# REGULARITY OF BINOMIAL EDGE IDEALS OF CERTAIN BLOCK GRAPHS

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Abstract. We obtain an improved lower bound for the regularity of the binomial edge ideals of trees. We then prove an upper bound for the regularity of the binomial edge ideals proposed by Saeedi Madani and Kiani for a subclass of block graphs called clique-path graphs. As a consequence we obtain sharp upper and lower bounds for the regularity of binomial edge ideals of the class of trees called lobsters.

## 1. Introduction

Let $G$ be a simple graph on the vertex set $[n]$. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables, where $K$ is a field. Then the ideal $J(G)$ generated by $\{x_iy_j - x_jy_i \mid (i,j) \text{ is an edge in } G\}$ is called the binomial edge ideal of $G$. This was introduced by Herzog et al., [9] and independently by Ohtani, [12]. Recently, there have been many results relating the algebraic properties of the binomial edge ideals and the combinatorial data of the graphs, see [1], [2], [4], [10], [13], [14], [15]. In particular, there have been active research on relating algebraic invariants of the binomial edge ideals such as Castelnuovo-Mumford regularity, depth, betti numbers etc. with combinatorial invariants associated with graphs such as length of maximal induced path, number of maximal cliques, matching number etc. Matsuda and Murai proved that $\ell \leq \operatorname{reg}(S/J_G) \leq n-1$, where $\ell$ is the length of the longest induced path in $G$, [10]. They conjectured that if $\operatorname{reg}(S/J_G) = n-1$, then $G$ is a path of length $n$. Ene and Zarojanu proved this conjecture in the case of closed graphs, [5]. A graph is said to be closed if the binomial edge ideal has a quadratic Gröbner basis. Choudhry et al. proved that if $T$ is a tree whose longest induced path has length $\ell$, then $\operatorname{reg}(S/J_T) = \ell$ if and only if $T$ is a caterpillar, [1]. In [13], Saeedi Madani and Kiani proved that for a closed graph $G$, $\operatorname{reg}(S/J_G) \leq c(G)$, where $c(G)$ is the number of maximal cliques in $G$. Later they generalized this result to case of binomial edge ideal of a pair of a closed graph and a complete graph, [14]. Further, Saeedi Madani and Kiani proposed that if $G$ is any graph, then $\operatorname{reg}(S/J_G) \leq c(G)$.

In this article, we show that the bound proposed by Saeedi Madani and Kiani holds for a subclass of block graphs which we call clique-path graphs, see Definition 2.2. We prove in Theorem 3.4 that if $G$ is a clique-path graph with a path $P$ of length $\ell$ and $r$ maximal cliques of size at least 3, then $\operatorname{reg}(S/J_G) \leq \ell + r$. Note that if $G$ is a clique-path graph as given above, then it has $\ell + r$ maximal cliques, namely $\ell$ edges of $P$ and $r$ maximal cliques of size at least 3. We then obtain, in Theorem 3.7, an upper bound for the regularity of the binomial edge ideals of lobsters (see Section 2 for definition). Lobsters are well studied objects in graph theory. They are the most natural generalization of the caterpillar graphs. Lobsters occur very often in the graph theory literature, especially in the context of well-known graceful tree conjecture, [11], [6]. We also obtain a precise expression for the regularity of binomial edge ideals of a subclass of lobsters in Theorem 3.9.

From [10], it follows that the regularity of the binomial edge ideal of a tree is bounded below by the diameter of the tree. We improve this bound to show that the regularity of the binomial edge ideal of a tree is lower bounded by the number of internal vertices, Theorem 3.3.

## 2. Preliminaries

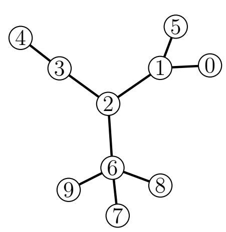
In this section, we set up the basic definitions and notation needed for the main results.

Let $T$ be a tree and $\mathrm{L}(T) = \{v \in V(T) | \deg(v) = 1\}$ be the set of all leaves of $T$. We say a tree $T$ is a *caterpillar* if $T \setminus \mathrm{L}(T)$ is either empty or is a path.

Similarly, a tree $T$ is said to be a *lobster*, if $T \setminus \mathrm{L}(T)$ is a caterpillar, [7]. Observe that every caterpillar is also a lobster. A longest path in a lobster is called a *spine* of the lobster. It is easy to see that given any spine, every edge of a caterpillar is incident to it. With respect to a fixed spine $P$, the pendant edges incident with $P$ are called *whiskers*. It is easy to see that every non-leaf vertex $u$ not incident on a fixed spine $P$ of a lobster forms the center of a star($K_{1,m}, m \geq 2$), and each such star is said to be a *limb* with respect to $P$. More generally, given a vertex $v$ on any simple path $P$, we can attach a star $(K_{1,m}, m \geq 2)$ with center $u$ by identifying exactly one of the leaves of the star with $v$. Such a star is called *a limb attached to* $P$.

Note that the limbs and whiskers depend on the spine. Whenever a spine in a graph is fixed, we will refer to them only as limb and whisker.

**Example 2.1.** *Let $G$ denote the given graph on 10 vertices:*




$G$

*In this example, $G$ has many longest induced paths. The path induced by the vertices $\{0, 1, 2, 3, 4\}$, $\{0, 1, 2, 6, 9\}$ are two such (there are more) paths. Let $P$ denote the path induced by the vertices $\{0, 1, 2, 3, 4\}$. Then $(1, 5)$ is a whisker with respect to $P$. Also the subgraph induced by the vertices $\{2, 6, 7, 8, 9\}$ is a limb with respect to $P$. If we consider $\{0, 1, 2, 6, 9\}$ as spine $P$, then $\{(1, 5), (6, 7), (6, 8)\}$ are whiskers with respect to $P$ and the path induced by $\{2, 3, 4\}$ is a limb.*

Here we define a new subclass of block graphs called clique-path graphs.

**Definition 2.2.** *A clique-path graph $G$ is the union $P \cup C_1 \cup \cdots \cup C_r$ where $P$ is a simple path on the vertices $\{v_0, \ldots, v_\ell\}$ and $C_1, \ldots, C_r$ are cliques such that*

1. $|C_i| \geq 3$ *for all* $i$;
2. $C_i \cap C_j \subset P$ *and* $|C_i \cap C_j| \leq 1$ *for every* $i \neq j$;
3. $|C_i \cap P| = 1$.

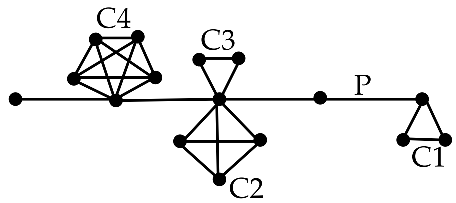



Figure 2. Clique-Path Graph

It follows from Proposition 1.2 in [9] that not all clique-path graphs are closed graphs. In fact, any clique-path graph that has at least one clique attached to an internal vertex of the path $P$ is not closed. See Figure 2 for an example.

## 3. Bounds on the regularity

In this section, we obtain sharp bounds on the regularity of the binomial edge ideals of certain classes of graphs. It was shown in [1] that if $G$ is a $\mathcal{C}_\ell$ graph, then $\operatorname{reg}(S/J_G) = \ell$. While $\mathcal{C}_\ell$ graph can be seen as a graph obtained by arranging cliques along a path, a clique-path graph is obtained by attaching cliques to a path. We first obtain a lower bound for the regularity of binomial edge ideals of trees. We then prove an upper bound for the regularity of binomial edge ideals of clique-path graphs. As a consequence, we obtain an upper bound to the regularity of the lobsters.

We begin by making a general observation about Betti numbers of quotients of graded ideals.

**Remark 3.1.** Let $I$ be a graded ideal of a polynomial ring $S = K[x_1, \ldots, x_n]$ and $f$ be a homogeneous element of $S$ of degree $d$ which is a regular element modulo $I$. Then we have an exact sequence

$$0 \longrightarrow S/I[-d] \xrightarrow{\mu_f} S/I \longrightarrow S/(I,f) \longrightarrow 0,$$

where $\mu_f$ denote the multiplication by $f$. Correspondingly, for each $m \geq 0$, there is a graded long exact sequence of the Tor functor

$$\begin{aligned} \cdots \quad &\longrightarrow \quad \operatorname{Tor}_i^S(K, S/I)_{m-d} \xrightarrow{\mu_f} \operatorname{Tor}_i^S(K, S/I)_m \longrightarrow \operatorname{Tor}_i^S(K, S/(I,f))_m \\ &\longrightarrow \quad \operatorname{Tor}_{i-1}^S(K, S/I)_{m-d} \xrightarrow{\mu_f} \cdots . \end{aligned}$$

Since the multiplication maps on Tor are zero, for each $i$, we have short exact sequences

$$0 \longrightarrow \operatorname{Tor}_i^S(K, S/I)_m \longrightarrow \operatorname{Tor}_i^S(K, S/(I,f))_m \longrightarrow \operatorname{Tor}_{i-1}^S(K, S/I)_{m-d} \longrightarrow 0$$

and consequently

$$\beta_{i,m}(S/(I,f)) = \beta_{i,m}(S/I) + \beta_{i-1,m-d}(S/I).$$

**Corollary 3.2.** *Let $G$ be a simple finite graph on $[n]$ with a free vertex, say $n$, and $G' = G \cup \{(n, n+1)\}$. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and $S' = S[x_{n+1}, y_{n+1}]$. Then $\beta_{i,i+j}(S'/J_{G'}) = \beta_{i,i+j}(S/J_G) + \beta_{i-1,i+j-2}(S/J_G)$. In particular,* $\operatorname{reg}(S'/J_{G'}) = \operatorname{reg}(S/J_G) +$ *1.*

*Proof.* Since $n$ is a free vertex in $G$, by Lemma 21 of [15], $x_n y_{n+1} - x_{n+1} y_n$ is a regular element on $S'/J_G$. Note that $\operatorname{Tor}_i^S(K, S/J_G) \cong \operatorname{Tor}_i^{S'}(K, S'/J_G)$. Hence by Remark 3.1, we get $\beta_{i,i+j}(S'/J_{G'}) = \beta_{i,i+j}(S/J_G) + \beta_{i-1,i+j-2}(S/J_G)$. Therefore it follows directly from the betti table that $\operatorname{reg}(S'/J_{G'}) = \operatorname{reg}(S/J_G) + 1$. $\square$

Now we obtain a lower bound on the regularity of the binomial edge ideals of trees in terms of the number of internal vertices. Given a tree $T$, it is easy to see that one can construct $T$ from the trivial graph by adding vertices $v_i$ to $T_{i-1}$ at step $i$ to get $T_i$ so that $v_i$ is a leaf in in the tree $T_i$. Such an ordering of vertices is called a *leaf ordering*.

**Theorem 3.3.** *If $G$ is a tree with $m$ internal vertices, then* $\operatorname{reg}(S/J_G) \geq m + 1$.

*Proof.* Let $v_1, \ldots, v_r$ be a leaf ordering of the vertices of $G$, and let $G_i$ be the subgraph of $G$ induced by $v_1, \ldots, v_i$. Let $m_i$ denote the number of internal vertices of $G_i$. We argue by induction on $i$. If $i = 2$, then $G_2$ is an edge and $\operatorname{reg}(S/J_{G_2}) = 1$. Therefore, the result holds. Assume the result for $G_i$. Then $G_{i+1}$ is obtained by adding a leaf $v_{i+1}$ to some vertex $v$ of $G_i$. If $v$ is a leaf in $G_i$, then $v$ is a free vertex in $G_i$, and hence by Corollary 3.2, $\operatorname{reg}(S/J_{G_{i+1}}) = \operatorname{reg}(S/J_{G_i}) + 1$. Further, $v$ becomes a new internal vertex in $G_i$, i.e., $m_{i+1} = m_i + 1$, and therefore the result holds. If $v$ is an internal vertex in $G_i$, then $m_{i+1} = m_i$ and since $G_i$ is an induced subgraph of $G_{i+1}$, $\operatorname{reg}(S/J_{G_{i+1}}) \geq \operatorname{reg}(S/J_{G_i}) \geq m_i + 1 = m_{i+1} + 1$ as required. $\square$

Now we prove an upper bound for the regularity of the binomial edge ideals of clique-path graphs.

**Theorem 3.4.** *Let $G$ be a* clique-path *graph having a path $P$ of length $\ell$ on vertices $\{v_0, \ldots, v_\ell\}$ and cliques $C_1, \ldots, C_r$ attached to $P$. Then* $\operatorname{reg}(S/J_G) \leq \ell + r$.

*Proof.* Let $G$ be a graph as given in the statement of the theorem. We prove the assertion by induction on $\ell + r$, i.e., the number of maximal cliques in $G$. If $r = 0$, then $G$ is a path and hence the result holds. Let $p = \min\{j \mid v_j \notin C_i \text{ for all } i\}$. If $p \leq \ell - 2$, then by removing $v_{p+2}, \ldots, v_\ell$, the resulting graph $G'$ satisfies the property $\operatorname{reg}(S/J_G) = \operatorname{reg}(S/J_{G'}) + \ell - p - 1$, by repeated application of Corollary 3.2. Therefore, we may assume that $p \geq \ell - 1$. Let $F_1, \ldots, F_m$ be a leaf-ordering of the maximal cliques in $G$ such that $F_m$ is either the edge $(v_{\ell-1}, v_\ell)$ or $F_m = C_i$ for some $i$ and $v_\ell \in C_i$. Let $F_{i_1}, \ldots, F_{i_q}$ be such that $F_{i_j} \cap F_m = \{v_k\}$, where $k \in \{\ell - 1, \ell\}$ and $F_m \cap P = \{v_k\}$. Let $G'$ be the graph obtained by inserting a clique on the vertices $F_m \cup F_{i_1} \cup \cdots \cup F_{i_q}$ and $G''$ be the graph induced by $[n] \setminus \{v\}$. Then as in the proof of [4, Theorem 1.1], we have the exact sequence:

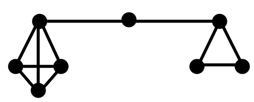
$$0 \longrightarrow \frac{S}{\operatorname{in}_<(J_G)} \longrightarrow \frac{S}{\operatorname{in}_<(J_{G'})} \oplus \frac{S}{(x_i, y_i) + \operatorname{in}_<(J_{G''})} \longrightarrow \frac{S}{(x_i, y_i) + \operatorname{in}_<(J_{G'})} \longrightarrow 0. \quad (1)$$

Note that $G'$ is a clique-path graph with number of maximal cliques $\ell + r - q$, $q \geq 1$. Therefore, by induction, $\operatorname{reg}(S/J_{G'}) \leq \ell + r - q \leq \ell + r - 1$. It can also be seen that $G''$ has $q + 2$ connected components, namely $F'_{i_1}, \ldots, F'_{i_q}, F'_m, G''_1$, where $F'_j$ is either an isolated vertex or $C_j \setminus \{v_k\}$ for some $j$ and $G''_1 = G \setminus (F_{i_1} \cup \cdots \cup F_{i_q} \cup F_m)$. Therefore $\operatorname{reg}(S/J_{G''}) = \sum_{j=1}^{q} \operatorname{reg}(S/J_{F'_{i_j}}) + \operatorname{reg}(S/J_{F'_m}) + \operatorname{reg}(S/J_{G''_1})$. It is easy to see that $G''_1$ is a clique-path graph with a path of length $\ell-1$ and with either $r-q$ or $r-q-1$ cliques of size at least three depending on whether $F_m$ is an edge or not. By induction hypothesis, $\operatorname{reg}(S/J_{G_1''}) \leq (\ell-1)+r-q$, if $F_m$ is an edge or $\operatorname{reg}(S/J_{G_1''}) \leq (\ell-1)+r-q-1$ otherwise, in which case $\operatorname{reg}(S/J_{F_m'}) = 0$. Therefore, $\operatorname{reg}(S/J_{G''}) \leq \ell + r$. Now it follows from the short exact sequence (1) that $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/\operatorname{in}_<(J_G)) \leq \ell + r$. ☐

**Corollary 3.5.** *If $G$ is a* clique-path *graph with a path $P$ of length $\ell$, $r$ maximal cliques of size at least* 3 *and $t$ limbs attached to $P$, then* $\operatorname{reg}(S/J_G) \leq \ell + r + t$.

*Proof.* We prove by induction on $t$. If $t = 0$, then the result follows from Theorem 3.4. Let $t \geq 1$. Let $v$ be a leaf on a limb and $N(v) = \{u\}$. Let $G'$ be the graph obtained by adding a clique on the vertices on $N(u) \cup \{u\}$ and $G''$ be the graph induced on $[n] \setminus \{u\}$. Note that $G'$ has $r+1$ maximal cliques of size at least three and $t-1$ limbs. By induction, $\operatorname{reg}(S/J_{G'}) \leq \ell + (r+1) + (t-1) = \ell + r + t$. Moreover, $G''$ has $r$ maximal cliques of size at least three and $t-1$ limbs. Therefore, by induction $\operatorname{reg}(S/J_{G''}) \leq \ell + r + t - 1$. By the exact sequence (1), we have $\operatorname{reg}(S/J_G) \leq \ell + r + t$. ☐

**Example 3.6.** *This is an example to show that the bound given in Theorem 3.4 is sharp.*



G

*Let $G$ be a graph as given here. Then $G$ is a* clique-path *graph with path $P$ of length $\ell = 2$ and $r = 2$ cliques. It can be seen that* $\operatorname{reg}(S/J_G) = 4 = \ell + r$.

Now we prove the regularity bound for a subclass of trees, namely the lobsters.

**Theorem 3.7.** *If $G$ is a lobster with spine $P$ of length $\ell$ with $t$ limbs and $r$ whiskers attached to the spine $P$, then* $\operatorname{reg}(S/J_G) \leq \ell + t + r + 2$.

*Proof.* Let $G$ be a graph with spine $P$ of length $\ell$ with $t$ limbs and $r$ whiskers attached to the spine $P$. Let $G'$ be the graph obtained from $G$ by extending each whisker to a limb by adding a new edge to a new distinct vertex. Then the spine length of $G'$ will be either $\ell$ or $\ell+1$ or $\ell+2$, depending upon whether there is a whisker attached to the penultimate vertices of each side of the spine. Note that $G'$ can have at most two new whiskers. Then we have the following cases:

Case I: Suppose $G'$ has no new whiskers. Then $G'$ is a graph with no whiskers and $t+r$ limbs. By Theorem 3.4, $\operatorname{reg}(S/J_{G'}) \leq \ell + t + r$. Since $G$ is an induced subgraph of $G$, $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/J_{G'})$.

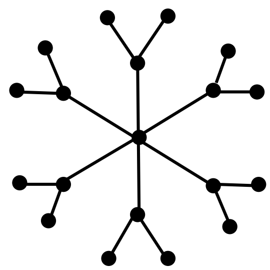
Case II: Suppose $G'$ has only one new whisker. Then extend this whisker to a limb as done above and obtain a new graph $G''$ with spine length $\ell+1$, $t+r$ limbs and no whiskers. Then by Theorem 3.4, $\operatorname{reg}(S/J_{G''}) \leq \ell + t + r + 1$. Since $G$ is an induced subgraph of $G''$, we have $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/J_{G''})$.

Case III: Suppose $G'$ has two new whiskers. As done earlier, extend these two whiskers to limbs to obtain $G''$. Then $G''$ has spine length $\ell + 2$, $t + r$ limbs and no whiskers.

Therefore by Theorem 3.4, $\operatorname{reg}(S/J_{G''}) \leq \ell + t + r + 2$. As $G$ is an induced subgraph of $G''$, we have $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/J_{G''})$.

Hence $\operatorname{reg}(S/J_G) \leq \ell + t + r + 2$. □

**Example 3.8.** *This is an example of a lobster which achieves the upper bound given in Theorem 3.7.*



$G$

*This graph $G$ has many different longest induced paths. Fixing any one of them, one can see that $G$ has spine length $\ell = 4$, $t = 4$ limbs and $r = 2$ whiskers attached to the spine. A computation on Macaulay2 shows that the* $\operatorname{reg}(S/J_G) = 12 = \ell + t + r + 2$.

Below, we obtain a precise expression for the regularity of the binmoial edge ideal of a subclass of lobsters.

**Theorem 3.9.** *If $G$ is a lobster with spine length $\ell$ and $t$ limbs of the form $K_{1,2}$ attached to the spine, then* $\operatorname{reg}(S/J_G) = \ell + t$.

*Proof.* Let $G$ be a lobster with spine $P$ of length $\ell$ and $t$ limbs of the form $K_{1,2}$ attached to the spine $P$. Let $G'$ be the caterpillar graph obtained by deleting all the leaves of $G$ which are not on the spine $P$ of $G$. Then $\operatorname{reg}(S/J_{G'}) = \ell$, [1]. Let $m$ be a leaf of $G$ and $(m, m+1)$ be an edge in $G$. Let $G''$ denote the graph $G' \cup \{(m, m+1)\}$. Then by Corollary 3.2, $\operatorname{reg}(S/J_{G''}) = \ell + 1$. Observe that $G$ is obtained by iterating the above procedure of attaching a vertex to a leaf $t$-times. Since a vertex is attached to a leaf, the regularity increases exactly by one at each step. Therefore, $\operatorname{reg}(S/J_G) = \ell + t$. □

**Corollary 3.10.** *Let $G$ be a lobster with spine $P$ of length $\ell$, $t$ limbs and $r$ whiskers. Then* $\ell + t \leq \operatorname{reg}(S/J_G) \leq \ell + t + r + 2$.

*Proof.* The upper bound is proved in Theorem 3.7. To prove the lower bound, note that $G$ has a subgraph $G'$ with spine $P$, $t$ pure limbs and without any whiskers as an induced subgraph. By Theorem 3.9, $\operatorname{reg}(S/J_{G'}) = \ell + t$ as required. □

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