WEAK DIMENSION OF FP-INJECTIVE MODULES OVER CHAIN RINGS

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ABSTRACT. It is proven that the weak dimension of each FP-injective module over a chain ring which is either Archimedean or not semicoherent is less or equal to 2. This implies that the projective dimension of any countably generated FP-injective module over an Archimedean chain ring is less or equal to 3.

By [7, Theorem 1], for any module G over a commutative arithmetical ring R the weak dimension of G is $0,\ 1,\ 2$ or ∞. In this paper we consider the weak dimension of almost FP-injective modules over a chain ring. This class of modules contains the one of FP-injective modules and these two classes coincide if and only if the ring is coherent. If G is an almost FP-injective module over a chain ring R then its weak dimension is possibly infinite only if R is semicoherent and not coherent. In the other cases the weak dimension of G is at most 2. Moreover this dimension is not equal to 1 if R is not an integral domain. Theorem 15 summarizes main results of this paper.

We complete this short paper by considering almost FP-injective modules over local fqp-rings. This class of rings was introduced in [1] by Abuhlail, Jarrar and Kabbaj. It contains the one of arithmetical rings. It is shown the weak dimension of G is infinite if G is an almost FP-injective module over a local fqp-ring which is not a chain ring (see Theorem 23).

All rings in this paper are unitary and commutative. A ring R is said to be a **chain ring**[1] if its lattice of ideals is totally ordered by inclusion. Chain rings are also called valuation rings (see [9]). If M is an R-module, we denote by w.d.(M) its **weak dimension** and p.d.(M) its **projective dimension**. Recall that w.d.$(M) \leq n$ if $\mathrm{Tor}^R_{n+1}(M,N) = 0$ for each R-module N. For any ring R, its **global weak dimension** w.gl.d(R) is the supremum of w.d.(M) where M ranges over all (finitely presented cyclic) R-modules. Its **finitistic weak dimension** f.w.d.(R) is the supremum of w.d.(M) where M ranges over all R-modules of finite weak dimension.

A ring is called **coherent** if all its finitely generated ideals are finitely presented. As in [14], a ring R is said to be **semicoherent** if $\mathrm{Hom}_R(E,F)$ is a submodule of a flat R-module for any pair of injective R-modules $E,\ F$. A ring R is said to be **IF** (**semi-regular** in [14]) if each injective R-module is flat. If R is a chain ring, we denote by P its maximal ideal, Z its subset of zerodivisors which is a prime ideal and $Q(= R_Z)$ its fraction ring. If x is an element of a module M over a ring R, we denote by $(0 : x)$ the annihilator ideal of x and by $\mathrm{E}(M)$ the injective hull of M.

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[1]we prefer "chain ring " to "valuation ring" to avoid confusion with "Manis valuation ring".

Some preliminary results are needed to prove Theorem 15 which is the main result of this paper.

Proposition 1. [7, Proposition 4]. *Let R be a chain ring. The following conditions are equivalent:*

1. *R is semicoherent;*
2. *Q is an IF-ring;*
3. *Q is coherent;*
4. *either $Z = 0$ or Z is not flat;*
5. *$\mathrm{E}(Q/Qa)$ is flat for each nonzero element a of Z;*
6. *there exists $0 \neq a \in Z$ such that $(0 : a)$ is finitely generated over Q.*

An exact sequence of R-modules $0 \to F \to E \to G \to 0$ is **pure** if it remains exact when tensoring it with any R-module. Then, we say that F is a **pure** submodule of E. When E is flat, it is well known that G is flat if and only if F is a pure submodule of E. An R-module E is **FP-injective** if $\mathrm{Ext}^1_R(F, E) = 0$ for any finitely presented R-module F. We recall that a module E is FP-injective if and only if it is a pure submodule of every overmodule. We define a module G to be **almost FP-injective** if there exist a FP-injective module E and a pure submodule D such that $G \cong E/D$. By [15, 35.9] the following theorem holds:

Theorem 2. *A ring R is coherent if and only if each almost FP-injective R-module is FP-injective.*

Proposition 3. *For any ring R the following conditions are equivalent:*

1. *R is self FP-injective;*
2. *each flat R-module is almost FP-injective.*

Proof. (1) $\Rightarrow$ (2). Each flat module G is of the form F/K where F is free and K is a pure submodule of F. Since F is FP-injective then G is almost FP-injective.

(2) $\Rightarrow$ (1). We use the fact that R is projective to show that R is a direct summand of a FP-injective module. □

Proposition 4. *Let R be a ring and $0 \to L \xrightarrow{v} M \to N \to 0$ a pure exact sequence of R-modules. Then* $\mathrm{w.d.}(M) = \max(\mathrm{w.d.}(L), \mathrm{w.d.}(N))$.

Proof. Let $m = \mathrm{w.d.}(M)$ and let G be an R-module. We consider the following flat resolution of G:

$$F_p \xrightarrow{u_p} F_{p-1} \dots F_2 \xrightarrow{u_2} F_1 \xrightarrow{u_1} F_0 \to G \to 0.$$

For each positive integer p let G_p be the image of u_p. We have the following exact sequence:

$$\mathrm{Tor}^R_1(G_m, M) \to \mathrm{Tor}^R_1(G_m, N) \to G_m \otimes_R L \xrightarrow{id_{G_m} \otimes v} G_m \otimes_R M.$$

Since $\mathrm{Tor}^R_1(G_m, M) \cong \mathrm{Tor}^R_{m+1}(G, M) = 0$ and $id_{G_m} \otimes v$ is a monomorphism we deduce that $\mathrm{Tor}^R_{m+1}(G, N) \cong \mathrm{Tor}^R_1(G_m, N) = 0$. So, $\mathrm{w.d.}(N) \leq m$. Now it is easy to show first, that $\mathrm{w.d.}(L) \leq m$ too, and then, if $q = \max(\mathrm{w.d.}(L), \mathrm{w.d.}(N))$, that $\mathrm{Tor}^R_{q+1}(G, M) = 0$ for each R-module G. Hence $m = q$. □

Proposition 5. *Let R be a chain ring. Then:*

1. $\mathrm{w.d.}(R/Z) = \infty$ *if R is semicoherent and $Z \neq 0$;*
2. $\mathrm{w.d.}(R/Z) = 1$ *if R is not semicoherent.*

Proof. We consider the following exact sequence: $0 \to R/Z \to Q/Z \to Q/R \to 0$. If $R \neq Q$ then $\mathrm{w.d.}(Q/R) = 1$.

(1). By [4, Propositions 8 and 14] applied to Q and Proposition 1 $\mathrm{w.d.}(Q/Z) = \infty$. By using the previous exact sequence when $R \neq Q$ we get $\mathrm{w.d.}(R/Z) = \infty$.

(2). By Proposition 1 $\mathrm{w.d.}(Q/Z) = 1$. When $R \neq Q$ we conclude by using the previous exact sequence. □

Proposition 6. *Let R be a chain ring. If R is IF, let $H = \mathrm{E}(R/rR)$ where $0 \neq r \in P$. If R is not IF, let $x \in \mathrm{E}(R/Z)$ such that $Z = (0 : x)$ and $H = \mathrm{E}(R/Z)/Rx$. Then:*

1. *H is FP-injective;*
2. *for each $0 \neq r \in R$ there exists $h \in H$ such that $(0 : h) = Rr$;*
3. *$\mathrm{w.d.}(H) = \infty$ if R is semicoherent and $R \neq Q$;*
4. *$\mathrm{w.d.}(H) = 2$ if R is not semicoherent.*

Proof. (1). When $R \neq Q$, H is FP-injective by [4, Proposition 6]. If $R = Q$ is not IF then $H \cong \mathrm{E}(R/rR)$ for each $0 \neq r \in P$ by [4, Proposition 14].

(2). Let $0 \neq r \in R$. Then $(0 : r) \subseteq Z = (0 : x)$. From the injectivity of $\mathrm{E}(R/Z)$ we deduce that there exists $y \in \mathrm{E}(R/Z)$ such that $x = ry$. Now, if we put $h = y + Rx$ it is easy to check that $(0 : h) = (Rx : y) = Rr$.

(3) and (4). By [4, Proposition 8] $\mathrm{E}(R/Z)$ is flat. We use the exact sequence $0 \to R/Z \to \mathrm{E}(R/Z) \to H \to 0$ and Proposition 5 to conclude. □

The following proposition is a slight modification of [4, Proposition 9].

Proposition 7. *Let R be a chain ring and G an injective module. Then there exists a pure exact sequence: $0 \to K \to I \to G \to 0$, such that I is a direct sum of submodules isomorphic to R or H.*

Proof. There exist a set Λ and an epimorphism $\varphi : L = R^{\Lambda} \to G$. We put $\Delta = \mathrm{Hom}_R(H, G)$ and $\rho : H^{(\Delta)} \to G$ the morphism defined by the elements of Δ. Thus ψ and ρ induce an epimorphism $\phi : I = R^{(\Lambda)} \oplus H^{(\Delta)} \to G$. Since, for every $r \in P, r \neq 0$, each morphism $g : R/Rr \to G$ can be extended to $H \to G$, we deduce that $K = \ker \phi$ is a pure submodule of I. □

Lemma 8. *Let G be an almost FP-injective module over a chain ring R. Then for any $x \in G$ and $a \in R$ such that $(0 : a) \subset (0 : x)$ there exists $y \in G$ such that $x = ay$.*

Proof. We have $G = E/D$ where E is a FP-injective module and D a pure submodule. Let $e \in E$ such that $x = e + D$. Let $b \in (0 : x) \setminus (0 : a)$. Then $be \in D$. So, we have $b(e - d) = 0$ for some $d \in D$. Whence $(0 : a) \subset Rb \subseteq (0 : e - d)$. From the FP-injectivity of E we deduce that $e - d = az$ for some $z \in E$. Hence $x = ay$ where $y = z + D$. □

Let M be a non-zero module over a ring R. We set:

$$M_\sharp = \{s \in R \mid \exists 0 \neq x \in M \text{ such that } sx = 0\} \quad \text{and} \quad M^\sharp = \{s \in R \mid sM \subset M\}.$$

Then $R \setminus M_\sharp$ and $R \setminus M^\sharp$ are multiplicative subsets of R. If M is a module over a chain ring R then $M_\sharp$ and $M^\sharp$ are prime ideals called respectively the **bottom prime ideal** and the **top prime ideal** associated with M.

Proposition 9. *Let G be a module over a chain ring R. Then:*

(1) $G_\sharp \subseteq Z$ *if* G *is flat;*
(2) $G^\sharp \subseteq Z$ *if* G *is almost FP-injective;*
(3) $G^\sharp \subseteq Z \cap G_\sharp$ *if* G *is FP-injective. So,* G *is a module over* $R_{G_\sharp}$*;*
(4) G *is flat and FP-injective if* G *is almost FP-injective and* $G_\sharp \cup G^\sharp \subset Z$*. In this case* $G^\sharp \subseteq G_\sharp$*.*

Proof. (1). Let $a \in G_\sharp$. There exists $0 \neq x \in G$ such that $ax = 0$. The flatness of G implies that $x \in (0 : a)G$. So, $(0 : a) \neq 0$ and $a \in Z$.

(2). Let $s \in R \setminus Z$. Then for each $x \in G$, $0 = (0 : s) \subseteq (0 : x)$. If G is FP-injective then $x = sy$ for some $y \in G$. If G is almost FP-injective then it is factor of a FP-injective module, so, the multiplication by s in G is surjective.

(3). Let $a \in R \setminus G_\sharp$ and $x \in G$. Let $b \in (0 : a)$. Then $abx = 0$, whence $bx = 0$. So, $(0 : a) \subseteq (0 : x)$. It follows that $x = ay$ for some $y \in G$ since G is FP-injective. Hence $a \notin G^\sharp$.

(4). Let $0 \to X \to E \to G \to 0$ be a pure exact sequence where E is FP-injective, and $L = G^\sharp \cup G_\sharp$. Then $0 \to X_L \to E_L \to G \to 0$ is a pure exact sequence and by [5, Theorem 3] E_L is FP-injective. By [4, Theorem 11] R_L is an IF-ring. Hence G is FP-injective and flat. □

Example 10. *Let* D *be a valuation domain. Assume that* D *contains a non-zero prime ideal* $L \neq P$ *and let* $0 \neq a \in L$*. By* [7, Example 11 and Corollary 9] $R = D/aD_L$ *is semicoherent and not coherent. Since* R *is not self FP-injective then* $\mathrm{E}_R(R/P)$ *is not flat by* [3, Proposition 2.4]*.*

Example 11. *Let* R *be a semicoherent chain ring which is not coherent and* G *an FP-injective module which is not flat. Then* G_Z *is almost FP-injective over* R *but not over* Q*, and* $(G_Z)_\sharp \cup (G_Z)^\sharp = Z$*.*

Proof. Let G' be the kernel of the canonical homomorphism $G \to G_Z$. By Proposition 9 the multiplication in G and G/G' by any $s \in R \setminus Z$ is surjective. So, $G_Z \cong G/G'$. For each $s \in P \setminus Z$ we put $G_{(s)} = G/(0 :_G s)$. Then $G_{(s)}$ is FP-injective because it is isomorphic to G. Since $G' = \cup_{s \in P\setminus Z}(0 :_G s)$ then $G_Z \cong \varinjlim_{s\in P\setminus Z} G_{(s)}$. By [15, 33.9(2)] G_Z is factor of the FP-injective module $\oplus_{s\in P\setminus Z} G_{(s)}$ modulo a pure submodule. Hence G_Z is almost FP-injective over R. Since Q is IF and G_Z is not FP-injective by [5, Theorem 3], from Theorem 2 we deduce that G_Z is not almost FP-injective over Q. We complete the proof by using Proposition 9(4). □

The following lemma is a consequence of [10, Lemma 3] and [13, Proposition 1.3].

Lemma 12. *Let* R *be a chain ring for which* $Z = P$*, and* A *an ideal of* R*. Then* $A \subset (0 : (0 : A))$ *if and only if* P *is faithful and there exists* $t \in R$ *such that* $A = tP$ *and* $(0 : (0 : A)) = tR$*.*

Proposition 13. *Let* R *be a chain ring for which* $Z \neq 0$*. Then* $\mathrm{w.d.}(G) \neq 1$ *for any almost FP-injective* R*-module* G*.*

Proof. By way of contradiction assume there exists an almost FP-injective R-module G with $\mathrm{w.d.}(G) = 1$. There exists an exact sequence $0 \to K \to F \to G \to 0$ which is not pure, where F is free and K is flat.

First we assume that $R = Q$, whence $P = Z$. So there exist $b \in R$ and $x \in F$ such that $bx \in K \setminus bK$. We put $B = (K : x)$. From $b \in B$ we deduce that $(0 : B) \subseteq (0 : b)$. We investigate the following cases:

1. $(0:B)\subset(0:b)$.

Let $a\in(0:b)\setminus(0:B)$. Then $(0:a)\subset(0:(0:B))$. If P is not faithful then $(0:a)\subset B$ by Lemma 12. If P is faithful then $B\neq(0:(0:B))$ if $B=Pt$ for some $t\in R$. But, in this case $(0:a)$ is not of the form Ps for some $s\in R$. So, $(0:a)\subset B$. By Lemma 8 there exist $y\in F$ and $z\in K$ such that $x=ay+z$. It follows that $bx=bz\in bK$. Whence a contradiction.

2.1. $(0:B)=(0:b)\subset P$.

Let $r\in P\setminus(0:b)$. Then $(0:r)\subset Rb$. Let $0\neq c\in(0:r)$. There exists $t\in P$ such that $c=tb$. So, $(0:b)\subset(0:c)$. If $cx\in cK$, then $tbx=tby$ for some $y\in K$. Since K is flat we get that $(bx-by)\in(0:t)K\subseteq bK$ ($c=bt\neq0$ implies that $(0:t)\subset bR$), whence $bx\in bK$, a contradiction. Hence $cx\notin cK$. So, if we replace b with c we get the case **1**.

2.2.1. $(0:B)=(0:b)=P=bR$.

Since $b^2=0$ we have $b(bx)=0$. The flatness of K implies that $bx\in bK$, a contradiction.

2.2.2. $(0:B)=(0:b)=P\neq bR$.

Let $c\in P\setminus bR$. Then $cx\notin K$ and there exists $s\in P$ such that $b=sc$. We have $(K:cx)=Rs\neq Rb$. So, if $scx\notin sK$ we get the case **2.1** by replacing b with s and x with cx. Suppose that $scx\in sK$. We get $s(cx-y)=0$ for some $y\in K$. The flatness of F implies that $(cx-y)\in(0:s)F\subset cF$. Whence $y=c(x+z)$ for some $z\in F$. If $y=cv$ for some $v\in K$ then $bx=sy=scv=bv\in bK$. This is false. Hence $c(x+z)\in K\setminus cK$ and $(0:(K:x+z))\subseteq(0:c)\subset Rs\subset P$. So, we get either the case **1** or the case **2.1** by replacing x with $(x+z)$ and b with c.

Now, we assume that $R\neq Q$. First, we show that G_Z is flat. Since K and F are flat then so are K_Z and F_Z. If Q is coherent, then K_Z is FP-injective. So, it is a pure submodule of F_Z and consequently G_Z is flat. If Q is not coherent, then Z is flat, and by using [5, Theorem 3] it is easy to show that G_Z is almost FP-injective. Since $\mathrm{w.d.}(G_Z)\leq1$, from above we deduce that G_Z is flat. Let G' be the kernel of the canonical homomorphism $G\to G_Z$. As in the proof of Proposition 9 we have $G_Z\cong G/G'$. So, G' is a pure submodule of G. For each $x\in G'$ there exists $s\in P\setminus Z$ such that $sx=0$. Hence G' is a module over R/Z. But if $0\neq a\in Z$, $(0:a)\subseteq Z\subset(0:x)$ for any $x\in G'$. By Lemma 8 $x=ay$ for some $y\in G$, and since G' is a pure submodule, we may assume that $y\in G'$. Hence $G'=0$, $G\cong G_Z$ and G is flat. □

Example 14. *Let D be a valuation domain. Let $0\neq a\in P$. By* [4, Theorem 11] *$R=D/aD$ is an IF-ring.*

Now it is possible to state and to prove our main result.

Theorem 15. *For any almost FP-injective module G over a chain ring R:*

(1) $\mathrm{w.d.}(G)=0$ *if R is an IF-ring;*
(2) *if R is a valuation domain which is not a field then:*
  (a) $\mathrm{w.d.}(G)=0$ *if $G_\sharp=0$ (G is torsionfree);*
  (b) $\mathrm{w.d.}(G)=1$ *if $G_\sharp\neq0$;*
(3) *if R is semicoherent but not coherent then;*
  (a) $\mathrm{w.d.}(G)=\infty$ *if $Z\subset G_\sharp$;*
  (b) $\mathrm{w.d.}(G)=0$ *if $G_\sharp\cup G^\sharp\subset Z$;*
  (c) *if $G_\sharp\cup G^\sharp=Z$ either* $\mathrm{w.d.}(G)=0$ *if G is FP-injective or* $\mathrm{w.d.}(G)=\infty$ *if G is not FP-injective;*

(4) *if R is not semicoherent then $Z\otimes_R G$ is flat and almost FP-injective, and:*
   (a) w.d.$(G)=2$ *if* $Z\subset G_\sharp$;
   (b) w.d.$(G)=0$ *if* $G_\sharp\cup G^\sharp\subset Z$;
   (c) *if* $G_\sharp\cup G^\sharp=Z$ *either* w.d.$(G)=0$ *or* w.d.$(G)=2$. *More precisely G is flat if and only if, for any $0\neq x\in G$, $(0:x)$ is not of the form Qa for some $0\neq a\in Z$.*

Proof. (1). Since R is IF then G is FP-injective and flat.

(2). It is an immediate consequence of the fact that w.gl.d$(R)=1$.

(3). From [7, Theorem 2] which asserts that f.w.d.$(R)=1$ in this case, we deduce that w.d.$(G)<\infty\Rightarrow$ w.d.$(G)\leq 1$ and by Proposition 13 w.d.$(G)=0$. We use Proposition 9 to complete the proof. (Example 11 shows there exist almost FP-injective modules G which are not FP-injective, with $Z=G_\sharp\cup G^\sharp$).

(4). First we assume that G is injective. By Proposition 7 there exists a pure exact sequence: $0\rightarrow K\rightarrow I\rightarrow G\rightarrow 0$, such that I is a direct sum of submodules isomorphic to R or H. If I is flat then so is G. We may assume that G is not flat. It follows that w.d.$(I)=2$ by Proposition 6. By Proposition 4 w.d.$(G)\leq 2$. Then, we assume that G is FP-injective. So, G is a pure submodule of its injective hull E. Again, by Proposition 4, w.d.$(G)\leq 2$. If G is almost FP-injective then $G\cong E/D$ where E is FP-injective and D a pure submodule. We again use Proposition 4 to get w.d.$(G)\leq 2$. We use Propositions 9 and 13 to complete the proof of (a), (b) and the first assertion of (c).

Let $0\neq x\in G$ and $A=(0:x)$. Since G is a Q-module A is an ideal of Q. Suppose that A is not a non-zero principal ideal of Q and let $0\neq r\in A$. Then $rQ\subset A$. It follows that $(0:A)\subset(0:r)$. Let $b\in(0:r)\setminus(0:A)$. Since Q is not coherent there exists $a\in(0:r)\setminus Qb$. Then $(0:a)\subset(0:b)\subseteq A$ by Lemma 12. By Lemma 8 there exists $y\in G$ such that $x=ay$, whence $x\in(0:r)G$. Now suppose that $A=Qr$ for some $0\neq r\in Z$. If $a\in(0:r)$ then $rQ\subset(0:a)$, whence $x\notin aG$. So, $x\notin(0:r)G$. This completes the proof of (c).

It remains to prove the first assertion. By [7, Theorem 2] $Z\otimes_Z G$ is flat. It is easy to check that Z is a Q-module, whence so is $Z\otimes_R G$. Since Q is self FP-injective we conclude that $Z\otimes_R G$ is almost FP-injective by Proposition 3. □

We say that a chain ring is **Archimedean** if its maximal ideal is the only non-zero prime ideal.

Corollary 16. *For any almost FP-injective module G over an Archimedean chain ring R:*

(1) w.d.$(G)=0$ *if R is an IF-ring;*
(2) w.d.$(G)\leq 1$ *if R is a valuation domain;*
(3) *either* w.d.$(G)=2$ *or* w.d.$(G)=0$ *if R is not coherent. More precisely G is flat if and only if, for any $0\neq x\in G$ $(0:x)$ is not of the form Ra for some $0\neq a\in P$. Moreover $P\otimes_R G$ is flat and almost FP-injective.*

Let us observe that w.d.$(G)\leq 2$ for any almost FP-injective module G over an Archimedean chain ring.

Example 17. *Let D be an Archimedean valuation domain. Let $0\neq a\in P$. Then D/aD is Archimedean and it is IF by Example 14.*

Corollary 18. *Let R be an Archimedean chain ring. For any almost FP-injective R-module G which is either countably generated or uniserial:*

(1) p.d.$(G) \leq 1$ *if* R *is an IF-ring;*
(2) p.d.$(G) \leq 2$ *if* R *is a valuation domain;*
(3) p.d.$(G) \leq 3$ *and* p.d.$(P \otimes_R G) \leq 1$ *if* R *is not coherent.*

Proof. By [6, Proposition 16] each uniserial module (ideal) is countably generated. So, this corollary is a consequence of Corollary 16 and [12, Lemmas 1 and 2]. □

Remark 19. *If* R *is an Archimedean valuation domain then* w.d.$(G) \leq 1$ *and* p.d.$(G) \leq 2$ *for any* R*-module* G.

Let R be a ring, M an R-module. A R-module V is M**-projective** if the natural homomorphism $\mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(V, M/X)$ is surjective for every submodule X of M. We say that V is **quasi-projective** if V is V-projective. A ring R is said to be an **fqp-ring** if every finitely generated ideal of R is quasi-projective.

The following theorem can be proven by using [1, Lemmas 3.8, 3.12 and 4.5].

Theorem 20. [8, Theorem 4.1]. *Let* R *a local ring and* N *its nilradical. Then* R *is a fqp-ring if and only if either* R *is a chain ring or* R/N *is a valuation domain and* N *is a divisible torsionfree* R/N*-module.*

Lemma 21. *Let* R *be a local ring and* P *its maximal ideal. Assume that* $(0 : P) \neq 0$. *If* R *is a FP-injective module then* $(0 : P)$ *is a simple module.*

Proof. Let $0 \neq a \in (0 : P)$. By [11, Corollary 2.5] $Ra = (0 : (0 : a))$. But $(0 : a) = P$. So, $Ra = (0 : P)$. □

Example 22. *Let* D *be a valuation domain which is not a field and* E *a non-zero divisible torsionfree* D*-module which is not uniserial. Then* $R = D \propto E$ *the trivial extension of* D *by* E *is a local fqp-ring which is not a chain ring by* [8, Corollary 4.3(2)].

Theorem 23. *For any non-zero almost FP-injective module* G *over a local fqp-ring* R *which is not a chain ring,* w.d.$(G) = \infty$.

Proof. By [8, Proposition 5.2] f.w.d.$(R) \leq 1$. If N is the nilradical of R then its quotient ring Q is R_N and this ring is primary.

First assume that $R = Q$. Then each flat module is free and f.w.d.$(R) = 0$ by [2, Theorems P and 6.3]. If G is free, it follows that R is FP-injective. By Lemma 21 this is possible only if N is simple, whence R is a chain ring. Hence w.d.$(G) = \infty$.

Now assume that $R \neq Q$ and w.d.$(G) \leq 1$. Then w.d.$(G_N) \leq 1$. Since f.w.d.$(Q) = 0$ it follows that G_N is flat. As in the proof of Proposition 13, with the same notations we show that $G_N \cong G/G'$ and G' is a module over R/N. But if $0 \neq a \in N$, $(0 : a) = N \subset (0 : x)$ for any $x \in G'$. As in the proof of Lemma 8 we show that $x = ay$ for some $y \in G$. Since G' is a pure submodule, we may assume that $y \in G'$. Hence $G' = 0$, $G \cong G_N$ and G is flat. We use the first part of the proof to conclude. □

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