On ground state of non local Schrödinger operators. *

Yuri Kondratiev† Stanislav Molchanov‡ Sergey Pirogov§
Elena Zhizhina¶

Abstract

We study a ground state of a non local Schrödinger operator associated with an evolution equation for the density of population in the stochastic contact model in continuum with inhomogeneous mortality rates. We found a new effect in this model, when even in the high dimensional case the existence of a small positive perturbation of a special form (so-called, small paradise) implies the appearance of the ground state. We consider the problem in the Banach space of bounded continuous functions $C_b(R^d)$ and in the Hilbert space $L^2(R^d)$.

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1 Introduction

The asymptotic behavior of stochastic infinite-particle systems in continuum can be studied in terms of evolution equations for correlation functions. For

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†Fakultat fur Mathematik, Universitat Bielefeld, 33615 Bielefeld, Germany (kondrat@math.uni-bielefeld.de).

‡Department of Mathematics and Statistics, UNC Charlotte(USA) and University Higher School of Economics, Russian Federation (smolchan@uncc.edu)

§Institute for Information Transmission Problems, Moscow, Russia (s.a.pirogov@bk.ru).

¶Institute for Information Transmission Problems, Moscow, Russia (ejj@iitp.ru).

the stochastic contact model in the continuum [2] the evolution equation for the first correlation function (i.e., the density of the system) is closed and it can be considered separately from equations for higher-order correlation functions [1]. In this case we have the following evolution problem for $u \in C([0,\infty);\mathcal{E})$ associated with a nonlocal diffusion generator L:

$$\frac{\partial u}{\partial t} = Lu, \quad u = u(t,x), \quad x \in \mathbb{R}^d, t \geq 0, \quad u(0,x) = u_0(x) \geq 0. \qquad (1)$$

in a proper functional space $\mathcal{E}$. As $\mathcal{E}$, we consider in our paper two spaces: $C_b(\mathbb{R}^d)$, the Banach space of bounded continuous functions on $\mathbb{R}^d$, and $L^2(\mathbb{R}^d)$. These spaces are corresponding to two different regimes in the contact model: systems with bounded density and ones essentially localized in the space.

The operator L has the following form:

$$Lu(x) = -m(x)u(x) + \int_{\mathbb{R}^d} a(x-y)u(y)dy, \qquad (2)$$

where $a(x) \geq 0,\ a \in L^1(\mathbb{R}^d)$ is an even continuous function such that:

$$\int_{\mathbb{R}^d} a(x)dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x)dx < \infty; \qquad (3)$$

$$\tilde{a}(p) = \int_{\mathbb{R}^d} e^{-i(p,x)} a(x)dx \in L^1(\mathbb{R}^d); \text{then } \tilde{a}(p) \in L^2(\mathbb{R}^d) \text{ since } |\tilde{a}(p)| \leq 1. \quad (4)$$

The function $a(x-y)$ is the dispersal kernel associated with birth rates in the contact model. The function $m(x)$ is related with mortality rates. We assume here that

$$m(x) \in C_b(\mathbb{R}^d), \quad 0 \leq m(x) \leq 1, \quad m(x) \to 1,\ |x| \to \infty. \qquad (5)$$

The contact model with homogeneous mortality rates $m(x) \equiv const$ has been studied in [1]. It was proved there that only in the case $m(x) \equiv 1$ there exists a family of stationary measures of the model (for $d \geq 3$); if $m(x) \equiv m \neq 1$, then the density of population either exponentially growing (supercritical regime: $m < 1$) or exponentially decaying (subcritical regime: $m > 1$).

Here we are interesting in local perturbations of the stationary regime, when $m(x)$ is an inhomogeneous in space non-negative function. We prove that local fluctuations of the mortality with respect to the critical value

$m(x) \equiv 1$ can push the system away from the stationary regime. As a result of such local perturbations, we will observe exponentially increasing density of population everywhere in the space. The goal of this paper is to obtain conditions on the mortality rates which give the existence of a positive discrete spectrum of the operator (2) in the spaces $C_b(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, and to prove the existence and uniqueness of a positive eigenfunction $\psi(x) > 0$ corresponding to the maximal eigenvalue $\lambda_0 > 0$ of the operator L. We call this function the ground state of the operator L.

We prove in the paper that in small dimensions $d = 1, 2$ a positive discrete spectrum of the operator L is nonempty for any (small) local positive fluctuation $V(x) = 1 - m(x)$ of the mortality $m(x)$ from the critical value. If $d \geq 3$, then a positive eigenvalue appears in two cases: if exists such a region of any (small) positive volume, where the fluctuation $V(x)$ is equal to 1, or if $V(x)$ is positive and less than 1 in a large enough region. We stress that the function $V(x)$ should be bounded from above by 1, since the mortality $m(x) \geq 0$ is a non-negative function. Thus in the high dimensional case $d \geq 3$ we observe crucially new effects different from those of the Shrödinger operators. We prove that small (in the integral sense) perturbations $V(x)$ of the mortality $m(x)$ from the critical value $m(x) \equiv 1$ imply the existence of positive eigenvalues of the non local operator (2). The analogous result is proved for the model in the subcritical regime.

2 Spectral properties of L

In this section we describe a general approach to study a discrete spectrum of the operator L in both spaces $C_b(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$. This approach is based on the analytic Fredholm theorem and the study of a spectral radius of compact operators.

The operator L can be rewritten as

$$Lu(x) \;=\; L_0 u(x) \;+\; V(x)u(x), \quad u(x) \in C_b(\mathbb{R}^d), \tag{6}$$

$$L_0 u(x) \;=\; \int_{\mathbb{R}^d} a(x-y)(u(y) - u(x))dy, \quad V(x) \;=\; 1 - m(x).$$

The potential $0 \leq V(x) \leq 1$ describes local (negative) fluctuations of the mortality, and we assume that

$$V(x) \in C_b(\mathbb{R}^d) \quad \text{and} \quad \lim_{|x| \to \infty} V(x) = 0.$$

The operator L_0 is bounded and dissipative in $C_b(\mathbb{R}^d)$:

$$\|(\lambda - L_0)f\| \ \geq \ \lambda\|f\| \quad \text{for} \quad \lambda \geq 0.$$

Lemma 1 *The operator L has only discrete spectrum in the half-plane*

$$\mathcal{D} = \{\lambda \in \mathbb{C} \mid \ Re\lambda > 0\}.$$

Proof is based on the analytic Fredholm theorem for analytic operator-valued functions, see [4], [5]. Since for any $\lambda \in \mathcal{D}$ the resolvent $(L_0 - \lambda)^{-1}$ is a bounded operator, we have

$$(\lambda - L_0 - V) = (\lambda - L_0)(1 - (\lambda - L_0)^{-1}V), \tag{7}$$

and

$$(\lambda - L_0 - V)^{-1} = (1 - (\lambda - L_0)^{-1}V)^{-1} \ (\lambda - L_0)^{-1}. \tag{8}$$

Using the Neumann series for the operator $(\lambda - L_0)^{-1}$ we get:

$$(\lambda - L_0)^{-1} \ = \ \frac{1}{\lambda + 1} \ + \ \frac{1}{\lambda + 1}A_\lambda, \tag{9}$$

where

$$A_\lambda = (\lambda + 1)(\lambda - L_0)^{-1} - 1 = \sum_{n=1}^{\infty} \frac{a^{*n}}{(\lambda + 1)^n}, \tag{10}$$

and A_λ is a bounded convolution operator when $\lambda \in \mathcal{D}$. The decomposition (9) for $(\lambda - L_0)^{-1}$ is the crucial point of our reasoning. The kernel of A_λ is

$$G_\lambda(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(p, x-y)} \frac{\tilde{a}(p) \ dp}{\lambda + 1 - \tilde{a}(p)} \ = \ \left(\sum_{n=1}^{\infty} \frac{a^{*n}}{(\lambda + 1)^n} \right) (x - y), \tag{11}$$

and, in particular,

$$G_\lambda(u) \in C_b(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d} G_\lambda(u) \ du \ < \ \infty, \quad \forall \lambda \in \mathcal{D}. \tag{12}$$

We denote by W_λ an operator of multiplication by the function

$$W_\lambda = 1 - \frac{V}{\lambda + 1}.$$

It is a bounded operator with the bounded inverse operator W_λ^{-1} when $\lambda \in \mathcal{D}$. Then (9) implies

$$1 - (\lambda - L_0)^{-1} V = W_\lambda - \frac{1}{\lambda + 1} A_\lambda V,$$

and we can rewrite (8) in the following way:

$$(\lambda - L_0 - V)^{-1} = (1 - Q_\lambda)^{-1} \ \left((\lambda - L_0) W_\lambda \right)^{-1}, \tag{13}$$

where

$$Q_\lambda = \frac{1}{\lambda + 1} W_\lambda^{-1} A_\lambda V. \tag{14}$$

The relations (11) - (12) for the kernel $G_\lambda(x-y)$ of the operator A_λ and the conditions on the potential $V(x)$ imply that Q_λ is the compact operator. Really, let us define the truncated operator of multiplication on the potential $V^{(r)}(x) = \chi_r(x) V(x)$ and the convolution operator $A_\lambda^{(r)}$ with the truncated kernel $G_\lambda^{(r)}(x) = \chi_r(x) G_\lambda(x)$, where $\chi_r(x)$ is the indicator function of the ball $B_r = \{x : \ |x| < r\}$. Obviously, the operators $A_\lambda^{(r)} V^{(r)}$ are compact for any $r > 0$, and $A_\lambda^{(r)}$, $V^{(r)}$ converge in norm to A_λ, V correspondingly. Thus we conclude that $A_\lambda V$ is the compact operator.

Consequently, Q_λ is an analytic operator-valued function, such that Q_λ is a compact operator for any $\lambda \in \mathcal{D}$. Then using the analytic Fredholm theorem [5] (Theorem VI.14) we get that the function $(1 - Q_\lambda)^{-1}$ is a meromorphic function in $\mathcal{D}$. Since $((\lambda - L_0) W_\lambda)^{-1}$ is a bounded operator, we can conclude that the operator L has only a discrete spectrum in $\mathcal{D}$. ■

Remark 2 *The representations (10)-(11) imply that*
1) A_λ *is a positivity improving operator for all* $\lambda > 0$, *since* $G_\lambda(x-y) \ > \ 0, \ \forall \, x, y \in \mathbb{R}^d$,
2) $G_\lambda(x-y)$ *is monotonically decreasing with respect to* $\lambda > 0$,
3) formulas (14), (10), (11) imply that $Q_\lambda, \ \lambda > 0$, *is a positivity improving compact integral operator in* $C_b(\mathbb{R}^d)$ *with the kernel*

$$\mathcal{Q}_\lambda(x, y) \ = \ \frac{G_\lambda(x-y) V(y)}{\lambda + 1 - V(x)}. \tag{15}$$

We study next the behavior of the spectral radius $r(Q_\lambda)$ of the operator Q_λ as a function of λ when $\lambda > 0$.

Remark 3 *From the known formula for the spectral radius it follows that if Q is a positive operator, and if there exists a function $\varphi(x) \in \mathcal{E}$, $\varphi \geq 0$, $\|\varphi\| = 1$, such that*

$$Q\varphi(x) \;\geq\; c_0\varphi(x), \tag{16}$$

then

$$r(Q) \;\geq\; c_0,$$

(see e.g. [3], Theorem 6.2)

Lemma 4 *The spectral radius $r(Q_\lambda)$ is continuous and monotonically decreasing with respect to $\lambda > 0$. Moreover, $r(Q_\lambda) \to 0$ for $\lambda \to +\infty$.*

Proof. The continuity of the spectral radius follows from the compactness and continuity in λ of the operator Q_λ.

The second statement of the lemma follows from the fact, that if A, B are positive operators, such that $A \leq B$ in the order sense defined by the cone of positive functions, then $r(A) \leq r(B)$. Really, it is easy to see that for both our spaces $\mathcal{E}$ norms of positive operators A, B are attained on non negative functions:

$$\|A\| \;=\; \sup_{f\geq 0} \frac{\|Af\|}{\|f\|},$$

and the same for B. Thus, $\|A\| \leq \|B\|$.

Analogously, for any $n \in N$ we have $\|A^n\| \leq \|B^n\|$, and consequently, $r(A) \leq r(B)$ due to the formula for the spectral radius $r(A) = \lim_{n\to\infty} \sqrt[n]{\|A^n\|}$. The positive kernel $\mathcal{Q}_\lambda(x, y)$ of the operator Q_λ is the monotonically decreasing function of $\lambda > 0$. Thus we have

$$0 \;\leq\; Q_{\lambda_1} \;\leq\; Q_{\lambda_2}, \quad \text{when} \quad \lambda_1 > \lambda_2,$$

and consequently, the spectral radius $r(Q_\lambda)$ is monotonically decreasing with respect to $\lambda > 0$. The convergence to 0 follows from (14). ■

As follows from (7), the equation on the eigenfunction ψ

$$(L_0 + V - \lambda)\psi \;=\; 0, \quad \lambda > 0, \tag{17}$$

can be written as

$$Q_\lambda\psi(x) \;=\; \psi(x). \tag{18}$$

where Q_λ is a compact positive operator defined by (14).

Using Lemma 4 we can conclude that if

$$\lim_{\lambda \to 0+} r(Q_\lambda) \ > \ 1, \tag{19}$$

then there exists such $\lambda > 0$ that

$$r(Q_\lambda) = 1, \quad \text{and} \quad r(Q_{\lambda'}) < 1 \ \text{ for } \ \lambda' > \lambda. \tag{20}$$

If $r(Q_\lambda) < 1$ for all $\lambda > 0$, then the positive spectrum of L is absent. For example, the positive spectrum of L is absent in $C_b(\mathbb{R}^d)$ if $d \geq 3$, $V \in L^1(\mathbb{R}^d)$, $0 \leq V(x) \leq 1 - \varepsilon < 1$, and L^1-norm of V is small enough.

From (20) it follows by the Krein-Rutman theorem ([3], Theorem 6.2) that 1 is the eigenvalue of Q_λ with a positive eigenfunction $\psi_\lambda(x) > 0$. Obviously, λ is the maximal positive eigenvalue of the operator L, and $\psi_\lambda(x)$ is the ground state of the operator L. The uniqueness of the ground state $\psi_\lambda(x) > 0$ of the operator L in the space $L^2(\mathbb{R}^d)$ follows from the positivity improving property of the semigroups e^{tL_0} and e^{tL}, see e.g. [6], Theorem XIII.44. The last semigroup is positivity improving due to the Feynman-Kac formula.

Lemma 5 *1. If the ground state $\psi_\lambda(x) \in C_b(\mathbb{R}^d)$, then $\psi_\lambda(x) \to 0$ as $|x| \to \infty$.*
2. If the ground state $\psi_\lambda(x) \in L^2(\mathbb{R}^d)$, then $\psi_\lambda(x) \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\psi_\lambda(x) \to 0$ as $|x| \to \infty$.
3. If the ground state $\psi_\lambda(x) \in C_b(\mathbb{R}^d)$ and $V(x) \in L^2(\mathbb{R}^d)$, then $\psi_\lambda(x) \in L^2(\mathbb{R}^d)$. In particular, if the potential $V(x) \in C_0(\mathbb{R}^d)$ then $\psi_\lambda(x) \in L^2(\mathbb{R}^d)$.

Proof. 1. The ground state $\psi_\lambda(x)$ satisfies the equation (18) with Q_λ defined by (14). Since $V(x) \to 0$ as $|x| \to \infty$ and $G_\lambda(x) \in L^1(\mathbb{R}^d)$ by (12), then from the Lebesgue convergence theorem it follows that the convolution of the functions $V\psi_\lambda$ and G_λ tends to 0 for $|x| \to \infty$. Thus, all eigenfunctions $\psi_\lambda(x) \in C_b(\mathbb{R}^d)$ for positive λ tend to 0 when $|x| \to \infty$.

2. We again use the equation (18) for the function $\psi_\lambda(x)$. Since $V(x)$ is bounded, then $(V\psi_\lambda)(x) \in L^2(\mathbb{R}^d)$ and $(\widetilde{V\psi_\lambda})(p) \in L^2(\mathbb{R}^d)$. Using (11) and (4) we conclude that $\widetilde{G_\lambda}(p) \in L^2(\mathbb{R}^d)$ $(\lambda > 0)$. Then $(\widetilde{A_\lambda V\psi_\lambda})(p) \in L^1(\mathbb{R}^d)$ as a product of two functions from $L^2(\mathbb{R}^d)$. Consequently, $(A_\lambda V\psi_\lambda)(x) \in C_b(\mathbb{R}^d)$, and $\psi_\lambda(x) = Q_\lambda \psi_\lambda(x) \in C_b(\mathbb{R}^d)$, since $W_\lambda^{-1} \in C_b(\mathbb{R}^d)$ for any $\lambda > 0$.

3. The statement directly follows from (18). ∎

3 Ground state in $C_b(\mathbb{R}^d)$

Now we formulate conditions on $V(x)$ which give the existence of the ground state $\psi(x)$. We omit the subscript λ in $\psi_\lambda(x)$ in the subsequent text.

Theorem 6 *Let $d = 1, 2$. Then for any $V \not\equiv 0$ the operator $L = L_0 + V$ has a positive eigenvalue $\lambda > 0$ with the corresponding positive eigenfunction $\psi(x) > 0$.*

Proof. Conditions on the function $a(x)$ imply that $\tilde{a}(p) = 1 - (Cp, p) + o(|p|^2)$ as $|p| \to 0$. Let us take a function $\varphi(x) \in C_0(\mathbb{R}^d),\ \varphi \geq 0,\ \|\varphi\| = 1$ such that $(\widetilde{V\varphi})(0) = \int V(x)\varphi(x)dx > 0$. Then for any $x \in supp\ \varphi$

$$(Q_\lambda \varphi)(x) \;=\; \frac{1}{\lambda + 1 - V(x)} \int_{\mathbb{R}^d} G_\lambda(x-y)V(y)\varphi(y)\ dy \;=$$

$$\frac{1}{\lambda + 1 - V(x)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ipx+ipy} \frac{\tilde{a}(p)\ dp}{\lambda + 1 - \tilde{a}(p)} V(y)\varphi(y)\ dy \;=$$

$$\frac{1}{\lambda + 1 - V(x)} \int_{\mathbb{R}^d} \frac{\tilde{a}(p)e^{-ipx}}{\lambda + 1 - \tilde{a}(p)} (\widetilde{V\varphi})(p)\ dp \to \infty.$$

The continuous functions $U_\lambda(x) = (Q_\lambda \varphi)^{-1}(x)$ tend to 0 monotonically as $\lambda \to 0+$ by Remark 2 and uniformly on *supp* φ by the Dini theorem. Thus for any c_0 there exists $\lambda > 0$ for which (16) is fulfilled. Consequently, $\lim_{\lambda \to 0+} r(Q_\lambda) = \infty$ and (19) holds. ■

For dimensions $d \geq 3$, the integral in (11) has a finite limit as $\lambda \to 0+$ and

$$\sup_\lambda \sup_{x,y} G_\lambda(x-y) \;\leq\; \sup_{x,y} G_0(x-y) \;\leq\; \int_{\mathbb{R}^d} \frac{|\tilde{a}(p)|\ dp}{|1 - \tilde{a}(p)|} \;=\; g_0 \;< \infty. \quad (21)$$

Thus the existence of the ground state in the case $d \geq 3$ depends on the properties of $V(x)$.

Theorem 7 (small paradise) *Let $d \geq 3$. Assume that there exists $\delta > 0$ such that $V(x) = 1$ when $x \in B_\delta$, where B_δ is a ball of a radius δ. Then the ground state of L exists.*

Proof. It is enough to show (19). Let us take for a given $0 < \delta < 1$ a "continuous approximation" of the indicator function $\hat{\chi}_{B_\delta}(x)$:

$$\hat{\chi}_{B_\delta}(x) = 0,\ x \in \mathbb{R}^d \backslash B_\delta, \quad \hat{\chi}_{B_\delta}(x) = 1,\ x \in B_{0.9\delta},$$

where $B_{0.9\delta} \subset B_\delta$, and $0 \le \hat{\chi}_{B_\delta}(x) \le 1,\ x \in B_\delta \backslash B_{0.9\delta}$. Then for any $\lambda \in (0,1)$ and any $x \in B_\delta$ we get:

$$Q_\lambda \hat{\chi}_{B_\delta}(x) \ = \ \int_{B_\delta} \mathcal{Q}_\lambda(x,y)\hat{\chi}_{B_\delta}(y)dy \ \ge \ \frac{1}{\lambda}\int_{B_{0.9\delta}} G_\lambda(x-y)dy \ \ge \ \frac{vol(B_{0.9\delta})}{\lambda}\kappa_1, \tag{22}$$

where $vol(B_{0.9\delta})$ is the volume of the ball $B_{0.9\delta} \subset B_\delta$, and κ_1 is defined as

$$\kappa_1 = \min_{x,y \in B_1} G_1(x-y) < \min_{\lambda \in (0,1)} \min_{x,y \in B_\delta} G_\lambda(x-y) \tag{23}$$

Thus $\lim_{\lambda \to 0+} r(Q_\lambda) = \infty$ for any $\delta > 0$. ■

Remark 8 *Let $d \ge 3$. For any $\delta > 0$ there is $\varepsilon > 0$ such that if $V(x) \ge 1-\varepsilon$ for $x \in B_\delta$, then the ground state of L exists.*

The proof of the statement follows the similar reasoning as above in Theorem 7.

Proposition 9 (dependence λ on δ) *Let $d \ge 3$. Denote by*

$$L(\delta) \ = \ L_0 \ + \ V^\delta, \quad \delta \in (0,1),$$

the family of operators of the form (2), where $V^\delta = V_1^\delta + V_2^\delta$ with

$$V_1^\delta = \chi_{B_\delta}(x), \quad V_2^\delta(x) \ge 0, \quad \text{and} \quad \int_{\mathbb{R}^d} V_2^\delta(x)dx \le c_0\delta^d.$$

Let $\lambda(\delta)$ be the maximal eigenvalue of $L(\delta)$. Then $\lambda(\delta) \to 0+$ as $\delta \to 0+$.

Proposition follows from two-sided estimate on the spectral radius $r(Q_\lambda)$:

$$\frac{c_1\delta^d}{\lambda} \ \le \ r(Q_\lambda) \ \le \ \frac{c_2\delta^d}{\lambda}. \tag{24}$$

The upper bound in (24) follows from the evident inequality:

$$r(Q_\lambda) \le \|Q_\lambda\| \le \sup_x \int_{\mathbb{R}^d} |\mathcal{Q}_\lambda(x,y)|dy$$

and the estimate (21). The lower bound follows from the estimate (22).

Theorem 10 *Assume that for some $\beta \in (0,1)$ there exists $R > 0$ such that*

$$\beta \;\le\; V(x) \;\le\; 1, \quad x \in B_R. \tag{25}$$

Then the ground state of the operator L exists for $R = R(\beta)$ sufficiently large.

Proof. We will prove that there exists a ground state ψ of the operator L, such that $\psi \in L^2(\mathbb{R}^d)$. Then from Lemma 5 it follows that $\psi \in C_b(\mathbb{R}^d)$.

To prove the existence of $\psi \in L^2(\mathbb{R}^d)$ it is sufficient to verify that the quadratic form (Lf, f) is positive for some $f \in L^2(\mathbb{R}^d)$. Let us take $f = \chi_{B_R}$. Then

$$(L\chi_{B_R}, \chi_{B_R}) = (L_0\chi_{B_R}, \chi_{B_R}) + (V\chi_{B_R}, \chi_{B_R}), \tag{26}$$

and

$$(V\chi_{B_R}, \chi_{B_R}) \;\ge\; \beta \; vol(B_R). \tag{27}$$

For the operator L_0 we have:

$$-(L_0 f, f) = \int\int a(y-x)(f(x)-f(y))f(x)dydx =$$

$$\frac{1}{2}\int\int a(y-x)(f(x)-f(y))^2 dydx.$$

Consequently, the first term in (26) can be written as

$$-(L_0\chi_{B_R}, \chi_{B_R}) = \int_{|x|<R}\int_{|y|>R} a(y-x)dxdy \le \int_{|x|<R}\int_{|x|+|z|>R} a(z)dxdz =$$

$$C_d \int_0^R r^{d-1}\int_{|z|>R-r} a(z)dzdr = C_d \int_{\mathbb{R}^d} a(z)\int_{(R-|z|)_+}^R r^{d-1}drdz =$$

$$\frac{C_d R^d}{d}\int_{\mathbb{R}^d} a(z)\left(1 - \left(1 - \frac{|z|}{R}\right)_+^d\right) dz.$$

Here $\frac{C_d R^d}{d} = vol(B_R)$. Thus,

$$\frac{1}{vol(B_R)}(L_0\chi_{B_R}, \chi_{B_R}) \;\to\; 0 \quad \text{as} \quad R \to \infty. \tag{28}$$

Finally, (26) - (28) imply that the operator L has a positive discrete spectrum and a ground state if R is taken sufficiently large. ■

4 Ground state in $L^2(\mathbb{R}^d)$

Now we can omit the condition of the continuity of $V(x)$. The statements of Lemmas 1,4 hold in the space $L^2(\mathbb{R}^d)$, since the operator Q_λ defined by (14) is a compact positive operator in the order sense in $L^2(\mathbb{R}^d)$. The further consideration of the problem in the space $L^2(\mathbb{R}^d)$ is simpler because the operator $L = L_0 + V$ is a bounded self-adjoint operator in $L^2(\mathbb{R}^d)$. The analysis of the operator L is based on transformations of equation (17), that are analogous to the transformations exploited in the theory of Schrödinger operators. As a result, the ground state problem can be reduced to the spectral analysis of a compact positive self-adjoint operator.

The equation (17) on the eigenfunction $\psi(x)$ can be rewritten in the following way:

$$V^{1/2}(\lambda - L_0)^{-1}V^{1/2}u \;=\; u, \tag{29}$$

where

$$u \;=\; V^{1/2}\psi, \quad \psi \;=\; (\lambda - L_0)^{-1}V^{1/2}u.$$

Inserting (9) to (29) we get that (29) is equivalent to the following equation:

$$S_\lambda u \;=\; u, \quad \text{where} \quad S_\lambda \;=\; \frac{1}{\lambda+1}W_\lambda^{-1}V^{1/2}A_\lambda V^{1/2}, \tag{30}$$

and A_λ is defined by (10). S_λ is a compact positive operator, and it is similar to the compact positive symmetric operator

$$\hat{S}_\lambda \;=\; \frac{1}{\lambda+1}W_\lambda^{-1/2}V^{1/2}A_\lambda V^{1/2}W_\lambda^{-1/2}.$$

Consequently the spectra of operator S_λ and self-adjoint operator $\hat{S}_\lambda$ are the same, and

$$r(S_\lambda) \;=\; r(\hat{S}_\lambda) \;=\; \|\hat{S}_\lambda\|_{L^2}. \tag{31}$$

The spectral radius $r(S_\lambda)$ of the operator S_λ has the same properties as in Lemma 4.

All statements of Theorems 6,7,10 are also valid in the space $L^2(\mathbb{R}^d)$. In the proof of Theorem 6 we have to use the variational principle for $\hat{S}_\lambda$ instead of Remark 3. The upper bound for the spectral radius follows from the inequality

$$r(\hat{S}_\lambda) = \|\hat{S}_\lambda\|_{L^2} \le \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{S}_\lambda^2(x,y)\,dx\,dy\right)^{1/2}.$$

Remark 11 *If the potential $V \in L^1(\mathbb{R}^d)$, then from (18) and Lemma 5 it follows that $\psi(x) \in L^1(\mathbb{R}^d)$ for any $\lambda > 0$.*

5 The subcritical regime

We consider in this section a local perturbation in the form of "small paradise" for the subcritical regime: $m(x) \equiv m > 1$. In this case the non local operator L has a form

$$Lu(x) \ = \ -\tilde{m}(x)u(x) \ + \ \int_{\mathbb{R}^d} a(x-y)u(y)dy. \tag{32}$$

It can be rewritten as

$$Lu(x) \ = \ L_0u(x) \ + \ D(x)u(x), \tag{33}$$

where

$$L_0u(x) \ = \ \int_{\mathbb{R}^d} a(x-y)(u(y)-u(x))dy, \tag{34}$$

$$D(x) \ = \ 1-\tilde{m}(x) \ = \ \tilde{V}(x)-h, \quad h=m-1>0, \tag{35}$$

and

$$\tilde{V}(x) \ = \ m-\tilde{m}(x), \quad 0 \le \tilde{V}(x) \le m, \quad \tilde{V}(x) \to 0 \ \text{ as } \ |x| \to \infty. \tag{36}$$

If $\tilde{V}(x) \equiv 0$, then from (33)-(35) it follows that

$$(Lu,u) \ \le \ -h(u,u), \quad h=m-1>0.$$

The equation on a positive eigenvalue of the operator (33) and the corresponding eigenfunction $\psi(x)$

$$L_0\psi \ + \ D\psi \ = \ \hat{\lambda}\psi, \quad \hat{\lambda} > 0,$$

is equivalent to the above equations (17) - (18) with

$$\mathcal{Q}_\lambda(x,y) \ = \ \frac{G_\lambda(x-y)\tilde{V}(y)}{\lambda+1-\tilde{V}(x)}, \quad 0 \le \tilde{V}(x) \le m$$

under additional condition: $\lambda > h$. Moreover, as follows from Lemma 5, in this case the ground state $\psi(x) \in L^2(\mathbb{R}^d)$.

Theorem 12 (small paradise in the subcritical regime) *For any $d \geq 1$, assume that there exists $\delta > 0$ such that $\tilde{V}(x) = m$ when $x \in B_\delta$, where B_δ is a ball of a radius δ. Then the ground state of $L = L_0 + D$ exists.*

Proof. It is enough to prove inequality similar to (19):

$$\lim_{\lambda \to h+} r(Q_\lambda) > 1.$$

Using the same reasoning as in Theorem 7 for the normalized indicator function $\phi_\delta(x) = \frac{1}{\sqrt{|B_\delta|}} \chi_{B_\delta}(x)$ we get for any $x \in B_\delta$:

$$Q_\lambda \phi_\delta(x) \;=\; \int_{B_\delta} \mathcal{Q}_\lambda(x,y)\phi_\delta(y)dy \;\geq\; \frac{m}{(\lambda + 1 - m)\;\sqrt{|B_\delta|}} \int_{B_\delta} G_\lambda(x-y)dy \;\to\; \infty$$

as $\lambda \to h+$. Thus $\lim_{\lambda \to h+} r(Q_\lambda) = \infty$ for any $\delta > 0$, and equation (18) has a solution which is the ground state $\psi(x) > 0$ corresponding to the maximal eigenvalue $\hat{\lambda} = \lambda - h > 0$ of the operator (33). ■

Theorem 12 means that any local perturbation in the form of small paradise in the subcritical regime produces a crucial change in the asymptotic behavior of the system: instead of the exponentially decreasing population density one can find an exponentially increasing population everywhere in the space, and the density profile is described by the corresponding ground state $\psi(x)$.

6 Concluding remarks

Existence and uniqueness of the ground state of the operator L in $L^2(\mathbb{R}^d)$ immediately implies the following asymptotic formulas on the solution $u(x,t)$ of the evolution problem (1).

Proposition 13 *Assume that there exists a unique ground state $\psi > 0$ of the operator L in $L^2(\mathbb{R}^d)$, and $\lambda > 0$ be the maximal eigenvalue.*

Then in $L^2(\mathbb{R}^d)$ the following asymptotic formula holds:

$$u(x,t) \;=\; e^{t\lambda} c_0 \psi(x)(1 + o(1)) \quad (t \to \infty), \tag{37}$$

where $c_0 = (u_0, \psi)_{L^2} > 0$ for any initial condition $u_0 \geq 0,\ u_0 \not\equiv 0$.

In $C_b(\mathbb{R}^d)$ for any $u_0(x) \ge 0,\ u_0 \not\equiv 0$, and any bounded domain $D \subset \mathbb{R}^d$ we have

$$\int_D u(x,t)dx \ \to\ \infty \qquad as \qquad t \to \infty.$$

Proof. The operator (2) is a bounded self-adjoint operator in $L^2(\mathbb{R}^d)$. Then the asymptotic (37) is a direct consequence of the spectral decomposition for the operator L.

Let us fix a ball B such that $B \cap\ supp\ u_0 \neq \emptyset$, and denote $u_0^B(x) = u_0(x) \cdot \chi_B(x)$. Then $u_0^B(x) \in L^2(\mathbb{R}^d),\ u_0^B \not\equiv 0$. Using the positivity of the semigroup e^{tL} we have for any bounded domain D

$$\int_D u(x,t)dx = \langle u(\cdot,t), \chi_D(\cdot)\rangle = \langle e^{tL}u_0, \chi_D\rangle \ge$$

$$\langle e^{tL}u_0^B, \chi_D\rangle \sim e^{\lambda t} c_0(\psi, \chi_D)_{L^2} \to \infty \quad (t \to \infty) \tag{38}$$

with $c_0 = (u_0^B, \psi)_{L^2} > 0$. ■

Conclusions. Since $u(t,x), x \in \mathbb{R}^d$ describes the density of the population at time t, then the asymptotic (38) means that, in the case when the operator L has a positive eigenvalue $\lambda > 0$, the population is exponentially increasing everywhere in the space. Moreover in the case $L^2(\mathbb{R}^d)$, for any initial density $u(0,x) = u_0(x) \ge 0$ the shape of $u(t,x)$ tends to the shape of the ground state $\psi(x)$ of the operator L up to the multiplication on $e^{\lambda t}$.

If $m(x) \equiv 0$, then the total mass grows as e^t. So the exponent λ of the growth in the case of existence of the ground state is in the interval $(0,1]$.

To observe the exponential growing of the density in small dimensions $d = 1, 2$, it is enough to have any small region where mortality is less than 1 (Theorem 6). If $d \ge 3$, then a positive eigenvalue appears in two cases:
1) if there is a region of a positive volume, where the mortality is equal to 0 (Theorem 7) (small paradise);
2) if the mortality has an upper bound less than 1 in a region, and the size of the region depends on the upper bound of the mortality (Theorem 10).

We found that perturbations in the form of small paradise are very powerful: they switch a subcritical regime in the system (with exponentially decreasing population density) to a supercritical regime with exponentially increasing population (Theorem 12).

Next natural question is what happens if the mortality $m(x)$ is greater than 1 (or even $m(x)$ is a growing function) outside of a region, where the mortality has a local negative fluctuations. Is it possible that an active growing inside of a bounded region can be stronger than the influence of a large mortality outside? We suppose to study this question in a forthcoming paper. We also plan to continue spectral analysis of the operator L in more details including the study of continuous spectrum and the study of a structure of the ground state.

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