

Improved pointwise iteration-complexity of a regularized ADMM and of a regularized non-Euclidean HPE framework

Max L.N. Gonçalves ^{*} Jefferson G. Melo ^{*} Renato D.C. Monteiro [†]

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Abstract

This paper describes a regularized variant of the alternating direction method of multipliers (ADMM) for solving linearly constrained convex programs. It is shown that the pointwise iteration-complexity of the new method is better than the corresponding one for the standard ADMM method and that, up to a logarithmic term, is identical to the ergodic iteration-complexity of the latter method. Our analysis is based on first presenting and establishing the pointwise iteration-complexity of a regularized non-Euclidean hybrid proximal extragradient framework whose error condition at each iteration includes both a relative error and a summable error. It is then shown that the new method is a special instance of the latter framework where the sequence of summable errors is identically zero when the ADMM stepsize is less than one or a nontrivial sequence when the stepsize is in the interval $[1, (1 + \sqrt{5})/2)$.

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1 Introduction

The goal of this paper is to present a regularized variant of the alternating direction method of multipliers (ADMM) for solving the linearly constrained convex problem

$$\inf\{f(y) + g(s) : Cy + Ds = c\} \tag{1}$$

where \mathcal{X} , \mathcal{Y} and \mathcal{S} are inner product spaces, $f : \mathcal{Y} \rightarrow (-\infty, \infty]$ and $g : \mathcal{S} \rightarrow (-\infty, \infty]$ are proper closed convex functions, $C : \mathcal{Y} \rightarrow \mathcal{X}$ and $D : \mathcal{S} \rightarrow \mathcal{X}$ are linear operators, and $c \in \mathcal{X}$.

^{*}Institute of Mathematics and Statistics, Federal University of Goiás, Campus II- Caixa Postal 131, CEP 74001-970, Goiânia-GO, Brazil. (E-mails: maxlmg@ufg.br and jefferson@ufg.br). This work was done while these authors were a postdoc at the School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA. The work of these authors was supported in part by CNPq Grant 444134/2014-0, 309370/2014-0, 200852/2014-0, 201047/2014-4 and FAPEG/GO.

[†]School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, 30332-0205. (email: monteiro@isye.gatech.edu). The work of this author was partially supported by NSF Grant CMMI-1300221.

The standard ADMM for solving (1) recursively computes a sequence $\{(s_k, y_k, x_k)\}$ as follows. Given (y_{k-1}, x_{k-1}) , it computes (s_k, y_k, x_k) as:

$$\begin{aligned} s_k &= \operatorname{argmin}_s \left\{ g(s) - \langle x_{k-1}, Ds \rangle_{\mathcal{X}} + \frac{\beta}{2} \|Cy_{k-1} + Ds - c\|_{\mathcal{X}}^2 \right\}, \\ y_k &= \operatorname{argmin}_y \left\{ f(y) - \langle x_{k-1}, Cy \rangle_{\mathcal{X}} + \frac{\beta}{2} \|Cy + Ds_k - c\|_{\mathcal{X}}^2 \right\}, \\ x_k &= x_{k-1} - \theta\beta [Cy_k + Ds_k - c] \end{aligned}$$

where $\beta > 0$ is a fixed penalty parameter and $\theta > 0$ is a fixed stepsize.

The ADMM was introduced in [8, 10] and is thoroughly discussed in [2, 9]. Recently, there has been some growing interest in this method (see for instance [1, 4, 6, 11, 13, 26, 32] and the references cited therein). To discuss the complexity results about the ADMM, we use the terminology weak pointwise or strong pointwise bounds to refer to complexity bounds relative to the best of the k first iterates or the last iterate, respectively, to satisfy a suitable termination criterion. The first iteration-complexity bound for the ADMM was established only recently in [25] under the assumption that C is injective. More specifically, its ergodic iteration-complexity is derived in [25] for any $\theta \in (0, 1]$ while a weak pointwise iteration-complexity easily follows from the approach in [25] for any $\theta \in (0, 1)$. Subsequently, without assuming that C is injective, [15] established the ergodic iteration-complexity of the ADMM with $\theta = 1$, and also of a variant of the ADMM method, namely, the split inexact Uzawa method [33]. Paper [14] establishes the weak pointwise and ergodic iteration-complexity of a class of ADMM type methods which includes the standard ADMM for any $\theta \in (0, (1 + \sqrt{5})/2)$. A strong pointwise iteration-complexity bound for the ADMM with $\theta \in (0, 1]$ is derived in [16]. Finally, a number of papers (see for example [4, 5, 7, 12, 13, 21] and references therein) have developed pointwise and/or ergodic iteration-complexity bounds for other variants of the ADMM which are quite similar to the corresponding ones for the standard ADMM.

Although different termination criteria are used in the aforementioned papers, their complexity results can be easily rephrased in terms of a simple termination, namely: for a given $\rho > 0$, terminate with a quadruple $(s, y, x, x') \in \mathcal{S} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{X}$ satisfying

$$\max\{\|Cy + Ds - c\|, \|x' - x\|\} \leq \rho, \quad 0 \in \partial_\rho g(s) - D^*x, \quad 0 \in \partial_\rho f(y) - C^*x'.$$

In terms of this termination, the best pointwise iteration-complexity bounds are $\mathcal{O}(\rho^{-2})$ while the best ergodic ones are $\mathcal{O}(\rho^{-1})$ but the pointwise results guarantee that above two inclusions hold with $\rho = 0$ (i.e., with ∂_ρ replaced by ∂). This paper presents a regularized ADMM whose strong pointwise iteration-complexity is $\mathcal{O}(\rho^{-1} \log(\rho^{-1}))$ for any stepsize $\theta \in (0, (1 + \sqrt{5})/2)$, and hence improves the aforementioned pointwise iteration complexity by an $\mathcal{O}(\rho \log(\rho^{-1}))$ factor.

It is well-known (e.g., see [25]) that the standard ADMM with $\theta \in (0, 1]$ and C injective can be viewed as an inexact proximal point (PP) method, more specifically, as an instance of the hybrid proximal extragradient (HPE) framework proposed by [30]. In contrast to the original Rockafellar's PP method which is based on a summable error condition, the HPE framework is based on a relative HPE error condition involving Euclidean distances. Convergence results for the HPE framework are studied in [30], and its weak pointwise and ergodic iteration-complexities are established in [23] (see also [24, 25]). Applications of the HPE framework to the iteration-complexity analysis of several zero-order (resp., first-order) methods for solving monotone variational inequalities and monotone inclusions (resp., saddle-point problems) are discussed in [17, 18, 23, 24, 25]. Paper [31]

describes and studies the convergence of a non-Euclidean HPE (NE-HPE) framework which essentially generalizes the HPE one to the context of general Bregman distances. The latter framework was further generalized in [20] where its ergodic iteration-complexity was established. More specifically, consider the monotone inclusion problem $0 \in T(z)$ where T is a maximal monotone operator and let w be a convex differentiable function. Recall that for a given pair $(z_-, \lambda) = (z_{k-1}, \lambda_k)$, the exact PP method computes the next iterate $z = z_k$ as the (unique) solution of the prox-inclusion $\lambda^{-1}[\nabla w(z_-) - \nabla w(z)] \in T(z)$. An instance of the NE-HPE framework described in [20] computes an approximate solution of this inclusion based on the following relative NE-HPE error criterion: for some tolerance $\sigma \in [0, 1]$, a triple $(\tilde{z}, z, \varepsilon) = (\tilde{z}_k, z_k, \varepsilon_k)$ is computed such that

$$r := \frac{1}{\lambda} [\nabla w(z_-) - \nabla w(z)] \in T^\varepsilon(\tilde{z}), \quad (dw)_z(\tilde{z}) + \lambda\varepsilon \leq \sigma(dw)_{z_-}(\tilde{z}) \quad (2)$$

where dw is the Bregman distance defined as $(dw)_z(z') = w(z') - w(z) - \langle \nabla w(z), z' - z \rangle$ for every z, z' and T^ε denotes the ε -enlargement [3] of T (it has the property that $T^\varepsilon(u) \supset T(u)$ for each u with equality holding when $\varepsilon = 0$). Clearly, if $\sigma = 0$ in (2), then $z = \tilde{z}$ and $\varepsilon = 0$, and the inclusion in (2) reduces to the prox-inclusion. Also, the HPE framework is the special case of the NE-HPE one in which $w(\cdot) = \|\cdot\|^2/2$ and $\|\cdot\|$ is the Euclidean norm.

Section 2 considers a MIP of the form $0 \in (S+T)(z)$ where T is monotone, S is μ -monotone with respect to w for some $\mu > 0$ (see condition **A1**) and w is a regular distance generating function (see Definition 2.2). It then presents and establishes the strong pointwise iteration-complexity of a variant of the NE-HPE framework for solving such a MIP in which the inclusion in (2) is strengthened to $r \in S(\tilde{z}) + T^\varepsilon(\tilde{z})$ but its error condition is weakened in that an additional nonnegative tolerance is added to the right hand side of the inequality in (2) which is τ -upper summable. This extension of the error condition will be useful in the analysis of the regularized ADMM of Section 4 with ADMM stepsizes $\theta > 1$.

Section 3 presents and establishes the strong pointwise iteration-complexity of a regularized NE-HPE framework which solves the inclusion $0 \in T(z)$ with no assumption on T other than monotonicity. The latter framework is based on the idea of invoking the above NE-HPE variant to solve perturbed MIPs of the form $0 \in (S+T)(z)$ where $S(\cdot) = \mu[\nabla w(\cdot) - \nabla w(z_0)]$ for some $\mu > 0$, point z_0 and regular distance generating function w .

Section 4 presents and establishes the $\mathcal{O}(\rho^{-1} \log(\rho^{-1}))$ strong pointwise iteration-complexity of a regularized ADMM whose description depends on β, θ (as the standard ADMM) and a regularization parameter μ . It is well-known that (1) can be reformulated as a monotone inclusion problem of the form $0 \in T(z)$ with $z = (x, y)$. The regularized ADMM can be viewed as a special instance of the regularized NE-HPE framework applied to the latter inclusion where: i) all stepsizes λ_k 's are equal one; ii) the distance generating function w depends on β, θ and operator C as in relation (57); and, iii) the sequence of τ -upper summable errors is zero when the ADMM stepsize $\theta \in (0, 1)$ and nontrivial (and hence nonzero) when $\theta \in [1, (1 + \sqrt{5})/2)$. Hence, the iteration complexity analysis of the regularized ADMM for the case in which $\theta \in [1, (1 + \sqrt{5})/2)$ requires both a combination of relative and τ -upper summable errors while the one for the case of $\theta \in (0, 1)$ requires only relative errors. Moreover, the distance generating function w is strongly convex only when C is injective but is always regular and hence fulfills the conditions required for the iteration-complexity results of Section 3 to hold.

This paper is organized as follows. Subsection 1.1 presents the notation and review some basic concepts about convexity and maximal monotone operators. Section 2 introduces the class of

regular distance generating functions and presents the aforementioned variant of the NE-HPE framework. Section 3 presents the regularized NE-HPE framework and its complexity analysis. Section 4 contains two subsections. Subsection 4.1 describes the regularized ADMM and its pointwise iteration-complexity result whose proof is given in Subsection 4.2. Finally, the appendix reviews some basic results about dual seminorms and existence of optimal solutions and/or Lagrange multipliers for linearly constrained convex programs, and presents the proofs of two results of Subsection 4.2.

1.1 Basic concepts and notation

This subsection presents some definitions, notation and terminology needed by our presentation.

The set of real numbers is denoted by \mathbb{R} . The set of non-negative real numbers and the set of positive real numbers are denoted by \mathbb{R}_+ and \mathbb{R}_{++} , respectively. For $t > 0$, we let $\log^+(t) := \max\{\log t, 0\}$.

Let \mathcal{Z} be a finite-dimensional real vector space with inner product denoted by $\langle \cdot, \cdot \rangle$ and let $\|\cdot\|$ denote an arbitrary seminorm in \mathcal{Z} . Its dual (extended) seminorm, denoted by $\|\cdot\|^*$, is defined as $\|\cdot\|^* := \sup\{\langle \cdot, z \rangle : \|z\| \leq 1\}$. Some basic properties of the dual seminorm are given in Proposition A.1 in Appendix A.

Given a set-valued operator $S : \mathcal{Z} \rightrightarrows \mathcal{Z}$, its domain is denoted by $\text{Dom}(S) := \{z \in \mathcal{Z} : S(z) \neq \emptyset\}$ and its inverse operator $S^{-1} : \mathcal{Z} \rightrightarrows \mathcal{Z}$ is given by $S^{-1}(v) := \{z : v \in S(z)\}$. The operator S is said to be monotone if

$$\langle z - z', s - s' \rangle \geq 0 \quad \forall z, z' \in \mathcal{Z}, \forall s \in S(z), \forall s' \in S(z').$$

Moreover, S is maximal monotone if it is monotone and, additionally, if T is a monotone operator such that $S(z) \subset T(z)$ for every $z \in \mathcal{Z}$ then $S = T$. The sum $S + T : \mathcal{Z} \rightrightarrows \mathcal{Z}$ of two set-valued operators $S, T : \mathcal{Z} \rightrightarrows \mathcal{Z}$ is defined by $(S + T)(x) := \{a + b \in \mathcal{Z} : a \in S(x), b \in T(x)\}$ for every $x \in \mathcal{Z}$. Given a scalar $\varepsilon \geq 0$, the ε -enlargement $T^{[\varepsilon]} : \mathcal{Z} \rightrightarrows \mathcal{Z}$ of a monotone operator $T : \mathcal{Z} \rightrightarrows \mathcal{Z}$ is defined as

$$T^{[\varepsilon]}(z) := \{v \in \mathcal{Z} : \langle v - v', z - z' \rangle \geq -\varepsilon, \quad \forall z' \in \mathcal{Z}, \forall v' \in T(z')\} \quad \forall z \in \mathcal{Z}. \quad (3)$$

Recall that the ε -subdifferential of a proper closed convex function $f : \mathcal{Z} \rightarrow [-\infty, \infty]$ is defined by $\partial_\varepsilon f(z) := \{v \in \mathcal{Z} : f(z') \geq f(z) + \langle v, z' - z \rangle - \varepsilon \quad \forall z' \in \mathcal{Z}\}$ for every $z \in \mathcal{Z}$. When $\varepsilon = 0$, then $\partial_0 f(x)$ is denoted by $\partial f(x)$ and is called the subdifferential of f at x . The operator ∂f is trivially monotone if f is proper. If f is a proper lower semi-continuous convex function, then ∂f is maximal monotone [28]. The conjugate f^* of f is the function $f^* : \mathcal{Z} \rightarrow [-\infty, \infty]$ defined as

$$f^*(v) = \sup_{z \in \mathcal{Z}} \langle v, z \rangle - f(z) \quad \forall v \in \mathcal{Z}.$$

2 A non-Euclidean HPE framework for a special class of MIPs

This section describes and derives convergence rate bounds for a non-Euclidean HPE framework for solving inclusion problems consisting of the sum of two operators, one of which is assumed to be μ -monotone with respect to a Bregman distance for some $\mu > 0$. The latter concept implies strong monotonicity of the operator when the Bregman distance is nondegenerate, i.e., its associated distance generating function is strongly monotone. However, it should be noted that when the Bregman distance is degenerate, the latter concept does not imply strong monotonicity of the operator.

We start by introducing the definition of a distance generating function and its corresponding Bregman distance adopted in this paper.

Definition 2.1. *Let $W \subset \mathcal{Z}$ be a convex set. A continuous convex function $w : W \rightarrow \mathbb{R}$ is called a distance generating function if $\text{int}(W) \neq \emptyset$ and w is continuously differentiable on $\text{int}(W)$. Moreover, w induces the Bregman distance $dw : W \times \text{int}(W) \rightarrow \mathbb{R}$ defined as*

$$(dw)(z'; z) := w(z') - w(z) - \langle \nabla w(z), z' - z \rangle \quad \forall z' \in W, \forall z \in \text{int}(W). \quad (4)$$

For simplicity, for every $z \in \text{int}(W)$, the function $(dw)(\cdot; z)$ will be denoted by $(dw)_z$ so that

$$(dw)_z(z') = (dw)(z'; z) \quad \forall z \in \text{int}(W), \forall z' \in W.$$

The following useful identities follow straightforwardly from (4):

$$\nabla(dw)_z(z') = -\nabla(dw)_{z'}(z) = \nabla w(z') - \nabla w(z) \quad \forall z, z' \in \text{int}(W), \quad (5)$$

$$(dw)_v(z') - (dw)_v(z) = \langle \nabla(dw)_v(z), z' - z \rangle + (dw)_z(z') \quad \forall z' \in W, \forall v, z \in \text{int}(W). \quad (6)$$

Our analyses of the non-Euclidean HPE frameworks presented in Sections 2 and 3 require an extra property of the distance generating function, namely, that of being regular with respect to a seminorm.

Definition 2.2. *For given positive constants α and L , closed convex set $Z \subset \text{int}(W)$ and seminorm $\|\cdot\|$ in \mathcal{Z} , a distance generating function $w : W \rightarrow \mathbb{R}$ is said to be (α, L) -regular with respect to $(Z, \|\cdot\|)$ if*

$$(dw)_z(z') \geq \frac{\alpha}{2} \|z - z'\|^2 \quad \forall z, z' \in Z, \quad (7)$$

$$\|\nabla w(z) - \nabla w(z')\|^* \leq L \|z - z'\| \quad \forall z, z' \in Z. \quad (8)$$

Note that if the seminorm in Definition 2.2 is a norm, then (7) implies that w is strongly convex, in which case the corresponding dw is said to be nondegenerate. However, since we are not necessarily assuming that $\|\cdot\|$ is a norm, our approach includes the case of w being not strongly convex, or equivalently, dw being degenerate (e.g., see Example 2.3(b) below).

Some examples of regular distance generating functions are as follows.

Example 2.3. *a) The distance generating function $w : \mathcal{Z} \rightarrow \mathbb{R}$ defined by $w(\cdot) := \langle \cdot, \cdot \rangle / 2$ is a $(1, 1)$ -regular with respect to $(\mathcal{Z}, \|\cdot\|)$ where $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$.*

b) Let $A : \mathcal{Z} \rightarrow \mathcal{Z}$ be a self-adjoint positive semidefinite linear operator. The distance generating function $w : \mathcal{Z} \rightarrow \mathbb{R}$ defined by $w(\cdot) := \langle A(\cdot), \cdot \rangle / 2$ is a $(1, 1)$ -regular with respect to $(\mathcal{Z}, \|\cdot\|)$ where $\|\cdot\| := \langle A(\cdot), \cdot \rangle^{1/2}$.

c) Let $\delta \in (0, 1]$ be given and define $W := \{x \in \mathbb{R}^n : x_i + \delta/n > 0, \forall i = 1, \dots, n\}$. The distance generating function $w : W \rightarrow \mathbb{R}$ defined by $w(x) := \sum_{i=1}^n (x_i + \delta/n) \log(x_i + \delta/n)$ for every $x \in W$ is a $(1/(1 + \delta), n/\delta)$ -regular with respect to $(Z, \|\cdot\|_1)$ where $Z = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \dots, n\}$.

The following result gives some useful properties of regular distance generating functions.

Lemma 2.4. *Let $w : W \rightarrow \mathbb{R}$ be an (α, L) -regular distance generating function with respect to $(Z, \|\cdot\|)$ as in Definition 2.2. Then,*

$$\left(1 + \frac{1}{t}\right)^{-1} (dw)_z(z') \leq \frac{L}{\alpha} [(dw)_z(\tilde{z}) + t(dw)_{\tilde{z}}(z')] \quad \forall t > 0, \forall z, z', \tilde{z} \in Z; \quad (9)$$

$$\|\nabla(dw)_{z'}(z)\|^* \leq \frac{\sqrt{2}L}{\sqrt{\alpha}} [(dw)_{z'}(z)]^{1/2} \quad \forall z, z' \in Z. \quad (10)$$

Proof. Using (4) and (8), it is easy to see that

$$(dw)_z(z') = w(z') - w(z) - \langle \nabla w(z), z' - z \rangle \leq \frac{L}{2} \|z - z'\|^2 \quad \forall z, z' \in Z. \quad (11)$$

To show (9), let $t > 0$ and $z, z', \tilde{z} \in Z$ be given. By (7), we have

$$(dw)_z(\tilde{z}) + t(dw)_{\tilde{z}}(z') \geq \frac{\alpha}{2} (\|z - \tilde{z}\|^2 + t\|\tilde{z} - z'\|^2). \quad (12)$$

Using the fact that

$$\min_{\gamma_1, \gamma_2} \{\gamma_1^2 + t\gamma_2^2 \mid \gamma_1, \gamma_2 \geq 0, \gamma_1 + \gamma_2 \geq \|z - z'\|\} = (1 + 1/t)^{-1} \|z - z'\|^2$$

and $(\gamma_1, \gamma_2) = (\|z - \tilde{z}\|, \|\tilde{z} - z'\|)$ is a feasible point for the above problem, we then conclude that

$$\|z - \tilde{z}\|^2 + t\|\tilde{z} - z'\|^2 \geq (1 + 1/t)^{-1} \|z - z'\|^2$$

which, together with (11) and (12), immediately yields (9). Finally, it is easy to see that (10) immediately follows from (5), (7) and (8). \square

Throughout this section, we assume that $w : W \rightarrow \mathbb{R}$ is an (α, L) -regular distance generating function with respect to $(Z, \|\cdot\|)$ where $Z \subset \text{int}(W)$ is a closed convex set and $\|\cdot\|$ is a seminorm in \mathcal{Z} . Our problem of interest in this section is the MIP

$$0 \in S(z) + T(z) \quad (13)$$

where $S, T : \mathcal{Z} \rightrightarrows \mathcal{Z}$ are monotone operators and the following conditions hold:

A0) $\text{Dom}(T) \subset Z$;

A1) S is μ -monotone on Z with respect to w , i.e., there exists a constant $\mu > 0$ such that

$$\langle z - z', s - s' \rangle \geq \mu [(dw)_z(z') + (dw)_{z'}(z)] \quad \forall z, z' \in Z, \forall s \in S(z), \forall s' \in S(z'); \quad (14)$$

A2) the solution set $(S + T)^{-1}(0)$ of (13) is nonempty.

We observe that when the seminorm $\|\cdot\|$ is a norm, then (14) implies that S is strongly monotone. However, the latter needs not be the case when $\|\cdot\|$ is not a norm.

We now state a non-Euclidean-HPE (NE-HPE) framework for solving (13) which generalizes the ones studied in [20, 31].

Framework 1 (A NE-HPE variant for solving (13)).

(0) Let $z_0 \in Z$, $\eta_0 \geq 0$, $\tau > 0$ and $\sigma \in [0, 1)$ be given, and set $k = 1$;

(1) choose $\lambda_k > 0$ and find $(\tilde{z}_k, z_k, \varepsilon_k, \eta_k) \in Z \times Z \times \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$r_k := \frac{1}{\lambda_k} \nabla(dw)_{z_k}(z_{k-1}) \in \left(S(\tilde{z}_k) + T^{[\varepsilon_k]}(\tilde{z}_k) \right), \quad (15)$$

$$(dw)_{z_k}(\tilde{z}_k) + \lambda_k \varepsilon_k + \eta_k \leq \sigma(dw)_{z_{k-1}}(\tilde{z}_k) + (1 - \tau)\eta_{k-1}; \quad (16)$$

(2) set $k \leftarrow k + 1$ and go to step 1.

end

We now make some remarks about Framework 1. First, it does not specify how to find λ_k and $(\tilde{z}_k, z_k, \varepsilon_k, \eta_k)$ satisfying (15) and (16). The particular scheme for computing λ_k and $(\tilde{z}_k, z_k, \varepsilon_k, \eta_k)$ will depend on the instance of the framework under consideration and the properties of the operators S and T . Second, if w is strongly convex on Z , $\sigma = 0$ and $\eta_0 = 0$, then (16) implies that $\varepsilon_k = 0$, $\eta_k = 0$ and $z_k = \tilde{z}_k$ for every k , and hence that $r_k \in (S + T)(z_k)$ in view of (15). Therefore, the HPE error conditions (15)-(16) can be viewed as a relaxation of an iteration of the exact non-Euclidean proximal point method, namely,

$$0 \in \frac{1}{\lambda_k} \nabla(dw)_{z_{k-1}}(z_k) + (S + T)(z_k).$$

Third, if w is strongly convex on Z , then it can be shown that the above inclusion has a unique solution z_k , and hence that, for any given $\lambda_k > 0$, there exists a quadruple $(\tilde{z}_k, z_k, \varepsilon_k, \eta_k)$ of the form $(z_k, z_k, 0, 0)$ satisfying (15)-(16) with $\sigma = 0$. Clearly, computing the quadruple in this (exact) manner is expensive, and hence computation of (inexact) quadruples satisfying the HPE (relative) error conditions with $\sigma > 0$ is more computationally appealing. Fourth, even though the definition of a regular distance generating function does not exclude the case in which w is constant, such a case is not interesting from an algorithmic analysis point of view. In fact, if $\eta_0 = 0$ and w is constant, then we have that \tilde{z}_1 is already a solution of (13) since it follows from (16) with $k = 1$ that $\varepsilon_1 = 0$, and hence that $0 \in (S + T)(\tilde{z}_1)$ in view of (15) with $k = 1$. Fifth, the more general HPE error condition (16) is clearly equivalent to

$$dw_{z_k}(\tilde{z}_k) + \lambda_k \varepsilon_k \leq \sigma dw_{z_{k-1}}(\tilde{z}_k) + \tilde{\eta}_k$$

where $\tilde{\eta}_k = (1 - \tau)\eta_{k-1} - \eta_k$. The consideration of this additional error $\{\tilde{\eta}_k\}$ will be useful in the analysis of the ADMM variant studied in Section 4. It is a simple verification to see that $\{\tilde{\eta}_k\}$ is ξ -upper summable, i.e.,

$$\limsup_{k \rightarrow \infty} \sum_{j=1}^k \frac{\tilde{\eta}_j}{(1 - \xi)^j} < \infty,$$

for any $\xi \in [0, \tau]$. Hence, if $\{\tilde{\eta}_k\}$ is nonnegative, then it is summable in the sense that $\sum_{k=1}^{\infty} \tilde{\eta}_k < \infty$.

We now make some remarks about the relationship of Framework 1 with the ones studied in [20, 22, 31]. First, Framework 1 with $S = 0$ and $\{\eta_k\}$ identically zero reduces to the one studied in [20] and also to the one in [31] if $\{\varepsilon_k\}$ is chosen to be identically zero. However, unless w is

constant, condition **A1** does not allow us to take $S = 0$, and hence the convergence rate results of this section do not apply to the setting of [20], and hence of [31]. Second, in contrast to [20], the regularity condition on w and the μ -monotonicity of S with respect to w allow us to establish a geometric (pointwise) convergence rate for the sequence $\{dw_{z_k}(z^*) + \eta_k\}$ for any $z^* \in (S + T)^{-1}(0)$ (see Proposition 2.6 below). Third, when w is the usual Euclidean distance generating function as in Example 2.3(a) and $\{\eta_k\}$ is identically zero, Framework 1 and the corresponding results derived in this section reduce to the ones studied in Subsection 2.2 of [22].

We also remark that the special case of Framework 1 in which $S(\cdot) = \mu \nabla dw_{z_0}(\cdot)$ for some $z_0 \in Z$ and $\mu > 0$ sufficiently small will be used in Section 3 as a way towards solving the inclusion $0 \in T(z)$. The resulting framework can then be viewed as a regularized NE-HPE framework in the sense that the operator T is slightly perturbed and regularized by the operator $\mu \nabla dw_{z_0}(\cdot)$.

In the remaining part of this section, we focus our attention on establishing convergence rate bounds for the sequence $\{dw_{z_k}(z^*) + \eta_k\}$ and the sequence of residual pairs $\{(r_k, \varepsilon_k)\}$ generated by any instance of Framework 1. We start by deriving a preliminary technical result.

Lemma 2.5. *For every $k \geq 1$, the following statements hold:*

(a) *for every $z \in W$, we have*

$$(dw)_{z_{k-1}}(z) - (dw)_{z_k}(z) = (dw)_{z_{k-1}}(\tilde{z}_k) - (dw)_{z_k}(\tilde{z}_k) + \lambda_k \langle r_k, \tilde{z}_k - z \rangle;$$

(b) *for every $z \in W$, we have*

$$(dw)_{z_{k-1}}(z) - (dw)_{z_k}(z) + (1 - \tau)\eta_{k-1} \geq (1 - \sigma)(dw)_{z_{k-1}}(\tilde{z}_k) + \lambda_k \langle r_k, \tilde{z}_k - z \rangle + \varepsilon_k + \eta_k;$$

(c) *for every $z^* \in (S + T)^{-1}(0)$, we have*

$$(dw)_{z_{k-1}}(z^*) - (dw)_{z_k}(z^*) + (1 - \tau)\eta_{k-1} \geq (1 - \sigma)(dw)_{z_{k-1}}(\tilde{z}_k) + \lambda_k \mu (dw)_{\tilde{z}_k}(z^*) + \eta_k.$$

Proof. (a) Using (6) twice and using the definition of r_k given by (15), we obtain

$$\begin{aligned} (dw)_{z_{k-1}}(z) - (dw)_{z_k}(z) &= (dw)_{z_{k-1}}(z_k) + \langle \nabla(dw)_{z_{k-1}}(z_k), z - z_k \rangle \\ &= (dw)_{z_{k-1}}(z_k) + \langle \nabla(dw)_{z_{k-1}}(z_k), \tilde{z}_k - z_k \rangle + \langle \nabla(dw)_{z_{k-1}}(z_k), z - \tilde{z}_k \rangle \\ &= (dw)_{z_{k-1}}(\tilde{z}_k) - (dw)_{z_k}(\tilde{z}_k) + \langle \nabla(dw)_{z_{k-1}}(z_k), z - \tilde{z}_k \rangle \\ &= (dw)_{z_{k-1}}(\tilde{z}_k) - (dw)_{z_k}(\tilde{z}_k) + \lambda_k \langle r_k, \tilde{z}_k - z \rangle. \end{aligned}$$

(b) This statement follows as an immediate consequence of item (a) and (16).

(c) Let $z^* \in (S + T)^{-1}(0)$ be. Then, there exists $a^* \in Z$ such that $a^* \in S(z^*)$ and $-a^* \in T(z^*)$. In view of (15), we can write r_k as $r_k = r_k^a + r_k^b$ where $r_k^a \in S(\tilde{z}_k)$ and $r_k^b \in T^{\varepsilon_k}(\tilde{z}_k)$. Since $a^* \in S(z^*)$ and $r_k^a \in S(\tilde{z}_k)$, condition **A1** implies that $\langle r_k^a - a^*, \tilde{z}_k - z^* \rangle \geq \mu(dw)_{\tilde{z}_k}(z^*)$. On the other hand, since $-a^* \in T(z^*)$ and $r_k^b \in T^{\varepsilon_k}(\tilde{z}_k)$, (3) implies that $\langle r_k^b + a^*, \tilde{z}_k - z^* \rangle \geq -\varepsilon_k$. Hence,

$$\langle r_k, \tilde{z}_k - z^* \rangle + \varepsilon_k = \langle r_k^a - a^*, \tilde{z}_k - z^* \rangle + \langle r_k^b + a^*, \tilde{z}_k - z^* \rangle + \varepsilon_k \geq \mu(dw)_{\tilde{z}_k}(z^*), \quad (17)$$

which together with item (b) with $z = z^*$ yields (c). \square

Under the assumption that the sequence of stepsizes $\{\lambda_k\}$ is bounded away from zero, the following result shows that the sequence $\{dw_{z_k}(z^*) + \eta_k\}$ converges geometrically to zero for every solution z^* of (13).

Proposition 2.6. *Assume that there exists $\lambda > 0$ such that $\lambda_k \geq \lambda$ for all k and define*

$$\underline{\tau} := \min \left\{ \frac{\alpha}{L} \left(\frac{1}{1-\sigma} + \frac{1}{\mu\lambda} \right)^{-1}, \tau \right\} \in (0, 1) \quad (18)$$

where μ is as in **A1**. Then, for every $k \geq 1$ and $z^* \in (S + T)^{-1}(0)$,

$$(dw)_{z_k}(z^*) + \eta_k \leq (1 - \underline{\tau})^k [(dw)_{z_0}(z^*) + \eta_0], \quad (19)$$

$$\|\nabla(dw)_{z^*}(\tilde{z}_k)\|^* \leq \frac{\sqrt{2}L}{\sqrt{\alpha}} \left[1 + \frac{1}{\sqrt{1-\sigma}} \right] (1 - \underline{\tau})^{(k-1)/2} [(dw)_{z_0}(z^*) + \eta_0]^{1/2}. \quad (20)$$

Proof. Let $z^* \in (S + T)^{-1}(0)$ be given. It follows from Lemma 2.5(c) and inequality (9) with $t = \mu\lambda_k/(1 - \sigma)$, $z = z_{k-1}$, $\tilde{z} = \tilde{z}_k$ and $z' = z^*$ that

$$\begin{aligned} (dw)_{z_k}(z^*) + \eta_k &\leq \left(1 - \frac{\alpha}{L} \left(\frac{1}{1-\sigma} + \frac{1}{\mu\lambda_k} \right)^{-1} \right) (dw)_{z_{k-1}}(z^*) + (1 - \tau)\eta_{k-1} \\ &\leq (1 - \underline{\tau}) [(dw)_{z_{k-1}}(z^*) + \eta_{k-1}] \end{aligned}$$

where the second inequality is due to the assumption that $\lambda_k \geq \lambda$ for all k and the definition of $\underline{\tau}$ in (18). Clearly, (19) follows from last inequality.

Now, Lemma 2.5(c) implies that

$$(1 - \sigma)(dw)_{z_{k-1}}(\tilde{z}_k) \leq (dw)_{z_{k-1}}(z^*) + \eta_{k-1},$$

which combined with (19) yields

$$(1 - \sigma)(dw)_{z_{k-1}}(\tilde{z}_k) \leq (1 - \underline{\tau})^{k-1} ((dw)_{z_0}(z^*) + \eta_0). \quad (21)$$

On the other hand, using (5), inequality (10) and the triangle inequality, we have

$$\begin{aligned} \|\nabla(dw)_{z^*}(\tilde{z}_k)\|^* &\leq \|\nabla(dw)_{z_{k-1}}(z^*)\|^* + \|\nabla(dw)_{z_{k-1}}(\tilde{z}_k)\|^* \\ &\leq \frac{\sqrt{2}L}{\sqrt{\alpha}} \left[((dw)_{z_{k-1}}(z^*))^{1/2} + ((dw)_{z_{k-1}}(\tilde{z}_k))^{1/2} \right] \end{aligned}$$

which together with (19), (21) and the fact that $\eta_{k-1} \geq 0$ imply (20). \square

The next result derives convergence rate bounds for the sequences $\{r_k\}$ and $\{\varepsilon_k\}$ generated by an instance of Framework 1 under the assumption that the sequence of stepsizes $\{\lambda_k\}$ is bounded away from zero.

Proposition 2.7. *Assume that $\lambda_k \geq \lambda > 0$ for all $k \geq 1$ and let $\underline{\tau}$ be as defined in (18). Then, for every $k \geq 1$, $r_k \in (S(\tilde{z}_k) + T^{[\varepsilon_k]}(\tilde{z}_k))$ and the convergence rate bounds*

$$\|r_k\|^* \leq \frac{2\sqrt{2}L}{\lambda\sqrt{\alpha}} (1 - \underline{\tau})^{(k-1)/2} \sqrt{\underline{d}_0 + \eta_0}, \quad \varepsilon_k \leq \frac{1}{\lambda(1 - \sigma)} (1 - \underline{\tau})^{k-1} [\underline{d}_0 + \eta_0] \quad (22)$$

hold where $\underline{d}_0 := \inf\{(dw)_{z_0}(z) : z \in (S + T)^{-1}(0)\}$.

Proof. The first statement of the proposition follows from (15). Let $z^* \in (S + T)^{-1}(0)$ be given. Using (5), (15), $\lambda_k \geq \lambda > 0$, the triangle inequality and inequality (10), we have

$$\begin{aligned} \|r_k\|^* &= \frac{1}{\lambda_k} \|\nabla(dw)_{z_k}(z_{k-1})\|^* \leq \frac{1}{\lambda} [\|\nabla(dw)_{z_k}(z^*)\|^* + \|\nabla(dw)_{z_{k-1}}(z^*)\|^*] \\ &\leq \frac{\sqrt{2}L}{\lambda\sqrt{\alpha}} [((dw)_{z_k}(z^*))^{1/2} + ((dw)_{z_{k-1}}(z^*))^{1/2}] \end{aligned}$$

which combined with (19) yields

$$\|r_k\|^* \leq \frac{\sqrt{2}L}{\lambda\sqrt{\alpha}} [1 + (1 - \underline{\tau})^{1/2}] (1 - \underline{\tau})^{(k-1)/2} [(dw)_{z_0}(z^*) + \eta_0]^{1/2}.$$

As $\underline{\tau} \in (0, 1]$ (see (18)) and z^* is arbitrary point in $(S + T)^{-1}(0)$, the bound on r_k follows from the definition of \underline{d}_0 .

Now, since $\lambda_k \geq \lambda > 0$, it follows from (16) that

$$\lambda\varepsilon_k \leq \lambda_k\varepsilon_k \leq \sigma(dw)_{z_{k-1}}(\tilde{z}_k) + (1 - \tau)\eta_{k-1}.$$

On the other hand, Lemma 2.5(c) implies that

$$(1 - \sigma)(dw)_{z_{k-1}}(\tilde{z}_k) \leq (dw)_{z_{k-1}}(z^*) + (1 - \tau)\eta_{k-1}.$$

Combining the last two inequities and the fact that $\sigma \in [0, 1)$, we obtain

$$\lambda\varepsilon_k \leq \frac{\sigma}{1 - \sigma}(dw)_{z_{k-1}}(z^*) + \frac{1 - \tau}{1 - \sigma}\eta_{k-1} \leq \frac{1}{1 - \sigma} [(dw)_{z_{k-1}}(z^*) + (1 - \tau)\eta_{k-1}],$$

which together with (19) and the fact that $\tau > 0$ imply that

$$\varepsilon_k \leq \frac{(1 - \underline{\tau})^{k-1}}{\lambda(1 - \sigma)} [(dw)_{z_0}(z^*) + \eta_0].$$

Since z^* is arbitrary point in $(S + T)^{-1}(0)$, the bound on ε_k follows from the definition of \underline{d}_0 . \square

3 A regularized NE-HPE framework for solving MIPs

This section describes and establishes the pointwise iteration-complexity of a regularized NE-HPE framework for solving MIPs which, specialized to the case of the Euclidean Bregman distance and error sequence $\{\eta_k\}$ identically zero, reduces to the regularized HPE framework of [22]. The latter framework has been shown in [22] to have better iteration-complexity than the one for the usual HPE framework derived in [23]. Moreover, the derived pointwise iteration-complexity bound for the case of a general Bregman distance is, up to a logarithm factor, the same as the ergodic iteration-complexity bound for the standard NE-HPE method obtained in [20].

Our problem of interest in this section is the MIP

$$0 \in T(z) \tag{23}$$

where $T : \mathcal{Z} \rightrightarrows \mathcal{Z}$ is a monotone operator such that the solution set $T^{-1}(0)$ of (23) is nonempty.

We assume in this section that, for given positive constants α and L , a closed convex set $Z \subset \text{int}(W)$ such that $\text{Dom}(T) \subset Z$, and seminorm $\|\cdot\|$ in \mathcal{Z} , $w : W \rightarrow \mathbb{R}$ is an (α, L) -regular distance generating function with respect to $(Z, \|\cdot\|)$. The regularized NE-HPE framework solves (23) based on the idea of solving the regularized MIP

$$0 \in T(z) + \mu \nabla(dw)_{z_0}(z) \quad (24)$$

for a fixed $z_0 \in Z$ and a sufficiently small $\mu > 0$. Hence, we also assume that the solution set of (24)

$$Z_\mu^* := \{z \in Z : 0 \in T(z) + \mu \nabla(dw)_{z_0}(z)\} \quad (25)$$

is nonempty for every $\mu > 0$. It can be shown that if T is maximal monotone and dw is strongly convex on Z , then the latter assumption always hold (see for example Corollary 12.44 and Proposition 12.54 of [29]).

Note that (24) is a special case of (13) with $S(\cdot) = \mu \nabla(dw)_{z_0}(\cdot)$, and from the above assumptions the operators S and T satisfy **A0** and **A2**. Moreover, this operator S together with w and Z satisfies **A1**. Indeed, using the definition of S and (5), we conclude that for every $z, z' \in Z$,

$$\begin{aligned} \langle S(z) - S(z'), z - z' \rangle &= \mu \langle \nabla(dw)_{z_0}(z) - \nabla(dw)_{z_0}(z'), z - z' \rangle \\ &= \mu \langle \nabla(dw)_{z'}(z), z - z' \rangle = \mu [(dw)_z(z') + (dw)_{z'}(z)] \end{aligned}$$

where the last equality is due to (6) with $v = z'$. Hence, we can use any instance of Framework 1 with $S(\cdot) = \mu \nabla(dw)_{z_0}(\cdot)$ to approximately solve the regularized inclusion (24), and hence (23) when $\mu > 0$ is sufficiently small.

For every $\mu > 0$, define

$$d_0 := \inf_{z \in T^{-1}(0)} (dw)_{z_0}(z), \quad d_\mu := \inf_{z \in Z_\mu^*} (dw)_{z_0}(z). \quad (26)$$

The following simple result establishes a crucial relationship between d_0 and d_μ .

Lemma 3.1. *For any $\mu > 0$ and $z_\mu^* \in Z_\mu^*$, there holds $(dw)_{z_0}(z_\mu^*) \leq d_0$. As a consequence, $d_\mu \leq d_0$.*

Proof. Let $\mu > 0$ and $z_\mu^* \in Z_\mu^*$ be given. Clearly, $-\mu \nabla(dw)_{z_0}(z_\mu^*) \in T(z_\mu^*)$. Hence, monotonicity of T implies that any $z^* \in T^{-1}(0)$ satisfies $\langle \nabla(dw)_{z_0}(z_\mu^*), z^* - z_\mu^* \rangle \geq 0$. The latter conclusion and relation (6) with $v = z_0$, $z' = z^*$ and $z = z_\mu^*$ then imply that

$$(dw)_{z_0}(z^*) - (dw)_{z_0}(z_\mu^*) = \langle \nabla(dw)_{z_0}(z_\mu^*), z^* - z_\mu^* \rangle + (dw)_{z_\mu^*}(z^*) \geq 0.$$

As $z^* \in T^{-1}(0)$ is arbitrary, the first part of the lemma follows from the definition of d_0 . The second part of the lemma follows from the first one and the definition of d_μ . \square

Note that, in view of Proposition 2.7, any instance of Framework 1 applied to (24) generates a triple $(\tilde{z}_k, r_k, \varepsilon_k)$ satisfying

$$\tilde{r}_k := r_k - \mu \nabla(dw)_{z_0}(\tilde{z}_k) \in T^{[\varepsilon_k]}(\tilde{z}_k)$$

where the residual pair (r_k, ε_k) satisfies the bounds in (22) with $\underline{d}_0 = d_\mu$, and hence converges to 0. Even though the sequence \tilde{r}_k does not necessarily converge to 0, it can be made sufficiently small, i.e., $\|\tilde{r}_k\|^* \leq \rho$ for some tolerance $\rho > 0$, by choosing $\mu = \rho/\mathcal{O}(d_0)$. Indeed, we will show later that

there exists a constant $\mathcal{D}_0 = \mathcal{O}(d_0)$ such that $\|\nabla(dw)_{z_0}(\tilde{z}_k)\|^* \leq \mathcal{D}_0$ for every k . Hence, choosing $\mu = \rho/\mathcal{D}$ for some $\mathcal{D} \geq 2\mathcal{D}_0$ and computing a residual r_k such that $\|r_k\|^* \leq \rho/2$ guarantees that

$$\|\tilde{r}_k\|^* \leq \|r_k\|^* + \mu\|\nabla(dw)_{z_0}(\tilde{z}_k)\|^* \leq \frac{\rho}{2} + \mu\mathcal{D}_0 = \rho \left(\frac{1}{2} + \frac{\mathcal{D}_0}{\mathcal{D}} \right) \leq \rho,$$

and hence that \tilde{r}_k is a sufficiently small residual for (23).

A technical difficulty of the above scheme is that the bound \mathcal{D}_0 depends on d_0 , which is generally not known. Our first framework below is essentially Framework 1 applied to (24) with an arbitrary guess of \mathcal{D} , and hence of $\mu = \rho/\mathcal{D}$. In view of the above discussion, it is guaranteed to work only for large values of \mathcal{D} , i.e., $\mathcal{D} \geq 2\mathcal{D}_0$ (see Theorem 3.2). Subsequently, we present a dynamic scheme (see Framework 3) which successively calls Framework 2 for a sequence of increasing values of \mathcal{D} . It is shown in Theorem 3.3 that the latter scheme has the same iteration-complexity as Framework 2 under the hypothetical assumption that \mathcal{D}_0 is known.

Framework 2 (A static regularized NE-HPE framework for solving (23)).

Input: $(z_0, \eta_0, \mathcal{D}) \in Z \times \mathbb{R}_+ \times \mathbb{R}_{++}$ and $(\sigma, \tau, \rho, \varepsilon) \in [0, 1) \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++}$;

(0) Set $\mu = \rho/\mathcal{D}$ and $k = 1$;

(1) choose $\lambda_k > 0$ and find $(z_k, \tilde{z}_k, \varepsilon_k, \eta_k) \in Z \times Z \times \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$r_k := \frac{1}{\lambda_k} \nabla(dw)_{z_k}(z_{k-1}) \in \left(\mu \nabla(dw)_{z_0}(\tilde{z}_k) + T^{[\varepsilon_k]}(\tilde{z}_k) \right), \quad (27)$$

$$(dw)_{z_k}(\tilde{z}_k) + \lambda_k \varepsilon_k + \eta_k \leq \sigma(dw)_{z_{k-1}}(\tilde{z}_k) + (1 - \tau)\eta_{k-1}; \quad (28)$$

(2) if $\|r_k\|^* \leq \rho/2$ and $\varepsilon_k \leq \varepsilon$, then stop and output $(\tilde{z}_k, r_k, \varepsilon_k)$; otherwise, set $k \leftarrow k + 1$ and go to step 1.

end

We now make two remarks about Framework 2. First, as mentioned above, it is the special case of Framework 1 in which $S(\cdot) = \mu \nabla(dw)_{z_0}(\cdot)$, and hence solves MIP (24). Second, since Section 2 only deals with convergence rate bounds, a stopping criterion was not added to Framework 1. In contrast, Framework 2 incorporates a stopping criterion (see step 2 above) based on which its iteration-complexity bound is obtained. Clearly, (27) together with the termination criteria $\|r_k\|^* \leq \rho/2$ and $\varepsilon_k \leq \varepsilon$ provides a certificate of the quality of \tilde{z}_k as an approximate solution of (24).

The next result establishes the pointwise iteration-complexity of Framework 2 and shows that any instance of Framework 2 also solves (23) when \mathcal{D} is sufficiently large.

Theorem 3.2. *Assume that $\lambda_k \geq \lambda > 0$ for all $k \geq 1$. Then, the following statements hold:*

(a) *Framework 2 terminates in at most*

$$\max \left\{ \frac{L}{\alpha} \left(\frac{\mathcal{D}}{\lambda\rho} + \frac{1}{1-\sigma} \right), \frac{1}{\tau} \right\} \left[2 + 2 \max \left\{ \log^+ \left(\frac{4\sqrt{2}L(d_0 + \eta_0)^{1/2}}{\lambda\rho\sqrt{\alpha}} \right), \log^+ \left(\frac{d_0 + \eta_0}{\lambda(1-\sigma)\varepsilon} \right)^{1/2} \right\} \right] \quad (29)$$

iterations with a triple $(\tilde{z}_k, r_k, \varepsilon_k)$ satisfying the following conditions

$$r_k - \mu \nabla(dw)_{z_0}(\tilde{z}_k) \in T^{[\varepsilon_k]}(\tilde{z}_k), \quad \|r_k\|^* \leq \rho/2, \quad \varepsilon_k \leq \varepsilon;$$

(b) for every $k \geq 1$,

$$\|\nabla(dw)_{z_0}(\tilde{z}_k)\|^* \leq \mathcal{D}_0 := \frac{\sqrt{2}L}{\sqrt{\alpha}} \left[2 + \frac{1}{\sqrt{1-\sigma}} \right] (d_0 + \eta_0)^{1/2};$$

(c) if $\mathcal{D} \geq 2\mathcal{D}_0$, then $\|r_k - \mu \nabla(dw)_{z_0}(\tilde{z}_k)\|^* \leq \rho$.

Proof. (a) Assume that Framework 2 has not terminated at the k -th iteration. Then, either $\|r_k\|^* > \rho/2$ or $\varepsilon_k > \varepsilon$. Assume first that $\|r_k\|^* > \rho/2$. Since Framework 2 is a special case of Framework 1 applied to MIP (24) with $S(z) = \mu \nabla(dw)_{z_0}(z)$, the latter assumption and Proposition 2.7 imply that

$$\frac{\rho}{2} < \|r_k\|^* \leq \frac{2\sqrt{2}L}{\lambda\sqrt{\alpha}} (1 - \underline{\tau})^{(k-1)/2} (d_\mu + \eta_0)^{1/2} \leq \frac{2\sqrt{2}L}{\lambda\sqrt{\alpha}} (1 - \underline{\tau})^{(k-1)/2} (d_0 + \eta_0)^{1/2},$$

where the last inequality is due to Lemma 3.1. Rearranging the last inequality, taking logarithms of both sides of the resulting inequality and using the fact that $\log(1 - \underline{\tau}) \leq -\underline{\tau}$, we conclude that

$$k < 1 + \underline{\tau}^{-1} \log \left(\frac{32L^2(d_0 + \eta_0)}{(\lambda\rho)^2\alpha} \right).$$

If, on the other hand, $\varepsilon_k > \varepsilon$, we conclude by using a similar reasoning that

$$k < 1 + \underline{\tau}^{-1} \log \left(\frac{d_0 + \eta_0}{(1 - \sigma)\lambda\varepsilon} \right).$$

The complexity bound in (a) now follows from the above observations, the definition of $\underline{\tau}$ in (18), and the fact that $\mu = \rho/\mathcal{D}$.

(b) It follows from Proposition 2.6 that

$$\|\nabla(dw)_{z_\mu^*}(\tilde{z}_k)\|^* \leq \frac{\sqrt{2}L}{\sqrt{\alpha}} \left(1 + \frac{1}{\sqrt{1-\sigma}} \right) ((dw)_{z_0}(z_\mu^*) + \eta_0)^{1/2}$$

for an arbitrary point $z_\mu^* \in Z_\mu^*$. Using (5), the triangle inequality, inequality (10), and the above inequality, we obtain

$$\begin{aligned} \|\nabla(dw)_{z_0}(\tilde{z}_k)\|^* &\leq \|\nabla dw_{z_0}(z_\mu^*)\|^* + \|\nabla(dw)_{z_\mu^*}(\tilde{z}_k)\|^* \\ &\leq \frac{\sqrt{2}L}{\sqrt{\alpha}} \left[((dw)_{z_0}(z_\mu^*))^{1/2} + \left(1 + \frac{1}{\sqrt{1-\sigma}} \right) ((dw)_{z_0}(z_\mu^*) + \eta_0)^{1/2} \right] \\ &\leq \frac{\sqrt{2}L}{\sqrt{\alpha}} \left[2 + \frac{1}{\sqrt{1-\sigma}} \right] ((dw)_{z_0}(z_\mu^*) + \eta_0)^{1/2} \end{aligned}$$

which implies the conclusion of (b), in view of Lemma 3.1 and the definition of \mathcal{D}_0 .

(c) This statement follows immediately from (a) and (b) (see the paragraph following Lemma 3.1). \square

We now make some remarks about Theorem 3.2. First, if $(1 - \sigma)^{-1}$ and τ^{-1} are $\mathcal{O}(1)$, and an input \mathcal{D} for Framework 2 satisfying $2\mathcal{D}_0 \leq \mathcal{D} = \mathcal{O}(L(d_0 + \eta_0)^{1/2}/\sqrt{\alpha})$ is known, then the complexity bound (29) becomes

$$\mathcal{O} \left(\frac{L}{\alpha} \left(\frac{L(d_0 + \eta_0)^{1/2}}{\lambda \rho \sqrt{\alpha}} + 1 \right) \left[1 + \max \left\{ \log^+ \left(\frac{L(d_0 + \eta_0)^{1/2}}{\lambda \rho \sqrt{\alpha}} \right), \log^+ \left(\frac{d_0 + \eta_0}{\lambda \varepsilon} \right)^{1/2} \right\} \right] \right). \quad (30)$$

Second, in general an upper bound \mathcal{D} as in the first remark is not known and, in such a case, bound (29) can be much worse than the one above, e.g., when $\mathcal{D} \gg 2\mathcal{D}_0$.

We now consider the case where an upper bound $\mathcal{D} \geq 2\mathcal{D}_0$ such that $\mathcal{D} = \mathcal{O}(L(d_0 + \eta_0)^{1/2}/\sqrt{\alpha})$ is not known and describe a scheme based on Framework 2 whose iteration-complexity order is equal to (30).

Framework 3 (A dynamic regularized NE-HPE framework for solving (23)).

- (0) Let $z_0 \in \mathcal{Z}$, $\eta_0 \geq 0$, $\tau > 0$, $\sigma \in [0, 1)$ and a tolerance pair $(\rho, \varepsilon) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ be given and set $\mathcal{D} = \rho$;
- (1) call Framework 2 with input $(z_0, \eta_0, \mathcal{D})$ and $(\sigma, \tau, \rho, \varepsilon)$ to obtain $(\tilde{z}, r, \tilde{\varepsilon})$ as output;
- (2) compute $\tilde{r} = r - (\rho/\mathcal{D})\nabla(dw)_{z_0}(\tilde{z})$; if $\|\tilde{r}\|^* \leq \rho$ then stop and output $(\tilde{z}, \tilde{r}, \tilde{\varepsilon})$; else, set $\mathcal{D} \leftarrow 2\mathcal{D}$ and go to step 1.

end

Each iteration of Framework 3 (referred to as an outer iteration) invokes Framework 2, which performs a certain number of iterations (called inner iterations) which in turn is bounded by (29). The following result gives the overall inner iteration-complexity of Framework 3.

Theorem 3.3. *Assume that $\lambda_k \geq \lambda > 0$ for all $k \geq 1$. Then, Framework 3 with input $(z_0, \eta_0) \in \mathcal{Z} \times \mathbb{R}_+$ and $(\tau, \sigma, \rho, \varepsilon) \in (0, 1) \times [0, 1) \times \mathbb{R}_{++} \times \mathbb{R}_{++}$ such that $(1 - \sigma)^{-1}$ and $1/\tau$ are $\mathcal{O}(1)$ finds a triple $(\tilde{z}, \tilde{r}, \tilde{\varepsilon})$ satisfying*

$$\tilde{r} \in T^{[\tilde{\varepsilon}]}(\tilde{z}), \quad \|\tilde{r}\|^* \leq \rho, \quad \tilde{\varepsilon} \leq \varepsilon$$

in at most

$$\mathcal{O} \left(\frac{L}{\alpha} \left(\frac{1}{\lambda} + 1 \right) \left(1 + \frac{L(d_0 + \eta_0)^{1/2}}{\rho \sqrt{\alpha}} \right) \left[1 + \max \left\{ \log^+ \left(\frac{L(d_0 + \eta_0)^{1/2}}{\lambda \rho \sqrt{\alpha}} \right), \log^+ \left(\frac{d_0 + \eta_0}{\lambda \varepsilon} \right)^{1/2} \right\} \right] \right) \quad (31)$$

inner iterations.

Proof. Note that at the k -th outer iteration of Framework 3, we have $\mathcal{D} = 2^{k-1}\rho$. Moreover, in view of Theorem 3.2(c), Framework 3 terminates in at most K outer iterations where K is the smallest integer $k \geq 1$ satisfying $2^{k-1}\rho \geq 2\mathcal{D}_0$. Thus,

$$K = 1 + \lceil \log^+ (2\mathcal{D}_0/\rho) \rceil.$$

In order to simplify the calculations, let us denote

$$\beta_1 := 2 + 2 \max \left\{ \log^+ \left(\frac{4\sqrt{2}L(d_0 + \eta_0)^{1/2}}{\lambda\rho\sqrt{\alpha}} \right), \log^+ \left(\frac{d_0 + \eta_0}{\lambda(1 - \sigma)\varepsilon} \right)^{1/2} \right\}. \quad (32)$$

In view of Theorem 3.2(a) and (32), we see that the overall number of inner iterations is bounded by

$$\tilde{K} := \beta_1 \sum_{k=1}^K \max \left\{ \frac{L}{\alpha} \left(\frac{2^{k-1}}{\lambda} + \frac{1}{1 - \sigma} \right), \frac{1}{\tau} \right\} \leq \beta_1 \left[\frac{L}{\alpha} \left(\frac{2^K - 1}{\lambda} + \frac{K}{1 - \sigma} \right) + \frac{K}{\tau} \right],$$

and hence

$$\tilde{K} \leq \frac{L\beta_1}{\alpha} \left[\frac{1}{\lambda} + \frac{1}{1 - \sigma} + \frac{1}{\tau} \right] 2^K. \quad (33)$$

To end the proof, it suffices to show that \tilde{K} is bounded by (31). If $K = 1$, then (32) combined with (33) and the fact that $(1 - \sigma)^{-1}$ and $1/\tau$ are $\mathcal{O}(1)$ shows that (31) trivially holds. Assume now that $K > 1$ and note that $k := K - 1$ violates the inequality $2^{k-1}\rho \geq 2\mathcal{D}_0$, and hence that $2^K < 8\mathcal{D}_0/\rho$. The latter estimate combined with inequality (33) implies that

$$\tilde{K} < \frac{4L\beta_1}{\alpha} \left[\frac{1}{\lambda} + \frac{1}{1 - \sigma} + \frac{1}{\tau} \right] \frac{\mathcal{D}_0}{\rho}$$

which together with (32), and the fact that $(1 - \sigma)^{-1}$ and $1/\tau$ are $\mathcal{O}(1)$ and $\mathcal{D}_0 = \mathcal{O}(L(d_0 + \eta_0)^{1/2}/\sqrt{\alpha})$, imply that \tilde{K} is bounded by (31). \square

Note that if $\varepsilon_k = 0$ for every k , then the complexity bound (31) becomes independent of the tolerance ε , namely, it reduces to

$$\mathcal{O} \left(\frac{L}{\alpha} \left(\frac{1}{\lambda} + 1 \right) \left(1 + \frac{L(d_0 + \eta_0)^{1/2}}{\rho\sqrt{\alpha}} \right) \left[1 + \log^+ \left(\frac{L(d_0 + \eta_0)^{1/2}}{\lambda\rho\sqrt{\alpha}} \right) \right] \right). \quad (34)$$

4 A regularized ADMM

The goal of this section is to present a regularized ADMM for solving linearly constrained convex programming problems which has a better pointwise iteration-complexity than the usual ADMM. It contains two subsections. The first one describes our setting, our assumptions, the regularized ADMM and its improved pointwise iteration-complexity bound. The second one is dedicated to the proof of the main result stated in the first subsection.

4.1 A regularized ADMM and its pointwise iteration-complexity

In this subsection, we let \mathcal{X} , \mathcal{Y} and \mathcal{S} be inner product spaces whose inner products are denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$, $\langle \cdot, \cdot \rangle_{\mathcal{Y}}$, $\langle \cdot, \cdot \rangle_{\mathcal{S}}$, respectively. Let us also consider the norm in \mathcal{X} given by $\|\cdot\|_{\mathcal{X}} := \langle \cdot, \cdot \rangle_{\mathcal{X}}^{1/2}$. Our problem of interest is

$$\inf \{ f(y) + g(s) : Cy + Ds = c \} \quad (35)$$

where $c \in \mathcal{X}$, $C : \mathcal{Y} \rightarrow \mathcal{X}$ and $D : \mathcal{S} \rightarrow \mathcal{X}$ are linear operators, and $f : \mathcal{Y} \rightarrow (-\infty, \infty]$ and $g : \mathcal{S} \rightarrow (-\infty, \infty]$ are proper closed convex functions.

The following assumptions are made throughout this section:

B1) problem (35) has an optimal solution (s^*, y^*) and an associated Lagrange multiplier x^* , or equivalently, there exists a triple (s^*, y^*, x^*) satisfying

$$0 \in \partial g(s^*) - D^* x^*, \quad 0 \in \partial f(y^*) - C^* x^*, \quad C y^* + D s^* = c; \quad (36)$$

B2) there exists $x \in \mathcal{X}$ such that $(C^* x, D^* x) \in \text{ri}(\text{dom } f^*) \times \text{ri}(\text{dom } g^*)$.

We now make a few remarks about the above assumptions. First, it follows from the last conclusion of Proposition A.2 in Appendix A that, if the solution set of (35) is nonempty and bounded, then **B2** holds. Second, by Proposition A.2(a), if the infimum in (35) is $< \infty$ and **B2** holds, then (35) has an optimal solution. Hence, **B2** together with the Slater condition that there exists a feasible pair (s, y) for (35) such that $(s, y) \in \text{ri}(\text{dom } g) \times \text{ri}(\text{dom } f)$ imply that **B1** holds (see Proposition A.2(c)). Third, **B2** implies that $\text{Im}(D^*) \cap \text{ri}(\text{dom } g^*) \neq \emptyset$, and hence that $\partial(g^* \circ D^*)(x) = D(\partial g^*(D^* x))$ for every $x \in \mathcal{X}$ (see Theorem 3.2.1 in Chap. XI of [19]). The latter observation and the fact that $(\partial g)^{-1} = \partial g^*$ then easily imply that a pair (y^*, x^*) is a solution of the inclusion problem

$$0 \in T(x, y) := \begin{bmatrix} C y - c + \partial(g^* \circ D^*)(x) \\ -C^* x + \partial f(y) \end{bmatrix} \quad (37)$$

if and only if there exists $s^* \in \mathcal{S}$ such that (s^*, y^*, x^*) satisfies (36).

We are ready to state the regularized ADMM for solving (35).

Dynamic regularized alternating direction method of multipliers (DR-ADMM):

(0) Let $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$, positive scalars β, θ and a tolerance $\rho > 0$ be given, and set $\mathcal{D} = \rho$.

(1) set $\mu = \rho/\mathcal{D}$, $\beta_1 = \beta\theta/(\theta + \mu)$, $\beta_2 = \beta(1 + \mu)$ and $k = 1$, and go to (a);

(a) set $\hat{x}_{k-1} = (\theta x_{k-1} + \mu x_0)/(\theta + \mu)$ and compute an optimal solution $s_k \in \mathcal{S}$ of the subproblem

$$\min_{s \in \mathcal{S}} \left\{ g(s) - \langle D^* \hat{x}_{k-1}, s \rangle_{\mathcal{S}} + \frac{\beta_1}{2} \|C y_{k-1} + D s - c\|_{\mathcal{X}}^2 \right\}; \quad (38)$$

(b) set $\tilde{x}_k, \hat{y}_{k-1}$ and u_k as

$$\tilde{x}_k = \hat{x}_{k-1} - \beta_1 (C y_{k-1} + D s_k - c) \quad (39)$$

$$\hat{y}_{k-1} = (y_{k-1} + \mu y_0)/(1 + \mu) \quad (40)$$

$$u_k = \tilde{x}_k + \beta_2 (C \hat{y}_{k-1} + D s_k - c) \quad (41)$$

and compute an optimal solution $y_k \in \mathcal{Y}$ of the subproblem

$$\min_{y \in \mathcal{Y}} \left\{ f(y) - \langle C^* u_k, y \rangle_{\mathcal{Y}} + \frac{\beta_2}{2} \|C y + D s_k - c\|_{\mathcal{X}}^2 \right\}; \quad (42)$$

(c) update x_k as

$$x_k = x_{k-1} - \theta \beta \left[C y_k + D s_k - c + \frac{\mu}{\theta \beta} (\tilde{x}_k - x_0) \right]; \quad (43)$$

(d) if

$$\left(\beta\theta \|p_k\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|q_k\|_{\mathcal{X}}^2 \right)^{1/2} \leq \rho/2 \quad (44)$$

where

$$p_k := \frac{1}{\beta\theta}(x_{k-1} - x_k), \quad q_k := \beta C(y_{k-1} - y_k), \quad (45)$$

then set $(s, y, \tilde{x}, p, q) = (s_k, y_k, \tilde{x}_k, p_k, q_k)$ and go to (2); else set $k \leftarrow k + 1$ and go to (a);

(2) compute

$$\tilde{p} := p - \frac{\mu}{\beta\theta}(\tilde{x} - x_0), \quad \tilde{q} := q - \mu\beta C(y - y_0); \quad (46)$$

if

$$\left(\beta\theta \|\tilde{p}\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|\tilde{q}\|_{\mathcal{X}}^2 \right)^{1/2} \leq \rho,$$

then stop and output $(s, y, \tilde{x}, \tilde{p}, \tilde{q})$; otherwise, set $\mathcal{D} \leftarrow 2\mathcal{D}$ and go to step 1.

end

We now make some remarks about the DR-ADMM. First, it can be shown that assumption **B.2** implies that both subproblems (38) and (42) have optimal solutions (see for example [25, Proposition 7.2]). Second, loop (a)-(d) with $\mu = 0$ is exactly the ADMM method with penalty parameter β and relaxation stepsize θ (see for example [9]) since in this situation we have $\beta_1 = \beta_2 = \beta$, $\hat{x}_{k-1} = u_k = x_{k-1}$ and $\hat{y}_{k-1} = y_{k-1}$. DR-ADMM should then be viewed as a regularized ADMM method which dynamically adjusts the regularization parameter $\mu > 0$, or equivalently, the parameter $\mathcal{D} > 0$. Third, as will be shown later on in this section, DR-ADMM is a special instance of Framework 3 in which $\varepsilon_k = 0$ for all $k \geq 1$. Indeed, steps 0, 1 and 2 of DR-ADMM correspond exactly to steps 0, 1 and 2 of Framework 3, respectively. A single execution of steps 1 and 2 is referred to as an outer iteration of DR-ADMM. A single execution of steps (a)-(d) is referred to as an inner iteration of DR-ADMM which, in the context of Framework 3, corresponds to an iteration of Framework 2 (see step 1 of Framework 3). The cycle of inner iterations consisting of (a)-(d) corresponds to the implementation of a special instance of Framework 2 in which $\varepsilon_k = 0$ for all $k \geq 1$. Moreover, the two residuals r and \tilde{r} computed at the end of steps 1 and 2 of Framework 3, respectively, correspond in the context of the DR-ADMM to the pairs (p, C^*q) and $(\tilde{p}, C^*\tilde{q})$, respectively (see Lemma 4.4).

The following result, which is the main one of this section, shows that the regularized ADMM has a better pointwise iteration-complexity than the usual ADMM. Its complexity bound depends on the quantity

$$d_0 := \inf \left\{ \frac{1}{2\beta\theta} \|x_0 - x^*\|_{\mathcal{X}}^2 + \frac{\beta}{2} \|C(y_0 - y^*)\|_{\mathcal{X}}^2 : (x^*, y^*) \text{ is solution of (37)} \right\}. \quad (47)$$

Theorem 4.1. *DR-ADMM with stepsize $\theta \in (0, (1 + \sqrt{5})/2)$ terminates in at most*

$$\mathcal{O} \left(\left(1 + \frac{\sqrt{d_0}}{\rho} \right) \left[1 + \log^+ \left(\frac{\sqrt{d_0}}{\rho} \right) \right] \right) \quad (48)$$

iterations with $(s, y, \tilde{x}, \tilde{p}, \tilde{q})$ satisfying

$$\tilde{p} = Cy + Ds - c, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{bmatrix} \partial g(s) - D^* \tilde{x} \\ \partial f(y) - C^*(\tilde{x} + \tilde{q}) \end{bmatrix} \quad (49)$$

and

$$\left(\beta \theta \|\tilde{p}\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|\tilde{q}\|_{\mathcal{X}}^2 \right)^{1/2} \leq \rho. \quad (50)$$

Before ending this subsection, we compare the iteration-complexity bound of Theorem 4.1 with the ones obtained in [16, 25]. A pointwise convergence rate for the standard ADMM when $\theta = 1$ is established in [16]. More specifically, it is implicitly shown that the latter method generates a sequence $\{(s_k, y_k, \tilde{x}_k, \tilde{p}_k, \tilde{q}_k)\}$ (which is the same as the one generated by a cycle of inner iterations of the DR-ADMM with $\mu = 0$) satisfying (49) and

$$\beta \|\tilde{p}_k\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|\tilde{q}_k\|_{\mathcal{X}}^2 \leq \frac{2d_0}{k+1} \quad \forall k \geq 1,$$

which, as a consequence, implies an $\mathcal{O}(d_0/\rho^2)$ pointwise iteration-complexity bound for finding a $(s, y, \tilde{x}, \tilde{p}, \tilde{q})$ satisfying (49) and (50) with $\theta = 1$. Hence, our bound (48) is better than the latter one by an $\mathcal{O}(\rho \log(\rho^{-1}))$ factor.

We now discuss the relationship of bound (48) with the ergodic iteration-complexity bound that immediately follows from [25, Theorem 7.5]). Given $\rho > 0$ and $\varepsilon > 0$, this result implies that the ADMM with $\theta = 1$ obtains, by averaging the first j elements of the sequence $\{(s_k, y_k, \tilde{x}_k, \tilde{p}_k, \tilde{q}_k)\}$ for some $j = \mathcal{O}(\max\{d_0/\rho, d_0^2/\varepsilon\})$, a quintuple $\{(s^a, y^a, \tilde{x}^a, \tilde{p}^a, \tilde{q}^a)\}$ satisfying

$$\tilde{p}^a = Cy^a + Ds^a - c, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{bmatrix} \partial g_\varepsilon(s^a) - D^* \tilde{x}^a \\ \partial f_\varepsilon(y^a) - C^*(\tilde{x}^a + \tilde{q}^a) \end{bmatrix}$$

and

$$\left(\beta \|\tilde{p}^a\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|\tilde{q}^a\|_{\mathcal{X}}^2 \right)^{1/2} \leq \rho.$$

Hence, the dependence of the above ergodic bound on ρ is $\mathcal{O}(d_0/\rho)$, which is better than the pointwise bound (48) by only a logarithmic factor. On the other hand, with respect to ε , bound (48) is better than the above ergodic bound since it does not depend on ε .

Finally, we end this subsection by mentioning that other iteration-complexity bounds have been established for the ADMM for any $\theta \in (0, (\sqrt{5}+1)/2)$ based on different but related stopping criteria (see for example [12, 14]) and to which the above comments apply in a similar manner.

4.2 Proof of Theorem 4.1

In this subsection, we establish Theorem 4.1 by first showing that DR-ADMM is an instance of Framework 3, and then translating Theorem 3.3 to the context of the DR-ADMM to obtain the complexity bound (48).

The first result below establishes, as a consequence of some useful relations, that (49) is essentially an invariance of the inner iterations of the DR-ADMM.

Lemma 4.2. *The k -th iterate $(s_k, x_k, y_k, \tilde{x}_k)$ obtained during a cycle of inner iterations satisfies*

$$0 \in \partial g(s_k) - D^* \tilde{x}_k, \quad (51)$$

$$0 \in \frac{1}{\theta\beta}(x_k - x_{k-1}) + \left[\partial(g^* \circ D^*)(\tilde{x}_k) + Cy_k - c + \frac{\mu}{\theta\beta}(\tilde{x}_k - x_0) \right], \quad (52)$$

$$0 \in \beta C^* C(y_k - y_{k-1}) + [\partial f(y_k) - C^* \tilde{x}_k + \mu\beta C^* C(y_k - y_0)], \quad (53)$$

$$\tilde{x}_k - x_{k-1} = \frac{x_k - x_{k-1}}{\theta} + \beta C(y_k - y_{k-1}) \quad (54)$$

where μ is the constant value of the smoothing parameter during this cycle. As a consequence, $(s, y, \tilde{x}, \tilde{p}, \tilde{q})$ obtained at the end of each cycle of inner iterations (see (46)) satisfies (49).

Proof. From the optimality condition for s_k in (38) and definition of \tilde{x}_k in (39), we have

$$D^* \tilde{x}_k = D^* [\hat{x}_{k-1} - \beta_1(Cy_{k-1} + Ds_k - c)] \in \partial g(s_k) \quad (55)$$

and then (51) holds. It also follows from (55) that $s_k \in \partial g^*(D^* \tilde{x}_k)$ which in turn yields $Ds_k \in D\partial g^*(D^* \tilde{x}_k) \subset \partial(g^* \circ D^*)(\tilde{x}_k)$. Therefore, the inclusion (52) follows from (43). Now, from the optimality condition for y_k in (42) and definition of u_k in (41), we obtain

$$\begin{aligned} 0 &\in \partial f(y_k) - C^* u_k + \beta_2 C^*(Cy_k + Ds_k - c) \\ &= \partial f(y_k) - C^* [\tilde{x}_k + \beta_2(C\hat{y}_{k-1} + Ds_k - c)] + \beta_2 C^*(Cy_k + Ds_k - c) \end{aligned}$$

which is equivalent to

$$0 \in \partial f(y_k) - C^* \tilde{x}_k + \beta_2 C^* C(y_k - \hat{y}_{k-1}). \quad (56)$$

On the other hand, the definitions of β_2 and \hat{y}_{k-1} give

$$\begin{aligned} \beta_2(y_k - \hat{y}_{k-1}) &= \beta(1 + \mu)(y_k - \hat{y}_{k-1}) = \beta(1 + \mu)y_k - \beta(y_{k-1} + \mu y_0) \\ &= \beta(y_k - y_{k-1}) + \mu\beta(y_k - y_0). \end{aligned}$$

Hence, (56) yields

$$0 \in \partial f(y_k) - C^* \tilde{x}_k + C^* C[\beta(y_k - y_{k-1}) + \mu\beta(y_k - y_0)]$$

which proves (53).

Let us now prove the relation (54). Using the definitions of \hat{x}_{k-1} , \tilde{x}_k and β_1 , we obtain

$$\begin{aligned} \tilde{x}_k - x_{k-1} + \frac{\mu}{\theta}(\tilde{x}_k - x_0) &= \frac{1}{\theta} [(\theta + \mu)\tilde{x}_k - (\theta x_{k-1} + \mu x_0)] = \frac{\theta + \mu}{\theta}(\tilde{x}_k - \hat{x}_{k-1}) \\ &= -\frac{(\theta + \mu)\beta_1}{\theta}(Cy_{k-1} + Ds_k - c) = -\beta(Cy_{k-1} + Ds_k - c). \end{aligned}$$

The previous identities yield

$$\tilde{x}_k - x_{k-1} = -\beta \left(Cy_{k-1} + Ds_k - c + \frac{\mu}{\theta\beta}(\tilde{x}_k - x_0) \right)$$

which combined with (43) proves (54). Now, the equality in (49) follows from (43), (45) and (46). The inclusion in (49) follows from (51), (53) and the definition of \tilde{q} in (46). \square

Since our approach towards proving Theorem 4.1 is via showing that DR-ADMM is a special instance of Framework 3, we need to introduce the elements required by the setting of Section 3, namely, the set Z , the seminorm $\|\cdot\|$ and the distance generating function $w : Z \rightarrow \mathbb{R}$. Towards this goal, endow the space $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}} + \langle \cdot, \cdot \rangle_{\mathcal{Y}}$, and define the set Z as $Z := \mathcal{Z}$, and the seminorm $\|\cdot\|$ and the function $w : Z \rightarrow \mathbb{R}$ as

$$\|(x, y)\| := \left(\frac{1}{\beta\theta} \|x\|_{\mathcal{X}}^2 + \beta \|Cy\|_{\mathcal{X}}^2 \right)^{1/2}, \quad w(x, y) := \frac{1}{2} \|(x, y)\|^2, \quad \forall z = (x, y) \in \mathcal{Z}. \quad (57)$$

Clearly, the Bregman distance associated with w is given by

$$(dw)_z(z') = \frac{1}{2\beta\theta} \|x' - x\|_{\mathcal{X}}^2 + \frac{\beta}{2} \|C(y' - y)\|_{\mathcal{X}}^2 \quad \forall z = (x, y) \in \mathcal{Z}, z' = (x', y') \in \mathcal{Z}. \quad (58)$$

The following result shows that Z , w and $\|\cdot\|$ defined above, as well as the operator T defined in (37), fulfill the assumptions of Section 3.

Lemma 4.3. *Let the set Z , function w and seminorm $\|\cdot\|$ be as defined above. Then, the following statements hold:*

- (a) *the function w is a $(1, 1)$ -regular distance generating function with respect to $(Z, \|\cdot\|)$;*
- (b) *the set Z_{μ}^* as in (25) where $z_0 = (x_0, y_0)$ and T is as in (37) is nonempty for every $\mu > 0$.*

Proof. (a) This statement follows directly from Example 2.3(b) with $A(x, y) = (x/(\beta\theta), \beta C^*Cy)$ for every $(x, y) \in \mathcal{Z}$.

(b) The proof of this statement is given in Appendix B. □

The next result gives a sufficient condition for the sequence generated by a cycle of inner iterations of the DR-ADMM to be an implementation of Framework 2.

Lemma 4.4. *Let $\sigma \in [0, 1)$ and $\tau \in (0, 1]$ be given and consider the operator T and Bregman distance dw as in (37) and (58), respectively. Let $\{(x_k, \tilde{x}_k, y_k, p_k, q_k)\}$ be the sequence generated during a cycle of inner iterations of DR-ADMM with parameter $\mathcal{D} > 0$ and define*

$$z_{k-1} = (x_{k-1}, y_{k-1}), \quad \tilde{z}_k = (\tilde{x}_k, y_k), \quad \lambda_k = 1, \quad \epsilon_k = 0 \quad \forall k \geq 1. \quad (59)$$

Then, the following statements hold:

- (a) *the sequence $\{(z_k, \tilde{z}_k, \lambda_k, \epsilon_k)\}$ satisfies inclusion (27) and the left hand side r_k of this inclusion in terms of p_k and q_k is given by $r_k = (p_k, C^*q_k)$;*
- (b) *if there exists a sequence $\{\eta_k\}$ such that $\{(p_k, q_k, \eta_k)\}$ satisfies*

$$[\sigma - (\theta - 1)^2] \frac{\beta \|p_k\|_{\mathcal{X}}^2}{2\theta} + [\sigma(1 + \theta) - 1] \frac{\|q_k\|_{\mathcal{X}}^2}{2\beta\theta} + \frac{(\sigma + \theta - 1)}{\theta} \langle p_k, q_k \rangle_{\mathcal{X}} \geq \eta_k - (1 - \tau)\eta_{k-1},$$

then the sequence $\{(z_k, \tilde{z}_k, \lambda_k, \epsilon_k, \eta_k)\}$ satisfies the error condition (28);

- (c) *condition (44) is equivalent to $\|r_k\|^* \leq \rho/2$ where r_k is as in (27) and $\|\cdot\|$ is the seminorm defined in (57).*

As a consequence, if the assumption of (b) is satisfied then the sequence $\{(z_k, \tilde{z}_k, \lambda_k, \varepsilon_k, \eta_k)\}$ is an implementation of Framework 2 with input $z_0 = (x_0, y_0)$ and (η_0, \mathcal{D}) .

Moreover, if every cycle of inner iterations of DR-ADMM satisfies the assumption of (b), then DR-ADMM is an instance of Framework 3.

Proof. (a) These statements follows from definitions of T and dw in (37) and (58), respectively, and relations (45), (52) and (53).

(b) Using (45) and (54), we obtain

$$x_{k-1} - \tilde{x}_k = q_k + \beta p_k, \quad x_k - \tilde{x}_k = q_k + (1 - \theta)\beta p_k. \quad (60)$$

Hence, it follows from (58) and (59) that

$$\begin{aligned} (dw)_{z_k}(\tilde{z}_k) + \lambda_k \varepsilon_k &= \frac{1}{2\beta\theta} \|\tilde{x}_k - x_k\|_{\mathcal{X}}^2 \\ &= \frac{1}{2\beta\theta} [\|q_k\|_{\mathcal{X}}^2 + \beta^2(\theta - 1)^2 \|p_k\|_{\mathcal{X}}^2 + 2(1 - \theta)\beta \langle q_k, p_k \rangle_{\mathcal{X}}]. \end{aligned}$$

On the other hand, using (58), (59), (60) and definition of q_k in (45), we obtain

$$\begin{aligned} (dw)_{z_{k-1}}(\tilde{z}_k) &= \frac{1}{2\beta\theta} \|x_{k-1} - \tilde{x}_k\|_{\mathcal{X}}^2 + \frac{\beta}{2} \|C(y_{k-1} - y_k)\|_{\mathcal{X}}^2 \\ &= \frac{1}{2\beta\theta} \|q_k + \beta p_k\|_{\mathcal{X}}^2 + \frac{1}{2\beta} \|q_k\|_{\mathcal{X}}^2 \\ &= \frac{(\theta+1)}{2\beta\theta} \|q_k\|_{\mathcal{X}}^2 + \frac{\beta}{2\theta} \|p_k\|_{\mathcal{X}}^2 + \frac{1}{\theta} \langle q_k, p_k \rangle_{\mathcal{X}}. \end{aligned}$$

Statement (b) now follows immediately from the above two identities.

(c) First of all, note that $\|\cdot\| = \langle A(\cdot), \cdot \rangle$ where $A(x, y) = (x/(\beta\theta), \beta C^* C y)$ for every $(x, y) \in \mathcal{Z}$ and define $\delta_k^x := x_{k-1} - x_k$ and $\delta_k^y := y_{k-1} - y_k$. Hence, it follows from the identity in (a), (45), the definition of A , Proposition A.1(a) and (57) that

$$\|r_k\|^* = \|(p_k, C^* q_k)\|^* = \|A(\delta_k^x, \delta_k^y)\|^* = \|(\delta_k^x, \delta_k^y)\| = \left(\beta\theta \|p_k\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|q_k\|_{\mathcal{X}}^2 \right)^{1/2}$$

from which statement (c) follows.

To show the last statement of the lemma, first note that (37), (45), (46) and (58) imply that $\tilde{r} = (\tilde{p}, C^* \tilde{q})$. Now, a similar argument as above using (45), (46), the definition of A , Proposition A.1(a) and (57) imply that $\|\tilde{r}\|^* = (\beta\theta \|\tilde{p}\|_{\mathcal{X}}^2 + \frac{1}{\beta} \|\tilde{q}\|_{\mathcal{X}}^2)^{1/2}$, from which the last statement of the lemma follows. \square

In view of Lemma 4.4, it suffices to show that DR-ADMM satisfies the assumption of Lemma 4.4(b) in order to show that it is an instance of Framework 3. We will prove the latter fact by considering two cases, namely, whether the stepsize θ is in $(0, 1)$ or in $[1, (\sqrt{5} + 1)/2]$. The next result consider the case in which $\theta \in (0, 1)$ and Lemma 4.7 below considers the other case.

Lemma 4.5. *Assume that the DR-ADMM stepsize $\theta \in (0, 1)$. Let $\{(x_k, \tilde{x}_k, y_k, p_k, q_k)\}$ be the sequence generated during a cycle of inner iterations of DR-ADMM with parameter $\mathcal{D} > 0$ and define $\eta_k = 0$ for every $k \geq 1$. Then, the sequence $\{(p_k, q_k, \eta_k)\}$ satisfies the assumption of Lemma 4.4(b) with $\sigma = \theta + (\theta - 1)^2 \in (0, 1)$ and any $\tau \in (0, 1]$.*

Proof. Using the definition of σ , we have

$$\begin{aligned} & [\sigma - (\theta - 1)^2] \frac{\beta \|p_k\|_{\mathcal{X}}^2}{2\theta} + [\sigma(1 + \theta) - 1] \frac{\|q_k\|_{\mathcal{X}}^2}{2\beta\theta} + \frac{(\sigma + \theta - 1)}{\theta} \langle p_k, q_k \rangle_{\mathcal{X}} \\ &= \frac{\beta \|p_k\|_{\mathcal{X}}^2}{2} + \frac{\theta^2}{2\beta} \|q_k\|_{\mathcal{X}}^2 + \theta \langle p_k, q_k \rangle_{\mathcal{X}} = \frac{1}{2\beta} \|\beta p_k + \theta q_k\|_{\mathcal{X}}^2 \geq 0. \end{aligned}$$

Hence the lemma follows due to the definition of $\{\eta_k\}$. \square

Before handling the other case in which $\theta \in [1, (\sqrt{5} + 1)/2)$, we first establish the following technical result.

Lemma 4.6. *Consider the sequence $\{(x_k, \tilde{x}_k, y_k, p_k, q_k)\}$ generated during a cycle of inner iterations of DR-ADMM with parameter $\mathcal{D} > 0$. Then, the following statements hold:*

(a) *if $\theta \in [1, 2)$, then*

$$\|p_1\|_{\mathcal{X}} \|q_1\|_{\mathcal{X}} \leq \frac{4\theta \max\{\beta, \beta^{-1}\}}{2 - \theta} d_0$$

where d_0 is as in (47);

(b) *for any $k \geq 2$, we have*

$$\langle p_k, q_k \rangle_{\mathcal{X}} \geq (1 - \theta) \langle p_{k-1}, q_k \rangle_{\mathcal{X}}. \quad (61)$$

Proof. The proof of this lemma is given in Appendix C. \square

In contrast to the case in which $\theta \in (0, 1)$, the following result shows that the case in which $\theta \in [1, (\sqrt{5} + 1)/2)$ requires a non-trivial choice of sequence $\{\eta_k\}$, and hence uses the full generality of the approach of Section 3.

Lemma 4.7. *Assume that the DR-ADMM stepsize $\theta \in [1, (\sqrt{5} + 1)/2)$ and consider the sequence $\{(p_k, q_k)\}$ generated during a cycle of inner iterations of DR-ADMM with parameter $\mathcal{D} > 0$. Then, there exist $\sigma, \tau \in (0, 1)$ such that the sequence $\{(p_k, q_k, \eta_k)\}$ where $\{\eta_k\}$ is defined as*

$$\eta_0 = \frac{4(\sigma + \theta - 1) \max\{\beta, \beta^{-1}\}}{(2 - \theta)(1 - \tau)} d_0, \quad \eta_k = \frac{[\sigma - (\theta - 1)^2] \beta}{2\theta} \|p_k\|_{\mathcal{X}}^2 \quad \forall k \geq 1$$

satisfies the assumption of Lemma 4.4(b).

Proof. Using the Cauchy-Schwarz inequality, definitions of η_0 and η_1 and Lemma 4.6(a), it is easy to see that the inequality in Lemma 4.4(b) with $k = 1$ holds for any $\theta \in [1, (\sqrt{5} + 1)/2)$ and $\sigma, \tau \in (0, 1)$. Note that for every $\theta \in [1, (\sqrt{5} + 1)/2)$, there exists $\sigma, \tau \in (0, 1)$ such that the matrix

$$M(\theta, \sigma, \tau) = \begin{bmatrix} (1 - \tau)[\sigma - (\theta - 1)^2] & (\sigma + \theta - 1)(1 - \theta) \\ (\sigma + \theta - 1)(1 - \theta) & \sigma(1 + \theta) - 1 \end{bmatrix}$$

is positive definite. Indeed, $M(\theta, 1, 0)$ can be easily seen to be positive definite and hence for σ close to 1 and τ close to 0 the matrix $M(\theta, \sigma, \tau)$ is still positive definite. Now, Lemma 4.6(b) and definition

of $\{\eta_k\}$ imply that a sufficient condition for the inequality in Lemma 4.4(b) with $k \geq 2$ to hold is that

$$(1 - \tau)[\sigma - (\theta - 1)^2] \frac{\beta \|p_{k-1}\|_{\mathcal{X}}^2}{2\theta} + (\sigma(1 + \theta) - 1) \frac{\|q_k\|_{\mathcal{X}}^2}{2\beta\theta} + \frac{(\sigma + \theta - 1)(1 - \theta)}{\theta} \langle p_{k-1}, q_k \rangle_{\mathcal{X}} \geq 0,$$

which trivially holds in view of the Cauchy-Schwarz inequality and the fact that the matrix $M(\theta, \sigma, \tau)$ is positive definite. \square

Now we are ready to proof Theorem 4.1.

Proof of Theorem 4.1: First note that (49) follows from the last statement in Lemma 4.2. Now, it follows by combining Lemmas 4.4, 4.5 and 4.7 that DR-ADMM with $\theta \in (0, (1 + \sqrt{5})/2)$ is an instance of Framework 3 applied to problem (37) in which $\{(z_k, \tilde{z}_k, \lambda_k, \varepsilon_k)\}$ is as in (59) and $\{\eta_k\}$ is as defined in Lemma 4.5 if $\theta \in (0, 1)$ or as in Lemma 4.7 if $\theta \in [1, (1 + \sqrt{5})/2)$. This conclusion together with Lemma 4.3(a) then imply that Theorem 3.3, as well as the observation following it, holds with $\lambda = \alpha = L = 1$. The remaining conclusion of the theorem now follows from the latter observation, definition of η_0 and Lemma 4.4(c).

Appendix

A Some basic technical results

The following result gives some properties of the dual seminorm whose simple proof is omitted.

Proposition A.1. *Let $A : \mathcal{Z} \rightarrow \mathcal{Z}$ be a self-adjoint positive semidefinite linear operator and consider the seminorm $\|\cdot\|$ in \mathcal{Z} given by $\|z\| = \langle Az, z \rangle^{1/2}$ for every $z \in \mathcal{Z}$. Then, the following statements hold:*

- (a) $\text{Dom}\|\cdot\|^* = \text{Im}(A)$ and $\|Az\|^* = \|z\|$ for every $z \in \mathcal{Z}$;
- (b) if A is invertible then $\|z\|^* = \langle A^{-1}z, z \rangle^{1/2}$ for every $z \in \mathcal{Z}$.

Next proposition discuss the existence of solutions of a problem related to (37).

Proposition A.2. *Let a linear operator $E : \mathcal{Z} \rightarrow \tilde{\mathcal{Z}}$, a vector $e \in \text{Im} E$ and a proper closed convex function $h : \mathcal{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ be such that*

$$\inf_{z \in \mathcal{Z}} \{h(z) : Ez = e\} < \infty. \tag{62}$$

Then, the following statements hold:

- (a) if $\text{ri}(\text{dom} h^*) \cap \text{Im}(E^*) \neq \emptyset$, then (62) has an optimal solution z^* ;
- (b) the optimal solution set of (62) is nonempty and bounded if and only if $0 \in \text{int}(\text{dom} h^* + \text{Im}(E^*))$;
- (c) if the assumption of (a) holds and (62) has a Slater point, i.e., a point $\bar{z} \in \text{ri}(\text{dom} h)$ such that $E\bar{z} = e$, then there exists a Lagrange pair $(z, x) = (z^*, x^*)$ satisfying

$$0 \in \partial h(z) - E^*x, \quad Ez = e.$$

As a consequence, if the set of optimal solutions of (62) is nonempty and bounded, then $\text{ri}(\text{dom } h^*) \cap \text{Im}(E^*) \neq \emptyset$.

Proof. Since h is a proper closed convex function, we have $(h^*)^* = h$. Hence, the proof of (a) and (b) follows from Lemma 2.2.2 in Chap. X of [19] with $A_0 = E^*$, $g = h^*$ and $s = e$, and the discussion following Theorem 2.2.3. The proof of (c) follows easily from [27, Corollary 28.2.2]. \square

B Proof of Lemma 4.3(b)

Let a scalar $\mu > 0$ and note that $(x, y) \in Z_\mu^*$ if and only if (x, y) satisfies

$$0 \in \left[\partial(g^* \circ D^*)(x) + Cy - c + \frac{\mu}{\theta\beta}(x - x_0) \right], \quad 0 \in [\partial f(y) - C^*x + \mu\beta C^*C(y - y_0)]. \quad (63)$$

Hence, the proof of Lemma 4.3(b) will follow if we show that (63) has a solution. Towards this goal, let us consider the problem

$$\inf_{(s, y, u, v)} \left\{ g(s) + f(y) + \frac{\beta\theta}{2\mu} \|u + [\mu/(\beta\theta)]x_0\|_{\mathcal{X}}^2 + \frac{\beta\mu}{2} \|v\|_{\mathcal{X}}^2 : Ds + Cy + u = c, Cy - v = Cy_0 \right\}. \quad (64)$$

It is easy to see that $(s, y, c - Ds - Cy, C(y - y_0))$ is a Slater point of the above problem for any $(s, y) \in \text{ri}(\text{dom } f) \times \text{ri}(\text{dom } g) \neq \emptyset$. Hence, since condition **B2** easily implies that the assumption of Proposition A.2(a) holds, it follows from Proposition A.2(c) that there exist $(\bar{s}, \bar{y}, \bar{u}, \bar{v}) \in \mathcal{S} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{X}$ and $(\bar{x}_1, \bar{x}_2) \in \mathcal{X} \times \mathcal{X}$ such that

$$0 \in [\partial g(\bar{s}) - D^*\bar{x}_1], \quad 0 \in [\partial f(\bar{y}) - C^*\bar{x}_1 - C^*\bar{x}_2], \quad 0 = \frac{\beta\theta}{\mu}(\bar{u} + [\mu/(\beta\theta)]x_0) - \bar{x}_1, \quad 0 = \beta\mu\bar{v} + \bar{x}_2, \quad (65)$$

$$D\bar{s} + C\bar{y} + \bar{u} = c, \quad C\bar{y} - \bar{v} = Cy_0. \quad (66)$$

Hence, from the first inclusion in (65), we have $\bar{s} \in \partial g^*(D^*\bar{x}_1)$ which yields $D\bar{s} \in \partial(g^* \circ D^*)(\bar{x}_1)$. The latter inclusion together with the first equality in (65) and the first one in (66) imply that the pair (\bar{x}_1, \bar{y}) satisfies the first inclusion in (63). Similarly, combining the second inclusion and the last equality in (65) with the last equality in (66), we conclude that the pair (\bar{x}_1, \bar{y}) satisfies the second inclusion in (63). We have then shown that (\bar{x}_1, \bar{y}) solves (63). \square

C Proof of Lemma 4.6

(a) Let a point $z_\mu^* := (x_\mu^*, y_\mu^*) \in Z_\mu^*$ (see Lemma 4.3(b)) and consider $(z_0, z_1, \tilde{z}_1, \lambda_1, \epsilon_1)$ as in (59). It follows from the definitions of Δ_1^x and Δ_1^y , $2ab \leq a^2 + b^2$ for all $a, b \geq 0$ and $\theta \geq 1$ that

$$\begin{aligned} \|p_1\|_{\mathcal{X}} \|q_1\|_{\mathcal{X}} &\leq \frac{1}{2\theta} (\|x_1 - x_0\|_{\mathcal{X}}^2 + \|C(y_1 - y_0)\|_{\mathcal{X}}^2) \\ &\leq \frac{1}{\theta} (\|x_1 - x_\mu^*\|_{\mathcal{X}}^2 + \|C(y_1 - y_\mu^*)\|_{\mathcal{X}}^2 + \|x_0 - x_\mu^*\|_{\mathcal{X}}^2 + \|C(y_0 - y_\mu^*)\|_{\mathcal{X}}^2) \\ &\leq 2 \max\{\beta, \beta^{-1}\} ((dw)_{z_1}(z_\mu^*) + (dw)_{z_0}(z_\mu^*)) \end{aligned} \quad (67)$$

where the last inequality is due to definitions of z_0, z_1 and dw (see (58)).

On the other hand, Lemma 4.4(a) implies that inclusion (27) is satisfied for $(z_0, z_1, \tilde{z}_1, \lambda_1, \epsilon_1)$ and hence it follows from Lemma 2.5(a) with $z = z_\mu^*$, $\lambda_1 = 1$ and the fact that $\langle r_1, \tilde{z}_1 - z_\mu^* \rangle \geq 0$ (see (17) with $k = 1$, $z^* = z_\mu^*$ and $\epsilon_1 = 0$) that

$$(dw)_{z_1}(z_\mu^*) \leq (dw)_{z_0}(z_\mu^*) + (dw)_{z_1}(\tilde{z}_1) - (dw)_{z_0}(\tilde{z}_1). \quad (68)$$

Using the definitions in (58), (59) and (60), we obtain

$$\begin{aligned} (dw)_{z_1}(\tilde{z}_1) - (dw)_{z_0}(\tilde{z}_1) &= \frac{1}{2\beta\theta} \|q_1 + (1-\theta)\beta p_1\|_{\mathcal{X}}^2 - \frac{1}{2\beta\theta} \|q_1 + \beta p_1\|_{\mathcal{X}}^2 - \frac{1}{2\beta} \|q_1\|_{\mathcal{X}}^2 \\ &= \frac{(\theta-1)\beta}{2} \|p_1\|_{\mathcal{X}}^2 - \frac{1}{2} \left(\sqrt{\beta} p_1 - \frac{q_1}{\sqrt{\beta}} \right)^2 \leq \frac{(\theta-1)\beta}{2} \|p_1\|_{\mathcal{X}}^2 \\ &\leq \frac{(\theta-1)}{\theta} \left(\frac{\|x_1 - x_\mu^*\|_{\mathcal{X}}^2}{\beta\theta} + \frac{\|x_0 - x_\mu^*\|_{\mathcal{X}}^2}{\beta\theta} \right) \\ &\leq \frac{2(\theta-1)}{\theta} [(dw)_{z_1}(z_\mu^*) + (dw)_{z_0}(z_\mu^*)] \end{aligned}$$

where the the second inequality is due to the definition of p_1 and the fact that $2ab \leq a^2 + b^2$ for all $a, b \geq 0$, and the last inequality is due to (58) and definitions of z_0, z_1 and z_μ^* . Hence, combining the last estimative with (68), we obtain

$$(dw)_{z_1}(z_\mu^*) \leq \frac{\theta}{2-\theta} \left(1 + \frac{2(\theta-1)}{\theta} \right) (dw)_{z_0}(z_\mu^*) = \frac{3\theta-2}{2-\theta} (dw)_{z_0}(z_\mu^*).$$

Therefore, statement (a) follows from (67), the last inequality and Lemma 3.1.

(b) From the inclusion (53) we see that

$$C^*(\tilde{x}_j - \beta C(y_j - y_{j-1})) \in \partial f_{\mu,\beta}(y_j) \quad \forall j \geq 1$$

where $f_{\mu,\beta}(y) := f(y) + (\beta\mu/2) \|C(y - y_0)\|_{\mathcal{X}}^2$ for every $y \in \mathcal{Y}$. Hence, using relation (54), we have

$$\frac{1}{\theta} C^*(x_j - (1-\theta)x_{j-1}) \in \partial f_{\mu,\beta}(y_j) \quad \forall j \geq 1. \quad (69)$$

For every $k \geq 2$, using the previous inclusion for $j = k-1$ and $j = k$, it follows from the monotonicity of the subdifferential of $f_{\mu,\beta}$ and the definitions of p_k and q_k that

$$\begin{aligned} 0 &\leq \frac{1}{\theta} \langle C^*(x_k - (1-\theta)x_{k-1}) - C^*(x_{k-1} - (1-\theta)x_{k-2}), y_k - y_{k-1} \rangle_{\mathcal{Y}} \\ &= \langle \beta C^*(p_k - (1-\theta)p_{k-1}), y_{k-1} - y_k \rangle_{\mathcal{Y}} = \langle p_k - (1-\theta)p_{k-1}, q_k \rangle_{\mathcal{Y}}, \end{aligned}$$

from which (b) follows, and then the proof is concluded. □

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