Minimal normal graph covers

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Abstract

A graph is normal if it admits a clique cover $\mathcal{C}$ and a stable set cover $\mathcal{S}$ such that each clique in $\mathcal{C}$ and each stable set in $\mathcal{S}$ have a vertex in common. The pair $(\mathcal{C}, \mathcal{S})$ is a normal cover of the graph. We present the following extremal property of normal covers. For positive integers c, s, if a graph with n vertices admits a normal cover with cliques of sizes at most c and stable sets of sizes at most s, then $c + s \geq \log_2(n)$. For infinitely many n, we also give a construction of a graph with n vertices that admits a normal cover with cliques and stable sets of sizes less than $0.87 \log_2(n)$. Furthermore, we show that for all n, there exists a normal graph with n vertices, clique number $\Theta(\log_2(n))$ and independence number $\Theta(\log_2(n))$.

When c or s are very small, we can describe all normal graphs with the largest possible number of vertices that allow a normal cover with cliques of sizes at most c and stable sets of sizes at most s. However, such extremal graphs remain elusive even for moderately small values of c and s.

1 Introduction

A graph G is said to be *normal* if it admits a set $\mathcal{C}$ of cliques and a set $\mathcal{S}$ of stable sets with the following properties:

(1) $\mathcal{C}$ is a cover of G, i.e., every vertex in G belongs to one of the cliques in $\mathcal{C}$;

(2) $\mathcal{S}$ is a cover of G, i.e., every vertex in G belongs to one of the stable sets in $\mathcal{S}$;

(3) Every clique in $\mathcal{C}$ and every stable set in $\mathcal{S}$ have a vertex in common.

We say that the pair $(\mathcal{C}, \mathcal{S})$ is a *normal cover* of G.

Clearly, a graph is normal if and only if its complement is normal. This property is reminiscent on the notion of perfect graphs, see [4, 6, 8]. Namely, normality is one of the basic properties that every perfect graph satisfies. Of course, normality is much weaker condition since every odd cycle of length at least 9 is normal. Actually, De Simone and Körner conjectured [3] that every graph without induced C_5, C_7, and the complement of C_7 is normal. Their conjecture was recently disproved [5]. Nevertheless, normal graphs make an interesting class of graphs. Their importance lies in their close relationship to the notion of graph entropy, one of central concepts in information theory; see Csiszár and Körner [1] or [2, 6, 7].

For $c, s \in \mathbb{N}$, we say that a normal cover is a (c, s)-*normal cover*, if the sizes of cliques in the cover are at most c and the sizes of stable sets in the cover are at most s. Furthermore, if a graph G admits a (c, s)-normal cover, we say it is (c, s)-*normal*. In this paper we consider the following extremal problem:

Problem 1. *How large can a (c, s)-normal graph be?*

We define $N(c, s)$ as the maximum number of vertices of a (c, s)-normal graph. The obtained results that are reminiscent on the basics of Ramsey theory are far from conclusive.

Note first that it is not completely obvious that $N(c, s)$ is always finite. However, in Section 3 we prove an upper bound given by the following theorem.

Theorem 1.1. *If G is a (c, s)-normal graph, then $|G| \leq 2^{c+s}$, i.e., $N(c, s) \leq 2^{c+s}$.*

To state the theorem in different words, if G has n vertices and is (c, s)-normal, then $c + s \geq \log n$ where the logarithm has base 2, as it is assumed for all logarithms in this paper. Hence, each normal cover of G must have at least one *element* (i.e., a stable set or a clique) of size $\Omega(\log n)$.

In analogy with perfect graphs, one might think that $N(c, s)$ should be closer to $c \cdot s$ (see [9, 10]). However, the bound in Theorem 1.1 must be exponential, because in Section 2.3 we present a construction that proves the following lower bounds.

Theorem 1.2. *For all integers $c \geq 6$, $N(c, c) \geq \sqrt{5}^c$. Additionally, $N(5, 5) \geq 55$ and $N(4, 4) \geq 25$.*

To compare this bound with the bound in Theorem 1.1, note that $0.86 < \frac{1}{\log \sqrt{5}} < 0.87$. Hence, every normal cover of a graph on n vertices must have a stable set or a clique of size at least $\frac{1}{2} \log n$, but for infinitely many n we can construct a normal graph with a normal cover where all cliques and stable sets from the cover are of size less than $0.87 \log n$.

To prove Theorem 1.2 we construct, for each positive integer c, a (c, c)-normal graph with roughly $\sqrt{5}^c$ vertices. We also present a different, simpler construction of a (c, c)-normal graph in Section 2.1 which gives a bit weaker lower bound for $N(c, c)$. The construction determines only some pairs of vertices to be connected by an edge or a non-edge. Any other pair of vertices can be adjacent or non-adjacent without affecting the (c, c)-normality. If for those pairs we choose independently with probability $\frac{1}{2}$ whether they are connected by an edge or not, then we argue that for large c, we get small independence and clique numbers with positive probability. This way, we obtain in Section 2.2 the following theorem.

Theorem 1.3. *For all n, there exists a normal graph G with n vertices that has clique number and independence number of order $\Theta(\log n)$.*

From Ramsey theory we know that every graph on n vertices must contain a stable set or a clique of size $\frac{1}{2} \log n$. Moreover, random graphs have independence number and clique number of order $\Theta(\log n)$. Thus, Theorem 1.3 shows that normal graphs can have clique number and independence number comparable to those of random graphs. In relation to this, it should be mentioned that partial motivation for this study came from the question whether random graphs $G(n, p)$ are normal or not with high probability. As of today, this question is still unresolved.

Finally, in Section 4 we describe, for small values of c and s ($c \leq 2$ or $s \leq 2$), all (c, s)-normal graphs with $N(c, s)$ vertices. For $c = 3$ (or $s = 3$) we derive a polynomial bound by showing that $N(3, s) = \Theta(s^3)$.

It is worth observing that the results of this paper are reminiscent to the basic facts about Ramsey numbers: the upper bound for $N(c, s)$ is exponential as is the lower bound for $N(c, c)$; the bases of exponents in the lower and upper bound are different and there is no clear evidence whether any of them is asymptotically tight. Small values of $N(c, s)$ are easy to obtain while the exact values for even a reasonably small values of c and s seem out of reach. Although our techniques are different from those in Ramsey theory, they exhibit the non-constructiveness familiar from Ramsey theory. While the lower bound is obtained by a direct, nonprobabilistic construction, the upper bound is proved by a non-combinatorial technique based on an algebraic argument using vector space dimension.

2 Constructions of normal graphs with small cliques and stable sets

In this section we prove Theorem 1.2 and Theorem 1.3. For each integer $c > 1$, we will first present a simple construction of a family of (c, c)-normal graphs with roughly 2^c vertices. Then we will continue in two directions. For the first direction, we will show with a probabilistic argument that there is some graph in the constructed family of (c, c)-normal graphs that has independence number and clique number of size $\Theta(c)$, which will prove Theorem 1.3. For the second direction, we will enhance the construction to get a (c, c)-normal graph with at least $\sqrt{5}^c$ vertices, which will prove Theorem 1.2.

2.1 The construction of G_c

A *red-blue graph* is a graph with each edge of color red or blue. When drawing such a graph, we will additionally make blue edges solid and red edges dashed. For an edge-colored graph H, we say that a graph G *is of type H* if

- $V(G) = V(H)$,

- each blue edge of H is an edge of G,

- no red edge of H is an edge of G.

For a red-blue graph H and for positive integers c, s, let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a cover of H with blue cliques of size at most c and let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover of H with red cliques of size at most s. If the clique C_i and the stable set S_j intersect for any i and j, then we say that the pair $(\mathcal{C}, \mathcal{S})$ is a (c, s)-*normal cover of H*. If H admits a (c, s)-normal cover, we say that it is (c, s)-*normal*. From the definition of graphs of type H it is clear that if $(\mathcal{C}, \mathcal{S})$ is a (c, s)-normal cover of H, then $(\mathcal{C}, \mathcal{S})$ is also a (c, s) normal cover of any graph of type H.
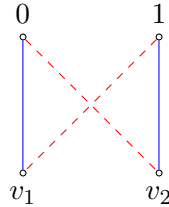

Figure 1: The red-blue graph G_2. Blue cliques $\{0, v_1\}, \{1, v_2\}$ and red cliques $\{0, v_2\}, \{1, v_1\}$ form a $(2, 2)$-normal cover.

For an integer $c \geq 1$, define a red-blue graph G_c recursively as $G_1 = K_1$ and, for $c \geq 2$, G_c is obtained from two copies of the graph G_{c-1}, denoted by $G_{c-1}(0)$ and $G_{c-1}(1)$, by adding two new vertices 0 and 1, such that each vertex of $G_{c-1}(0)$ is connected to 0 by a blue edge and to 1 by a red edge and each vertex of $G_{c-1}(1)$ is connected to 1 by a blue edge and to 0 by a red edge (see Figure 2).

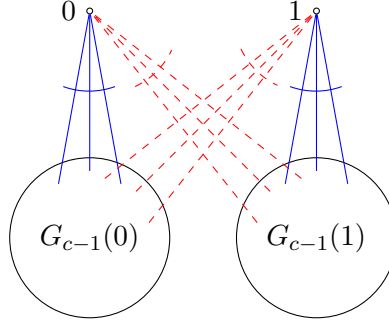The graphs G_2, G_3 and G_4 are presented in Figures 1, 3 and 4, respectively.

Figure 2: The recursive construction of G_c.

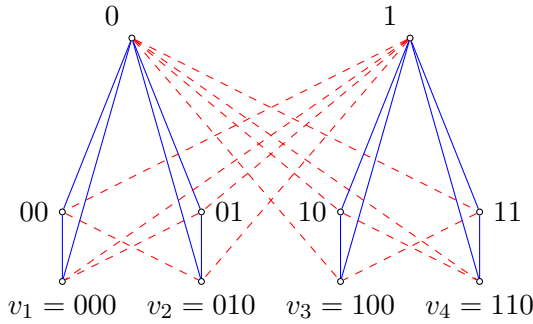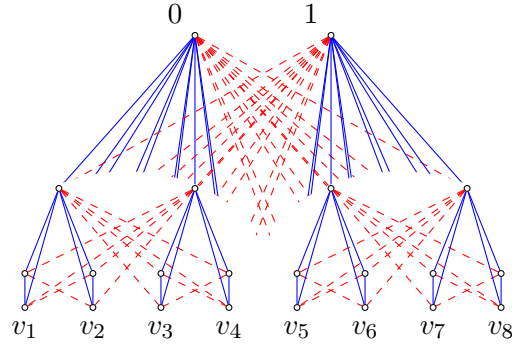Figure 3: The graph G_3 with vertices labeled as binary sequences.

Figure 4: The graph G_4. To make the picture more clear, the edges from 0 and 1 are not drawn completely.

Next, we give an alternative, non-recursive description of G_c. Let $\mathrm{V}(G_c)$ be the set of all binary sequences of length at least 1 and at most c under the condition that all binary sequences of length exacly c have to end with 0. Let $a = a_1 a_2 \ldots a_k \in \{0,1\}^k$ and $b = b_1 b_2 \ldots b_l \in \{0,1\}^l$ be two vertices of G_c. We may assume $k \leq l$. If $k = l$, then a and b are not adjacent. Otherwise, ab is an edge if $b_1 = a_1$, $b_2 = a_2$,..., $b_{k-1} = a_{k-1}$. The edge is blue if $a_k = b_k$ and red if $a_k \neq b_k$. The graph G_3 in Figure 3 is labeled according to this description. The proof that both descriptions of G_c give the same graph is left to the reader.

Proposition 2.1. *For each positive integer c, the graph G_c is (c,c)-normal and has $3 \cdot 2^{c-1} - 2$ vertices.*

Proof. From the description of G_c by binary sequences, it is clear that

$$|\mathrm{V}(G_c)| = 2 \cdot 2^{c-1} + 2^{c-2} + 2^{c-3} + \cdots + 2 = 3 \cdot 2^{c-1} - 2.$$

Let $m = 2^{c-1}$ and let $v_1, v_2, \ldots, v_m$ be the vertices that correspond to all binary sequences of length c in $\mathrm{V}(G_c)$. We will use these vertices as "generators" for a (c,c)-normal cover of G_c. For each vertex $v_i = a_{i,1} a_{i,2} \ldots a_{i,c}$, define the set C_i as the set of all beginnings of the sequence v_i, i.e., $C_i = \{a_{i,1} a_{i,2} \ldots a_{i,k} \mid 1 \leq k \leq c\}$. It is clear from the definition of G_c that C_i is a blue clique of size c. Similarly, for each vertex v_i, define the set $S_i = \{a_{i,1} a_{i,2} \ldots a_{i,k-1} \bar{a}_{i,k} \mid 1 \leq k < c\} \cup \{v_i\}$, where we put a bar over $a_{i,k}$ to indicate $a_{i,k} \neq \bar{a}_{i,k}$. In other words, S_i is the set of all sequences that are either v_i or have the same beginning as v_i, but then differ from the beginning of v_i by the last bit. From the definition of G_c, we see that S_i is a red clique of size c.

It is clear that each vertex of G_c is in some blue clique C_i and in some red clique S_j. Moreover, each blue clique and each red clique intersect. To see this, take a blue clique C_i and a red clique S_j. If $i = j$, then

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C_i and S_j bot contain v_i. If $i \neq j$, then let k be the first index where the elements of sequences v_i and v_j are different. It follows that the sequence of the first k elements of v_i is in both C_i and S_j. Hence, the blue cliques C_i and the red cliques S_j form a (c, c)-normal cover of the graph G_c, which is thus (c, c)-normal. $\quad\square$

For the later use, the (c, c)-normal cover of the graph G_c as described in the proof of Proposition 2.1 will be referred to as the *standard normal cover of G_c*.

2.2 Normal graphs with small clique number and independence number

In the proof of Theorem 1.3 we will use the following two lemmas.

Lemma 2.2. *For any large enough integer c, every subset U of vertices of G_c of size $4c$ has at least $4.1c^2$ pairs of vertices that are not connected by a blue edge and at least $4.1c^2$ pairs of vertices that are not connected by a red edge.*

Proof. We will give the proof only for blue edges since the red-blue graph obtained from G_c by interchanging red and blue edges is isomorphic to G_c (this follows easily by inspection of the recursive construction). In the proof, we will identify verices of G_c with the corresponding binary sequences and we will regularly use the term *height of a vertex* to refer to the length of the sequence corresponding to this vertex. We say that a vertex $v \in \mathrm{V}(G_c)$ is a *predecessor* of a vertex $u \in \mathrm{V}(G_c)$, if u and v are connected by a blue edge and the height of v is smaller than the height of u. We also say that u is an ancestor of v. If v is a predecessor of u and has height one less than u, we say that v is a *direct predecessor* of u and u is a *direct ancestor* of v.

Let U be a subset of vertices of G_c that induces the most blue edges among all subsets of vertices of G_c of size $4c$. We can assume the following three properties for the set U (we may need to change U to get these properties, but U will remain a set with $4c$ vertices and it will induce the same number of blue edges as the current U).

a) *For each $u \in U$, all predecessors of u are also in U.* If there is some predecessor v of u that is not in U, then because the neighbors (we only consider blue edges) of u are also neighbors of v, we can replace u by v in U without decreasing the number of blue edges in it. After we do such a transformation once, the sum of heights of vertices in U strictly decreases so after finitely many such transformations all vertices of U will have all their predecessors in U.

b) *If a vertex $v \in U$ of height k has no direct ancestors in U, then all other vertices of U of height at least k have all their ancestors in U.* If this was not the case, then there exists a vertex $u \notin U$ of height at least $k + 1$ that is a direct ancestor of some vertex from U. By a), u has strictly more neighbors in U than v, thus if we substitute v for u in U we get strictly more blue edges in U, which is a contradiction with the choice of U.

c) *For each k, $1 \leq k < c - 1$, U has at most one vertex of height k that does not have two direct ancestors.* If, for some k, there are two such vertices in U, say u and v, then by b) it must be the case that u and v each has exactly one direct ancestor in U. Hence, we can replace the direct ancestor of v in U and all of its ancestors in U with the direct ancestor of u which is not in U and with its corresponding ancestors. With this trasformation of U, the number of vertices in U as well as the number of blue edges induced by U is preserved. Furthermore, the assumption a) is preserved and, for each $l \neq k$, $1 \leq l < c - 1$, the number of vertices of height l in U that do not have two direct ancestors is also preserved. Because v now has no direct ancestors in U, it must be the case that all other vertices of height k in U have exactly two direct ancestors in U by b). Because we can do the same transformation for all k, we proved that c) can indeed be assumed.

For $i = 1, 2, \ldots, c$, let x_i be the number of vertices of height i in U. With assumptions b) and c) in hand, we can argue that x_c is large. Let a vertex $v \in U$ be a lowest vertex from U that has no direct ancestors and let k be the height of v. If $k \geq c - 1$, then using b) and c) we can prove by induction on $c - i$ that, for all $1 \leq i < c$, it holds

$$x_i \leq \left\lceil \frac{x_c + 1}{2^{c-i-1}} \right\rceil.$$

If $k < c - 1$, then using induction on $c - i$ we get

$$x_i = \frac{x_c}{2^{c-i-1}} \qquad \text{if } k < i < c \qquad \text{(use a) and b) to prove that)}$$

and

$$x_i \leq \left\lceil \frac{x_{k+1}/2 + 1}{2^{k-i}} \right\rceil \qquad \text{if } i \leq k \qquad \text{(use b) and c) to prove that).}$$

In the second case, it furthermore holds

$$x_i \leq \left\lceil \frac{x_{k+1}/2 + 1}{2^{k-i}} \right\rceil \leq \left\lfloor \frac{x_{k+1}}{2^{k+1-i}} \right\rfloor + 1. \tag{1}$$

The second inequality in (1) is easy to see using the fact that x_{k+1} is even, which follows from b). In any case, we have

$$x_i \leq \frac{x_c}{2^{c-i-1}} + 1$$

for all i, which gives

$$4c \leq x_c + \left(x_c + 1 \right) + \left(\frac{x_c}{2} + 1 \right) + \left(\frac{x_c}{4} + 1 \right) + \cdots + \left(\frac{x_c}{2^{c-2}} + 1 \right) < 3x_c + c$$

and hence $x_c > c$.

So we know that there are at least c vertices in U of height c, which implies that there are at least c vertices in U of height $c - 1$ and $\lceil \frac{c}{2} \rceil$ vertices of height $c - 2$ etc. Let $X \subseteq U$ be the set of all vertices in U of height at least $c - 4$. Because each vertex in X has at most 22 possible ancestors, the set X induces only $O(c)$ blue edges. Hence, there are at least

$$\binom{2c + \lceil \frac{c}{2} \rceil + \lceil \frac{c}{4} \rceil + \lceil \frac{c}{8} \rceil}{2} - O(c) \geq 4.13c^2 - O(c)$$

pairs of vertices from X not connected by a blue edge. Hence, for large enough c, we have at least $4.1c^2$ pairs of vertices in U that are not connected by a blue edge. This completes the proof. $\square$

Lemma 2.3. *For every large enough integer c, there exists a graph G of type G_c such that $\alpha(G) < 4c$ and $\omega(G) < 4c$.*

Proof. For a given integer c, let G be a random graph of type G_c which is obtained from G_c by adding each missing edge of G_c with probability $\frac{1}{2}$ independently for each edge, removing red edges and uncoloring blue edges. Let n be the number of vertices of G. By Lemma 2.2, we get that, for large enough c,

$$\Pr\left[\omega(G) \geq 4c\right] \leq \sum_{\substack{U \subseteq V(G) \\ |U| = 4c}} \Pr\left[U \text{ induces a clique}\right]$$

$$\leq \binom{n}{4c} 2^{-4.1c^2}.$$

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By Proposition 2.1 we have $c \geq \log n - 1$. Therefore,

$$\Pr\left[\omega(G) \geq 4c\right] \leq n^{4c} 2^{-4.1c \log n + 4.1c} = 2^{4.1c} n^{-0.1c},$$

which converges to 0 when c tends to infinity. Hence, for large enough c,

$$\Pr\left[\omega(G) \geq 4c\right] < \frac{1}{2}.$$

From the symmetry between blue and red edges in the recursive definition of G_c it follows that

$$\Pr\left[\alpha(G) \geq 4c\right] < \frac{1}{2}.$$

Hence, for large enough c,

$$\Pr\left[G \text{ has a stable set or a clique of size } 4c\right] < 1.$$

This means that there exists a graph G of type G_c that has no clique or stable set of size $4c$. $\qquad\square$

Proof of Theorem 1.3. Let n be a positive integer and let c be such that $|V(G_{c-1})| < n \leq |V(G_c)|$. By Lemma 2.3, there exists a graph G of type G_c such that $\alpha(G) = O(c)$ and $\omega(G) = O(c)$. We will make a serries of removals of vertices of G until we get a graph with n, $n-1$ or $n-2$ vertices. Each time we will remove three vertices as can be seen in Figure 5. It is not hard to see that removing three by three vertices this way preserves normality of the graph.
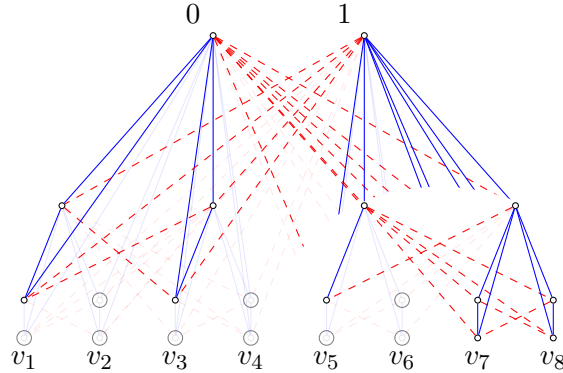

Figure 5: The graph G_4 with three triples of vertices (circled) removed. If we would also remove the fourth triple, we would get G_3.

Let G' be the resulting graph with n, $n-1$ or $n-2$ vertices. Since G' is an induced subgraph of G, it holds $\alpha(G') = O(c)$ and $\omega(G') = O(c)$. Next, we add at most two new vertices C and D to the graph G', together with all possible edges from C and D except of the edge CD, to get an n-vertex graph G''. Clearly, this can increase α or ω by at most 2. We leave it to the reader to show that G'' is a normal graph. By Proposition 2.1 we know that $c = \Theta(\log n)$. By our construction of the graph G'' it follows that $\alpha(G'') = \Omega(\log n)$ and $\omega(G'') = \Omega(\log n)$, hence $\alpha(G'') = \Theta(\log n)$ and $\omega(G'') = \Theta(\log n)$. $\qquad\square$

2.3 An improved lower bound for $N(c, c)$

We have shown in Proposition 2.1 that $N(c, c) \geq 3 \cdot 2^{c-1} - 2$. Here we improve this bound.

For a positive integer c, define a red-blue graph F_c as $F_1 = G_1$, $F_2 = G_2$ and, for $c > 2$, connect five copies of F_{c-2} as shown in Figure 6, where each edge between a copy of F_{c-2} and a vertex in $\{A, B, C, D, E\}$

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indicates the full red/blue join. Using induction on c, we can verify that the graphs F_c are (c, c)-normal: a (c, c)-normal cover of F_c consists of the union of normal covers of graphs F_{c-2} where we add two vertices to each clique and two vertices to each stable set (which vertices we add can be seen from Figure 6, there is only one option). We leave it to the reader to show that this is indeed a normal cover. We call it the *standard normal cover of F_c*.
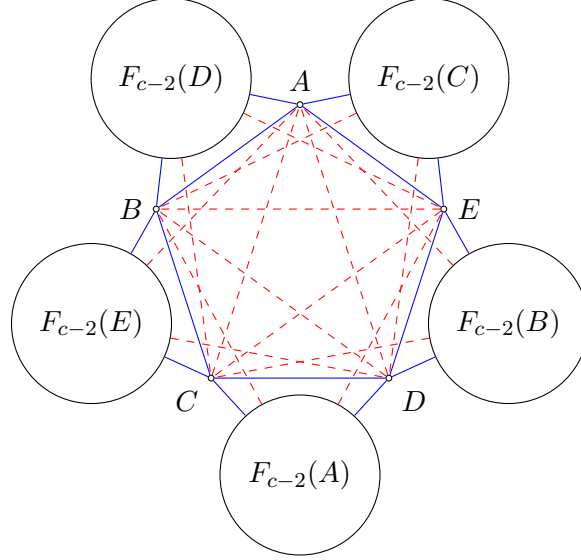

Figure 6: The recursive construction of F_c from five copies of F_{c-2}. The blue edge between A and $F_{c-2}(C)$ represents all possible blue edges between A and vertices in $F_{c-2}(C)$. The same holds for any other edge shown in the picture.

Now we can prove Theorem 1.2, which states:

For all integers $c \geq 6$, $N(c, c) \geq \sqrt{5}^c$. Additionally, $N(5, 5) \geq 55$ and $N(4, 4) \geq 25$.

Proof of Theorem 1.2. If we denote $f_c = |V(F_c)|$, then we have the recursion $f_1 = 1$, $f_2 = 4$, $f_c = 5f_{c-2} + 5$. This shows that $N(5, 5) \geq 55$ and $N(4, 4) \geq 25$. For $c \geq 6$, the inequality $f_c \geq \sqrt{5}^c$ follows easily from $f_c = 5f_{c-2} + 5$ by induction (the basis is for $c = 6$ and $c = 7$). $\square$

To conclude this section, let us discuss a special property that tells something about the optimality of the construction of graphs F_c and prevents a simple operation that would improve the lower bound of Theorem 1.2.

Proposition 2.4. *For each positive integer c, if a subset of $V(F_c)$ of size at most c does not induce a red edge, but intersects all red cliques of the standard normal cover of F_c, then it is of size c and has at least $c - 1$ common vertices with some blue clique of the standard normal cover of F_c.*

Note that Proposition 2.4 does not hold for graphs G_c. Take, for example, G_5 and let $X = \{0, 1\}$ and $Y = \{00, 01, 10, 11\}$ be vertices of height 1 and 2, respectively. Note that X and Y both intersect all red and all blue cliques of the standard normal cover, while they do not induce any edge. We can connect vertices of X with a blue edge and vertices of Y with red edges and add a new vertex v to G_c, connect it to X with blue edges and to Y with red edges. The new red-blue graph is also $(5, 5)$-normal and has more vertices than G_5. It is clear that we could add in a similar way even two more vertices to increase the blue clique $\{0, 1, v\}$ to size 5 and the graph would still be $(5, 5)$-normal.

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Proof. For $c = 1$ and $c = 2$, the proposition is trivially true. Actually, as we will see below, the case $c = 2$ is essentially the only reason for why it can happen that a set of size at most c that does not induce a red edge, but intersects all red cliques of the standard normal cover of F_c, can have only $c - 1$ common vertices with some blue clique of the standard normal cover of F_c.

Assume $c > 2$ and assume that the proposition holds for F_{c-2}. Let U be a subset of $V(F_c)$ of size at most c that does not induce a red edge, but intersects all red cliques of the standard normal cover of F_c. Using notation from Figure 6, U contains at most two vertices from the set $\{A, B, C, D, E\}$, because any three vertices from this set induce at least one red edge.

If U contains no vertex in $\{A, B, C, D, E\}$, then we have to cover all red cliques of each of five copies of F_{c-2} independently. Hence, there exists a copy of F_{c-2} for which we can cover all of its red cliques with a set of size at most $c/5$ that does not induce a red edge. Because the proposition holds for F_{c-2}, we have $c/5 \geq c - 2$, which means $c \leq 2$ (which is false). We conclude that U contains at least one vertex from $\{A, B, C, D, E\}$.

If U contains exactly one vertex from $\{A, B, C, D, E\}$, then we see in Figure 6 that there exist at least three copies of F_{c-2} for which U has to cover all their red cliques with at most $c - 1$ vertices, meaning that there exists a copy of F_{c-2} for which U has to cover all of its red cliques with at most $(c - 1)/3$ vertices (not inducing a red edge). But we assumed that the proposition holds for F_{c-2}, hence $(c - 1)/3 \geq c - 2$, which means $c \leq 2$, which is false.

It follows that U contains exactly 2 vertices from $\{A, B, C, D, E\}$. Because of the symmetry we may assume that U contains A and B. By Figure 6, $U \backslash \{A, B\}$ has to cover all stable sets of $F_{c-2}(D)$ and we know that it does not induce a red edge. By induction hypothesis, $U \backslash \{A, B\}$ is of size $c - 2$ and has at least $c - 3$ vertices in some blue clique Q of the standard normal cover of $F_{c-2}(D)$. It follows that U is of size c and it has at least $c - 1$ common vertices with the blue clique $Q \cup \{A, B\}$ that belongs to the standard normal cover of F_c. $\qquad\square$

3   An upper bound on $N(c, s)$

The red-blue graphs constructed in the previous section all admit (c, c)-normal covers for some $c = \Theta(\log n)$. In this section we prove that we cannot do better than that. More specifically, we prove Theorem 1.1 confirming that $N(c, s) \leq 2^{c+s}$.

Before going to the proof, we need some preparation. Let G be a (c, s)-normal graph with a (c, s)-normal cover $(\mathcal{C}, \mathcal{S})$. For a clique $C \in \mathcal{C}$, we say that a vertex $v \in C$ is *private to C* if it is in no other clique from $\mathcal{C}$. Similarly, for a stable set $S \in \mathcal{S}$, we say that a vertex $v \in S$ is *private to S* if it belongs to no other stable set from $\mathcal{S}$. We say that a (c, s)-normal cover $(\mathcal{C}, \mathcal{S})$ of G is *minimal* if no proper subset of $\mathcal{C}$ is a clique cover and no proper subset of $\mathcal{S}$ is a stable set cover of G. Note that $(\mathcal{C}, \mathcal{S})$ is minimal if and only if each clique in $\mathcal{C}$ has a private vertex and each stable set in $\mathcal{S}$ has a private vertex. Clearly, each (c, s)-normal graph has a minimal (c, s)-normal cover.

If we have a minimal normal cover of G, we can use the algorithm C-REDUCE to reduce the size of G without affecting normality. Note that there is no condition in the algorithm on the input cover to be minimal.

Claim 3.1. *The cover $(\mathcal{C}, \mathcal{S})$ remains a normal cover of the resulting graph after the execution of* C-REDUCE.

Proof. If $|P| = 1$, then it is clear that $\mathcal{C}$ remains a cover of $G - v$ by cliques. Because the clique C was the only clique from $\mathcal{C}$ that included v, $(\mathcal{C}, \mathcal{S})$ remains a normal cover.

If $|P| > 1$, then because v and v' were private to C, $\mathcal{C}$ remains a cover of G by cliques. Furthermore, all stable sets in $\mathcal{S}$ remain stable sets, since none of them contains both v and $\tilde{v}$, which are adjacent in G, and all edges from $\{v, v'\}$ to vertices out of C have been removed. $\qquad\square$

Algorithm 1: C-REDUCE

Input: A graph G with a normal cover $(\mathcal{C}, \mathcal{S})$, a clique $C \in \mathcal{C}$ and a vertex v that is private to C.

1 Let P be the set of all private vertices of C.

2 if $|P| = 1$ **then**

3 $\quad$ Remove v from G and from all stable sets in $\mathcal{S}$.

4 $\quad$ Remove C from $\mathcal{C}$.

5 else

6 $\quad$ Because $v \in P$ and $|P| \neq 1$, there exists $v' \in P \setminus \{v\}$.

7 $\quad$ Remove all edges in G that join v or v' with a vertex outside C.

8 $\quad$ Contract the edge vv'.

Let G be a (c, s)-normal graph with a minimal (c, s)-normal cover $(\mathcal{C}, \mathcal{S})$. We say that vertices $v_1, v_2, \ldots, v_{|\mathcal{C}|} \in \mathrm{V}(G)$ are *clique generators* for $\mathcal{C}$ if there is a bijective correspondence between vertices v_i and cliques $C_i \in \mathcal{C}$ such that for each i, v_i is private to C_i. We define *stable set generators* for $\mathcal{S}$ analogously.

In what follows, we prove Theorem 1.1 by induction on $c + s$. The strategy is as follows. For a (c, s)-normal graph G, we first take a minimal (c, s)-normal cover $(\mathcal{C}, \mathcal{S})$. Then we find a set of clique generators and we run C-REDUCE for all these generators, i.e., we repeat the reduction $|\mathcal{C}|$ times[1]. Then we are left with a $(c - 1, s)$-normal graph G'. If G' has at least half of the original vertices, we can use induction hypothesis to show that $N(c, s) \leq 2N(c - 1, s) \leq 2^{c+s}$.

In order for G not to lose too many vertices during the just described reduction of clique generators, we want to prove that in any minimal normal cover we do not have too many stable sets and cliques. To prove this, we will use the following lemma.

Lemma 3.2. *Let $(\mathcal{C}, \mathcal{S})$ be a minimal (c, s)-normal cover of a (c, s)-normal graph G on $n > 1$ vertices. If $U \subseteq \mathrm{V}(G)$ is a set of clique generators that is also a set of stable set generators of $(\mathcal{C}, \mathcal{S})$, then $n \geq 2|U|$.*

Proof. From the definition of generators it follows that $|\mathcal{C}| = |\mathcal{S}| = |U|$. If we denote $t = |U|$, let $U = \{v_1, v_2, \ldots, v_t\}$ and let the remaining vertices of G be $v_{t+1}, v_{t+2}, \ldots, v_n$. Furthermore, let the (unique) clique and the (unique) stable set from $(\mathcal{C}, \mathcal{S})$ that contains v_i be denoted by C_i and S_i, respectively. For $i = 1, \ldots, n$, we define column vectors $c_i, s_i \in \{0, 1\}^t$ as

$$c_i(j) = \begin{cases} 1 & v_i \in C_j \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad s_i(j) = \begin{cases} 1 & v_i \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

This gives us, for each i,

$$c_i \cdot s_i^T[j, k] = \begin{cases} 1 & \text{if } C_j \cap S_k = \{v_i\} \\ 0 & \text{otherwise.} \end{cases}$$

Because $(\mathcal{C}, \mathcal{S})$ is a normal cover, we have

$$\sum_{i=1}^{n} c_i \cdot s_i^T = J_t,$$

where J_t denotes the all-ones $t \times t$ matrix. Furthermore,

$$\sum_{i=1}^{t} c_i \cdot s_i^T = I_t,$$

---

[1] In the formal proof of Theorem 1.1 below, we reduce clique generators only if we have at least as many stable sets as cliques. Else, we reduce stable set generators.

where I_t denotes the identity matrix. It follows that

$$\sum_{i=t+1}^{n} c_i \cdot s_i^T = J_t - I_t.$$

On the left we have a sum of $n - t$ matrices of rank 1 and on the right we have a matrix of rank t. Here we used the assumption that $n > 1$, which implies that $t > 1$. It follows that $n - t \geq t$ which proves the lemma. □

Now we can prove that no minimal normal cover contains too many stable sets or cliques.

Lemma 3.3. *Let $(\mathcal{C}, \mathcal{S})$ be a minimal (c, s)-normal cover of a graph G of order n. Then $|\mathcal{C}| + |\mathcal{S}| \leq n + 1$.*

Proof. We may assume without loss of generality that $|\mathcal{C}| \geq |\mathcal{S}|$. We will prove the lemma by induction on n, starting with $n = 1$ where the lemma clearly holds. For $n > 1$, exactly one of the following two options has to hold.

Suppose first that each clique $C_i \in \mathcal{C}$ contains a vertex v_i which is private to C_i and is also private to a stable set, which we denote by $S_i \in \mathcal{S}$, such that v_i is the unique private vertex of S_i. If we denote by U the set of vertices v_i, then U is a set of clique generators for $(\mathcal{C}, \mathcal{S})$. Because $|\mathcal{C}| \geq |\mathcal{S}|$, the set U is also a set of stable set generators for $(\mathcal{C}, \mathcal{S})$ and, in particular, $|\mathcal{S}| = |U|$. This implies that $2|U| \leq n$ by Lemma 3.2 and hence $|\mathcal{C}| + |\mathcal{S}| \leq n$.

Suppose now that there exists a clique $C \in \mathcal{C}$ such that all of its private vertices are covered with stable sets, each of which has some private vertex outside C. If we run C-REDUCE on the clique C and one of its private vertices, we get a normal graph G with one vertex less. Because the stable sets that contain private vertices of C had at least one private vertex outside C, the number of stable sets in the new minimal normal cover is not reduced while the number of cliques is reduced by at most one. The result follows by induction. □

We are ready to prove Theorem 1.1. Actually, we will prove a slightly stronger version in order to use induction.

Lemma 3.4. $N(c, s) \leq 2^{c+s} - 1$.

Proof. The proof is by induction on $c + s$. Let G be a (c, s)-normal graph of order n. If $c = 1$ or $s = 1$, then G is an empty graph or a complete graph, hence $n = c + s - 1 \leq 2^{c+s} - 1$. So suppose $c, s \geq 2$ and let $(\mathcal{C}, \mathcal{S})$ be a minimal (c, s)-normal cover of G. By Lemma 3.3 we have $|\mathcal{C}| + |\mathcal{S}| \leq n + 1$ and hence $|\mathcal{C}| \leq \frac{n+1}{2}$ or $|\mathcal{S}| \leq \frac{n+1}{2}$. We may assume that $|\mathcal{C}| \leq |\mathcal{S}|$ and therefore $|\mathcal{C}| \leq \frac{n+1}{2}$ because if this is not the case, then we can replace G by the complement of G. Next, we run C-REDUCE $|\mathcal{C}|$-times, once for each clique in $\mathcal{C}$ and for some private vertex of this clique. By Claim 3.1 we are left with a $(c - 1, s)$-normal graph on at least $\frac{n-1}{2}$ vertices which implies that $\frac{n-1}{2} \leq N(c - 1, s)$. By the induction hypothesis, we have $N(c - 1, s) \leq 2^{c+s-1} - 1$ which implies $n \leq 2^{c+s} - 1$. □

4   Some exact values of $N(c, s)$

We start by a simple observation.

Claim 4.1. *If $(\mathcal{C}, \mathcal{S})$ is a minimal (c, s)-normal cover of a graph G with $N(c, s)$ vertices, then each clique in $\mathcal{C}$ has c vertices and each stable set in $\mathcal{S}$ has s vertices.*

Proof. Because of the symmetry between cliques and stable sets, it suffices to prove the claim for cliques. Suppose that there is a clique $C \in \mathcal{C}$ of size strictly less than c with a private vertex v. If we add a new vertex u to G and connect it to all vertices of C, we can increase the clique C by adding the vertex u. Furthermore, if we clone each stable set from $\mathcal{S}$ that contains v and replace v with u in one of the clones, the resulting cover is clearly a (c, s)-normal cover of the (new) graph G. Hence, we have a (c, s)-normal graph with more than $N(c, s)$ vertices, a contradiction. $\qquad\square$

For integers $r \geq 2$ and $k \geq 1$, let a red-blue graph $G_{r,k}$ be the following graph on $n = r(k+1)$ vertices. The blue subgraph of $G_{r,k}$ is the disjoint union of r stars, each isomorphic to $K_{1,k}$. The vertices of degree k in these stars are called the *roots*. For each star, add red edges to form a red clique on the non-root vertices of the star, and for each root vertex u, add red edges to every non-root vertex in all other stars. If $r \geq 3$, we also add red edges between each pair of roots.

It is easy to see that the graph $G_{r,k}$ is $(2, k+r-1)$-normal: take each blue edge for a clique of the cover and for each root, take all of its blue neighbors together with all of the remaining roots for a stable set. If we denote by $m = k+r-1$ the size of stable sets in our cover, we have $n = (k+1)(m-k+1) = \left(\frac{m}{2}+1\right)^2 - (k - \frac{m}{2})^2$. Hence, for $k = \lfloor \frac{m}{2} \rfloor$ and $k = \lceil \frac{m}{2} \rceil$, we have $n = \lfloor \left(\frac{m}{2}+1\right)^2 \rfloor$.

Next, we will compute $N(c, s)$ and we will describe (c, s)-normal graphs with $N(c, s)$ vertices for small values of c and s. Note that $N(c, s) = N(s, c)$ and for every (c, s)-normal graph with $N(c, s)$ vertices, its complement is an (s, c)-normal graph with $N(s, c)$ vertices.

Let G be a (c, s)-normal graph with $N(c, s)$ vertices and minimum number of edges. Let $(\mathcal{C}, \mathcal{S})$ be a minimal (c, s)-normal cover of G. Together with Claim 4.1 we therefore have the following properties of G:

1. Every clique in $\mathcal{C}$ has c vertices.

2. Every stable set in $\mathcal{S}$ has s vertices.

3. Every edge of G is in one of the cliques in $\mathcal{C}$.

4. Every clique in $\mathcal{C}$ has a private vertex. The degree of every vertex that is private to a clique in $\mathcal{C}$ is $c - 1$.

5. Every stable set in $\mathcal{S}$ has a private vertex.

Proposition 4.2. *Let $m \geq 1$ be an integer.*

a) *$N(1, m) = N(m, 1) = m$ and the only $(m, 1)$-normal graph with $N(m, 1)$ vertices is the complete graph K_m.*

b) *$N(2, m) = N(m, 2) = \lfloor \left(\frac{m}{2}+1\right)^2 \rfloor$ and $(2, m)$-normal graphs with $N(2, m)$ vertices are precisely graphs of type $G_{r,k}$, where $r = m + 1 - k$ and $k = \lfloor \frac{m}{2} \rfloor$ or $k = \lceil \frac{m}{2} \rceil$.*

c) *$N(3, 3) = 10$ and every $(3, 3)$-normal graph with $N(3, 3)$ vertices is of type G_3 or F_3.*

d) *For every $s \geq 4$ we have*

$$\tfrac{1}{27}s^3(1 + o(1)) \leq N(3, s) \leq \tfrac{4}{27}s^3(1 + o(1)).$$

Proof. a) Clearly, in a graph that has a normal cover with stable sets of size 1, every clique must contain all vertices, so the graph is complete. Thus, $N(1, m) = N(m, 1) = m$.

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b) Let $(\mathcal{C}, \mathcal{S})$ be a minimal $(2, m)$-normal cover of a graph G with $n = N(2, m)$ vertices, let Q be a set of clique generators and let $R = V(G) \backslash Q$ be the set of the remaining vertices. Let v be a vertex in R that is covered by the largest number of cliques from $\mathcal{C}$, say by k cliques. Since each of these k cliques is of size 2, any stable set in the cover that does not contain v must cover k private vertices of cliques that cover v together with at least $|R| - 1$ additional vertices to intersect each clique from $\mathcal{C}$, hence $m \geq k + |R| - 1$. Because no vertex is covered with more than k cliques, we have $n \leq |R| + k|R|$ which gives

$$n \leq (k+1)(m+1-k) = \left(\frac{m}{2} + 1\right)^2 - \left(k - \frac{m}{2}\right)^2.$$

The right-hand side is maximal only for $k = \lfloor \frac{m}{2} \rfloor$ or $k = \lceil \frac{m}{2} \rceil$ in which case we get $n \leq \lfloor \left(\frac{m}{2} + 1\right)^2 \rfloor$. The graph $G_{r,k}$ with parameters as in the statement of the proposition shows that $N(2, m) \geq \lfloor \left(\frac{m}{2} + 1\right)^2 \rfloor$ which proves $N(2, m) = \lfloor \left(\frac{m}{2} + 1\right)^2 \rfloor$. It is now clear that every $(2, m)$-normal graph with $N(2, m)$ vertices is of type $G_{r,k}$ (and vice versa).

c) The graphs of type G_3 (see Figure 3) and the graphs of type F_3 (see Figure 6) are examples of $(3, 3)$-normal graphs with 10 vertices. In what follows, we will prove that that a $(3, 3)$-normal graph cannot have more than 10 vertices and if it has 10 vertices, then it is of type G_3 or F_3. Let us first introduce an auxiliary graph H and some notation that will also be used in the proof of d).

For $s \geq 3$, let G be a $(3, s)$-normal graph with $N(3, s)$ vertices, and let $(\mathcal{C}, \mathcal{S})$ be a minimal $(3, s)$-normal cover of G. We remove all edges from G except those induced by cliques from $\mathcal{C}$. Note that doing so we did not lose the information about the "normal structure of G", which is hidden in the normal cover $(\mathcal{C}, \mathcal{S})$. For each $C \in \mathcal{C}$ we select one vertex that is private to C and denote it by p_C. Let $P = \{p_C \mid c \in \mathcal{C}\}$. By the assumption outlined above, vertices in P have degree 2 in G and it is easy to see that they form an independent set in G. Let $H = G - P$. Note that H has no isolated vertices. Each edge $e \in E(H)$ corresponds to at least one clique in $\mathcal{C}$. We denote by $m(e) \geq 1$ the number of cliques in $\mathcal{C}$ containing e, and call it the *multiplicity* of e in H. Clearly, H and its multiplicity function determine G up to an isomorphism. Moreover,

$$|G| = N(3, s) = |H| + \sum_{e \in E(H)} m(e). \tag{2}$$

For each vertex v of H define $\lambda(v) = \max\{m(e) \mid e \in E(H) \text{ is incident with } v\}$. For a vertex $v \in V(G)$, let $u \in V(G)$ be such that $\lambda(v) = m(vu)$. Consider a stable set $S \in \mathcal{S}$ containing the private vertex w in a clique $\{u, v, w\}$. Since S intersects all cliques in $\mathcal{C}$ and is independent, it must contain $m(uv)$ private vertices of the cliques containing u and v, and must contain a different vertex for each neighbor $u' \neq u$ of v in H (either contains u' or the corresponding private vertices). This shows that

$$\deg_H(v) + \lambda(v) - 1 \leq s. \tag{3}$$

Let us now consider the case $s = 3$. Note that $n = |G| \geq 10$ and all cliques in $\mathcal{C}$ and all stable sets in $\mathcal{S}$ are of size 3. By (3), the multiplicities in H are at most 3. If an edge uv has multiplicity 3, then considering a stable set that covers the private vertex of a clique from $\mathcal{C}$ that contains u and v, it follows that G has just 5 vertices. So, this cannot happen.

Suppose now that $m(uv) = 2$. Then there is a vertex z of H that is incident with all edges of $H - uv$. If H had just 3 vertices, (2) would imply that G has at most 9 vertices. Thus, there is a vertex $z' \notin \{u, v, z\}$. Since H has no isolated vertices and z covers all edges in $H - uv$, the vertex z' is adjacent to z and has degree 1 in H. The edges vz and uz (if they are present in H) have multiplicity 1 since no vertex covers all edges of $H - uz$ or $H - vz$. Using (2) and $|G| \geq 10$, we see that H must have another vertex or both

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edges uz and vz. Hence, the edge zz' must also have multiplicity 1, because neither u nor v (nor any other vertex) covers all edges of $H - zz'$. This implies that z has another neighbor z'', which is also of degree 1 in H. Hence, z is not adjacent to u, v because there cannot exist a stable set of size 3 that would cover the private vertex of the edge uz or vz and would intersect all cliques from $\mathcal{C}$. Considering (2) and $|G| \geq 10$, we see that z has yet another neighbor z'''. However, this is impossible since any stable set that would cover the private vertex corresponding to the clique containing zz' and would intersect all cliques from $\mathcal{C}$, would be of size at least 4. The conclusion is that all edges in H have multiplicity 1.

By (3), the degree of every vertex in H is at most 3. If a vertex v has degree 3, then H has only one connected component and no other vertex than v and its neighbors. If there was another vertex in H adjacent to a neighbor u of v, then a stable set that would cover the private vertex of the clique corresponding to uv could not cover all cliques from $\mathcal{C}$. Thus by (2), H needs to have at least 6 edges in order for G to have at least 10 vertices. This implies that $H = K_4$ which is clearly impossible. Hence, all vertices of H have degree 2 or 1. It follows that H is a disjoint union of cycles and paths.

If H has a connected component K_2, then G has at most $3 + N(3, 2) = 9$ vertices, which is not the case. It is clear that H has at most 3 connected components and because 3 connected components imply that G is a disjoint union of three triangles and has at most 9 vertices, H has at most 2 connected components. If H has two connected components, we know that none is K_2. Hence, a stable set from $\mathcal{S}$ needs to cover all edges of one component with just one vertex. This implies that both components are paths of length 2, which gives that G is of type G_3.

If H has only one component, we know that it is a path or a cycle. If H is a path of length at most 4, than G has at most 9 vertices, which is false. If H is a path of length at least 5, then the stable set that covers the private vertex corresponding to the second edge in the path cannot cover all cliques from $\mathcal{C}$. Hence, H is a cycle. It is easy to see that it has to be of length 5, which gives that G is of type F_3.

d) To prove the lower bound, let $d = \left\lceil \frac{s}{3} \right\rceil$. Consider the graph H consisting of $s - 2d + 2$ copies of the star $K_{1,d}$, each edge of which has multiplicity d. The corresponding graph G – see the proof in c) – is $(3, s)$-normal. By (2), it has $(s - 2d + 2)(d^2 + d + 1)$ vertices, which gives the desired lower bound.

As for the upper bound, define H and its multiplicity function m as described above. Let λ be the largest multiplicity over all edges of H; suppose it is attained for the edge uv. Consider a stable set containing private vertices of the cliques corresponding to uv. As discussed for the bound (3), it contains $s - \lambda$ other vertices $u_1, \ldots, u_{s-\lambda}$. Suppose that exactly first k of them are private vertices of cliques in $\mathcal{C}$ that have been removed when defining H. The vertices $u_{k+1}, \ldots, u_{s-\lambda}$ cover all edges of H except uv and possibly the edges that correspond to private vertices $u_1, \ldots, u_k$. Since H has no isolated vertices we conclude that all vertices of H except possibly u and v are one of u_i or their neighbors in G. By using this fact and (3), we obtain the following bound:

$$
\begin{aligned}
|H| &\leq 2(k+1) + \sum_{i=k+1}^{s-\lambda} (\deg_H(u_i) + 1) \\
&\leq 2(k+1) + \sum_{i=k}^{s-\lambda} (s - \lambda(u_i) + 2) \\
&= \mathrm{O}(s^2).
\end{aligned}
$$

A similar count holds for the sum of edge-multiplicities in H as shown next. By (2) and (3), it follows

that:

$$|G| \le |H| + \lambda + k + \sum_{i=k+1}^{s-\lambda} \lambda(u_i) \deg_H(u_i)$$

$$\le |H| + 2s + \sum_{i=k+1}^{s-\lambda} \lambda(u_i)(s - \lambda(u_i) + 1)$$

$$\le \mathrm{O}(s^2) + \sum_{i=k+1}^{s-\lambda} \lambda(u_i)(s - \lambda(u_i)).$$

If $\lambda \ge \frac{s}{2}$, then

$$\sum_{i=k+1}^{s-\lambda} \lambda(u_i)(s - \lambda(u_i)) \le \sum_{i=1}^{s-\lambda} \frac{s^2}{4} \le \frac{s^3}{8}.$$

Otherwise, for each $i = k+1, \ldots, s - \lambda$, we have $\lambda(u_i)(s - \lambda(u_i)) \le \lambda(s - \lambda)$. Consequently,

$$\sum_{i=k+1}^{s-\lambda} \lambda(u_i)(s - \lambda(u_i)) \le \lambda(s - \lambda)^2 \le \frac{4}{27} s^3,$$

where the last inequality is attained for $\lambda = \frac{s}{3}$ which maximizes the expression $\lambda(s - \lambda)^2$ for $0 \le \lambda \le s$. Because $\frac{1}{8} < \frac{4}{27}$, we have

$$|G| \le \frac{4}{27} s^3 + \mathrm{O}(s^2),$$

which establishes the upper bound.

$\square$

With a more delicate analysis, we can slightly improve the upper bound in Proposition 4.2 d) and replace the leading constant $\frac{4}{27}$ with a smaller constant. As the truth seems to be closer to the lower bound, this does not seem to be worth the effort of making the proof longer.

It would be interesting to prove that, for every fixed c, $N(c, s)$ is bounded above by a polynomial in s.

5 Acknowledgements

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