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On spectral measures of random Jacobi matrices

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Abstract

The paper studies the limiting behaviour of spectral measures of random Jacobi matrices of Gaussian, Wishart and MANOVA beta ensembles. We show that the spectral measures converge weakly to a limit distribution which is the semicircle distribution, Marchenko-Pastur distributions or the arcsine distribution, respectively. Regard that convergence as the law of large number, a central limit theorem is then derived.

Keywords: spectral measure; random Jacobi matrix; Gaussian beta ensemble; Wishart beta ensemble; MANOVA beta ensemble.
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1 Introduction

Three classical random matrix ensembles on the real line, Gaussian beta ensembles, Wishart beta ensembles and MANOVA beta ensembles, are now realized as eigenvalues of certain random Jacobi matrices. For instance, the following random Jacobi matrices whose components are independent and distributed as

$$H_{n,\beta} = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1} & a_n \end{pmatrix} \sim \frac{1}{\sqrt{n\beta}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}$$

are matrix models of (scaled) Gaussian beta ensembles for any $\beta > 0$. Here $\mathcal{N}(\mu, \sigma^2)$ denotes the normal (or Gaussian) distribution with mean μ and variance σ^2, and χ_k denotes the chi distribution with k degrees of freedom. Namely, the eigenvalues of $H_{n,\beta}$ are distributed as Gaussian beta ensembles,

$$(\lambda_1, \dots, \lambda_n) \propto |\Delta(\lambda)|^\beta \exp\left(-\frac{n\beta}{4} \sum_{i=1}^n \lambda_i^2 \right),$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ denotes the Vandermonde determinant.

The limiting behaviour of the empirical distributions of the three beta ensembles has been well studied. For Gaussian beta ensembles, as $n \to \infty$, their empirical distributions

$$L_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$$

converge weakly, almost surely, to the semicircle distribution, which is well known as Wigner's semicircle law. The limit distributions are Marchenko-Pastur distributions and the arcsine distribution in Wishart and MANOVA cases, respectively. The convergence means that for any bounded continuous function f on $\mathbb{R}$,

$$\langle L_n, f \rangle = \frac{1}{n} \sum_{i=1}^n f(\lambda_i) \to \langle \mu_\infty, f \rangle \text{ almost surely as } n \to \infty.$$

Here μ_∞ stands for the corresponding limit distribution. Regard it as the law of large number, the central limit theorem has also been investigated. It turns out that for a 'nice' test function f, the fluctuation around the limit converges in distribution to a normal random variable,

$$n(\langle L_n, f\rangle - \langle \mu_\infty, f\rangle) = \sum_{i=1}^{n}(f(\lambda_i) - \langle \mu_\infty, f\rangle) \overset{d}{\to} \mathcal{N}(0, a_f^2).$$

Here '$\overset{d}{\to}$' denotes the convergence in distribution or the weak convergence of random variables. See [4] for Gaussian and Wishart cases and [7] for a generalization of Gaussian beta ensembles with general potential.

The spectral measures of random Jacobi matrices associated with those beta ensembles have been investigated recently. The weak convergence to a limit distribution, the central limit theorem for moments and large deviations have been established, see [2, 6, 9]. The spectral measure of a finite Jacobi matrix, a symmetric tridiagonal matrix of the form,

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1} & a_n \end{pmatrix}, (a_i \in \mathbb{R}, b_i > 0),$$

is defined to be a unique probability measure μ on $\mathbb{R}$ satisfying

$$\langle \mu, x^k\rangle = \langle J^k e_1, e_1\rangle = J^k(1,1), k = 0, 1, \dots,$$

where $e_1 = (1, 0, \dots, 0)^t \in \mathbb{R}^n$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues of J and $(v_1, \dots, v_n)$ be the corresponding eigenvectors which are chosen to be an orthonormal basis of $\mathbb{R}^n$. Then the spectral measure μ can be written as

$$\mu = \sum_{i=1}^{n} q_j^2 \delta_{\lambda_i}.$$

It is known that the eigenvalues $\{\lambda_i\}$ are distinct, the weights $\{q_j^2\}$ are all positive and that a finite Jacobi matrix of size n is one-to-one correspondence with a probability measure supported on n real points.

It is the purpose of this paper to reconsider the limiting behaviour of the spectral measures μ_n related to those beta ensembles to see how nature it is. In all three cases, it turns out very interesting that the weights $\{w_i\} = \{q_i^2\}$ are independent of eigenvalues and have Diriclet distribution with parameters $(\beta/2, \dots, \beta/2)$, that is,

$$(w_1, \dots, w_n) \propto \prod_{i=1}^{n} w_i^{\frac{\beta}{2}-1} \mathbf{1}_{\{w_1+\cdots+w_{n-1}<1, w_i>0\}} dw_1 \cdots dw_{n-1},$$

where $w_n = 1 - (w_1 + \cdots + w_{n-1})$. Thus, by a direct calculation, we obtain

$$\mathbb{E}[\langle \mu_n, f\rangle] = \sum_{i=1}^{n} \mathbb{E}[q_i^2]\mathbb{E}[f(\lambda_i)] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[f(\lambda_i)] = \mathbb{E}[\langle L_n, f\rangle] = \langle \bar{L}_n, f\rangle.$$

Here $\bar{L}_n$ is the mean of L_n, which is defined to be a probability measure satisfying $\langle \bar{L}_n, f\rangle = \mathbb{E}[\langle L_n, f\rangle]$ for all bounded continuous function f. Similarly, we can derive a formula for the variance of $\langle \mu_n, f\rangle$ as

$$\mathrm{Var}[\langle \mu_n, f\rangle] = \frac{\beta n}{\beta n + 2}\mathrm{Var}[\langle L_n, f\rangle] + \frac{2}{n\beta + 2}(\langle \bar{L}_n, f^2\rangle - \langle \bar{L}_n, f\rangle^2).$$

Consequently, as $n \to \infty$,

$$\frac{n\beta}{2} \operatorname{Var}[\langle \mu_n, f \rangle] \to \langle \mu_\infty, f^2 \rangle - \langle \mu_\infty, f \rangle^2, \tag{1}$$

provided that $n \operatorname{Var}[\langle L_n, f \rangle] \to 0$, which is obvious true for a 'nice' function f. It follows that the spectral measures converge weakly to the same limit distribution μ_∞. It also suggests that the central limit theorem should hold with the scaling factor $(\sqrt{n\beta}/\sqrt{2})$ and the limit variance is given by (1). Now we can state the main result of this paper.

Theorem 1.1. (i) *The spectral measures μ_n converge weakly, in probability, to the same limit distribution μ_∞ as $n \to \infty$, that is, for any bounded continuous function f,*

$$\langle \mu_n, f \rangle = \sum_{i=1}^n q_i^2 f(\lambda_i) \to \langle \mu_\infty, f \rangle \text{ in probability as } n \to \infty;$$

(ii) *For a bounded continuous function for which $n \operatorname{Var}[\langle L_n, f \rangle] \to 0$,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}} (\langle \mu_n, f \rangle - \mathbb{E}[\langle \mu_n, f \rangle]) \xrightarrow{d} \mathcal{N}(0, \sigma^2(f)) \text{ as } n \to \infty,$$

where $\sigma^2(f) = \langle \mu_\infty, f^2 \rangle - \langle \mu_\infty, f \rangle^2$.

The paper is organized as follows. In the next section, we consider general random Jacobi matrices and derive the weak convergence of spectral measures as well as the central limit theorem for polynomial test functions. Applications to Gaussian, Wishart and MANOVA beta ensembles are then investigated in turn. The last section is devoted to extend the central limit theorem to a larger class of test functions.

2 Limiting behaviour of spectral measures of random Jacobi matrices

Let us begin by introducing some spectral properties of non random Jacobi matrices. A semi-infinite Jacobi matrix is a symmetric tridiagonal matrix of the form

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \text{ where } a_i \in \mathbb{R}, b_i > 0.$$

To a Jacobi matrix J, there exists a probability measure μ such that

$$\langle \mu, x^k \rangle = \int_{\mathbb{R}} x^k d\mu = \langle J^k e_1, e_1 \rangle, k = 0, 1, \dots,$$

where $e_1 = (1, 0, \dots,)^t \in \ell^2$. Then μ is unique, or μ is determined by its moments, if and only if, J is an essentially self-adjoint operator on ℓ^2. If the parameters $\{a_i\}$ and $\{b_i\}$ are bounded, or more generally, if $\sum b_i^{-1} = \infty$, then J is essentially self-adjoint, [12, Corollary 3.8.9]. In case of uniqueness, we call μ the spectral measure of J, or of (J, e_1).

As mentioned in the introduction, when J is a finite Jacobi matrix of order n, then μ is a probability measure supported on n eigenvalues $\{\lambda_i\}$ with weight $\{q_i^2\} = \{v_i(1)^2\}$,

$$\mu = \sum_{i=1}^n q_i^2 \delta_{\lambda_i}.$$

Here $(v_1, \dots, v_n)$ are the corresponding eigenvectors which are chosen to be an orthogonal basis of $\mathbb{R}^n$.

The following results are useful in this paper. We will omit their proofs.

Lemma 2.1. *Assume that* $\{\mu_n\}_{n=1}^{\infty}$ *and* μ *are probability measures on* $\mathbb{R}$ *such that for all* $k = 0, 1, \dots,$

$$\langle \mu_n, x^k \rangle \to \langle \mu, x^k \rangle \text{ as } n \to \infty.$$

Assume further that the measure μ *is determined by its moments. Then* μ_n *converges weakly to* μ *as* $n \to \infty$*. Moreover, if a continuous function* f *is bounded by some polynomial* P*, that is,* $|f(x)| \le P(x)$ *for all* $x \in \mathbb{R}$*, then we also have*

$$\langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ as } n \to \infty.$$

Lemma 2.2. *Let* $\{\mu_n\}_{n=1}^{\infty}$ *be a sequence of random probability measures and* μ *be a non-random probability measure which is determined by its moments. Assume that any moment of* μ_n *converges almost surely to that of* μ*, that is, for any* $k = 0, 1, \dots,$

$$\langle \mu_n, x^k \rangle \to \langle \mu, x^k \rangle \text{ a.s. as } n \to \infty.$$

Then as $n \to \infty$*, the sequence of measures* $\{\mu_n\}$ *converges weakly, almost surely, to* μ*, namely, for any bounded continuous function* f*,*

$$\langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ a.s. as } n \to \infty.$$

An analogous result holds for convergence in probability.

Let us now explain the main idea of this paper. Consider the sequence of random Jacobi matrices

$$J_n = \begin{pmatrix} a_1^{(n)} & b_1^{(n)} & & \\ b_1^{(n)} & a_2^{(n)} & b_2^{(n)} & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1}^{(n)} & a_n^{(n)} \end{pmatrix},$$

and let μ_n be the spectral measure of (J_n, e_1). Assume that each entry of J_n converges almost surely to a non random limit as $n \to \infty$, that is, for any fixed i, as $n \to \infty$,

$$a_i^{(n)} \to \bar{a}_i; \quad b_i^{(n)} \to \bar{b}_i \text{ a.s.} \tag{2}$$

Here we require that $\bar{a}_i$ and $\bar{b}_i$ are non random and $\bar{b}_i > 0$. Assume further that the spectral measure of (J_∞, e_1), denoted by μ_∞, is unique, where J_∞ is the infinite Jacobi matrix consisting of $\{\bar{a}_i\}$ and $\{\bar{b}_i\}$,

$$J_\infty = \begin{pmatrix} \bar{a}_1 & \bar{b}_1 & & \\ \bar{b}_1 & \bar{a}_2 & \bar{b}_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then the measure μ_∞ is determined by its moments, and hence we get the following result.

Theorem 2.3. *The spectral measures* μ_n *converge weakly, almost surely, to the limit measure* μ_∞ *as* $n \to \infty$*.*

Remark 2.4. If in the assumption (2), convergence in probability is assumed instead of almost sure convergence, then the spectral measures μ_n converge weakly, in probability, to μ_∞ as $n \to \infty$.

Proof. Let p be a polynomial of degree m. Then when n is large enough, $\langle \mu_n, p\rangle = p(J_n)(1,1)$ is a polynomial of $\{a_i^{(n)}, b_i^{(n)}\}_{i=1,\dots,\lceil \frac{m}{2}\rceil}$. Therefore, as $n\to\infty$,

$$\langle \mu_n, p\rangle \to \langle \mu_\infty, p\rangle \text{ a.s.},$$

which implies the weak convergence of μ_n by Lemma 2.2. □

Next, we consider the central limit theorem for polynomial test functions. It turns out that the central limit theorem for polynomial test functions is a direct consequence of a joint central limit theorem for entries of Jacobi matrices. Indeed, assume that there are random variables $\{\eta_i\}$ and $\{\zeta_i\}$ defined on the same probability space such that for some fixed $r>0$, for any i, as $n\to\infty$,

$$\begin{aligned}\tilde{a}_i^{(n)} &= n^r(a_i^{(n)} - \bar{a}_i) \xrightarrow{d} \eta_i,\\ \tilde{b}_i^{(n)} &= n^r(b_i^{(n)} - \bar{b}_i) \xrightarrow{d} \zeta_i.\end{aligned} \tag{3}$$

Moreover, we assume that the joint weak convergence holds. This means that any finite linear combination of $\tilde{a}_i^{(n)}$ and $\tilde{b}_i^{(n)}$ converges weakly to the corresponding linear combination of η_i and ζ_i as $n\to\infty$, namely, for any real numbers c_i and d_i,

$$\sum_{finite} (c_i\tilde{a}_i^{(n)} + d_i\tilde{b}_i^{(n)}) \xrightarrow{d} \sum_{finite} (c_i\eta_i + d_i\zeta_i) \text{ as } n\to\infty.$$

For now on, both conditions (2) and (3) will be written in a compact form

$$J_n \approx \begin{pmatrix} \bar{a}_1 & \bar{b}_1 & & \\ \bar{b}_1 & \bar{a}_2 & \bar{b}_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{1}{n^r}\begin{pmatrix} \eta_1 & \zeta_1 & & \\ \zeta_1 & \eta_2 & \zeta_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let f be a polynomial of $2k$ variables $(a_1,\dots,a_k,b_1,\dots,b_k)$. For simplicity, we write $f(a_i,b_i)$ instead of $f(a_1,\dots,a_k,b_1,\dots,b_k)$.

Lemma 2.5. (i) *As* $n\to\infty$,

$$n^r\left(f(a_i^{(n)}, b_i^{(n)}) - f(\bar{a}_i,\bar{b}_i)\right) - \sum_{i=1}^k \left(\frac{\partial f}{\partial a_i}(\bar{a}_i,\bar{b}_i)\tilde{a}_i^{(n)} + \frac{\partial f}{\partial b_i}(\bar{a}_i,\bar{b}_i)\tilde{b}_i^{(n)}\right) \xrightarrow{P} 0.$$

Here '$\xrightarrow{P}$' denotes the convergence in probability.

(ii) *As* $n\to\infty$,

$$n^r\left(f(a_i^{(n)}, b_i^{(n)}) - f(\bar{a}_i,\bar{b}_i)\right) \xrightarrow{d} \sum_{i=1}^k \left(\frac{\partial f}{\partial a_i}(\bar{a}_i,\bar{b}_i)\eta_i + \frac{\partial f}{\partial b_i}(\bar{a}_i,\bar{b}_i)\zeta_i\right).$$

Proof. Write

$$a_i^{(n)} = \bar{a}_i + \frac{1}{n^r}\tilde{a}_i^{(n)};\, b_i^{(n)} = \bar{b}_i + \frac{1}{n^r}\tilde{b}_i^{(n)}.$$

Then use the Taylor expansion of $f(a_i^{(n)}, b_i^{(n)})$ at $(\bar{a}_i,\bar{b}_i)$ with noting that the Taylor expansion of a polynomial consists of finitely many terms,

$$f(a_i^{(n)}, b_i^{(n)}) = f(\bar{a}_i,\bar{b}_i) + \frac{1}{n^r}\sum_{i=1}^k \left(\frac{\partial f}{\partial a_i}(\bar{a}_i,\bar{b}_i)\tilde{a}_i^{(n)} + \frac{\partial f}{\partial b_i}(\bar{a}_i,\bar{b}_i)\tilde{b}_i^{(n)}\right) + \sum{}^*.$$

Each term in the finite sum $\sum^*$ has the following form,

$$c(\alpha,\beta)\prod_{i=1}^{k}(a_i^{(n)}-\bar{a}_i)^{\alpha_i}(b_i^{(n)}-\bar{b}_i)^{\beta_i},$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are non negative integers and $\sum_{i=1}^{k}(\alpha_i+\beta_i)\ge 2$. Therefore, when we multiple that term by n^r, it converges to zero in distribution, and hence, in probability by Slutsky's theorem.

By using Slutsky's theorem again, we see that (ii) is a consequence of (i). The proof is completed, □

Let p be a polynomial of degree $m>0$. Then there is a polynomial of $2\lceil \frac{m}{2}\rceil$ variables such that for $n>m/2$,

$$\langle \mu_n, p\rangle = p(J_n)(1,1) = f(a_i^{(n)}, b_i^{(n)}).$$

Therefore, by Lemma 2.5, we obtain the central limit theorem for polynomial test functions.

Theorem 2.6. *For any polynomial p, $n^r\left(\langle\mu_n,p\rangle - \langle\mu_\infty,p\rangle\right)$ converges weakly to a limit as $n\to\infty$.*

Since we do not assume that all moments of $\{a_i^{(n)}\}$ and $\{b_i^{(n)}\}$ are finite, even the expectation of $\langle\mu_n,p\rangle$, $\mathbb{E}[\langle\mu_n,p\rangle]$ may not exist. Thus we need further assumptions to ensure the convergence of mean and variance in the central limit theorem above. Our assumptions are based on the following basic result in probability theory.

Lemma 2.7. *Assume that the sequence $\{X_n\}$ converges weakly to a random variable X. If for some $\delta>0$,*

$$\sup_n \mathbb{E}[|X_n|^{2+\delta}]<\infty,$$

then $\mathbb{E}[X_n]\to\mathbb{E}[X]$ and $\mathrm{Var}[X_n]\to\mathrm{Var}[X]$ as $n\to\infty$.

We make the following assumptions

(i) all moments of $\{a_i^{(n)}\}$ and $\{b_i^{(n)}\}$ are finite and the convergences in (2) hold in L_q for all $q<\infty$, which is equivalent to the following conditions

$$\sup_n \mathbb{E}[|a_i^{(n)}|^k]<\infty, \quad \sup_n \mathbb{E}[|b_i^{(n)}|^k]<\infty, \text{ for all } k=1,2,\dots; \tag{4}$$

(ii) $\mathbb{E}[\eta_i]=0, \mathbb{E}[\zeta_i]=0$, and for some $\delta>0$,

$$\sup_n \mathbb{E}[|\tilde{a}_i^{(n)}|^{2+\delta}]<\infty, \quad \sup_n \mathbb{E}[|\tilde{b}_i^{(n)}|^{2+\delta}]<\infty. \tag{5}$$

Lemma 2.8. *As $n\to\infty$,*

$$\mathbb{E}\left[\left(f(a_i^{(n)},b_i^{(n)})-f(\bar{a}_i,\bar{b}_i)\right)^2\right]\to \mathrm{Var}\left[\sum_{i=1}^{k}\left(\frac{\partial f}{\partial a_i}(\bar{a}_i,\bar{b}_i)\eta_i+\frac{\partial f}{\partial b_i}(\bar{a}_i,\bar{b}_i)\zeta_i\right)\right], \tag{6}$$

$$n^r\left(\mathbb{E}[f(a_i^{(n)},b_i^{(n)})]-f(\bar{a}_i,\bar{b}_i)\right)\to 0, \tag{7}$$

$$\mathrm{Var}\left[f(a_i^{(n)},b_i^{(n)})\right]\to \mathrm{Var}\left[\sum_{i=1}^{k}\left(\frac{\partial f}{\partial a_i}(\bar{a}_i,\bar{b}_i)\eta_i+\frac{\partial f}{\partial b_i}(\bar{a}_i,\bar{b}_i)\zeta_i\right)\right]. \tag{8}$$

Proof. It is just a direct consequence of Lemma 2.7. ☐

Now we state a slightly different form of the central limit theorem for polynomial test functions.

Theorem 2.9. *Under assumptions* (4) *and* (5), *for any polynomial* p, *as* $n \to \infty$,

$$n^r(\langle \mu_n, p\rangle - \mathbb{E}[\langle \mu_n, p\rangle]) \xrightarrow{d} \xi_\infty(p).$$

Here $\xi_\infty(p)$ *denotes the limit distribution. Moreover,* $\mathbb{E}[\xi_\infty(p)] = 0$ *and*

$$\mathrm{Var}[\langle \mu_n, p\rangle] \to \mathrm{Var}[\xi_\infty(p)] \text{ as } n \to \infty.$$

3 Gaussian beta ensembles or β-Hermite ensembles

Let $H_{n,\beta}$ be a random Jacobi matrix whose elements are independent (up to the symmetric constraint) and are distributed as

$$H_{n,\beta} = \frac{1}{\sqrt{n\beta}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}.$$

Then the eigenvalues $\{\lambda_i\}$ of $H_{n,\beta}$ have Gaussian beta ensembles [3], that is,

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \propto |\Delta(\lambda)|^\beta \exp\left(-\frac{n\beta}{4}\sum_{i=1}^n \lambda_j^2\right).$$

The weights $\{w_i\} = \{q_i^2\}$ are independent of $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \dots, \beta/2)$.

Lemma 3.1. (i) *As* $k \to \infty$,

$$\frac{\chi_k}{\sqrt{k}} \to 1 \text{ in probability and in } L_q \text{ for all } q < \infty.$$

(ii) *As* $k \to \infty$,

$$\sqrt{k}\left(\frac{\chi_k}{\sqrt{k}} - 1\right) = (\chi_k - \sqrt{k}) \xrightarrow{d} \mathcal{N}(0, \frac{1}{2}).$$

Since the elements of $H_n(\beta)$ are independence, it follows that joint convergence in distribution holds, namely, we can write

$$H_{n,\beta} \approx \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{1}{\sqrt{\beta n}} \begin{pmatrix} \mathcal{N}(0,2) & \mathcal{N}(0,\frac{1}{2}) & & \\ \mathcal{N}(0,\frac{1}{2}) & \mathcal{N}(0,2) & \mathcal{N}(0,\frac{1}{2}) & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Note that the spectral measure of the non random Jacobi matrix part in the above expression is the semicircle distribution, a probability measure on $[-2,2]$ with density

$$sc(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, (-2 \le x \le 2).$$

Consequently, we get the following result.

Theorem 3.2. *The spectral measure* μ_n *of* $H_{n,\beta}$ *converges weakly, in probability, to the semicircle law as* $n \to \infty$. *Moreover, for any polynomial* p, *as* $n \to \infty$,

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p\rangle - \langle sc, p\rangle) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2).$$

4 Wishart beta ensembles or β-Laguerre ensembles

For $m \in \mathbb{N}$ and $n > m-1$, let B_β be a bidiagonal matrix whose elements are independent and are distributed as

$$B_\beta = \frac{1}{\sqrt{n\beta}} \begin{pmatrix} \chi_{\beta n} & & & \\ \chi_{\beta(m-1)} & \chi_{\beta n-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{\beta n-\beta(m-1)} \end{pmatrix}.$$

Let $L_{m,n,\beta} = B_\beta B_\beta^t$. Then $L_{m,n,\beta}$ becomes a random Jacobi matrix whose eigenvalues are distributed as Wishart beta ensembles [3], namely,

$$(\lambda_1, \dots, \lambda_m) \propto |\Delta(\lambda)|^\beta \prod_{i=1}^m \lambda_i^a \exp\left(-\frac{n\beta}{2}\sum_{i=1}^m \lambda_i\right),$$

where $a = (\beta/2)(n-m+1)-1$. The weights $\{w_i\} = \{q_i^2\}$ are independent of $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \dots, \beta/2)$.

It is well known that as $m/n \to \gamma \in (0,1)$, the empirical distribution of Wishart beta ensembles converges weakly to the Marchenko-Pastur distribution with parameter $\gamma \in (0,1)$, a probability measure with density

$$mp_\gamma(x) = \frac{1}{2\pi\gamma x}\sqrt{(\lambda_+ - x)(x-\lambda_-)}, (\lambda_- < x < \lambda_+),$$

where $\lambda_\pm = (1 \pm \sqrt{\gamma})^2$.

It also know that the Jacobi matrix of the Marchenko-Pastur distribution with parameter $\gamma \in (0,1)$ is given by

$$MP_\gamma = \begin{pmatrix} 1 & \sqrt{\gamma} & & \\ \sqrt{\gamma} & 1+\gamma & \sqrt{\gamma} & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \sqrt{\gamma} & 1 & & \\ & \sqrt{\gamma} & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \sqrt{\gamma} & & \\ & 1 & \sqrt{\gamma} & \\ & & 1 & \sqrt{\gamma} \\ & & & \ddots & \ddots \end{pmatrix}.$$

Denote by $\{c_i\}_{i=1}^m$ and $\{d_j\}_{j=1}^{m-1}$ the diagonal and the sub-diagonal of the matrix $\sqrt{n\beta}B_\beta$. Then

$$L_{m,n,\beta} = \frac{1}{n\beta}\begin{pmatrix} c_1^2 & c_1 d_1 & & \\ c_1 d_1 & c_2^2 + d_1^2 & c_2 d_2 & \\ & \ddots & \ddots & \ddots \\ & & c_{m-1}d_{m-1} & c_m^2 + d_{m-1}^2 \end{pmatrix}.$$

Lemma 4.1. *For fixed k, as $m \to \infty$ and $m/n \to \gamma \in (0,1)$,*

$$\begin{aligned} \frac{c_k}{\sqrt{n\beta}} &\approx \frac{\chi_{n\beta}}{\sqrt{n\beta}} \approx 1 + \frac{1}{\sqrt{n\beta}}\eta_k, \quad \eta_k \sim \mathcal{N}(0, \frac{1}{2}), \\ \frac{d_k}{\sqrt{n\beta}} &\approx \frac{\chi_{m\beta}}{\sqrt{n\beta}} \approx \sqrt{\gamma} + \frac{1}{\sqrt{n\beta}}\zeta_k, \quad \zeta_k \sim \mathcal{N}(0, \frac{1}{2}), \\ \frac{c_k^2}{n\beta} &\approx 1 + \frac{1}{\sqrt{n\beta}}2\eta_k, \quad \frac{d_k^2}{n\beta} \approx \gamma + \frac{1}{\sqrt{n\beta}}2\zeta_k, \quad \frac{c_k d_k}{n\beta} \approx \sqrt{\gamma} + \frac{1}{\sqrt{n\beta}}(\sqrt{\gamma}\eta_k + \zeta_k). \end{aligned}$$

Consequently, as $m \to \infty$ and $m/n \to \gamma \in (0,1)$, we can write

$$L_{m,n,\beta} \approx \begin{pmatrix} 1 & \sqrt{\gamma} & & \\ \sqrt{\gamma} & 1+\gamma & \sqrt{\gamma} & \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{\sqrt{\gamma}}{\sqrt{m\beta}} \begin{pmatrix} 2\eta_1 & \sqrt{\gamma}\eta_1 + \zeta_1 & & \\ \sqrt{\gamma}\eta_1 + \zeta_1 & 2(\eta_2 + \zeta_1) & \sqrt{\gamma}\eta_2 + \zeta_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Here $\{\eta_i\}$ and $\{\zeta_i\}$ are two i.i.d. sequences of $\mathcal{N}(0, \frac{1}{2})$ random variables.

Theorem 4.2. *The spectral measure μ_m of $L_{m,n,\beta}$ conveges weakly, in probability, to the the Marchenko-Pastur distribution with parameter γ as $m \to \infty$, $m/n \to \gamma \in (0,1)$. Moreover for any polynomial p,*

$$\frac{\sqrt{m\beta}}{\sqrt{2}}(\langle \mu_m, p\rangle - \langle mp_\gamma, p\rangle) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2).$$

5 MANOVA beta ensembles or β-Jacobi ensembles

Let $a, b > -1$. Let $p_1, \dots, p_{2n-1}$ be independent random variables distributed as

$$p_k \sim \begin{cases} \text{Beta}\left(\frac{2n-k}{4}\beta, \frac{2n-k-2}{4}\beta + a + b + 2\right), & k \text{ even}, \\ \text{Beta}\left(\frac{2n-k-1}{4}\beta + a + 1, \frac{2n-k-1}{4}\beta + b + 1\right), & k \text{ odd}. \end{cases}$$

Here Beta(α, β) denotes the beta distribution with parameters α, β. Define

$$\begin{aligned} a_k &= p_{2k-2}(1 - p_{2k-3}) + p_{2k-1}(1 - p_{2k-2}), \\ b_k &= \sqrt{p_{2k-1}(1-p_{2k-2})p_{2k}(1-p_{2k-1})}, \end{aligned}$$

where $p_{-1} = p_0 = 0$, and form a random Jacobi matrix $J_{n,\beta}$ as

$$J_{n,\beta} = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1} & a_n \end{pmatrix}.$$

Then the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $J_{n,\beta}$ are distributed as MANOVA beta ensembles (cf. [8]),

$$(\lambda_1, \dots, \lambda_n) \propto |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^a (1-\lambda_i)^b, \quad \lambda_i \in [0,1].$$

The weights $\{w_i\} = \{q_i^2\}$ are independent of $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \dots, \beta/2)$.

We need the following properties of beta distributions.

Lemma 5.1. (i) *As $k \to \infty$,*

$$\text{Beta}(\frac{k}{2}, \frac{k}{2}) \to \frac{1}{2} \textit{ in probability and in } L_q \textit{ for all } q < \infty.$$

(ii) *As $k \to \infty$,*

$$2\sqrt{k}\left(\text{Beta}(\frac{k}{2}, \frac{k}{2}) - \frac{1}{2}\right) \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Let X_k and Y_k be two independent random variables having χ_k^2 distribution. Then it is know that

$$\text{Beta}(\frac{k}{2}, \frac{k}{2}) \stackrel{d}{=} \frac{X_k}{X_k + Y_k}.$$

For chi-squared distribution, we have

$$\frac{\chi_k^2}{k} \to 1 \text{ in probability as } k \to \infty.$$

Therefore

$$\frac{X_k}{X_k + Y_k} = \frac{\frac{X_k}{k}}{\frac{X_k}{k} + \frac{Y_k}{k}} \to \frac{1}{2} \text{ in probability as } k \to \infty.$$

The convergence in L_p is clear because beta distributions are bounded by 1.

Next we consider the central limit theorem for beta distributions. It also follows from the following central limit theorem for chi-squared distribution

$$\frac{\chi_k^2 - k}{\sqrt{k}} \stackrel{d}{\to} \mathcal{N}(0, 2) \text{ as } k \to \infty.$$

Indeed, if we write

$$2\sqrt{k}\left(\text{Beta}(\frac{k}{2}, \frac{k}{2}) - \frac{1}{2}\right) \stackrel{d}{=} 2\sqrt{k}\left(\frac{X_k}{X_k + Y_k} - \frac{1}{2}\right) = \frac{\frac{X_k - k}{\sqrt{k}} + \frac{k - Y_k}{\sqrt{k}}}{\frac{X_k}{k} + \frac{Y_k}{k}},$$

then as $k \to \infty$, the numerator converges in distribution to $\mathcal{N}(0, 4)$ because X_k and Y_k are independent while the denominator converges in probability to 2. Thus we obtain (ii). $\square$

Lemma 5.2. *As $n \to \infty$,*

$$\begin{aligned}
a_1^{(n)} &= p_1 \approx \frac{1}{2} + \frac{1}{2\sqrt{n\beta}} \mathcal{N}(0, 1), \\
b_1^{(n)} &= \sqrt{p_1 p_2 (1 - p_1)} \approx \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{n\beta}} \mathcal{N}(0, \frac{1}{8}), \\
a_k^{(n)} &= p_{2k-2}(1 - p_{2k-3}) + p_{2k-1}(1 - p_{2k-2}) \approx \frac{1}{2} + \frac{1}{2\sqrt{n\beta}} \mathcal{N}(0, \frac{1}{2}), \quad k \ge 2, \\
b_k^{(n)} &= \sqrt{p_{2k-1}(1 - p_{2k-2}) p_{2k}(1 - p_{2k-1})} \approx \frac{1}{4} + \frac{1}{2\sqrt{n\beta}} \mathcal{N}(0, \frac{1}{8}), \quad k \ge 2.
\end{aligned}$$

The joint asymptotic also holds.

Proof. For fixed k, as $n \to \infty$, it is clear that

$$p_k \approx \text{Beta}(\frac{n\beta}{2}, \frac{n\beta}{2}) \approx \frac{1}{2} + \frac{1}{2\sqrt{n\beta}} \mathcal{N}(0, 1).$$

Then the asymptotic for $a_k^{(n)}$ follows from Lemma 2.5 because it is a polynomial of $\{p_{2k-3}, p_{2k-2}, p_{2k-1}\}$.

The asymptotic for $b_k^{(n)}$ is a consequence of the following fact. If $X_n \to c \ne 0$ in probability and $\sqrt{n}(X_n - c) \stackrel{d}{\to} \mathcal{N}(0, \sigma^2)$ as $n \to \infty$, then

$$\sqrt{n}(\sqrt{X_n} - \sqrt{c}) = \frac{\sqrt{n}(X_n - c)}{\sqrt{X} + \sqrt{c}} \stackrel{d}{\to} \frac{\mathcal{N}(0, \sigma^2)}{2\sqrt{c}} = \mathcal{N}(0, \frac{\sigma^2}{4c}).$$

The joint asymptotic is clear. $\square$

Note that the arcsine distribution, a probability measure on $(0,1)$ with density

$$arcsine(x) = \frac{1}{\pi\sqrt{x(1-x)}}, (0 < x < 1),$$

is the spectral measure of the infinite Jacobi matrix

$$J_\infty = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & & & \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{4} & & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Therefore, we obtain the limiting behaviour of the spectral measures of MANOVA beta ensembles when the parameters a, b are fixed.

Theorem 5.3. *The spectral measure μ_n of $J_{n,\beta}$ converges weakly, in probability, to the arcsine distribution. For any polynomial p,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p\rangle - \langle arcsine, p\rangle) \to \mathcal{N}(0, \sigma_p^2).$$

Remark 5.4. Here we consider MANOVA beta ensembles with fixed parameters a, b. In [10], the author considers the limiting behaviour of empirical distributions and spectral measures when one or both parameters a, b grows with n. It turns out that in that regime, the limit distribution is the Marchenko-Pastur distribution or the semicircle distribution. See also [5] for the limiting behaviour of empirical distributions of β-Jacobi ensembles when parameters a, b also vary with n.

6 Extend the central limit theorem to large class of test function

Recall that by Theorem 2.9, for any polynomial p, as $n \to \infty$,

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p\rangle - \mathbb{E}[\langle \mu_n, p\rangle]) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2),$$

where

$$\sigma_p^2 = \lim_{n\to\infty} \frac{n\beta}{2} \operatorname{Var}[\langle \mu_n, p\rangle].$$

For all three beta ensembles in this paper, the spectral measure μ_n can be written as

$$\mu_n = \sum_{i=1}^n w_i \delta_{\lambda_i},$$

where the weights $\{w_i\}$ are independent of the eigenvalues $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \dots, \beta/2)$. One can easily show that

$$\mathbb{E}[w_i] = \frac{1}{n}, \mathbb{E}[w_i^2] = \frac{\beta+2}{n(n\beta+2)}, \mathbb{E}[w_i w_j] = \frac{\beta}{n(n\beta+2)}, (1 \le i \ne j \le n).$$

Therefore for any test function f, we can derive the following relations

$$\mathbb{E}[\langle \mu_n, f\rangle] = \mathbb{E}[\langle L_n, f\rangle], \tag{9}$$

$$\operatorname{Var}[\langle \mu_n, f\rangle] = \frac{\beta n}{\beta n + 2} \operatorname{Var}[\langle L_n, f\rangle] + \frac{2}{n\beta + 2}\left(\mathbb{E}[\langle \mu_n, f^2\rangle] - \mathbb{E}[\langle \mu_n, f\rangle]^2\right). \tag{10}$$

The mean of a random measure μ, denoted by $\bar{\mu}$, is defined to be a probability measure satisfying

$$\langle \bar{\mu}, f \rangle = \mathbb{E}[\langle \mu, f \rangle],$$

for all bounded continuous function f. However, the above relation still holds for a continuous function f with $\mathbb{E}[\langle \mu, |f| \rangle] < \infty$. Denote by $C(\mathbb{R})$ the set of continuous function on $\mathbb{R}$ and let

$$\mathcal{D} = \{ f \in C(\mathbb{R}) : n \operatorname{Var}[\langle L_n, f \rangle] \to 0, \langle \bar{\mu}_n, f \rangle \to \langle \mu_\infty, f \rangle, \langle \bar{\mu}_n, f^2 \rangle \to \langle \mu_\infty, f^2 \rangle \}.$$

Then $\mathcal{D}$ is a linear space containing all polynomials. It follows from the relation (10) that for $f \in \mathcal{D}$,

$$\lim_{n\to\infty} \frac{n\beta}{2} \operatorname{Var}[\langle \mu_n, f \rangle] = \langle \mu_\infty, f^2 \rangle - \langle \mu_\infty, f \rangle^2 =: \sigma^2(f).$$

Next, we use the following result to extend the central limit theorem to any test function in $\mathcal{D}$.

Lemma 6.1 ([1, Theorem 3.2]). *Let $\{Y_n\}_n$ and $\{X_{n,k}\}_{k,n}$ be real-valued random variables. Assume that*

(i) $X_{n,k} \xrightarrow{d} X_k$ *as* $n \to \infty$;

(ii) $X_k \xrightarrow{d} X$ *as* $k \to \infty$;

(iii) *for any* $\varepsilon > 0$, $\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(|X_{n,k} - Y_n| \ge \varepsilon) = 0$.

Then $Y_n \xrightarrow{d} X$ *as* $n \to \infty$.

Theorem 6.2. *For $f \in \mathcal{D}$,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}} \big(\langle \mu_n, f \rangle - \mathbb{E}[\langle \mu_n, f \rangle] \big) \xrightarrow{d} \mathcal{N}(0, \sigma(f)^2),$$

where $\sigma^2(f) = \langle \mu_\infty, f^2 \rangle - \langle \mu_\infty, f \rangle^2$.

Proof. Let $f \in \mathcal{D}$. Then since μ_∞ has a compact support, we can find a sequence of polynomials $\{p_k\}$ converging to f uniformly in the support of μ_∞. Thus

$$\sigma^2(p_k) \to \sigma^2(f) \text{ as } k \to \infty.$$

Let

$$Y_n = \frac{\sqrt{n\beta}}{\sqrt{2}} (\langle \mu_n, f \rangle - \mathbb{E}[\langle \mu_n, f \rangle]),$$
$$X_{n,k} = \frac{\sqrt{n\beta}}{\sqrt{2}} (\langle \mu_n, p_k \rangle - \mathbb{E}[\langle \mu_n, p_k \rangle]).$$

We only need to check three conditions in Lemma 6.1. Conditions (i) and (ii) are clear. For the condition (iii), note that $(f - p_k) \in \mathcal{D}$, and thus

$$\lim_{n\to\infty} \operatorname{Var}[X_{n,k} - Y_n] = \langle \mu_\infty, (f - p_k)^2 \rangle - \langle \mu_\infty, (f - p_k) \rangle^2,$$

which tends to zero as $k \to \infty$. Therefore, for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(|X_{n,k} - Y_n| \ge \varepsilon) \le \lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{\varepsilon^2} \operatorname{Var}[X_{n,k} - Y_n] = 0.$$

The theorem is proved. ∎

Remark 6.3. (i) The class $\mathcal{D}$ contains all test functions f for which the central limit theorem holds for the linear statistics $n(\langle L_n, f\rangle - \langle \mu_\infty, f\rangle)$. See [7] for a class of test functions in the case of Gaussian beta ensembles.

(ii) For Gaussian orthogonal ensembles and Gaussian unitary ensembles, the above central limit theorem was established for continuous bounded function f with bounded derivative [11, Theorem 3.3.3].

References

[1] Patrick Billingsley. *Convergence of probability measures.* John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[2] Holger Dette and Jan Nagel. Distributions on unbounded moment spaces and random moment sequences. *Ann. Probab.*, 40(6):2690–2704, 2012.

[3] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *J. Math. Phys.*, 43(11):5830–5847, 2002.

[4] Ioana Dumitriu and Alan Edelman. Global spectrum fluctuations for the β-Hermite and β-Laguerre ensembles via matrix models. *J. Math. Phys.*, 47(6):063302, 36, 2006.

[5] Ioana Dumitriu and Elliot Paquette. Global fluctuations for linear statistics of β-Jacobi ensembles. *Random Matrices Theory Appl.*, 1(4):1250013, 60, 2012.

[6] Fabrice Gamboa and Alain Rouault. Large deviations for random spectral measures and sum rules. *Appl. Math. Res. Express. AMRX*, (2):281–307, 2011.

[7] Kurt Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.*, 91(1):151–204, 1998.

[8] Rowan Killip and Irina Nenciu. Matrix models for circular ensembles. *Int. Math. Res. Not.*, (50):2665–2701, 2004.

[9] Jan Nagel. Moderate deviations for spectral measures of random matrix ensembles. arXiv:1308.5516.

[10] Jan Nagel. Nonstandard limit theorems and large deviations for the Jacobi beta ensemble. *Random Matrices Theory Appl.*, 3(3):1450012, 20, 2014.

[11] Leonid Pastur and Mariya Shcherbina. *Eigenvalue distribution of large random matrices*, volume 171 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2011.

[12] Barry Simon. *Szegő's theorem and its descendants.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 2011. Spectral theory for L^2 perturbations of orthogonal polynomials.

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