsilon_h} \|\nabla u - \mathbf{P}_{k-1}^e(\nabla u)\|_{0,e}\ \|[|\delta|] - \mathbf{P}_0^e([|\delta|])\|_{0,e} \\ &\leq C\sum_T h^k \epsilon \|u\|_{k+1,T}|\delta|_{1,T} \\ &\leq Ch^k(\sum_T \epsilon\|u\|_{k+1,T}^2)^{1/2}(\sum_T \epsilon|\delta|_{1,T}^2)^{1/2} \\ &\leq Ch^k(\sum_T \epsilon\|u\|_{k+1,T}^2)^{1/2}|||\delta||| \end{aligned} \tag{67}$$

Using same technique as described for estimation (67) we can bound

$$
\begin{aligned}
\sum_T \int_{\partial T} (\vec{b}.\mathbf{n}) u\delta &= \sum_e \int_e (\vec{b}.\mathbf{n}) u[|\delta|] \\
&\le C|\vec{b}.\mathbf{n}|_\infty \sum_e \|u - \mathbf{P}_{k-1}(u)\|_{0,e} \; \|[|\delta|] - \mathbf{P}_0([|\delta|])\|_{0,e} \\
&\le C \sum_T h_T^{k+1} |u|_{k+1,T} |\delta|_{1,T} \\
&\le Ch^k (\sum_T (\frac{h_T^2}{\epsilon}) |u|^2_{k+1,T})^{1/2} (\sum_T \epsilon |\delta|^2_{1,T})^{1/2} \\
&\le Ch^k (\sum_T (\frac{h_T^2}{\epsilon}) |u|^2_{k+1,T})^{1/2} \; |||\delta||| \qquad (68)
\end{aligned}
$$

6.3. Right-hand side estimation

In this paragraph we describe construction of external force term f.$H^{1,nc}(\tau_h; 2)$ is the space with minimum patch test to ensure the estimation (69) & (70).Let $\mathbf{P}_k^T : L^T \to \mathbb{P}^k$ be orthogonal L^2 projection and $(f_h)|_T := P^T_{k-2}(f)$. We consider locally $f|_T \in H^1(T)$ .Using L^2 projection operator , Cauchy-Schwarz inequality and standard approximation result we have

$$
\begin{aligned}
\Big| < f, v_h > - < f_h, v_h > \Big| &= \Big| \sum_T \int_T (f - \mathbf{p}^T_{k-2}(f))(v_h - \mathbf{p}^T_0(v_h)) \Big| \\
&\le \sum_T \|f - \mathbf{p}^T_{k-2}(f)\|_{0,T} \; \|v_h - \mathbf{p}^T_0(v_h)\|_{0,T} \\
&\le Ch^{\min(k-1,s-1)} (\sum_T (\frac{h_T^2}{\epsilon}) |f|^2_{s-1,T})^{1/2} (\sum_T \epsilon |v_h|^2_{1,T})^{1/2} \\
&= Ch^{\min(k-1,s-1)} (\sum_T (\frac{h_T^2}{\epsilon}) |f|^2_{s-1,T})^{1/2} |||v_h||| \qquad (69)
\end{aligned}
$$

Using L^2 projection, Cauchy-Schwarz inequality and standard approximation result we can bound second consistency error term

$$
\begin{aligned}
&\Big| < f, \delta_T(\vec{b}.\nabla v_h) > - < f_h, \delta_T(\vec{b}.\nabla v_h) > \Big| \\
&= \Big| \sum_T \int_T (f - \mathbf{P}^T_{k-2}(f))\delta_T[\vec{b}.\nabla v_h - \mathbf{P}^T_0(\vec{b}.\nabla v_h)] \Big| \\
&\le C h^{\min(k-1,s-1)} (\sum_T \delta_T |f|^2_{s-1})^{1/2} (\sum_T \delta_T \|\vec{b}.\nabla v_h\|^2_{0,T})^{1/2} \\
&= C h^{\min(k-1,s-1)} (\sum_T \delta_T |f|^2_{s-1})^{1/2} \ |||v_h||| \qquad (70)
\end{aligned}
$$

Remark: In the above estimation (69) and (70) constant C depend on $\frac{h_T^2}{\epsilon}$ and δ_T .In convection dominated case ϵ is very small quantity and so $\frac{1}{\epsilon}$ is very large quantity.But this does not hamper convergence analysis since as $h_T \to 0$, h_T^2 dominates ϵ.

7. Convergence Analysis

Theorem 7.1. *Let $u \in H^{k+1}(\Omega)$ be weak solution of the bilinear form (2).Let $v_h \in V_h^k$ the be discrete solution of the bilinear form (7).Let $v_h|_T \in H^2(T)$, $f|_T \in H^s(T), (s \ge 1)$ locally and the control parameter δ_T satisfies the condition (36) .Then the discrete solution satisfy*

$$
\begin{aligned}
|||u - u_h||| \ &\le \ C h^k (\sum_T \Gamma |u|^2_{k+1,T})^{1/2} + C h^k (\sum_T \epsilon \|u\|^2_{k+1,T})^{1/2} \\
&+ \ C h^{min(k-1,s-1)} (\sum_T \delta_T |f|^2_{s-1})^{1/2} \\
&+ \ C h^{min(k-1,s-1)} (\sum_T (\frac{h_T^2}{\epsilon}) |f|^2_{s-1,T})^{1/2}
\end{aligned}
$$

where norm $|||.|||$ is defined in (35), and Γ is defined by $\Gamma := \epsilon + \delta_T + h_T^2 + (\frac{h_T^2}{\delta}) + (\frac{h_T^2}{\epsilon}) + \delta_T^2(\frac{h_T^2}{\epsilon})$.

Proof. Let u_I be interpolation approximation of u in V_h^k .We consider VE space satisfies the estimations(12). Introducing u_I we divide $|||u - u_h|||$ into two parts as

$$
\begin{aligned}
|||u - u_h||| \ &= \ |||u - u_I + u_I - u_h||| \\
&\le \ |||u - u_I||| + |||u_I - u_h|||
\end{aligned}
$$

We first bound second term $|||u_I - u_h|||$ using discrete coercivity of $A_h(.,.)$, lemma(6.2) and lemma (6.1). Let us denote $\delta := u_h - u_I$. Using discrete coercivity of $A_h(.,.)$ we can write

$$\begin{aligned}
\alpha|||\delta|||^2 &\leq A_h(\delta,\delta) \\
&= A_h(u_h,\delta) - A_h(u_I,\delta) \\
&= < f_h,\delta > + < f_h, \delta_T \vec{b}.\nabla\delta > -A_h(u_I,\delta) \\
&= < f_h,\delta > + < f_h, \delta_T \vec{b}.\nabla\delta > - \sum_T A_h^T(u_I,\delta)
\end{aligned} \tag{71}$$

Adding and subtracting $A_h^T(u_\Pi,\delta)$ and $A^T(u_\Pi,\delta)$ we get

$$\begin{aligned}
A_h^T(u_I,\delta) &= A_h^T(u_I - u_\Pi,\delta) + A_h^T(u_\Pi,\delta) \\
&= A_h^T(u_I - u_\Pi,\delta) + A_h^T(u_\Pi,\delta) - A^T(u_\Pi,\delta) + A^T(u_\Pi,\delta) \\
&= A_h^T(u_I - u_\Pi,\delta) + A_h^T(u_\Pi,\delta) - A^T(u_\Pi,\delta) \\
&+ A^T(u_\Pi - u,\delta) + A^T(u,\delta)
\end{aligned} \tag{72}$$

We have shown that $A_h^T(u_h,v_h)$ is polynomial consistent in(), hence $A_h^T(u_\Pi,\delta) = A^T(u_\Pi,\delta)$

Therefore the estimation (72) reduces

$$A_h^T(u_I,\delta) = A_h^T(u_I - u_\Pi,\delta) + A^T(u_\Pi - u,\delta) + A^T(u,\delta) \tag{73}$$

We first estimate $\sum_T A^T(u,\delta)$. Applying Green's theorem on each element T we get

$$\begin{aligned}
&\frac{1}{2}(\int_T (\vec{b}.\nabla u)\delta - \int_T (\vec{b}.\nabla\delta)u - \int_T (\nabla.\vec{b})u\delta) \\
&= \int_T (\vec{b}.\nabla u)\delta - \frac{1}{2}\int_{\partial T} (\vec{b}.\mathbf{n})u\delta
\end{aligned} \tag{74}$$

summing up the estimation (74) over all element $T \in \tau_h$ we get additional term $\sum_T \frac{1}{2}\int_{\partial T}(\vec{b}.\mathbf{n})u\delta$ which is described in the following estimation-

$$\begin{aligned}
\sum_T A^T(u,\delta) &= \sum_T < (-\epsilon\Delta u + \vec{b}.\nabla u + cu), \delta >_T \\
&+ \sum_T \int_T \epsilon\nabla u.\mathbf{n}\delta - \sum_T \frac{1}{2}\int_{\partial T}(\vec{b}.\mathbf{n})u\delta \\
&+ \sum_T < (-\epsilon\Delta u + \vec{b}.\nabla u + cu), \delta_T\vec{b}.\nabla\delta >_T \\
&= \sum_T < f, \delta >_T + \sum_T < f, \delta_T\vec{b}.\nabla\delta > \\
&+ \sum_e \int_e (\epsilon\nabla u.\mathbf{n})[|\delta|] - \frac{1}{2}\sum_e \int_e (\vec{b}.\mathbf{n})u[|\delta|] \qquad (75)
\end{aligned}$$

Before putting(75) into (71) we first bound $A_h^T(u_I - u_\Pi, \delta)$ and $A^T(u_\Pi - u, \delta)$.

Using lemma(6.2) we can write

$$\left|\sum_T A_h^T(u_I - u_\Pi, \delta)\right| \leq Ch^k(\sum_T \zeta|u|^2_{k+1,T})^{1/2}\ |||\delta||| \qquad (76)$$

where $\zeta = \epsilon + \delta_T + h_T^2 + (\frac{h_T^2}{\epsilon}) + \delta_T^2(\frac{h_T^2}{\epsilon})$

Using lemma(6.1) we can write

$$\left|\sum_T A^T(u_\Pi - u, \delta)\right| \leq Ch^k(\sum_T \eta|u|^2_{k+1,T})^{1/2}|||\delta||| \qquad (77)$$

where $\eta = \epsilon + h_T^2 + \delta_T + (\frac{h_T^2}{\epsilon}) + (\frac{h_T^2}{\delta})$

We have seen that we got two consistency error terms in estimation(75). Now we shall bound consistency error terms.

Using estimation (67) we can write

$$\left|\sum_{e\in\varepsilon_h}\int_e (\epsilon\nabla u.\mathbf{n})[|\delta|]\right| \leq Ch^k(\sum_T \epsilon\|u\|^2_{k+1,T})^{1/2}|||\delta||| \qquad (78)$$

Using estimation (68) we can write

$$\left|\sum_T \int_{\partial T}(\vec{b}.\mathbf{n})u\delta\right| \leq Ch^k(\sum_T (\frac{h_T^2}{\epsilon}))|u|^2_{k+1,T})^{1/2}|||\delta||| \qquad (79)$$

After putting (75) into (71) we get two terms $| < f, \delta > - < f_h, \delta > |$ and $| \sum_T < f, \delta_T(\vec{b}.\nabla\delta) >_T - < f_h, \delta_T(\vec{b}.\nabla\delta) >_T |$.

Using estimation(69) we can write

$$\left| < f, \delta > - < f_h, \delta > \right| \leq Ch^{\min(k-1,s-1)}(\sum_T (\frac{h_T^2}{\epsilon})|f|_{s-1,T}^2)^{1/2}|||\delta||| \qquad (80)$$

Using estimation (70) we can write-

$$\left| \sum_T < f, \delta_T(\vec{b}.\nabla\delta) >_T - < f_h, \delta_T(\vec{b}.\nabla\delta) >_T \right|$$
$$Ch^{\min(k-1,s-1)}(\sum_T \delta_T |f|_{s-1}^2)^{1/2}|||\delta||| \qquad (81)$$

Putting estimation(76), (77), (78), (79), (80) and (81) in (71) we finally obtain

$$\begin{aligned} \alpha|||\delta||| &\leq Ch^k(\sum_T \Gamma|u|_{k+1,T}^2)^{1/2} + Ch^k(\sum_T \epsilon\|u\|_{k+1,T}^2)^{1/2} \\ &+ Ch^{\min(k-1,s-1)}(\sum_T \delta_T|f|_{s-1}^2)^{1/2} \\ &+ Ch^{\min(k-1,s-1)}(\sum_T (\frac{h_T^2}{\epsilon})|f|_{s-1,T}^2)^{1/2} \qquad (82) \end{aligned}$$

where $\Gamma = \epsilon + \delta_T + h_T^2 + (\frac{h_T^2}{\delta}) + (\frac{h_T^2}{\epsilon}) + \delta_T^2(\frac{h_T^2}{\epsilon})$

Again using triangle inequality we write

$$\begin{aligned} |||u - u_h||| &= |||u - u_I + u_I - u_h||| \\ &\leq |||u - u_I||| + |||u_I - u_h||| \end{aligned}$$

Using estimation(12) we can write

$$\begin{aligned} |||u - u_I||| &= [\sum_T \epsilon|u - u_I|_{1,T}^2 + c_0\|u - u_I\|_{0,T}^2 + \sum_T \delta_T\|\vec{b}.\nabla(u - u_I)\|_{0,T}^2]^{1/2} \\ &\leq Ch^k[\sum_T \epsilon|u|_{k+1,T}^2 + h_T^2\sum_T |u|_{k+1,T}^2 + \sum_T \delta_T|u|_{k+1,T}^2]^{1/2} \\ &\leq Ch^k(\sum_T \Gamma|u|_{k+1,T}^2)^{1/2} \qquad (83) \end{aligned}$$

Using inequality (83) we finally obtain required estimation

$$
\begin{aligned}
|||u-u_h||| &\leq Ch^k(\sum_T \Gamma|u|^2_{k+1,T})^{1/2} + Ch^k(\sum_T \epsilon\|u\|^2_{k+1,T})^{1/2} \\
&+ Ch^{\min(k-1,s-1)}(\sum_T \delta_T |f|^2_{s-1})^{1/2} \\
&+ Ch^{\min(k-1,s-1)}(\sum_T (\frac{h_T^2}{\epsilon})|f|^2_{s-1,T})^{1/2} \qquad (84)
\end{aligned}
$$

□

7.1. computation issue

The L^2- orthogonal projection operator Π_k^0 can be computed using elliptic projection operator Π_k^∇

$$\int_T p\Pi_k^\nabla(v_h)dT = \int_T pv_h dT$$

If degree $(p) < k-2$ then $\int_T pv_h dT$ is computable. If degree $(p) = k-1, k$ then we consider $\int_T pv_h dT = \int_T p\Pi_k^\nabla(v_h)$

diffusion part:

$$
\begin{aligned}
\int_T \epsilon\Pi_{k-1}(\nabla p)\Pi_{k-1}(\nabla v_h) &= \int_T \epsilon\nabla p\Pi_{k-1}(\nabla v_h) \\
&= \int_T \epsilon\nabla p\nabla v_h \\
&= \int_T -\epsilon\Delta p v_h + \int_{\partial T} \epsilon(\nabla p.\mathbf{n})v_h \qquad (85)
\end{aligned}
$$

$\int_T \Delta p v_h$ is computable using degrees of freedom of v_h on triangle since degree$(\Delta p) < k-2$.

$\int_{\partial T}(\nabla p.\mathbf{n})v_h$ is computable using degrees of freedom of v_h on boundary of triangle since degree$(\nabla p) < k-1$.

$$
\begin{aligned}
\int_T \vec{b}.\Pi_{k-1}(\nabla p)\Pi_k(v_h) &= \int_T \vec{b}.\nabla p\Pi_k(v_h) \\
&= \int_T (\vec{b}.\nabla p)v_h \qquad (86)
\end{aligned}
$$

Right hand side $\int_T(\vec{b}.\nabla p)v_h$ is computable using elliptic projection operator Π_k^{∇}

$$\begin{aligned}\int_T \vec{b}.\Pi_{k-1}(\nabla v_h)\Pi_k(p) &= \int_T \vec{b}.\Pi_{k-1}(\nabla v_h)p \\ &= \int_T (\vec{b}.\nabla v_h)p + \int_T (\vec{b}.\Pi_{k-1}(\nabla v_h) - \vec{b}.\nabla v_h)p \\ &\approx \int_T (\vec{b}.\nabla v_h)p \\ &= -\int_T \nabla.(\vec{b}p)v_h + \int_{\partial T}(\vec{b}.\mathbf{n})pv_h \qquad (87)\end{aligned}$$

error

$$\left| \int_T (\vec{b}.\Pi_{k-1}(\nabla v_h) - \vec{b}.\nabla v_h)p \right| \leq C\|\vec{b}\|_{0,\infty}\|p\|_{0,T}h_T|\nabla v_h|_{1,T} \qquad (88)$$

First part of right hand side $\int_T \nabla.(\vec{b}p)v_h$ is computable using elliptic projection operator.

For second part we will take $k-1th$ polynomial approximation of p, i.e. we will compute $\int_T(\vec{b}.\mathbf{n})\Pi_{k-1}(P)v_h$. Corresponding error

$$\left| \int_{\partial T}(\vec{b}.\mathbf{n})pv_h - \int_{\partial T}(\vec{b}.\mathbf{n})\Pi_{k-1}(p)v_h \right| \leq C\|\vec{b}.\mathbf{n}\|_{0,T}h_T^{1/2}|p|_{1,T}\|v_h\|_{0,T} \qquad (89)$$

$$\begin{aligned}\int_T \vec{b}\Pi_k(p)\Pi_k(v_h) &= \int_T \vec{b}(p)\Pi_k(v_h) \\ &= \int_T \vec{b}pv_h \qquad (90)\end{aligned}$$

Right hand $\int_T \vec{b}pv_h$ is computable using elliptic projection operator for polynomial p of degrees $k-1, k$.

similarly reaction part $\int_T c\Pi_k(p)\Pi_k(v_h)$ computable

Stabilization part

$$\begin{aligned}\int_T -\epsilon\Pi_{k-2}(\Delta p)\delta_T\vec{b}.\Pi_{k-1}(\nabla v_h) &= \int_T -\epsilon\Delta p\delta_T(\vec{b}.\nabla v) \\ &= \int_T \epsilon\delta_k\nabla.(\vec{b}\Delta p)v_h \\ &- \int_{\partial T}\epsilon(\vec{b}.\mathbf{n})\delta_T v_h\Delta p \qquad (91)\end{aligned}$$

Both term of the right hand side is computable.

$$\int_T c\Pi_k(p)\delta_T\vec{b}.\Pi_{k-1}(\nabla v_h) = \int_T cp\delta_T\vec{b}.\Pi_{k-1}(\nabla v_h) \tag{92}$$

This part is computable using same technique as (87)

$$\begin{aligned}\int_T \vec{b}.\Pi_{k-1}(\nabla p)\delta_T\vec{b}.\Pi_{k-1}(\nabla v_h) &= \int_T \vec{b}.\nabla p\delta_T\vec{b}.\nabla v_h \\ &= \int_T -\nabla.(\vec{b}(\vec{b}.\nabla p))v_h \\ &+ \int_{\partial T}(\vec{b}.\mathbf{n})(\vec{b}.\nabla p)v_h \end{aligned} \tag{93}$$

Both part of right hand side is computable using degrees of freedom of v_h.

8. Conclusion

In this paper we have introduced SDFEM type nonconforming VEM framework for convection dominated convection-diffusion reaction equation. To prove polynomial consistency we have assumed higher regularity of approximate solution, i.e, $v_h|_T \in H^2(T)$ where $v_h \in V_h^k$. The presented framework is not convergent in linear nonconforming virtual element space, i.e. it requires piecewise higher order polynomial function($k \geq 2$) and $f|_T \in H^s(T)$ where $s \geq 1$ and T is an arbitrary element, which may be considered as light drawback of this framework.

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