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STUDY OF q-GARNIER SYSTEM BY PADÉ METHOD

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Abstract. We give a simple form of the evolution equations and the scalar Lax pair for the q-Garnier system. Some degenerations to the q-Painlevé equations and the autonomous case as a generalized QRT system are discussed. Using two kinds of Padé problems on differential grid and q-grid, we derive some special solutions of the q-Garnier system in terms of the q-Appell Lauricella function and the generalized q-hypergeometric function.

1. Introduction

The Garnier system [1, 4] is known as an important extension of the Painlevé equations to multi-variables. Its q-difference analog, the q-Garnier system, was formulated by H.Sakai in [17].

There exists a simple method to study the Painlevé/Garnier equations using Padé approximation [25]. In this method, one can obtain the evolution equation, the Lax pair and some special solutions simultaneously, starting from a suitable Padé approximation (or interpolation) problem. This method has been applied [3, 11, 12, 14, 28] to various cases of discrete Painlevé equations [7, 16]. Our aim is to study the q-Garnier system applying the Padé method. We study both the usual (i.e. differential) Padé approximation and the Padé interpolation on q-grid, and obtain two kinds of special solutions written in terms of q-Appell Lauricella function and the generalized q-hypergeometric functions.

In section 2.1, we introduce a scalar Lax pair and derive the q-Garnier equation as the necessary condition for the compatibility. In section 2.2, the relation to the Sakai's matrix form is considered, and the sufficiency for the compatibility is proved. In section 2.3, we rewrite the q-Garnier system into more explicit (but nonbirational) form. In section 2.4, we discuss the degenerations to the q-Painlevé equations of types $E_7^{(1)}$, $E_6^{(1)}$ and $D_5^{(1)}$. The $E_7^{(1)}$ case is new and $E_6^{(1)}$, $D_5^{(1)}$ cases are known [17, 19].

In section 3, we formulate a hyper-elliptic generalization of the QRT system [15, 23]. Then the generalized QRT system is identified as the autonomous limit of the q-Garnier system.

In section 4, we study certain Padé problem on differential grid. In section 4.1, we show that the solutions of the Padé problem give special solutions of the Lax equation and q-Garnier system. In section 4.2, we derive the explicit expressions of the special solutions in terms of the q-Appell Lauricella function [18].

Similarly, in section 5 we study certain Padé problem on q-grid and obtain special solutions in terms of the generalized q-hypergeometric functions. We note that the higher order q-Painlevé system given by Suzuki [22] also has special solutions given in terms of the generalized q-hypergeometric functions.

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2. A simple form of the q-Garnier system

In this section, we give a reformulation of the evolution equations and the scalar Lax equations of the q-Garnier system.

2.1. Lax pairs and the q-Garnier equation. Fix a positive integer N and a complex parameter q $(0<|q|<1)$. Let $a_1,\dots,a_{N+1}$, $b_1,\dots,b_{N+1}$, c_1,c_2,d_1,d_2 be complex parameters with a constraint $\prod_{j=1}^{2N+1}\frac{a_i}{b_i}=q\prod_{i=1}^{2}\frac{c_i}{d_i}$ and $T_a: a\mapsto qa$ be the q-shift operator of parameter a.

In this section, we put

$$T:=T_{a_1}^{-1}T_{b_1}^{-1}, \tag{2.1}$$

and the corresponding shifts are denoted as $\overline{X}:=T(X)$, $\underline{X}:=T^{-1}(X)$. The operator T plays the role of time evolution of the q-Garnier system. Though one can choose any a_i,b_j instead of a_1,b_1, we consider the case $(i,j)=(1,1)$ for notational simplicity.

For an unknown function $y(x)$, we consider two linear equations: $L_2(x)$ between $y(x),y(qx),\overline{y}(x)$ and $L_3(x)$ between $y(x),\overline{y}(x),\overline{y}(x/q)$ defined as follows:

$$\begin{aligned} &L_2(x):=F(f,x)\overline{y}(x)-A_1(x)y(qx)+(x-b_1)G(g,x)y(x)=0,\\ &L_3(x):=F(\overline{f},x/q)y(x)+(x-a_1)G(g,x/q)\overline{y}(x)-qc_1c_2B_1(x/q)\overline{y}(x/q)=0,\end{aligned} \tag{2.2}$$

where

(2.3)

$$\begin{aligned} &A(x):=\prod_{j=1}^{N+1}(x-a_j),\quad B(x):=\prod_{j=1}^{N+1}(x-b_j),\quad A_1(x):=\frac{A(x)}{x-a_1},\quad B_1(x):=\frac{B(x)}{x-b_1},\\ &F(f,x):=\sum_{j=0}^{N}f_jx^j\quad G(g,x):=\sum_{j=0}^{N-1}g_jx^j,\end{aligned}$$

and $f_0,\dots,f_N,\overline{f}_0,\dots,\overline{f}_N,g_0,\dots,g_{N-1}$ are some variables independent of x.

Proposition 2.1. *The compatibility of L_2 and L_3 (2.2) gives the following conditions:*

$$c_1c_2A_1(x)B_1(x)-(x-a_1)(x-b_1)G(g,x)G(\underline{g},x)=0\quad\text{for}\quad F(f,x)=0, \tag{2.4}$$

$$qc_1c_2A_1(x)B_1(x)-F(f,x)F(\overline{f},x)=0\quad\text{for}\quad G(g,x)=0, \tag{2.5}$$

$$f_N\overline{f}_N=q(g_{N-1}-c_1)(g_{N-1}-c_2),\qquad f_0\overline{f}_0=a_1b_1(g_0-e_1)(g_0-e_2), \tag{2.6}$$

where $e_i:=d_i\nu/a_1b_1$, $\nu:=\prod_{j=1}^{N+1}(-a_j)$.

Proof. Under the condition $F(f,x)=0$, eliminating $y(x),y(qx)$ from $L_2(x)=\underline{L}_3(qx)=0$, we obtain eq.(2.4). Similarly, for $G(g,x)=0$ eliminating $y(qx),\overline{y}(x)$ from $L_2(x)=L_3(qx)=0$, we have eq.(2.5) Considering the highest coefficients of $L_2(x)$ and $L_3(x)$, we have the first equation of (2.6). Similarly, considering the lowest coefficients of $L_2(x)$ and $L_3(x)$, we have the second equation of (2.6). □

We remark that similar computations based on the contiguity type Lax pair have been done in [3, 14, 28].

Though eqs.(2.4)−(2.6) are given as equation for $2N+1$ variables $f_0,\dots,f_N,g_0,\dots,g_{N-1}$, they can be reduced to equations for $2N$ variables $\frac{f_1}{f_0},\dots,\frac{f_N}{f_0},g_0,\dots,g_{N-1}$ (see eqs.(2.17)−(2.19) in section 2.3). In section 2.2, eqs.(2.4)−(2.6) will be proved to be sufficient for the compatibility of L_2, L_3.

The most fundamental object is the linear three term equation $L_1(x)$ between $y(qx)$, $y(x)$, $y(x/q)$. Eliminating $\overline{y}(x), \overline{y}(\frac{x}{q})$ from $L_2(x), L_2(\frac{x}{q}), L_3(x)$ (2.2), we have the following expression for the three term equation $L_1(x)$:

$$\begin{aligned} L_1(x) &:= A(x)F(f,\frac{x}{q})y(qx) + qc_1c_2B(\frac{x}{q})F(f,x)y(\frac{x}{q}) \\ &-\Big\{(x-a_1)(x-b_1)F(f,\frac{x}{q})G(g,x) + \frac{F(f,x)}{G(g,\frac{x}{q})}V_1(f,\overline{f},\frac{x}{q})\Big\}y(x) = 0, \end{aligned} \tag{2.7}$$

where

$$V_1(f,\overline{f},x) := qc_1c_2A_1(x)B_1(x) - F(f,x)F(\overline{f},x). \tag{2.8}$$

Lemma 2.2. *The linear equation $L_1(x)$ (2.7) has the following properties: (i) it is a polynomial of degree $2N+1$ in x, (ii) the exponents are d_1, d_2 (at $x = 0$) and c_1, c_2 (at $x = \infty$), (iii) the N points x with $F(f,x) = 0$ are the apparent singularities (i.e., the solutions are regular there) such that*

$$\frac{y(qx)}{y(x)} = \frac{G(g,x)(x-b_1)}{A_1(x)} \quad \textit{for} \quad F(f,x) = 0. \tag{2.9}$$

Moreover, the coefficient of $y(x)$ in equation $L_1(x)$ is uniquely characterized by these properties once the coefficients of $y(qx)$ and $y(x/q)$ are given in the equation $L_1(x)$.

Proof. The properties (i)−(iii) follows by computation using the eqs.(2.5), (2.6). The polynomiality of the coefficient of $y(x)$ follows from eq.(2.5). The second half can easily be confirmed by counting the number of coefficients. □

2.2. **Correspondence to Sakai's Lax form.** In [17], Sakai formulated the q-Garnier system as a multivariable extension of the sixth q-Painlevé equation, by using the connection preserving deformation of a linear q-difference equation as follows.

$$Y(qx) = \mathcal{A}(x)Y(x), \quad \mathcal{A}(x) := \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}, \quad Y(x) := \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}. \tag{2.10}$$

The coefficient matrix $\mathcal{A}(x)$ are defined by the following conditions: (i) $\mathcal{A}(x) := \sum_{i=0}^{N+1} A_i x^i$, (ii) $A_{N+1} := \mathrm{diag}(\kappa_1, \kappa_2)$ and A_0 has eigenvalues θ_1, θ_2. (iii) $\det\mathcal{A}(x) = \kappa_1\kappa_2 \prod_{i=1}^{2N+2}(x - \alpha_i)$, such that $\kappa_1\kappa_2 \prod_{i=1}^{2N+2} \alpha_i = \theta_1\theta_2$. The conditions (i),(ii),(iii) determine the matrix $\mathcal{A}(x)$ up to $2N+1$ free parameters. $2N$ of them are the dependent variables of the q-Garnier system, and one natural choice of them are given by $\{\lambda_i, \mu_i\}_{i=1}^N$ where $b(\lambda_i) = 0$, $\mu_i = a(\lambda_i) = y_1(q\lambda_i)/y_1(\lambda_i)$ (a kind of Sklyanin's "magic recipe", see [20], [21] for example). The remaining one (the normalization of the polynomial $b(x)$) is a gauge parameter.

The system (2.10) can be equivalently described by the following scalar equation for the first component $y_1(x)$:

$$b(x/q)y_1(qx) - \{b(x/q)a(x) + b(x)d(x/q)\}y_1(x) + b(x)\det\mathcal{A}(x/q)y_1(x/q) = 0. \tag{2.11}$$

Here N zeros of $b(x)$ are apparent singularities .

Proposition 2.3. *The linear three term equation L_1 (2.7) is equivalent to eq.(2.11) up to a gauge transformation and a change of parameters.*

Proof. By a gauge transformation: $y_1(x) = H(x)\tilde{y}_1(x)$ with $H(qx)/H(x) = \prod_{i=1}^{N+1}(x-\alpha_i)$, the system (2.11) can be written as

$$\begin{aligned}&\textstyle\prod_{i=1}^{N+1}(x-\alpha_i)b(x/q)\tilde{y}_1(qx) - \{b(x/q)a(x)+b(x)d(x/q)\}\tilde{y}_1(x)\\ &+\kappa_1\kappa_2\textstyle\prod_{i=N+2}^{2N+2}(x/q-\alpha_i)b(x)\tilde{y}_1(x/q) = 0.\end{aligned} \tag{2.12}$$

Here, (i) the coefficient of $\tilde{y}_1(x)$ is a polynomial of degree $2N+1$ in x, (ii) the exponents are $\theta_1/\prod_{i=1}^{N+1}(-\alpha_i), \theta_2/\prod_{i=1}^{N+1}(-\alpha_i)$ (at $x=0$) and $\kappa_1, q^{-1}\kappa_2$ (at $x=\infty$), (iii) N zeros of $b(x)$ are apparent singularities. Then, these conditions determine the equation (2.12) up to $2N$ free parameters. Due to Lemma 2.2, we see that $L_1(x)$ (2.7) is equivalent to eq.(2.12) up to a change of parameters. □

As we will show below, eqs.(2.4)−(2.6) are sufficient for the compatibility of L_1, L_2 (or L_3). Hence eqs.(2.4)−(2.6) can be regarded as the q-Garnier system.

To prove the sufficiency, we first study the linear three term equation $L_1^*(x)$ between $\overline{y}(qx)$, $\overline{y}(x)$, $\overline{y}(x/q)$. Eliminating $y(x), y(qx)$ from $L_2(x), L_3(x), L_3(qx)$ (2.2), we have the following expression:

$$\begin{aligned}L_1^*(x) &:= \overline{A}(x)F(\overline{f},\frac{x}{q})\overline{y}(qx) + qc_1c_2\overline{B}(\frac{x}{q})F(\overline{f},x)\overline{y}(\frac{x}{q})\\ &-\frac{1}{q}\Big\{(x-a_1)(x-b_1)F(\overline{f},x)G(g,\frac{x}{q}) + \frac{F(\overline{f},\frac{x}{q})}{G(g,x)}V_1(f,\overline{f},x)\Big\}\overline{y}(x) = 0,\end{aligned} \tag{2.13}$$

where $V_1(f,\overline{f},x)$ is given in eq.(5.11).

The following can be proved in the similar way as Lemma 2.2.

Lemma 2.4. *The linear equation $L_1^*(x)$(2.13) has the following properties: (i) it is a polynomial of degree $2N+1$ in x, (ii) exponents are d_1, d_2 (at $x=0$) and c_1, c_2 (at $x=\infty$), (iii) the points x with $F(\overline{f},x)=0$ are the apparent singularities such that*

$$\frac{\overline{y}(qx)}{\overline{y}(x)} = \frac{B_1(x)}{c_1c_2G(g,x)(x-a_1/q)} \quad \textit{for} \quad F(\overline{f},x)=0. \tag{2.14}$$

Moreover, the coefficient of $\overline{y}(x)$ in equation $L_1^*(x)$ can be characterized by these properties once the coefficients of $\overline{y}(qx)$ and $\overline{y}(x/q)$ are given in the equation $L_1^*(x)$.

Theorem 2.5. *The linear equations L_1 (2.7) and L_2 (2.2) are compatible if and only if equations (2.4)−(2.6) for variables f, g are satisfied.*

Proof. The compatibility means that $T(L_1) = L_1^*$, i.e. the commutativity of the following:

$$\begin{array}{ccc} L_1^* \text{ (Lemma 2.4)} & = & L_1^* \text{ (2.13)}\\ \uparrow & & \uparrow\\ T\text{-shift} & & L_2, L_3 \text{ (2.2)}\\ \uparrow & & \downarrow\\ L_1 \text{ (Lemma 2.2)} & = & L_1 \text{ (2.7)}.\end{array}$$

This can be checked by the characterizations of L_1 and L_1^* in Lemma 2.2 and 2.4, and the relation: T(2.9)= (2.14), which follows from eq.(2.4). □

2.3. **Expressions in terms of roots.** Introduce variables λ_i, μ_i $(i=1,\dots,N)$ such that $F(f,x)=\Lambda(x):=f_N\prod_{i=1}^N(x-\lambda_i)$ and $\mu_i=\frac{y(q\lambda_i)}{y(\lambda_i)}$. We note that the variables $\{\mu_i\}$ are related to $\{g_i\}$ as $\mu_i=\frac{(x-b_1)G(g,x)}{A_1(x)}|_{x=\lambda_i}$. Then the linear three term equation L_1 (2.7) can be written as

$$\begin{aligned}L_1(x)=&\frac{A(x)}{x\Lambda(x)}y(qx)+\frac{qc_1c_2B(x/q)}{x\Lambda(x/q)}y(x/q)\\&-\Big[\frac{\nu(d_1+d_2)}{f_0x}+\frac{(c_1+c_2)}{f_N}+\sum_{i=1}^N\frac{1}{\lambda_i\Lambda'(\lambda_i)}\Big(\frac{A(\lambda_i)\mu_i}{x-\lambda_i}+\frac{qc_1c_2B(\lambda_i)}{(x-q\lambda_i)\mu_i}\Big)\Big]y(x)=0,\end{aligned}\tag{2.15}$$

where $f_0=f_N\prod_{i=1}^N(-\lambda_i)$ and $\nu=\prod_{j=1}^{N+1}(-a_j)$. As a curve in (λ_1,μ_1), the equation $L_1=0$ has the following characterization: (i) It is a polynomial of bidegree $(N+2,2)$, (ii) passing through the following $3N+9$ points $(0,d_i)_{i=1}^2$, $(\infty,c_i)_{i=1}^2$, $(b_i,0)_{i=1}^{N+1}$, $(a_i,\infty)_{i=1}^{N+1}$, $(x,0)$, $(\frac{x}{q},\infty)$, $(x,\frac{y(qx)}{y(x)})$, $(\frac{x}{q},\frac{y(x)}{y(\frac{x}{q})})$, $(\lambda_i,\mu_i)_{i=2}^N$ $(i\neq 1)$. By the symmetry, there exist similar characterizations for the other variables (λ_i,μ_i) also. This is a generalization of the geometric characterizations for the discrete Painlevé equations as a curve of bidegree $(3,2)$ passing through 12 points [7, 26, 27].

The linear equations L_2, L_3 (2.2) can be also written as

$$\begin{aligned}L_2(x)&=\Lambda(x)\overline{y}(x)-A_1(x)y(qx)+(x-b_1)\Xi(x)y(x)=0,\\L_3(x)&=\overline{\Lambda}(x/q)y(x)+(x-a_1)\Xi(x/q)\overline{y}(x)-qc_1c_2B_1(x/q)\overline{y}(x/q)=0,\end{aligned}\tag{2.16}$$

where $G(g,x)=\Xi(x):=g_{N-1}\prod_{i=1}^{N-1}(x-\xi_i)$. Then the evolution equations (2.4)–(2.6) can be written as

$$\Xi(\lambda_i)\underline{\Xi}(\lambda_i)=c_1c_2\frac{A_1(\lambda_i)B_1(\lambda_i)}{(\lambda_i-a_1)(\lambda_i-b_1)}\quad(i=1,\dots,N),\tag{2.17}$$

$$\frac{\Lambda(\xi_i)\overline{\Lambda}(\xi_i)}{f_N\overline{f}_N}=c_1c_2\frac{A_1(\xi_i)B_1(\xi_i)}{(g_{N-1}-c_1)(g_{N-1}-c_2)}\quad(i=1,\dots,N-1),\tag{2.18}$$

$$\prod_{i=1}^N\lambda_i\overline{\lambda}_i=\frac{a_1b_1}{q}\frac{(g_0-e_1)(g_0-e_2)}{(g_{N-1}-c_1)(g_{N-1}-c_2)},\tag{2.19}$$

where $g_0=g_{N-1}\prod_{i=1}^{N-1}(-\xi_i)$. The eqs.(2.17)–(2.19) are the q-Garnier equation in terms of $2N$ variables $\lambda_1,\dots,\lambda_N,\xi_1,\dots,\xi_{N-1},g_0$ (or g_{N-1}).

2.4. **Degeneration to the q-Painlevé equations.** We give a few comments on lower cases $N=1,2,3$. In [17], the q-Painlevé equation of type $D_5^{(1)}$ has appeared as a case for the q-Garnier system with $N=1$. This is easily seen from eqs.(2.4)–(2.6). For $N=2$ case, it is known [19] that the q-Painlevé equation of type $E_6^{(1)}$ appears as a particular case for the q-Garnier system with $N=2$. In fact, we have

Proposition 2.6. *For the case $N=2$ with a constraint $c_1=c_2$, the q-Garnier equation (2.4)–(2.6) admit the following reduction*

$$\begin{aligned}(fg-1)(f\underline{g}-1)&=\frac{(1-a_2f)(1-a_3f)(1-b_2f)(1-b_3f)}{(1-a_1f)(1-b_1f)},\\(fg-1)(f\overline{g}-1)&=\frac{(1-g/a_2)(1-g/a_3)(1-g/b_2)(1-g/b_3)}{(1-c_1g/e_1)(1-c_1g/e_2)},\end{aligned}\tag{2.20}$$

where e_i is the same as in Proposition 2.1.

Proof. Under the constraint, eqs.(2.4)−(2.6) admit a specialization $f_2 = 0$ and $g_1 = c_1$. Then we obtain the results where $f = -f_1/f_0$, $g = -g_0/c_1$. □

The eqs.(2.20) is the q-Painlevé equation of type $E_6^{(1)}$ [7, 16].

For $N = 3$ case, the q-Painlevé equation of type $E_7^{(1)}$ appears as a particular case for the q-Garnier system with $N = 3$. In fact, we have

Proposition 2.7. *For the case $N = 3$ with constraints $d_1 = d_2$ and $c_1 = c_2$, the q-Garnier equation (2.4)−(2.6) admit the following reduction*

(2.21)

$$\left\{g+(f+\tfrac{e_1}{c_1 f})\right\}\left\{\underline{g}+(f+\tfrac{qe_1}{c_1 f})\right\} = \frac{(f-a_2)(f-a_3)(f-a_4)(f-b_2)(f-b_3)(f-b_4)}{f^2(f-a_1)(f-b_1)},$$

$$\frac{(1-x_1/f)(1-x_1/\overline{f})}{(1-x_2/f)(1-x_2/\overline{f})} = \frac{x_2^2(x_1-a_2)(x_1-a_3)(x_1-a_4)(x_1-b_2)(x_1-b_3)(x_1-b_4)}{x_1^2(x_2-a_2)(x_2-a_3)(x_2-a_4)(x_2-b_2)(x_2-b_3)(x_2-b_4)},$$

where e_i is the same as in Proposition 2.1 and $x = x_1, x_2$ are solutions of the equation $g + (x + \frac{e_1}{c_1 x}) = 0$.

Proof. Under the constraint, eqs.(2.4)−(2.6) admit a specialization $f_0 = f_3 = 0$, $g_0 = e_1$ and $g_2 = c_1$. Then we obtain the results where $f = -f_1/f_2$, $g = g_1/c_1$. □

The eqs.(2.21) for the variables f, g is a kind of the q-Painlevé equation of type $E_7^{(1)}$, but the direction of the time evolution is different from the standard one [7, 16]. The relation of them will be discussed in [13].

2.5. **Correspondence of parameters and variables in §2, §4, §5.** In this paper, parameters a_i, b_i $(i = 1, \dots, N+1)$, c_i, d_i $(i = 1, 2)$, m, n and variables f_i, g_i $(i = 0, \dots, N-1)$, w_0, w_1 are used in slightly different means in §2, §4, §5. Their relations are given as follows:

(2.22)

$$\begin{aligned}
&a_i^{\S2} = 1/a_i^{\S4} = 1/a_i^{\S5} \quad (i=1,\dots,N+1), \quad a_{N+1}^{\S2} = 1/a_{N+1}^{\S4}, \quad a_{N+1}^{\S5} = 1/(q^{m+n})^{\S5},\\
&b_i^{\S2} = 1/b_i^{\S4} = 1/b_i^{\S5} \quad (i=1,\dots,N+1), \quad b_{N+1}^{\S2} = 1/b_{N+1}^{\S4}, \quad b_{N+1}^{\S5} = q,\\
&c_1^{\S4} = (q^m)^{\S4}, \quad c_1^{\S5} = (q^m)^{\S5}, \quad c_2^{\S4} = (q^n \textstyle\prod_{j=1}^{N+1} \frac{b_j}{a_j})^{\S4}, \quad c_2^{\S5} = (cq^n \prod_{j=1}^{N} \frac{b_j}{a_j})^{\S5},\\
&d_1^{\S4} = d_1^{\S5} = 1, \quad d_2^{\S4} = (q^{m+n+1})^{\S4}, \quad d_2^{\S5} = c^{\S5},\\
&f_0^{\S2} = (\tfrac{-a_1}{\nu} w_0)^{\S4} = (\tfrac{-a_1}{\nu} w_0)^{\S5}, \quad \overline{f}_0^{\S2} = (\tfrac{-b_1 d_1 d_2}{\nu} w_1)^{\S4} = (\tfrac{-b_1 d_1 d_2}{\nu} w_1)^{\S5},\\
&f_i^{\S2} = (\tfrac{-a_1}{\nu} w_0 f_i)^{\S4} = (\tfrac{-a_1}{\nu} w_0 f_i)^{\S5} \quad (i=0,\dots,N), \quad f_0^{\S4} = f_0^{\S5} = 1,\\
&\overline{f}_i^{\S2} = (\tfrac{-b_1 d_1 d_2}{\nu} w_1 \overline{f}_i)^{\S4} = (\tfrac{-b_1 d_1 d_2}{\nu} w_1 \overline{f}_i)^{\S5} \quad (i=0,\dots,N), \quad \overline{f}_0^{\S4} = \overline{f}_0^{\S5} = 1\\
&w_0^{\S4} = 1 - (\tfrac{g_0}{d_1})^{\S4} \quad w_1^{\S4} = 1 - (\tfrac{g_0}{d_2})^{\S4}, \quad w_0^{\S5} = 1 - (\tfrac{g_0}{d_1})^{\S5} \quad w_1^{\S5} = 1 - (\tfrac{g_0}{d_2})^{\S5},\\
&g_i^{\S2} = (\tfrac{a_1 b_1}{\nu} g_i)^{\S4} = (\tfrac{a_1 b_1}{\nu} g_i)^{\S5} \quad (i=0,\dots,N-1),
\end{aligned}$$

where $\nu = \prod_{j=1}^{N+1}(-a_j)$.

3. Autonomous case

In this section, we define a generalization of the QRT system [15] (see also [23]) for hyperelliptic curves and discuss its relation to the q-Garnier system.

3.1. **Generalization of the QRT map for hyperelliptic curve.** Let C be a curve of bidegree $(N+1,2)$ on $\mathbb{P}^1\times\mathbb{P}^1$ which passes through given $2N+5$ points $P_1,\dots,P_{2N+5}$. The number of free parameters of the defining polynomial is $3(N+2)-(2N+5)=N+1$, hence the curves C form an N dimensional family, generically of genus N.

The dynamical variables of the generalized QRT mapping is a set (a divisor) of N points $\{Q_1,\dots,Q_N\}$ on the curve C. Following Mumford [10] we represent it by a pair of functions $\Phi(x),\Psi(x)$, where $\Phi(x)$ is a polynomial of degree N and $\Psi(x):=\frac{S(x)}{R(x)}$ is a rational function of degree N, such that $Q_i=(x_i,\Psi(x_i))$, $\Phi(x_i)=0$, $(i=1,\dots,N)$. Note that the normalization of $\Phi(x)$ is irrelevant. A generalized QRT map is defined as follows:

(1) Fix N free parameters of the curve C so that it passes the initial points $Q_1,\dots,Q_N$. We represent the resulting curve as $C_0:\varphi(x,y):=\alpha(x)y^2+\beta(x)y+\gamma(x)=0$.

(2) Take a subset of indices $I\subset\{1,\dots,2N+5\}$ with $|I|=N+1$ and determine the rational function $\Psi(x)=\frac{S(x)}{R(x)}$ uniquely by the condition that the curve $y=\Psi(x)$ passes through the points $P_i=(x_{P_i},y_{P_i})(i\in I)$ and $Q_1,\cdots,Q_N$. By definition $\varphi_0(x,\Psi(x))$ is divisible by $\prod_{i\in I}(x-x_{P_i})\Phi(x)$ and we can define an involution $\iota_x^I:(\Phi,\Psi)\mapsto(\tilde{\Phi},\Psi)$ by the relation

$$R(x)^2\varphi(x,\Psi(x))=\prod_{i\in I}(x-x_{P_i})\Phi(x)\tilde{\Phi}(x). \tag{3.1}$$

(3) Since the polynomial $\varphi(x,y)$ is of degree 2 in y, the other involution $\iota_y:(x,y)\mapsto(x,\tilde{y})$ can be defined simply as $y\tilde{y}=\frac{\gamma(x)}{\alpha(x)}$. Namely we have $\iota_y:(\Phi,\Psi)\mapsto(\Phi,\tilde{\Psi})$ where

$$\Psi(x)\tilde{\Psi}(x)\alpha(x)=\gamma(x),\quad \text{for } \Phi(x)=0. \tag{3.2}$$

(4) We have the generalized QRT map is defined by the iteration $T_I:=\iota_y\iota_x^I$ or $T_I^{-1}:=\iota_x^I\iota_y$. They are the commutativity $T_IT_J=T_JT_I$ since they are translations on the Jacobian of the curve C_0.

3.2. **Relation to the q-Garnier system.** In order to apply the algorithm in previous subsection to the q-Garnier system (2.4)−(2.6), we consider the case where the points $P_1,\dots,P_{2N+5}$ are in the configuration $(a_i,\infty)_{i=1}^{N+1}$, $(b_i,0)_{i=1}^{N+1}$, $(\infty,c_i)_{i=1}^{2}$ and $(0,d_i)_{i=1}^{2}$. Here we included an additional point P_{2N+6} whose position is determined by the constraint $\prod_{i=1}^{N+1}\frac{a_i}{b_i}=\prod_{i=1}^{2}\frac{c_i}{d_i}$. Then the curve $C_0:\varphi(x,y)=0$ of bidegree $(N+1,2)$ can be written as

$$\begin{aligned}&\varphi(x,y):=A(x)y^2-U(x)y+c_1c_2B(x)=0,\\ &U(x):=\nu(d_1+d_2)+\sum_{i=1}^{N}u_ix^i+(c_1+c_2)x^{N+1}.\end{aligned} \tag{3.3}$$

where $A(x):=\prod_{i=1}^{N+1}(x-a_i),B(x):=\prod_{i=1}^{N+1}(x-b_i)$, $\nu:=\prod_{i=1}^{N+1}(-a_i)$. Note that the lowest/highest terms of $\varphi(x,y)$ in x are given by $\varphi|_{x^0}=\nu(y-d_1)(y-d_2)$ and $\varphi|_{x^{N+1}}=(y-c_1)(y-c_2)$. The parameters (conserved quantities) $u_1,\cdots,u_N$ are determined by the condition $\varphi(Q_i)$ for the initial points: $Q_1,\cdots,Q_N$.

To adjust the formulation given above to that in section 2, we take the index set I as $\{P_i|i\in I\}=\{(a_i,\infty)_{i=2}^{N+1},(b_1,0)\}$, and put $\Phi(x):=F(x)$ and $\Psi(x):=\frac{(x-b_1)G(x)}{A_1(x)}$ where $F(x):=\sum_{i=0}^{N}f_ix^i$ and $G(x):=\sum_{i=0}^{N-1}g_ix^i$. Then, the ι_x^I-flip defined by eq.(3.1) takes the form

$$(x-a_1)(x-b_1)G(x)^2-U(x)G(x)+c_1c_2A_1(x)B_1(x)=F(x)\tilde{F}(x), \tag{3.4}$$

where $A_1(x) := \frac{A(x)}{x-a_i}$ and $B_1(x) := \frac{B(x)}{x-b_i}$. This relation determines the polynomial $\tilde{F}(x)$ by

$$F(x)\tilde{F}(x) = c_1c_2A_1(x)B_1(x), \quad \text{for } G(x) = 0, \tag{3.5}$$

$$f_N\tilde{f_N} = (g_{N-1} - c_1)(g_{N-1} - c_2), \quad f_0\tilde{f_0} = a_1b_1(g_0 - e_1)(g_0 - e_2), \tag{3.6}$$

where $e_i := d_i\nu/a_1b_1$. On the other hand, the ι_y-flip (3.2) gives

$$(x - a_1)(x - b_1)G(x)\tilde{G}(x) = c_1c_2A_1(x)B_1(x), \quad \text{for } F(x) = 0. \tag{3.7}$$

It is easy to see that

Proposition 3.1. *The eqs.(3.5)−(3.7) correspond to the autonomous ($q = 1$) version of the q-Garnier system (2.4)−(2.6).*

4. The Padé problem on differential grid

In this section, we will study certain Padé approximation problem and solve it explicitly. As a result, we will obtain some special solutions of q-Garnier system in terms of q-Appell Lauricella function.

4.1. **Lax pairs and the q-Garnier equation.** In this subsection, starting the Padé approximation problem (4.3), we will derive the three term relations (4.5), (4.16) and nonlinear difference relations (4.13)−(4.15). We put

$$\psi(x) := \prod_{i=1}^{N+1} \frac{(a_ix)_\infty}{(b_ix)_\infty}. \tag{4.1}$$

Here and in what follows, we use the standard q-Pochhammer symbols defined as

$$(z)_\infty := \prod_{i=0}^{\infty}(1 - q^iz), \quad (z)_s := \frac{(z)_\infty}{(zq^s)_\infty}, \quad (z_1, z_2, \dots, z_k)_s := (z_1)_s(z_2)_s \dots (z_k)_s. \tag{4.2}$$

Define polynomials $P(x)$ and $Q(x)$ of degree m and $n \in \mathbb{Z}_{\geq 0}$ by the following Padé approximation condition:

$$\psi(x) = \frac{P(x)}{Q(x)} + O(x^{m+n+1}). \tag{4.3}$$

Here the common normalizations of the polynomials $P(x), Q(x)$ are fixed as $P(0) = 1$ tentatively.

In this section, due to the change of parameters in eq (2.22), the shift T of the parameters are given by

$$T := T_{a_1}T_{b_1}. \tag{4.4}$$

Let us consider two linear three term relations: $L_2(x)$ between $y(x), y(qx), \overline{y}(x)$ and $L_3(x)$ between $y(x), \overline{y}(x), \overline{y}(x/q)$ satisfied by the functions $y(x) = P(x)$ and $y(x) = \psi(x)Q(x)$. The following Proposition shows that these equations can be regarded as L_2 and L_3 equations for q-Garnier system.

Proposition 4.1. *The linear three term relations L_2 and L_3 can be written as follows:*

$$\begin{aligned} &L_2(x) := (b_1x)_1G(g, x)y(x) - A_1(x)y(qx) + (g_0)_1F(f, x)\overline{y}(x) = 0, \\ &L_3(x) := (rg_0)_1F(\overline{f}, x/q)y(x) + r(a_1x)_1G(g, x/q)\overline{y}(x) - B_1(x/q)\overline{y}(x/q) = 0, \end{aligned} \tag{4.5}$$

where

$$A(x) := \prod_{j=1}^{N+1} (a_j x)_1, \quad B(x) := \prod_{j=1}^{N+1} (b_j x)_1, \quad F(f,x) := 1 + \sum_{j=1}^{N} f_j x^j,$$
$$A_1(x) := \frac{A(x)}{(a_1 x)_1}, \quad B_1(x) := \frac{B(x)}{(b_1 x)_1}, \quad G(g,x) := \sum_{j=0}^{N-1} g_j x^j. \tag{4.6}$$

Here $r := q^{-(m+n+1)}$ *and* $f_1, \dots, f_N, \overline{f}_1, \dots, \overline{f}_N, g_0, \dots, g_{N-1}$ *are some constants depending on parameters* a_i, b_i, m, n *but independent of* x.

Proof. By the definition of the linear relations L_2 and L_3, they can be written as

$$L_2(x) = \begin{vmatrix} y(x) & y(qx) & \overline{y}(x) \\ P(x) & P(qx) & \overline{P}(x) \\ \psi(x)Q(x) & \psi(qx)Q(qx) & \overline{\psi}(x)\overline{Q}(x) \end{vmatrix} = 0, \tag{4.7}$$

$$L_3(x) = \begin{vmatrix} y(x) & \overline{y}(x) & \overline{y}(x/q) \\ P(x) & \overline{P}(x) & \overline{P}(x/q) \\ \psi(x)Q(x) & \overline{\psi}(x)\overline{Q}(x) & \overline{\psi}(x/q)\overline{Q}(x/q) \end{vmatrix} = 0. \tag{4.8}$$

Setting $\mathbf{y}(x) := \begin{bmatrix} P(x) \\ \psi(x)Q(x) \end{bmatrix}$, define Casorati determinants $D_1(x)$, $D_2(x)$ and $D_3(x)$ by

$$D_1(x) := \det[\mathbf{y}(x), \mathbf{y}(qx)], \quad D_2(x) := \det[\mathbf{y}(x), \overline{\mathbf{y}}(x)], \quad D_3(x) := \det[\mathbf{y}(qx), \overline{\mathbf{y}}(x)]. \tag{4.9}$$

Then, the linear relations L_2 and L_3 take the following forms:

$$L_2(x) = D_1(x)\overline{y}(x) - D_2(x)y(qx) + D_3(x)y(x) = 0,$$
$$L_3(x) = \overline{D}_1(x/q)y(x) + D_3(x/q)\overline{y}(x) - D_2(x)\overline{y}(x/q) = 0. \tag{4.10}$$

The determinants (4.9) can be computed by the condition (4.3) and the relations

$$\frac{\psi(qx)}{\psi(x)} = \prod_{i=1}^{N+1} \frac{(b_i x)_1}{(a_i x)_1}, \quad \frac{\overline{\psi}(x)}{\psi(x)} = \frac{(b_1 x)_1}{(a_1 x)_1}. \tag{4.11}$$

The results are given as

$$D_1(x) = \frac{\psi(x)}{A(x)} \{B(x)P(x)Q(qx) - A(x)P(qx)Q(x)\} =: C_0 \frac{\psi(x)x^{m+n+1}}{A(x)} F(f,x),$$
$$D_2(x) = \frac{\psi(x)}{(a_1 x)_1} \left\{(b_1 x)_1 P(x)\overline{Q}(x) - (a_1 x)_1 \overline{P}(x)Q(x)\right\} =: \frac{C_1 \psi(x) x^{m+n+1}}{(a_1 x)_1},$$
$$D_3(x) = \frac{\psi(x)}{A(x)} \left\{(b_1 x)_1 A_1(x) P(qx)\overline{Q}(x) - B(x)\overline{P}(x)Q(qx)\right\} =: \frac{C_1 \psi(x) x^{m+n+1}}{A(x)} (b_1 x)_1 G(g,x), \tag{4.12}$$

with some constant C_0, C_1. Substituting eqs.(4.12) into eq.(4.10), we obtain eq.(4.5), where the constants C_0, C_1 were fixed as $C_0 = (g_0)_1$, $C_1 = (rg_0)_1$ by the condition that eq.(4.5) have a solution such as $y(0) = P(0) = 1$. □

The following Proposition can be proved in the similar way as Proposition 2.1.

Proposition 4.2. *The compatibility of L_2 and L_3 (4.5) gives the following conditions:*

$$A_1(x)B_1(x) - r(a_1x, b_1x)_1 G(g,x)G(\underline{g},x) = 0 \quad \text{for} \quad F(f,x)=0, \tag{4.13}$$

$$A_1(x)B_1(x) - (g_0, rg_0)_1 F(f,x)F(\overline{f},x) = 0 \quad \text{for} \quad G(g,x)=0, \tag{4.14}$$

$$(g_0, rg_0)_1 f_N \overline{f}_N = \Big(qra_1 g_{N-1} + \frac{\prod_{i=2}^{N+1}(-b_i)}{s}\Big)\Big(b_1 g_{N-1} + s\prod_{i=2}^{N+1}(-a_i)\Big), \tag{4.15}$$

where r ($s := q^m$, resp.) is one of the exponents of the linear equation $L_1(x)$ (4.16) at $x = 0$ ($x = \infty$, resp.). These equations are regarded as the q-Garnier system (2.4)–(2.6).

We derive the three term relation $L_1(x)$ between $y(qx)$, $y(x)$, $y(x/q)$ satisfied by the functions $y(x) = P(x)$ and $y(x) = \psi(x)Q(x)$. The following Proposition can be given by the similar proof as in Lemma 2.2.

Proposition 4.3. *The three term relation $L_1(x)$ can be written in the form*

$$\begin{aligned} L_1(x) &:= rA(x)F(f,\frac{x}{q})y(qx) + B(\frac{x}{q})F(f,x)y(\frac{x}{q}) \\ &-\Big\{r(a_1x,b_1x)_1F(f,\frac{x}{q})G(g,x) + \frac{F(f,x)}{G(g,\frac{x}{q})}V_1(f,\overline{f},\frac{x}{q})\Big\}y(x) = 0, \end{aligned} \tag{4.16}$$

where

$$V_1(f,\overline{f},x) := A_1(x)B_1(x) - (g_0, rg_0)_1 F(f,x)F(\overline{f},x). \tag{4.17}$$

The equation $L_1(x)$ has the following properties: (i) it is a polynomial of degree $2N+1$ in x, (ii) the exponents are $1, r^{-1}$ (at $x = 0$) and $s, \frac{1}{qrs}\prod_{i=1}^{N+1}\frac{b_i}{a_i}$ (at $x = \infty$), (iii) the N points x with $F(f,x) = 0$ are the apparent singularities (i.e., the solutions are regular there) such that

$$\frac{y(qx)}{y(x)} = \frac{(b_1x)_1G(x)}{A_1(x)}. \tag{4.18}$$

Moreover, the coefficient of $y(x)$ in equation $L_1(x)$ is uniquely characterized by these properties once the coefficients of $y(qx)$ and $y(x/q)$ are given in the equation $L_1(x)$.

Proof. Similar to the proof of Lemma 2.2. □

4.2. **Special solutions.** We derive the explicit forms (4.26)–(4.28) of variables $\{f_i, g_i\}$ appearing in the Casorati determinants D_1 and D_3 (4.12). They are interpreted as the special solutions for q-Garnier system (4.13)–(4.15) due to the results of previous subsection.

Proposition 4.4. *For any given function $\psi(x) = \sum_{k=0}^{\infty} p_k x^k$, ($p_i = 0, i < 0$), the polynomials $P(x)$ and $Q(x)$ of degree m and n for the approximation condition (4.3) are given by*

$$P(x) = \sum_{i=0}^{m} s_{(m^n,i)}x^i, \quad Q(x) = \sum_{i=0}^{n} s_{((m+1)^i,m^{n-i})}(-x)^i, \tag{4.19}$$

where $m^n := (\underbrace{m, m, \ldots, m}_{n})$ *and* s_λ *is the Schur function defined by the Jacobi Trudi formula*

$$s_{(\lambda_1,\ldots,\lambda_l)} := \det(p_{\lambda_i-i+j})_{i,j=1}^{l}. \tag{4.20}$$

For the proof, see section 2 of [25].

Lemma 4.5. *The polynomials $P(x)$ and $Q(x)$ in proposition 4.4 can be expressed in terms of a single determinant as*

$$P(x) = x^m s_{(m^{n+1})}\big|_{p_i \to \sum_{j=0}^{i} x^{-j} p_{i-j}}, \quad Q(x) = (-x)^n s_{((m+1)^n)}\big|_{p_i \to p_i - x^{-1} p_{i-1}}. \tag{4.21}$$

Proof. Direct computation of the right hand side of eqs.(4.21). □

Note that the normalization of the polynomials in eqs.(4.19),(4.21) are different from the convention $P(0) = 1$. However, this difference does not affect the results in the following Proposition 4.7, since the common normalization factors of $P(x)$ and $Q(x)$ are cancels in eqs.(4.29)−(4.31).

Then, we apply the general results described above to the function $\psi(x)$ in eq.(4.1) which can be written as

$$\psi(x) = \sum_{k=0}^{\infty} p_k x^k = \exp\Big(\sum_{k=1}^{\infty}\sum_{s=1}^{N+1} \frac{b_s^k - a_s^k}{k(1-q^k)} x^k\Big). \tag{4.22}$$

We note that this kind of expression (4.22) has already appeared in [24].

By definition (4.1), it is easy to see the properties for p_k as follows:

$$\begin{aligned} &T_{a_s}^{-1}(p_i) = p_i - \frac{1}{q} a_s p_{i-1}, \quad T_{b_s}(p_i) = p_i - b_s p_{i-1}, \\ &T_{a_s}(p_i) = \sum_{j=0}^{i} a_s^j p_{i-j}, \quad T_{b_s}^{-1}(p_i) = \sum_{j=0}^{i} (b_s/q)^j p_{i-j}, \end{aligned} \tag{4.23}$$

for $s = 1, \ldots, N+1$.

Proposition 4.6. *The polynomials $P(x)$ and $Q(x)$ have the following special values:*

$$\begin{aligned} &P\Big(\frac{1}{a_s}\Big) = \Big(\frac{1}{a_s}\Big)^m T_{a_s}(\tau_{m,n+1}), \quad Q\Big(\frac{q}{a_s}\Big) = \Big(-\frac{q}{a_s}\Big)^n T_{a_s}^{-1}(\tau_{m+1,n}) \\ &P\Big(\frac{q}{b_s}\Big) = \Big(\frac{q}{b_s}\Big)^m T_{b_s}^{-1}(\tau_{m,n+1}), \quad Q\Big(\frac{1}{b_s}\Big) = \Big(-\frac{1}{b_s}\Big)^n T_{b_s}(\tau_{m+1,n}), \end{aligned} \tag{4.24}$$

for $s = 1, \ldots, N+1$. Where $\tau_{m,n}$ is defined as

$$\tau_{m,n} := s_{(m^n)} = \det(p_{m-i+j})_{i,j=1}^{n}. \tag{4.25}$$

Proof. Follows from (4.23) and the formula (4.21). □

Proposition 4.7. *The polynomials $F(f,x)$ and $G(g,x)$ are determined as follows:*

$$\frac{F(f,\frac{1}{a_i})}{F(f,\frac{1}{b_j})} = -q^{n-m} \frac{a_i}{b_j} \frac{B(\frac{1}{a_i})}{A(\frac{1}{b_j})} \frac{T_{a_i}(\tau_{m,n+1}) T_{a_i}^{-1}(\tau_{m+1,n})}{T_{b_j}^{-1}(\tau_{m,n+1}) T_{b_j}(\tau_{m+1,n})} \quad (i,j = 1,\ldots,N+1), \tag{4.26}$$

$$G(g,\frac{1}{a_i}) = -q^n \frac{a_i}{a_1} \frac{B_1(\frac{1}{a_i})}{(b_1/a_1)_1} \frac{T_{a_i}(\overline{\tau}_{m,n+1}) T_{a_i}^{-1}(\tau_{m+1,n})}{T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\overline{\tau}_{m+1,n})} \quad (i = 2,\ldots,N+1). \tag{4.27}$$

$$G(g,\frac{1}{b_i})=-q^m\frac{b_i}{b_1}\frac{A_1(\frac{1}{b_i})}{(a_1/b_1)_1}\frac{T_{b_i}^{-1}(\tau_{m,n+1})T_{b_i}(\overline{\tau}_{m+1,n})}{T_{b_1}^{-1}(\overline{\tau}_{m,n+1})T_{b_1}(\tau_{m+1,n})}\quad (i=2,\dots,N+1), \tag{4.28}$$

Proof. From the first equation of (4.12), we have

$$\frac{F(f,\frac{1}{a_i})}{F(f,\frac{1}{a_j})}=-\Big(\frac{a_i}{b_j}\Big)^{m+n+1}\frac{B(\frac{1}{a_i})}{A(\frac{1}{b_j})}\frac{P(\frac{1}{a_i})Q(\frac{q}{a_i})}{P(\frac{q}{b_j})Q(\frac{1}{b_j})}\quad (i,j=1,\dots,N+1). \tag{4.29}$$

From the second and third equations of (4.12), we have

$$G(g,\tfrac{1}{a_i})=-\Big(\frac{a_i}{a_1}\Big)^{m+n+1}\frac{B_1(\frac{1}{a_i})}{(b_1/a_1)_1}\frac{\overline{P}(\frac{1}{a_i})Q(\frac{q}{a_i})}{P(\frac{1}{a_1})\overline{Q}(\frac{1}{a_1})},\quad (i=2,\dots,N+1) \tag{4.30}$$

$$G(g,\tfrac{1}{b_i})=-\Big(\frac{b_i}{b_1}\Big)^{m+n+1}\frac{A_1(\frac{1}{b_i})}{(a_1/b_1)_1}\frac{P(\frac{q}{b_i})\overline{Q}(\frac{1}{b_i})}{\overline{P}(\frac{1}{b_1})Q(\frac{1}{b_1})}.\quad (i=2,\dots,N+1) \tag{4.31}$$

Substituting the special values (4.24) into the expressions (4.29)−(4.31) respectively, we obtain eqs.(4.26)−(4.28). □

We remark that the function p_k can be written in terms of the q-Appell Lauricella function φ_D [2] as follows:

Proposition 4.8. *The function p_k can be explicitly written as*

$$p_k=\frac{b_{N+1}^k\left(\frac{a_{N+1}}{b_{N+1}}\right)_k}{(q)_k}\varphi_D\Big(q^{-k},\frac{a_1}{b_1},\dots,\frac{a_N}{b_N},q^{-k+1}\frac{b_{N+1}}{a_{N+1}};q\frac{b_1}{a_{N+1}},\dots,q\frac{b_N}{a_{N+1}}\Big), \tag{4.32}$$

$$\varphi_D(\alpha,\beta_1,\dots,\beta_N,\gamma;z_1,\dots,z_N):=\sum_{m_i\geq 0}\frac{(\alpha)_{|m|}(\beta_1)_{m_1}\dots(\beta_N)_{m_N}}{(\gamma)_{|m|}(q)_{m_1}\dots(q)_{m_N}}z_1^{m_1}\dots z_N^{m_N}, \tag{4.33}$$

where $|m|:=m_1+\ldots+m_N$.

Proof. By the definition of $\psi(x)$ (4.1) and the q-binomial theorem, we have

$$\psi(x)=\sum_{m_i\geq 0}\frac{\left(\frac{a_1}{b_1}\right)_{m_1}\dots\left(\frac{a_{N+1}}{b_{N+1}}\right)_{m_{N+1}}}{(q)_{m_1}\dots(q)_{m_{N+1}}}b_1^{m_1}\dots b_{N+1}^{m_{N+1}}x^{m_1+\dots+m_{N+1}}. \tag{4.34}$$

Note that for $k\geq m_{N+1}$, we have

$$\frac{\left(\frac{a_{N+1}}{b_{N+1}}\right)_{m_{N+1}}}{(q)_{m_{N+1}}}=\frac{\left(\frac{a_{N+1}}{b_{N+1}}\right)_k(q^{-k})_{k-m_{N+1}}}{(q)_k\left(q^{-k+1}\frac{b_{N+1}}{a_{N+1}}\right)_{k-m_{N+1}}}\Big(q\frac{b_{N+1}}{a_{N+1}}\Big)^{k-m_{N+1}}. \tag{4.35}$$

Substituting eq.(4.35) with $k=|m|+m_{N+1}$ into eq.(4.34), we obtain eq.(4.32). □

In [18], a hypergeometric solution of the q-Garnier system is given in terms of the q-Appell Lauricella function φ_D (4.33). Our result corresponds to its determinantal generalization in terminating case. For the differential Garnier system, a more general determinant formula applicable also to the transcendental solutions is derived by applying the (Hermite-)Padé approximation [8, 9].

5. The Padé problem on q-grid

In this section, we will study certain Padé interpolation problem and solve it explicitly. As a result, we will obtain some special solutions in terms of the generalized q-hypergeometric function.

5.1. **Lax pairs and the q-Garnier system.** In this subsection, starting the Padé interpolation problem (5.2), we will derive the three term relations (5.3), (5.10) and the nonlinear difference relations (5.7)−(5.9).

For complex parameters $a_1, \ldots, a_N, b_1, \ldots, b_N, c \in \mathbb{C}^\times$, we put

$$\psi(x) := c^{\log_q x} \prod_{i=1}^{N} \frac{(a_i x, b_i)_\infty}{(a_i, b_i x)_\infty}. \tag{5.1}$$

Define polynomials $P(x)$ and $Q(x)$ of degree m and $n \in \mathbb{Z}_{\geq 0}$ by the following Padé approximation condition:

$$\psi(x_s) = \frac{P(x_s)}{Q(x_s)} \quad (x_s = q^s, s = 0, 1, \ldots m+n) \tag{5.2}$$

The common normalizations of the polynomials $P(x), Q(x)$ are fixed as $P(0) = 1$ tentatively. In this subsection, the shift T is given as (4.4).

Proposition 5.1. *For $y(x) = P(x)$ and $y(x) = \psi(x)Q(x)$, we have the following relations:*

(5.3)

$$\begin{aligned} L_2(x) &:= (b_1 x)_1 G(g, x) y(x) - (x/q^{m+n})_1 A_1(x) y(qx) + (g_0)_1 F(f, x) \overline{y}(x) = 0, \\ L_3(x) &:= (g_0/c)_1 F(\overline{f}, x/q) y(x) + \frac{1}{c}(a_1 x)_1 G(g, x/q) \overline{y}(x) - (x)_1 B_1(x/q) \overline{y}(x/q) = 0, \end{aligned}$$

where

$$\begin{aligned} &A(x) := \prod_{j=1}^{N} (a_j x)_1, \quad B(x) := \prod_{j=1}^{N} (b_j x)_1, \quad F(f, x) := 1 + \sum_{j=1}^{N} f_j x^j, \\ &A_1(x) := \frac{A(x)}{(a_1 x)_1}, \quad B_1(x) := \frac{B(x)}{(b_1 x)_1}, \quad G(g, x) := \sum_{j=0}^{N-1} g_j x^j, \end{aligned} \tag{5.4}$$

and $f_0, \ldots, f_N, \overline{f}_1, \ldots, \overline{f}_N, g_0, \ldots, g_{N-1}$ are some constants depending on parameters a_i, b_i, c, m, n but independent of x.

Proof. By the definition of the linear relations L_2 and L_3, they can be written as (4.7). Define Casorati determinants $D_1(x)$, $D_2(x)$ and $D_3(x)$ by (4.9). Then, the linear relations L_2 and L_3 can take the forms (4.10). The determinants (4.9) can be computed by the condition (5.2) and the relations

$$\frac{\psi(qx)}{\psi(x)} = c\frac{B(x)}{A(x)}, \quad \frac{\overline{\psi}(x)}{\psi(x)} = \frac{(a_1, b_1 x)_1}{(a_1 x, b_1)_1}. \tag{5.5}$$

The results are given as

$$
\begin{aligned}
D_1(x) &= \frac{\psi(x)}{A(x)}\{cB(x)P(x)Q(qx) - A(x)P(qx)Q(x)\} \\
&=: C_0\frac{\psi(x)}{A(x)}\prod_{i=0}^{m+n-1}(x/q^i)_1F(f,x), \\
D_2(x) &= \frac{\psi(x)}{(a_1x,b_1)_1}\left\{(a_1,b_1x)_1P(x)\overline{Q}(x) - (a_1x,b_1)_1\overline{P}(x)Q(x)\right\} \\
&=: \frac{C_1\psi(x)\prod_{i=0}^{m+n}(x/q^i)_1}{(a_1x,b_1)_1}, \\
D_3(x) &= \frac{\psi(x)}{A(x)(b_1)_1}\left\{(a_1,b_1x)_1A_1(x)P(qx)\overline{Q}(x) - c(b_1)_1B(x)\overline{P}(x)Q(qx)\right\} \\
&=: \frac{C_1\psi(x)\prod_{i+0}^{m+n-1}(x/q^i)_1}{A(x)(b_1)_1}(b_1x)_1G(g,x),
\end{aligned}
\tag{5.6}
$$

with some constant C_0, C_1. Substituting eqs.(5.6) into eq.(4.10), we obtain eq.(5.3), where the constants C_0, C_1 were fixed as $C_0 = (g_0)_1, C_1 = (g_0/c)_1$ by the condition that eq.(5.3) have a solution such as $y(0) = P(0) = 1$. □

Proposition 5.2. *The compatibility of L_2 and L_3 (5.3) gives the following conditions:*

$$
(qx, \frac{x}{q^{m+n}})_1A_1(x)B_1(x) - \frac{1}{c}(a_1x,b_1x)_1G(g,x)G(\underline{g},x) = 0 \quad \text{for} \quad F(f,x) = 0,
\tag{5.7}
$$

$$
(qx, \frac{x}{q^{m+n}})_1A_1(x)B_1(x) - (g_0, \frac{g_0}{c})_1F(f,x)F(\overline{f},x) = 0 \quad \text{for} \quad G(g,x) = 0,
\tag{5.8}
$$

$$
(g_0, \frac{g_0}{c})_1f_N\overline{f}_N = \Big(\frac{qa_1}{c}g_{N-1} - \frac{\prod_{i=2}^{N}(-b_i)}{q^{m-1}}\Big)\Big(b_1g_{N-1} - \frac{\prod_{i=2}^{N}(-a_i)}{q^n}\Big),
\tag{5.9}
$$

where c (q^m, resp.) is one of the exponents of the linear equation $L_1(x)$ at $x = 0$ ($x = \infty$, resp.). These equations are regarded as the q-Garnier system (2.4)–(2.6).

Proof. Similar to the proof of Proposition 2.1. □

The following Proposition shows that the equation has the properties as $L_1(x)$ equation for q-Garnier system.

Proposition 5.3. *The three term relation $L_1(x)$ between $y(qx)$, $y(x)$, $y(x/q)$ satisfied by the functions $y(x) = P(x)$ and $y(x) = \psi(x)Q(x)$ can be written in the form*

$$
\begin{aligned}
L_1(x) &:= \frac{1}{c}(\frac{x}{q^{m+n}})_1A(x)F(f,\frac{x}{q})y(qx) + (x)_1B(\frac{x}{q})F(f,x)y(\frac{x}{q}) \\
&-\Big\{\frac{1}{c}(a_1x,b_1x)_1F(f,\frac{x}{q})G(g,x) + \frac{F(f,x)}{G(g,\frac{x}{q})}V_1(f,\overline{f},\frac{x}{q})\Big\}y(x) = 0,
\end{aligned}
\tag{5.10}
$$

where

$$
V_1(f,\overline{f},x) := (qx, \frac{x}{q^{m+n}})_1A_1(x)B_1(x) - (g_0, \frac{g_0}{c})_1F(f,x)F(\overline{f},x).
\tag{5.11}
$$

The equation $L_1(x)$ has the following properties: (i) it is a polynomial of degree $2N+1$ in x, (ii) the exponents are $1, c$ (at $x = 0$) and $q^m, cq^n\prod_{i=1}^{N}\frac{b_i}{a_i}$ (at $x = \infty$), (iii) the N

points x with $F(f,x)=0$ are the apparent singularities (i.e., the solutions are regular there) such that

$$\frac{y(qx)}{y(x)}=\frac{(b_1x)_1G(x)}{A_1(x)(x/q^{m+n})_1}. \tag{5.12}$$

Moreover, the coefficient of $y(x)$ in equation $L_1(x)$ is uniquely characterized by these properties once the coefficients of $y(qx)$ and $y(x/q)$ are given in the equation $L_1(x)$.

Proof. Similar to the proof of Lemma 2.2. □

5.2. **Special solutions.** We derive the explicit forms (5.20)−(5.22) of variables $\{f_i, g_i\}$ appearing in the Casorati determinants D_1 and D_3 (5.6). They are interpreted as the special solutions for q-Garnier system (5.7)−(5.9).

Proposition 5.4. [5] *For a given sequence ψ_s, the polynomials $P(x)$ and $Q(x)$ of degree m and n for an interpolation problem*

$$\psi_s=P(x_s)/Q(x_s)\quad (s=0,1,\dots,m+n) \tag{5.13}$$

are given by the following determinant expressions:

$$P(x)=F(x)\det\Big[\sum_{s=0}^{m+n}u_s\frac{x_s^{i+j}}{x-x_s}\Big]_{i,j=0}^{n},\quad Q(x)=\det\Big[\sum_{s=0}^{m+n}u_sx_s^{i+j}(x-x_s)\Big]_{i,j=0}^{n-1}, \tag{5.14}$$

where $u_s:=\psi_s/F'(x_s)$ and $F(x):=\prod_{i=0}^{m+n}(x-x_i)$.

Lemma 5.5. *In the q-grid case of problem (5.13) (i.e., $x_s=q^s$), the formulae (5.14) take the following form:*

$$\begin{aligned} P(x)&=\frac{F(x)}{(q)_{m+n}^{n+1}}\det\Big[\sum_{s=0}^{m+n}\psi_s\frac{(q^{-(m+n)})_s}{(q)_s}\frac{q^{s(i+j+1)}}{x-q^s}\Big]_{i,j=0}^{n},\\ Q(x)&=\frac{1}{(q)_{m+n}^{n}}\det\Big[\sum_{s=0}^{m+n}\psi_s\frac{(q^{-(m+n)})_s}{(q)_s}q^{s(i+j+1)}(x-q^s)\Big]_{i,j=0}^{n-1}. \end{aligned} \tag{5.15}$$

Proof. In the derivation of (5.15), we have used the relations

$$F'(x_s)=(q)_s(q)_{m+n}/q^s(q^{-(m+n)})_s. \tag{5.16}$$

Substituting the value of $F'(x_s)$ (5.16) into the formulae (5.14), one obtains the determinant formulae (5.15). □

The normalization of the polynomials in eqs.(5.14), (5.15) are different from the convention $P(0)=1$ in section 5.1 and 5.2. As before, this difference does not affect the results in the following Proposition 5.7.

Proposition 5.6. *The polynomials $P(x)$ and $Q(x)$ defined in section 5.1 have the following special values:*

$$\begin{aligned} P(1/a_s)&=\frac{(a_s)_{m+n+1}}{a_s^m(a_s)_1^{n+1}(q)_{m+n}^{n+1}}T_{a_s}(\tau_{m,n}),\quad Q(q/a_s)=\frac{q^n(a_s/q)_1^n}{a_s^n(q)_{m+n}^n}T_{a_s}^{-1}(\tau_{m+1,n-1})\\ P(q/b_s)&=\frac{q^m(b_s/q)_{m+n+1}}{b_s^m(b_s/q)_1^{n+1}(q)_{m+n}^{n+1}}T_{b_s}^{-1}(\tau_{m,n}),\quad Q(1/b_s)=\frac{(b_s)_1^n}{b_s^n(q)_{m+n}^n}T_{b_s}(\tau_{m+1,n-1}), \end{aligned} \tag{5.17}$$

for $s = 1,\dots,N$. Here $\tau_{m,n}$ is defined as

$$\tau_{m,n} := \det\left[{}_{N+1}\varphi_N\left(\begin{matrix} b_1,\dots,b_N,q^{-(m+n)} \\ a_1,\dots,a_N \end{matrix}, cq^{i+j+1}\right)\right]_{i,j=0}^{n}, \tag{5.18}$$

and the q-HGF (the q-hypergeometric functions [2]) is defined by

$$ {}_k\varphi_l\left(\begin{matrix} \alpha_1, & \dots, & \alpha_k \\ \beta_1, & \dots, & \beta_l \end{matrix}, x\right) := \sum_{s=0}^{\infty} \frac{(\alpha_1,\dots,\alpha_k)_s}{(\beta_1,\dots,\beta_l,q)_s}\left[(-1)^s q^{\binom{s}{2}}\right]^{1+l-k} x^s, \tag{5.19}$$

with $\binom{s}{2} := s(s-1)/2$.

Proof. Follows from the formula (5.15) and the sequence $\psi_s = c^s \prod_{i=1}^N \frac{(b_i)_s}{(a_i)_s}$. □

Proposition 5.7. *The polynomials $F(f,x)$ and $G(g,x)$ are determined as follows:*

$$\frac{F(f,1/a_i)}{F(f,1/b_j)} = \alpha \frac{T_{a_i}(\tau_{m,n})T_{a_i}^{-1}(\tau_{m+1,n-1})}{T_{b_j}^{-1}(\tau_{m,n})T_{b_j}(\tau_{m+1,n-1})} \quad (i,j=1,\dots,N), \tag{5.20}$$

$$G(g,1/a_i) = \beta \frac{T_{a_i}(\overline{\tau}_{m,n})T_{a_i}^{-1}(\tau_{m+1,n-1})}{T_{a_1}(\tau_{m,n})T_{a_1}^{-1}(\overline{\tau}_{m+1,n-1})} \quad (i=2,\dots,N), \tag{5.21}$$

$$G(g,1/b_i) = \gamma \frac{T_{b_i}^{-1}(\tau_{m,n})T_{b_i}(\overline{\tau}_{m+1,n-1})}{T_{b_1}^{-1}(\overline{\tau}_{m,n})T_{b_1}(\tau_{m+1,n-1})} \quad (i=2,\dots,N), \tag{5.22}$$

where

$$\begin{aligned} \alpha &= -cq^{n-m}\frac{(a_iq^{m+n})_1(b_j/q)_1^n(a_i/q)_1^n}{(a_i)_1^{n+1}(b_j)_1^n}\frac{B(1/a_i)}{A(1/b_j)}, \\ \beta &= c\frac{(b_1,a_iq^{m+n})_1(a_i/q)_1^n B_1(1/a_i)}{a_1q^m(b_1/a_1)_1(a_i)_1^{n+1}}, \quad \gamma = \frac{(a_1)_1(b_i)_1^n A_1(1/b_i)}{b_1q^n(a_1/b_1)_1(b_i/q)_1^n}. \end{aligned} \tag{5.23}$$

Proof. Taking the ratio $D_1(1/a_i)/D_1(1/b_j)$ (5.6), we have

$$\frac{F(f,a_i)}{F(f,a_j)} = -c\prod_{s=0}^{m+n-1}\frac{(1/b_jq^s)_1}{(1/a_iq^s)_1}\frac{B(1/a_i)}{A(1/b_j)}\frac{P(1/a_i)Q(q/a_i)}{P(q/b_j)Q(1/b_j)} \quad (i,j=1,\dots,N). \tag{5.24}$$

Taking the ratio $D_3(1/a_i)/D_2(1/a_1)$ (5.6), we have

$$G(g,a_i) = -\frac{c(b_1)_1B_1(1/a_i)}{(a_1,b_1/a_1)_1}\frac{\prod_{s=0}^{m+n}(1/a_1q^s)_1}{\prod_{s=0}^{m+n-1}(1/a_iq^s)_1}\frac{\overline{P}(1/a_i)Q(q/a_i)}{P(1/a_1)\overline{Q}(1/a_1)} \quad (i=2,\dots,N) \tag{5.25}$$

Taking the ratio $D_3(1/a_i)/D_2(1/a_1)$ (5.6), we have

$$G(g,b_i) = -\frac{\prod_{s=0}^{m+n}(1/b_1q^s)_1}{\prod_{s=0}^{m+n-1}(1/b_iq^s)_1}\frac{(a_1)_1A_1(1/b_i)}{(a_1/b_1,b_1)_1}\frac{P(q/b_i)\overline{Q}(1/b_i)}{\overline{P}(1/b_1)Q(1/b_1)} \quad (i=2,\dots,N). \tag{5.26}$$

Substituting the special values (5.17) into the expressions (5.24)−(5.26) respectively, we obtain the values (5.20)−(5.22). □

In [22], some special solution of the higher order q-Painlevé system is given in terms of the q-hypergeometric function ${}_{N+1}\varphi_N$. Our results suggest the relation between the system in [22] and q-Garnier system. In fact, it turns out that these two are equivalent as will be shown in [13].

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