

Symmetry problems on stationary isothermic surfaces in Euclidean spaces *

Shigeru Sakaguchi[†]

January 7, 2016

Abstract

Let S be a smooth hypersurface properly embedded in \mathbb{R}^N with $N \geq 3$ and consider its tubular neighborhood \mathcal{N} . We show that, if a heat flow over \mathcal{N} with appropriate initial and boundary conditions has S as a stationary isothermic surface, then S must have some sort of symmetry.

Key words. heat equation, Cauchy problem, initial-boundary value problem, tubular neighborhood, stationary isothermic surface, symmetry.

AMS subject classifications. Primary 35K05 ; Secondary 35B40, 35K15, 35K20.

1 Introduction

The stationary isothermic surfaces of solutions of the heat equation have been much studied, and it has been shown that the existence of a stationary isothermic surface forces the problems to have some sort of symmetry (see [MPeS, MPrS, MS2, MS3, MS5, MS6, MS7, S]). A balance law for stationary zeros of temperature introduced by [MS1] plays a key role in the proofs. To be more precise, the balance law gives us that for any pair of points x and y in the stationary isothermic surface the heat contents of two balls centered at x and y respectively with an equal radius are equal for every time. In the above papers, always are dealt with the cases where each ball touches the boundary only at one point eventually.

*This research was partially supported by Grants-in-Aid for Scientific Research (B) (# 26287020) of Japan Society for the Promotion of Science.

[†]Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai, 980-8579, Japan. (sigersak@m.tohoku.ac.jp).

Then by studying the initial behavior of the heat content of each ball the authors extract some information of the principal curvatures of the boundary at the touching point.

We emphasize that in the present paper are dealt with the cases where each ball touches the boundary exactly at two points. Another new point of the present paper is to introduce a stationary isothermic surface as the leading character on behalf of the boundary of a given domain, for in the above mentioned papers a stationary isothermic surface of course plays a key role but it works as a strong supporting character.

Let us make our setting. Let Ω be a C^2 domain in \mathbb{R}^N with $N \geq 3$, whose boundary $\partial\Omega$ is connected and not necessarily bounded. Namely, $\partial\Omega$ is a C^2 hypersurface properly embedded in \mathbb{R}^N . Assume that there exists a number $R > 0$ satisfying:

(A-1) : The principal curvatures $\kappa_1(x), \dots, \kappa_{N-1}(x)$ of $\partial\Omega$ at $x \in \partial\Omega$ with respect to the outward normal direction to $\partial\Omega$ satisfy

$$\max_{1 \leq j \leq N-1} |\kappa_j(x)| < \frac{1}{R} \text{ for every } x \in \partial\Omega.$$

(A-2) : The tubular neighborhood \mathcal{N}_R of $\partial\Omega$ given by

$$\mathcal{N}_R = \{x \in \mathbb{R}^N : \text{dist}(x, \partial\Omega) < R\},$$

is a C^2 domain in \mathbb{R}^N and its boundary $\partial\mathcal{N}_R$ consists of two connected components Γ_+, Γ_- each of which is diffeomorphic to $\partial\Omega$.

Let us introduce two C^2 domains Ω_+, Ω_- in \mathbb{R}^N with $\partial\Omega_+ = \Gamma_+, \partial\Omega_- = \Gamma_-$, respectively, such that three domains $\Omega_+, \Omega_-, \mathcal{N}_R$ are disjoint, $\Omega_- \subset \Omega$, and $\Omega_+ \cup \Omega_- \cup \overline{\mathcal{N}_R} = \mathbb{R}^N$. Denote by $\mathcal{X}_{\Omega_+}, \mathcal{X}_{\Omega_-}$ the characteristic functions of the sets Ω_+, Ω_- , respectively. Consider the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathcal{N}_R \times (0, +\infty), \quad (1.1)$$

$$u = 1 \quad \text{on } \partial\mathcal{N}_R \times (0, +\infty), \quad (1.2)$$

$$u = 0 \quad \text{on } \mathcal{N}_R \times \{0\}, \quad (1.3)$$

and the Cauchy problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty) \text{ and } u = \mathcal{X}_{\Omega_+} + \mathcal{X}_{\Omega_-} \quad \text{on } \mathbb{R}^N \times \{0\}. \quad (1.4)$$

We have the following theorem.

Theorem 1.1 *Let $N = 3$ and let u be the unique bounded solution of either problem (1.1)-(1.3) or problem (1.4). Assume that there exists a function $a(t)$ satisfying*

$$u(x, t) = a(t) \quad \text{for every } (x, t) \in \partial\Omega \times (0, +\infty). \quad (1.5)$$

Then, either if $\partial\Omega$ has an umbilical point $p \in \partial\Omega$, that is, $\kappa_1(p) = \kappa_2(p)$, or if $\partial\Omega$ is unbounded and there exists a sequence of points $\{p_j\} \subset \partial\Omega$ with $\lim_{j \rightarrow \infty} \kappa_1(p_j) = \lim_{j \rightarrow \infty} \kappa_2(p_j) \in \mathbb{R}$, $\partial\Omega$ must be either a plane or a sphere.

The Hopf-Poincaré theorem [H, Theorem II, p. 113] says that the sum of the indices of all the isolated umbilical points equals the Euler number $\chi(\partial\Omega) (= 2 - 2 \times \text{genus})$ of $\partial\Omega$ and hence if the genus of $\partial\Omega$ does not equal 1 then $\partial\Omega$ must have at least one umbilical point. Therefore we have the following direct corollary.

Corollary 1.2 *Let $N = 3$ and let u be the unique bounded solution of either problem (1.1)-(1.3) or problem (1.4). Assume that (1.5) holds for some function $a(t)$. Then, if $\partial\Omega$ is bounded and the genus of $\partial\Omega$ does not equal 1, $\partial\Omega$ must be a sphere.*

We next consider the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathcal{N}_R \times (0, +\infty), \quad (1.6)$$

$$u = 1 \quad \text{on } \Gamma_+ \times (0, +\infty), \quad (1.7)$$

$$u = -1 \quad \text{on } \Gamma_- \times (0, +\infty), \quad (1.8)$$

$$u = 0 \quad \text{on } \mathcal{N}_R \times \{0\}, \quad (1.9)$$

and the Cauchy problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u = \mathcal{X}_{\Omega_+} - \mathcal{X}_{\Omega_-} \quad \text{on } \mathbb{R}^N \times \{0\}. \quad (1.10)$$

Then we have

Theorem 1.3 *Let $N \geq 3$ and let u be the unique bounded solution of either problem (1.6)-(1.9) or problem (1.10). Assume that (1.5) holds for some function $a(t)$. Then, the following hold:*

- (1) *If $\partial\Omega$ is bounded, then $\partial\Omega$ must be a sphere.*
- (2) *If $N = 3$ and $\partial\Omega$ is an entire graph over \mathbb{R}^2 , then $\partial\Omega$ must be a plane.*

By using the asymptotic formula of the heat content $\int_{B_R(x)} u(z, t) dz$ for $x \in \partial\Omega$ as $t \rightarrow +0$ introduced by [MS4] together with the balance law given by [MS1], we prove Theorems 1.1 and 1.3, where $B_R(x)$ denotes an open ball centered at $x \in \mathbb{R}^N$ with radius $R > 0$. Moreover Aleksandrov's sphere theorem and Bernstein's theorem for the minimal surface equation are needed to prove Theorem 1.3. In sections 2 and 3, we prove Theorems 1.1 and 1.3, respectively. The final section 4 gives several remarks and problems.

2 Proof of Theorem 1.1

The proofs of Theorems 1.1 and 1.3 have common ingredients. Therefore we begin with general dimensions N for later use, although Theorem 1.1 assumes that $N = 3$.

Let u be the unique bounded solution of either problem (1.1)-(1.3) or problem (1.4). Denote by $u^\pm = u^\pm(x, t)$ the unique bounded solutions of the initial-boundary value problems for the heat equation:

$$u_t = \Delta u \quad \text{in } (\mathbb{R}^N \setminus \overline{\Omega_\pm}) \times (0, +\infty), \quad (2.1)$$

$$u = 1 \quad \text{on } \Gamma_\pm \times (0, +\infty), \quad (2.2)$$

$$u = 0 \quad \text{on } (\mathbb{R}^N \setminus \overline{\Omega_\pm}) \times \{0\}, \quad (2.3)$$

respectively, or of the Cauchy problems for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u = \mathcal{X}_{\Omega_\pm} \quad \text{on } \mathbb{R}^N \times \{0\}, \quad (2.4)$$

respectively. Notice that $u = u^+ + u^-$ when u is the solution of problem (1.4). Then, by a result of Varadhan [V](see also [MS7, Theorem A, p. 2024]), we see that

$$-4t \log(u^\pm(x, t)) \rightarrow \text{dist}(x, \Gamma_\pm)^2 \quad \text{as } t \rightarrow +0 \quad (2.5)$$

uniformly on every compact sets in $\mathbb{R}^N \setminus \overline{\Omega_\pm}$.

By the assumptions (A-1) and (A-2), every point $x \in \partial\Omega$ determines two points $x_+ \in \Gamma_+$ and $x_- \in \Gamma_-$ satisfying

$$\partial B_R(x) \cap \Gamma_+ = \{x_+\} \quad \text{and} \quad \partial B_R(x) \cap \Gamma_- = \{x_-\},$$

respectively. Moreover, by letting $\kappa_1^\pm(x_\pm), \dots, \kappa_{N-1}^\pm(x_\pm)$ denote the principal curvatures of Γ_\pm at x_\pm with respect to the inward normal direction to $\partial\mathcal{N}_R$, respectively, we observe that

$$1 - R\kappa_j^+(x_+) = \frac{1}{1 - R\kappa_j(x)} > 0 \quad \text{and} \quad 1 - R\kappa_j^-(x_-) = \frac{1}{1 + R\kappa_j(x)} > 0 \quad (2.6)$$

for every $x \in \partial\Omega$ and every $j = 1, \dots, N-1$.

On the other hand, it follows from the balance law (see [MS1, Theorem 4, p. 704] or [MS2, Theorem 2.1, pp. 934-935]) that (1.5) gives

$$\int_{B_R(x)} u(z, t) dz = \int_{B_R(y)} u(z, t) dz \quad \text{for } t > 0 \quad (2.7)$$

for every $x, y \in \partial\Omega$. Moreover, by virtue of (2.6), an asymptotic formula given by [MS4] (see also [MS7, Theorem B, pp. 2024-2025]) yields that

$$\lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u^\pm(z, t) dz = c(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j^\pm(x_\pm) \right] \right\}^{-\frac{1}{2}}, \quad (2.8)$$

respectively. Here, $c(N)$ is a positive constant depending only on N and of course $c(N)$ depends on the problems (2.1)-(2.3) or (2.4). Then we have

Lemma 2.1 *Let u be the unique bounded solution of either problem (1.1)-(1.3) or problem (1.4). Assume that (1.5) holds for some function $a(t)$. Then there exists a constant $c > 0$ satisfying*

$$\left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j(x)) \right\}^{\frac{1}{2}} + \left\{ \prod_{j=1}^{N-1} (1 + R\kappa_j(x)) \right\}^{\frac{1}{2}} = c \quad \text{for every } x \in \partial\Omega, \quad (2.9)$$

where $\kappa_1(x), \dots, \kappa_{N-1}(x)$ denote the principal curvatures of $\partial\bar{\Omega}$ given in (A.1).

Proof. Let u be the unique bounded solution of problem (1.4). Then we have that $u = u^+ + u^-$. Hence, combining (2.7) with (2.8) yields that there exists a constant $c > 0$ satisfying

$$\left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j^+(x_+)) \right\}^{-\frac{1}{2}} + \left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j^-(x_-)) \right\}^{-\frac{1}{2}} = c \quad (2.10)$$

for every $x \in \partial\Omega$. Therefore (2.6) gives the conclusion.

Let u be the solution of problem (1.1)-(1.3). It follows from the comparison principle that

$$\max\{u^+, u^-\} \leq u \leq u^+ + u^- \quad \text{in } \mathcal{N}_R \times (0, \infty).$$

Therefore, in view of (2.5) and (2.8), we notice that for every $x \in \partial\Omega$

$$\begin{aligned}
& c(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j^+(x_+) \right] \right\}^{-\frac{1}{2}} + c(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j^-(x_-) \right] \right\}^{-\frac{1}{2}} \\
&= \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u^+(z, t) dz + \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u^-(z, t) dz \\
&= \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x) \setminus \Omega} u^+(z, t) dz + \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x) \cap \Omega} u^-(z, t) dz \\
&= \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x) \setminus \Omega} u(z, t) dz + \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x) \cap \Omega} u(z, t) dz \\
&= \lim_{t \rightarrow +0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u(z, t) dz.
\end{aligned}$$

Hence, with the aid of (2.7), we obtain (2.10) which yields the conclusion by (2.6). \square

Proof of Theorem 1.1: Let us complete the proof. Set $N = 3$ in (2.9). With the aid of the arithmetic-geometric mean inequality, we obtain from (2.9) that

$$c = \sqrt{(1 - R\kappa_1)(1 - R\kappa_2)} + \sqrt{(1 + R\kappa_1)(1 + R\kappa_2)} \leq \frac{2 - R(\kappa_1 + \kappa_2)}{2} + \frac{2 + R(\kappa_1 + \kappa_2)}{2} = 2$$

where $\kappa_j = \kappa_j(x)$ with $j = 1, 2$. By the assumption, $\partial\Omega$ has an umbilical point $p \in \partial\Omega$, that is, $\kappa_1(p) = \kappa_2(p)$, or $\partial\Omega$ is unbounded and there exists a sequence of points $\{p_j\} \subset \partial\Omega$ with $\lim_{j \rightarrow \infty} \kappa_1(p_j) = \lim_{j \rightarrow \infty} \kappa_2(p_j) \in \mathbb{R}$. Then we conclude that $c = 2$ and the equality holds in the above inequality. Hence $\kappa_1 = \kappa_2$ on $\partial\Omega$, that is, $\partial\Omega$ is called totally umbilical. Thus by the classical differential geometry $\partial\Omega$ must be either a plane or a sphere (see [H, Remark, p. 124] or [MoR, Theorem 3.30, p. 84] for instance). \square

3 Proof of Theorem 1.3

Let us use the auxiliary functions $u^\pm = u^\pm(x, t)$ given in section 2. We begin with the following lemma:

Lemma 3.1 *Let u be the unique bounded solution of either problem (1.6)-(1.9) or problem (1.10). Assume that (1.5) holds for some function $a(t)$. Then there exists a constant c satisfying*

$$\left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j(x)) \right\}^{\frac{1}{2}} - \left\{ \prod_{j=1}^{N-1} (1 + R\kappa_j(x)) \right\}^{\frac{1}{2}} = c \text{ for every } x \in \partial\Omega, \quad (3.1)$$

where $\kappa_1(x), \dots, \kappa_{N-1}(x)$ denote the principal curvatures of $\partial\Omega$ given in (A.1).

Proof. Let u be the solution of problem (1.10). Then we have that $u = u^+ - u^-$. Therefore the conclusion follows from the same argument as in the proof of Lemma 2.1.

Let u be the solution of problem (1.6)-(1.9). It follows from the comparison principle that

$$\max\{-u^-, u^+ - 2u^-\} \leq u \leq \min\{u^+, 2u^+ - u^-\} \text{ in } \mathcal{N}_R \times (0, \infty).$$

With the aid of these inequalities, in view of (2.5) and (2.8), by carrying out calculations similar to those in the proof of Lemma 2.1 for every $x \in \partial\Omega$, we can reach the conclusion.

□

Proof of Theorem 1.3: Let us complete the proof. Set

$$\Phi(\kappa_1, \dots, \kappa_{N-1}) = - \text{ the left-hand side of (3.1).}$$

Then we have that $\frac{\partial\Phi}{\partial\kappa_j} > 0$ for $j = 1, \dots, N-1$. Then, by introducing local coordinates, the condition $\Phi(\kappa_1, \dots, \kappa_{N-1}) = \text{constant}$ on the surface $\partial\Omega$ can be converted into a second order partial differential equation which is of elliptic type. Hence, if $\partial\Omega$ is bounded, then $\partial\Omega$ must be a sphere by Aleksandrov's sphere theorem [A]. Thus proposition (1) is proved.

Let us proceed to proposition (2). Set $N = 3$ in (3.1). Then

$$\sqrt{(1 - R\kappa_1)(1 - R\kappa_2)} - \sqrt{(1 + R\kappa_1)(1 + R\kappa_2)} = c, \quad (3.2)$$

where $\kappa_j = \kappa_j(x)$ with $j = 1, 2$, and hence

$$-4RH = c \left(\sqrt{(1 - R\kappa_1)(1 - R\kappa_2)} + \sqrt{(1 + R\kappa_1)(1 + R\kappa_2)} \right), \quad (3.3)$$

where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature of $\partial\Omega$. We distinguish three cases:

- (i) $c = 0$, (ii) $c > 0$, (iii) $c < 0$.

In case (i), by (3.3) we have $H = 0$ on $\partial\Omega$ and hence $\partial\Omega$ is the minimal entire graph of a function over \mathbb{R}^2 . Therefore, by Bernstein's theorem for the minimal surface equation, $\partial\Omega$ must be a plane. This gives the conclusion desired. (See [GT, G] for Bernstein's theorem.)

In case (ii), by (3.3) we have $H < 0$ on $\partial\Omega$. Suppose that there exists a sequence of points $\{p_n\}$ with $\lim_{n \rightarrow \infty} H(p_n) = 0$. Since $R\kappa_1(p_n), R\kappa_2(p_n) \in [-1, 1]$, by the Bolzano-Weierstrass theorem, by taking a subsequence if necessary, we may assume that $\{R\kappa_1(p_n)\}, \{R\kappa_2(p_n)\}$ converge to numbers $\alpha, -\alpha$, respectively, for some $\alpha \in [-1, 1]$. Hence by (3.2) we get $c = 0$ which is a contradiction. Therefore, there exists a number $\delta > 0$ such that

$$H \leq -\delta \text{ on } \partial\Omega,$$

which contradicts the fact that $\partial\Omega$ is an entire graph over \mathbb{R}^2 with the aid of the divergence theorem as in the proof of [MS3, Theorem 3.3, pp. 2732–2733]. The remaining case (iii) can be dealt with in a similar manner. Thus proposition (2) is proved. \square

Remark 3.2 *In section 2 we did not use the same argument as in section 3, for by introducing local coordinates, the condition (2.9) on the surface $\partial\Omega$ can not be converted into a second order partial differential equation which is of elliptic type.*

4 Concluding Remarks and Problems

In this final section, we mention several remarks and problems.

Concerning Theorem 1.1, spherical cylinders satisfy the assumption (1.5). Therefore, as in [MPeS], a theorem including a spherical cylinder as a conclusion is expected. Corollary 1.2 excludes closed surfaces with genus 1, but this might be technical. Concerning Theorem 1.3, right helicoids satisfy the assumption (1.5). Therefore, a theorem including a right helicoid as a conclusion is expected.

Let us set $N = 3$ both in (2.9) and in (3.1) and assume that $\partial\Omega$ is a minimal surface properly embedded in \mathbb{R}^3 . Then (2.9) yields that the Gauss curvature is constant and hence $\partial\Omega$ must be a plane. On the other hand, (3.1) holds true for every minimal surface by setting $c = 0$.

Concerning technical points in the theory of partial differential equations, (2.9) is not of elliptic type but (3.1) is of elliptic type, as is mentioned in section 3. Therefore, for (3.1) in general dimensions, Liouville-type theorems characterizing hyperplanes are expected as in [MS5, S].

References

- [A] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large V, Vestnik Leningrad Univ. 13 (1958), 5–8 (English translation: Trans. Amer. Math. Soc. 21 (1962), 412–415).
- [GT] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, (Second Edition.), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.
- [G] E. Giusti, Minimal Surfaces and Functions of Bounded Variations, Birkhäuser, Boston, Basel, Stuttgart, 1984.
- [H] H. Hopf, Lectures on differential geometry in the large, Second Edition, Lecture Notes in Math. 1000, Springer-Verlag, Berlin-NewYork, 1989.
- [MPeS] R. Magnanini, D. Peralta-Salas and S. Sakaguchi, Stationary isothermic surfaces in Euclidean 3-space, Math. Annalen, in press.

- [MPrS] R. Magnanini, J. Prajapat, and S. Sakaguchi, Stationary isothermic surfaces and uniformly dense domains, *Trans. Amer. Math. Soc.*, 358 (2006), 4821–4841.
- [MS1] R. Magnanini and S. Sakaguchi, Spatial critical points not moving along the heat flow II : The centrosymmetric case, *Math. Z.*, 230 (1999), 695–712, Corrigendum 232 (1999), 389–389.
- [MS2] R. Magnanini and S. Sakaguchi, Matzoh ball soup: Heat conductors with a stationary isothermic surface, *Ann. of Math.* 156 (2002), 931–946.
- [MS3] R. Magnanini and S. Sakaguchi, Stationary isothermic surfaces for unbounded domains, *Indiana University Mathematics Journal*, 56 (2007), 2723–2738.
- [MS4] R. Magnanini and S. Sakaguchi, Interaction between degenerate diffusion and shape of domain, *Proceedings Royal Soc. Edinburgh, Section A*, 137 (2007), 373–388.
- [MS5] R. Magnanini and S. Sakaguchi, Stationary isothermic surfaces and some characterizations of the hyperplane in the N -dimensional Euclidean space, *J. Differential Equations* 248 (2010), 1112–1119.
- [MS6] R. Magnanini and S. Sakaguchi, Nonlinear diffusion with a bounded stationary level surface, *Ann. Inst. Henri Poincaré - (C) Anal. Non Linéaire* 27 (2010), 937–952.
- [MS7] R. Magnanini and S. Sakaguchi, Matzoh ball soup revisited: the boundary regularity issue, *Mathematical Methods in Applied Sciences*, 36 (2013), 2023–2032.
- [MoR] S. Montiel and A. Ros, *Curves and Surfaces*, Graduate Studies in Math. vol. 69, Amer. Math. Soc., 2005.
- [S] S. Sakaguchi, Stationary level surfaces and Liouville-type theorems characterizing hyperplanes, in “ Geometric Properties of Parabolic and Elliptic PDE’s ”, Springer INdAM Series, Vol. 2, 2013, 269–282.
- [V] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Appl. Math.* 20 (1967), 431–455.