Monomial Difference Ideals

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Abstract

In this paper, basic properties of monomial difference ideals are studied. We prove the finitely generated property of well-mixed difference ideals generated by monomials. Furthermore, a finite prime decomposition of radical well-mixed monomial difference ideals is given. As a consequence, we prove that every strictly ascending chain of radical well-mixed monomial difference ideals in a difference polynomial ring is finite, which answers a question raised by E. Hrushovski in the monomial case. Moreover, the Alexander Duality for monomial ideals is generalized to the monomial case.

Keywords. Monomial difference ideal, well-mixed difference ideal, decomposition of monomial difference ideal, Hrushovski's question.

1 Introduction

Monomial ideals in a polynomial ring have been extendedly studied because of their connections with combinatorics since 1970s. Another reason to study monomial ideals is the fact that they appear as initial ideals of arbitrary ideals. Stanley was the first to use squarefree monomial ideals to study simplicial complexes ([6]). Since then, the study of squarefree monomial ideals has become an active research area in combinatorial commutative algebra. In this paper, we study the basic properties of monomial difference (abbr. σ-) ideals, and hope that they will play similar role in the study of general σ-ideals in a σ-polynomial ring.

It is well-known that Hilbert's basis theorem does not hold for σ-ideals in a σ-polynomial ring. Instead, we have Ritt-Raudenbush basis theorem which asserts that every perfect σ-ideal in a σ-polynomial ring has a finite basis. It is naturally to ask if the finitely generated property holds for more σ-ideals. Let k be a σ-field and R a finitely σ-generated k-σ-algebra. In [2, Section 4.6], Ehud Hrushovski raised the question whether a radical well-mixed σ-ideal in R is finitely generated. The question is also equivalent to whether the ascending chain condition holds for radical well-mixed σ-ideals in R. For the sake of convenience, let us state it as a conjecture:

Conjecture 1.1 *Every strictly ascending chain of radical well-mixed σ-ideals in R is finite.*

Also in [2, Section 4.6], Ehud Hrushovski proved that the answer is yes under some additional assumptions on R. In [5], Alexander Levin showed that the ascending chain condition does not hold if we drop the radical condition. The counter example given by Levin is a well-mixed σ-ideal generated by binomials. In [8, Section 9], Michael Wibmer showed that if R can be equipped with the structure of a k-σ-Hopf algebra, then Conjecture 1.1 is valid.

The main result of this paper is that a well-mixed σ-ideals generated by monomials in a σ-polynomial ring is finitely generated. Furthermore, we give a finite prime decomposition of radical well-mixed monomial difference ideals. As a consequence, Conjucture 1.1 is valid for radical well-mixed monomial σ-ideals in a σ-polynomial ring.

The paper will be organized as follows. In section 2, we list some basic facts from difference algebra. In section 3, we prove some basic properties about monomial σ-ideals. In section 4, we will give a counter example which shows that the well-mixed closure of a monomial σ-ideal may not be a monomial σ-ideal and prove the finitely generated property of well-mixed σ-ideal generated by monomials. In section 5, we will give a finite prime decomposition of radical well-mixed monomial σ-ideals. In section 6, we give a reflexive prime decomposition of perfect monomial σ-ideals. At last, in section 7, we will generalize the Alexander Duality for monomial ideals to the difference case.

2 Preliminaries

In this section, we list some basic notions and facts from difference algebra. For more details please refer to [7]. All rings in this paper will be assumed to be commutative and unital.

A difference ring or σ-ring for short (R, σ), is a ring R together a ring endomorphism $\sigma\colon R \to R$. If R is a field, then we call it a difference field, or a σ-field for short. We usually omit σ from the notation, simply refer to R as a σ-ring or a σ-field. In this paper, k is always assumed to be a σ-field of characteristic 0.

Following [3], we use the notation of symbolic exponents. Let x be an algebraic indeterminate and $p = \sum_{i=0}^{s} c_i x^i \in \mathbb{N}[x]$. For a in a σ-ring R, denote $a^p = \prod_{i=0}^{s} (\sigma^i(a))^{c_i}$. It is easy to check that $\forall p, q \in \mathbb{N}[x], a^{p+q} = a^p a^q, a^{pq} = (a^p)^q$.

Definition 2.1 *Let R be a σ-ring. An ideal I of R is a σ-ideal if $\forall a \in I \Rightarrow a^x \in I$. Suppose I is a σ-ideal of R, then I is called*

- *reflexive if $a^x \in I \Rightarrow a \in I$ for $a \in R$;*
- *well-mixed if $ab \in I \Rightarrow ab^x \in I$ for $a, b \in R$;*
- *perfect if $a^g \in I \Rightarrow a \in I$ for $a \in R, g \in \mathbb{N}[x]\backslash\{0\}$;*
- *σ-prime if I is reflexive and a prime ideal as an algebraic ideal.*

Lemma 2.2 *(1) A σ-ideal is perfect if and only if it is reflexive, radical, and well-mixed;*

(2) A σ-prime ideal is perfect;

(3) A prime σ-ideal is radical well-mixed.

Proof: It is easy. □

Lemma 2.3 *Let R be a σ-ring. A σ-ideal I of R is perfect if and only if $a^{x+1} \in I \Rightarrow a \in I$ for $a \in R$.*

Proof: For the proof, please refer to [7, p.16]. □

Let R be a σ-ring. If $F \subseteq R$ is a subset of R, denote the minimal ideal containing F by (F), the minimal σ-ideal containing F by $[F]$ and denote the minimal radical σ-ideal, the minimal reflexive σ-ideal, the minimal well-mixed σ-ideal, the minimal radical well-mixed σ-ideal, the minimal perfect σ-ideal containing F by $\sqrt{F}, F^*, \langle F\rangle$, $\langle F\rangle_r$, $\{F\}$ respectively, which are called the radical closure, the reflexive closure, the well-mixed closure, the radical well-mixed closure, the perfect closure of F respectively.

It can be checked that $\sqrt{F}^* = \sqrt{F^*}$ and $\{F\} = \langle F\rangle_r^*$.

Let k be a σ-field. Suppose $y = \{y_1, \ldots, y_n\}$ is a set of σ-indeterminates over k. Then the σ-polynomial ring over k in y is the polynomial ring in the variables $y, \sigma(y), \sigma^2(y), \ldots$. It is denoted by

$$k\{y\} = k\{y_1, \ldots, y_n\}$$

and has a natural k-σ-algebra structure.

3 Basic Properties of Monomial Difference Ideals

In the rest of this paper, unless otherwise specified, R always refers to the σ-polynomial ring $k\{y_1, \ldots, y_n\}$. Denote $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $\mathbb{N}[x]^* = \mathbb{N}[x]\backslash\{0\}$.

Definition 3.1 *A monomial in R is a product $\mathbb{Y}^{\mathbf{u}} = y_1^{u_1} \ldots y_n^{u_n}$ for $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}[x]^n$. A σ-ideal $I \subseteq R$ is called a monomial σ-ideal if it is generated by monomials.*

As a vector space over k, we can write the σ-polynomial ring R as

$$R = k[\mathbb{N}[x]^n] = \oplus_{\mathbf{u}\in\mathbb{N}[x]^n} R_{\mathbf{u}} = \oplus_{\mathbf{u}\in\mathbb{N}[x]^n} k\mathbb{Y}^{\mathbf{u}},$$

where $R_{\mathbf{u}} = k\mathbb{Y}^{\mathbf{u}}$ is the vector subspace of R spanned by the monomial $\mathbb{Y}^{\mathbf{u}}$. Since $R_{\mathbf{u}} \cdot R_{\mathbf{v}} \subseteq R_{\mathbf{u}+\mathbf{v}}$, we see that R is an $\mathbb{N}[x]^n$-graded ring. A monomial σ-ideal I defined above is just a graded σ-ideal of R, which means there exists a subset $S \subseteq \mathbb{N}[x]^n$ such that $I = k[S] := \oplus_{\mathbf{u}\in S} k\mathbb{Y}^{\mathbf{u}}$. Such S is called the *support set* of I.

For a subset $F \subseteq R$, we denote $C(F) = \{\mathbf{u} \in \mathbb{N}[x]^n \mid \mathbb{Y}^{\mathbf{u}} \in F\}$. And for a subset $S \subseteq \mathbb{N}[x]^n$, we denote $M(S) = \{\mathbb{Y}^{\mathbf{u}} \mid \mathbf{u} \in S\}$.

The following lemma is clear.

Lemma 3.2 *A subset $S \subseteq \mathbb{N}[x]^n$ is the support set of some monomial σ-ideal if and only if S satisfies*

(1) $\forall \mathbf{u} \in S, \mathbf{v} \in \mathbb{N}[x]^n$, $\mathbf{u} + \mathbf{v} \in S$;

(2) $\forall \mathbf{u} \in S$, $x\mathbf{u} \in S$.

A subset $S \subseteq \mathbb{N}[x]^n$ satisfying the above conditions is called a *character set*.

If $a = \sum a_u \mathbb{Y}^{\mathbf{u}} \in R, a_u \in k$, then

$$\mathrm{supp}(a) = \{\mathbb{Y}^{\mathbf{u}} \mid a_u \neq 0\}$$

is called the *support* of a.

Lemma 3.3 *Let I be a σ-ideal of R. Then the followings are equivalent:*

(a) I is a monomial σ-ideal;

(b) $\forall a \in R$, $a \in I$ if and only if $\mathrm{supp}(a) \subset I$.

Proof: It is obvious. □

Lemma 3.4 *If $I_1 = k[S_1]$ and $I_2 = k[S_1]$ are monomial σ-ideals. Then $I_1 + I_2$ and $I_1 \cap I_2$ are monomial σ-ideals.*

Proof: $I_1 + I_2 = k[S_1 \cup S_2]$, $I_1 \cap I_2 = k[S_1 \cap S_2]$. □

If $I = k[S]$ is a monomial σ-ideal, then the conditions for I to be radical, reflexive, perfect and prime can be described using the support set S. To show this, we first define an order on $\mathbb{N}[x]^n$. Let $f = \sum_{i=0}^{l} f_i x^i, g = \sum_{i=0}^{m} g_i x^i \in \mathbb{N}[x]$. Suppose $k > \max\{l, m\}$, and set $f_i = 0$ for $l+1 \leqslant i \leqslant k$, $g_i = 0$ for $m+1 \leqslant i \leqslant k$. Then define $f < g$ if there exists r such that $f_i = g_i$ for $i = r+1, \ldots, k$ and $f_r < g_r$. Extend $<$ to $\mathbb{N}[x]^n$ by comparing $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}[x]^n$ with respect to the lexicographic order. Obviously, this is a total order on $\mathbb{N}[x]^n$ and has the following properties.

Lemma 3.5 *The order $<$ defined above satisfies:*

(1) $\mathbf{u}_1 < \mathbf{v}_1, \mathbf{u}_2 \leqslant \mathbf{v}_2 \Rightarrow \mathbf{u}_1 + \mathbf{u}_2 < \mathbf{v}_1 + \mathbf{v}_2$*;*

(2) $\mathbf{u} < \mathbf{v} \Rightarrow x\mathbf{u} < x\mathbf{v}$.

Let $a \in R$. Then define $\deg(a)$ to be the maximal element in $C(\mathrm{supp}(a))$.

Proposition 3.6 *Let $I = k[S]$ be a monomial σ-ideal of R. Then:*

(1) I is radical if and only if $\forall \mathbf{u} \in \mathbb{N}[x]^n, \forall m \in \mathbb{N}^*$, $m\mathbf{u} \in S \Rightarrow \mathbf{u} \in S$*;*

(2) I is reflexive if and only if $\forall \mathbf{u} \in \mathbb{N}[x]^n$, $x\mathbf{u} \in S \Rightarrow \mathbf{u} \in S$*;*

(3) I is perfect if and only if $\forall \mathbf{u} \in \mathbb{N}[x]^n, \forall g \in \mathbb{N}[x]^*$, $g\mathbf{u} \in S \Rightarrow \mathbf{u} \in S$*;*

(4) I is prime if and only if $\forall \mathbf{u}, \mathbf{v} \in \mathbb{N}[x]^n$, $\mathbf{u} + \mathbf{v} \in S \Rightarrow \mathbf{u} \in S$ *or* $\mathbf{v} \in S$.

Proof:

(1) "$\Rightarrow$" follows from the definition of radical ideals.

"$\Leftarrow$". Suppose $a=\sum_{i=1}^k a_i\mathbb{Y}^{\mathbf{u}_i}, a_i\neq 0$, $\mathbf{u}_1<\ldots<\mathbf{u}_k$ and $a^m\in I$. To show $a\in I$, do induction on the number of terms of a. When $k=1$, $a^m=(a_1\mathbb{Y}^{\mathbf{u}_1})^m=a_1^m\mathbb{Y}^{m\mathbf{u}_1}\in I$. Therefore, $\mathbb{Y}^{m\mathbf{u}_1}\in I$ and hence $m\mathbf{u}_1\in S$. So $\mathbf{u}_1\in S$ or equivalently $\mathbb{Y}^{\mathbf{u}_1}\in I$ which implies $a\in I$. Assume that the conclusion is already correct for the case $<k$. Now for the case $=k$. Note $a^m=a_1^m\mathbb{Y}^{m\mathbf{u}_1}$+the other terms. Since $m\mathbf{u}_1$ is minimal in the set of all possible combinations of $\mathbf{u}_{i_1}+\ldots+\mathbf{u}_{i_m}$, the monomial $\mathbb{Y}^{m\mathbf{u}_1}$ cannot be cancelled in the expression of a^m and hence belongs to $\mathrm{supp}(a^m)$. Since I is a monomial σ-ideal, $\mathrm{supp}(a^m)\subseteq I$ and hence $\mathbb{Y}^{m\mathbf{u}_1}\in I$ or equivalently $m\mathbf{u}_1\in S$. So $\mathbf{u}_1\in S$ and $\mathbb{Y}^{\mathbf{u}_1}\in I$. Consider $a'=a-a_1\mathbb{Y}^{\mathbf{u}_1}$ with $k-1$ terms. Since $(a')^m=(a-a_1\mathbb{Y}^{\mathbf{u}_1})^m=a^m-\mathbb{Y}^{\mathbf{u}_1}\cdot *\in I$, by the induction hypothesis, $a'\in I$. Thus $a=a'+a_1\mathbb{Y}^{\mathbf{u}_1}\in I$.

(2) "$\Rightarrow$" follows from the definition of reflexive ideals.

"$\Leftarrow$". Suppose $a=\sum_{i=1}^k a_i\mathbb{Y}^{\mathbf{u}_i}, a_i\neq 0$ and $a^x\in I$. Since $a^x=\sum_{i=1}^k a_i^x\mathbb{Y}^{x\mathbf{u}_i}$ and I is a monomial σ-ideal, it follows $\mathbb{Y}^{x\mathbf{u}_i}\in I$ for every i. Therefore, $x\mathbf{u}_i\in S$ and hence $\mathbf{u}_i\in S$ for every i which implies $\mathbb{Y}_i^{\mathbf{u}}\in S$ for every i. Thus $a\in I$.

(3) "$\Rightarrow$" follows from the definition of perfect ideals.

"$\Leftarrow$". Suppose $a=\sum_{i=1}^k a_i\mathbb{Y}^{\mathbf{u}_i}, a_i\neq 0$, $\mathbf{u}_1<\ldots<\mathbf{u}_k$ and $a^{x+1}\in I$. To show $a\in I$, do induction on the number of terms of a. When $k=1$, $a^{x+1}=(a_1\mathbb{Y}^{\mathbf{u}_1})^{x+1}=a_1^{x+1}\mathbb{Y}^{(x+1)\mathbf{u}_1}\in I$. Therefore, $\mathbb{Y}^{(x+1)\mathbf{u}_1}\in I$ and hence $(x+1)\mathbf{u}_1\in S$. So $\mathbf{u}_1\in S$ or equivalently $\mathbb{Y}^{\mathbf{u}_1}\in I$ which implies $a\in I$. Assume that the conclusion is already correct for the case $<k$. Now for the case $=k$. Note $a^{x+1}=a_1^{x+1}\mathbb{Y}^{(x+1)\mathbf{u}_1}$+the other terms. Since $\mathbf{u}_1<\ldots<\mathbf{u}_k$ and $x\mathbf{u}_1<\ldots<x\mathbf{u}_k$, $(x+1)\mathbf{u}_1$ is minimal in the set of all possible combinations of $\mathbf{u}_i+x\mathbf{u}_j$. So the monomial $\mathbb{Y}^{(x+1)\mathbf{u}_1}$ cannot be cancelled in the expression of a^{x+1} and hence belongs to $\mathrm{supp}(a^{x+1})$. Since I is a monomial σ-ideal, $\mathrm{supp}(a^{x+1})\subseteq I$ and hence $\mathbb{Y}^{(x+1)\mathbf{u}_1}\in I$ or equivalently $(x+1)\mathbf{u}_1\in S$. So $\mathbf{u}_1\in S$ and $\mathbb{Y}^{\mathbf{u}_1}\in I$. Consider $a'=a-a_1\mathbb{Y}^{\mathbf{u}_1}$ with $k-1$ terms. Since $(a')^{x+1}=a^{x+1}-a_1a^x\mathbb{Y}^{\mathbf{u}_1}-aa_1^x\mathbb{Y}^{x\mathbf{u}_1}+a_1^{x+1}\mathbb{Y}^{(x+1)\mathbf{u}_1}\in I$, by the induction hypothesis, $a'\in I$. Thus $a=a'+a_1\mathbb{Y}^{\mathbf{u}_1}\in I$.

(4) "$\Rightarrow$" follows from the definition of prime ideals.

"$\Leftarrow$". Suppose it's not true, so there exists a and b in R such that $a\cdot b\in I$ but $a\notin I$ and $b\notin I$. Let a and b are such a pair such that $\deg(a)+\deg(b)$ is minimal. Since I is a monomial σ-ideal, $\mathrm{supp}(ab)\subset I$. In particular, the highest degree term is in I. The highest degree term is just the product of the leading terms of a and b, which we'll call $\mathrm{ld}(a)$ and $\mathrm{ld}(b)$. So $\mathrm{ld}(a)\cdot\mathrm{ld}(b)\in I$, and since they are monomials, we see that either $\mathrm{ld}(a)$ or $\mathrm{ld}(b)$ in I. Without loss of generality, assume it's $\mathrm{ld}(a)$. In that case $(a-\mathrm{ld}(a))\cdot b\in I$, but neither $a-\mathrm{ld}(a)$ nor b is in I, and this violates the minimality of the pair a and b.

□

Suppose $I=k[S]$ is a monomial σ-ideal. If I is radical, reflexive, well-mixed, perfect, or prime, then we call the corresponding support set S radical, reflexive, well-mixed, perfect, prime respectively.

Let S be a subset of $\mathbb{N}[x]^n$. Denote

$$[S] = \{x^i\mathbf{u} + t \mid \mathbf{u} \in S, i \in \mathbb{N}, t \in \mathbb{N}[x]^n\}$$
$$= \{g\mathbf{u} + t \mid \mathbf{u} \in S, g \in \mathbb{N}[x]^*, t \in \mathbb{N}[x]^n\}$$

and

$$\sqrt{S} = \{\mathbf{u} \in \mathbb{N}[x]^n \mid m\mathbf{u} \in [S], m \in \mathbb{N}^*\},$$
$$S^* = \{\mathbf{u} \in \mathbb{N}[x]^n \mid x^m\mathbf{u} \in [S], m \in \mathbb{N}\},$$
$$\{S\} = \{\mathbf{u} \in \mathbb{N}[x]^n \mid g\mathbf{u} \in [S], g \in \mathbb{N}[x]^*\}.$$

One can check that $[\mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in S] = k[[S]]$ and if $I = k[S]$ is a monomial σ-ideal, then $[S] = S$.

Proposition 3.7 *Let $I = k[S]$ be a monomial σ-ideal of R. Then $\sqrt{I} = k[\sqrt{S}]$, $I^* = k[S^*]$, and $\{I\} = k[\{S\}]$.*

Proof: Clearly, $k[\sqrt{S}] \subseteq \sqrt{I}$, $k[S^*] \subseteq I^*$, and $k[\{S\}] \subseteq \{I\}$. Just need to show $k[\sqrt{S}]$, $k[S^*]$, $k[\{S\}]$ are a radical σ-ideal, a reflexive σ-ideal, a perfect σ-ideal respectively.

Suppose $\mathbf{u} \in \sqrt{S}$ and $\mathbf{v} \in \mathbb{N}[x]^n$, then there exists $m \in \mathbb{N}^*$ such that $m\mathbf{u} \in S$. So $m(\mathbf{u}+\mathbf{v}) = m\mathbf{u} + m\mathbf{v} \in S, m(x\mathbf{u}) = x(m\mathbf{u}) \in S$ and hence $\mathbf{u}+\mathbf{v}, x\mathbf{u} \in \sqrt{S}$. Therefore, $k[\sqrt{S}]$ is a σ-ideal. Suppose $m \in \mathbb{N}^*$ and $m\mathbf{u} \in \sqrt{S}$, then $\exists m' \in \mathbb{N}^*$ such that $m'm\mathbf{u} \in S$, it follows $\mathbf{u} \in \sqrt{S}$ and thus $\sqrt{S}$ is radical.

Suppose $\mathbf{u} \in S^*$ and $\mathbf{v} \in \mathbb{N}[x]^n$, then there exists $m \in \mathbb{N}$ such that $x^m\mathbf{u} \in S$. So $x^m(\mathbf{u}+\mathbf{v}) = x^m\mathbf{u} + x^m\mathbf{v} \in S, x^m(x\mathbf{u}) = x^{m+1}\mathbf{u} \in S$ and hence $\mathbf{u}+\mathbf{v}, x\mathbf{u} \in S^*$. Therefore, $k[S^*]$ is a σ-ideal. Suppose $m \in \mathbb{N}$ and $x^m\mathbf{u} \in S^*$, then $\exists m' \in \mathbb{N}$ such that $x^{m'+m}\mathbf{u} \in S$, it follows $\mathbf{u} \in S^*$ and thus S^* is reflexive.

Suppose $\mathbf{u} \in \{S\}$ and $\mathbf{v} \in \mathbb{N}[x]^n$, then there exists $g \in \mathbb{N}[x]^*$ such that $g\mathbf{u} \in S$. So $g(\mathbf{u}+\mathbf{v}) = g\mathbf{u}+g\mathbf{v} \in S, g(x\mathbf{u}) = x(g\mathbf{u}) \in S$ and hence $\mathbf{u}+\mathbf{v}, x\mathbf{u} \in \{S\}$. Therefore, $k[\{S\}]$ is a σ-ideal. Suppose $g \in \mathbb{N}[x]^*$ and $g\mathbf{u} \in \{S\}$, then $\exists g' \in \mathbb{N}[x]^*$ such that $g'g\mathbf{u} \in S$, it follows $\mathbf{u} \in \{S\}$ and thus $\{S\}$ is perfect. □

4 Properties of Well-Mixed σ-Ideals Generated by Monomials

Unlike the radical closure, the reflexive closure, or the perfect closure of a monomial σ-ideal is still a monomial σ-ideal, the well-mixed closure of a monomial σ-ideal may not be a monomial σ-ideal. More precisely, it relies on the action of the difference operator. We will give a counter example. First let us give a concrete description of the well-mixed closure of a σ-ideal. Suppose F is a subset of any σ-ring R. Let $F' = \{a\sigma(b) \mid ab \in F\}$. Note that $F \subset F'$. Let $F^{[0]} = F$ and recursively define $F^{[k]} = (F^{[k-1]})'(k = 1, 2, \ldots)$. One can check that the well-mixed closure of F is

$$\langle F \rangle = \cup_{k=0}^{\infty} F^{[k]}.$$

Example 4.1 *Let $k = \mathbb{C}$ and $R = \mathbb{C}\{y_1, y_2\}$. Consider the σ-ideal $I = \langle y_1^2, y_2^2\rangle$ of R. If the difference operator on $\mathbb{C}$ is the identity map, we will show that I is not a monomial σ-ideal. Owing to the above process of obtaining the well-mixed closure, we see that y_1, y_2, y_1y_2 cannot appear in $\mathrm{supp}(a)$ for any $a \in I$. Suppose $a = a_1y_1 + a_2y_2 + *$ and $b = b_1y_1 + b_2y_2 + *$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$, $*$ represents terms of order larger than zero or of degree larger than one, such that $ab \in I$ and $y_1y_2^x \in \mathrm{supp}(ab^x)$. Since $y_1y_2 \notin \mathrm{supp}(ab)$, we have $a_1b_2 + a_2b_1 = 0$. So $ab^x = a_1b_2y_1y_2^x + a_2b_1y_1^xy_2 + * = a_1b_2(y_1y_2^x - y_1^xy_2) + *$. It follows that $y_1y_2^x - y_1^xy_2$ always appears in $c \in I$ as a whole and hence $y_1y_2^x$ cannot solely appear in $\mathrm{supp}(c)$ for $c \in I$. Thus I is not a monomial σ-ideal.*

On the other hand, if the difference operator on $\mathbb{C}$ is the conjugation map, that is $\sigma(i) = -i$, the situation is totally changed. Since $y_1^2-y_2^2 = (y_1+y_2)(y_1-y_2) \in I$, $(y_1+y_2)(y_1-y_2)^x = y_1^{x+1}+y_1^xy_2-y_1y_2^x-y_2^{x+1} \in I$ and hence $y_1^xy_2-y_1y_2^x \in I$. Since $y_1^2+y_2^2 = (y_1+iy_2)(y_1-iy_2) \in I$, $(y_1 + iy_2)(y_1 - iy_2)^x = y_1^{x+1} + iy_1^xy_2 + iy_1y_2^x - y_2^{x+1} \in I$ and hence $y_1^xy_2 + y_1y_2^x \in I$. So $y_1^xy_2, y_1y_2^x \in I$. Thus $I = [y_1^u, y_1^{w_1}y_2^{w_2}, y_2^v : 2 \preceq u, v, x + 1 \preceq w_1 + w_2]$($\preceq$ is defined below). In this case, $I = \langle y_1^2, y_2^2\rangle$ is indeed a monomial σ-ideal.

In the rest of this section, we will prove that a well-mixed σ-ideal generated by monomials could be generated by finitely many monomials as a well-mixed σ-ideal. For the proof, we need a new order on $\mathbb{N}[x]^n$ and some lemmas.

Definition 4.2 *Let $f = \sum_{i=0}^l f_ix^i, g = \sum_{i=0}^m g_ix^i \in \mathbb{N}[x]$. Suppose $k > \max\{l, m\}$, and set $f_i = 0$ for $l + 1 \leqslant i \leqslant k$, $g_i = 0$ for $m + 1 \leqslant i \leqslant k$. Then define $f \preceq g$ if $\sum_{j=i}^k f_j \leqslant \sum_{j=i}^k g_j$ for $i = 0, \ldots, k$. Note that $\preceq$ is a partial order on $\mathbb{N}[x]$. Extend $\preceq$ to $\mathbb{N}[x]^n$ by defining $\mathbf{u} = (u_1, \ldots, u_n) \preceq \mathbf{v} = (v_1, \ldots, v_n)$ if and only if $u_i \preceq v_i$ for $i = 1, \ldots, n$.*

Let $\mathbf{u}, \mathbf{v} \in \mathbb{N}[x]^n$. It is easy to see that if $\mathbf{u} \preceq \mathbf{v}$, then $\mathbb{Y}^{\mathbf{v}} \in \langle \mathbb{Y}^{\mathbf{u}}\rangle$. Moreover, the partial order $\preceq$ has the following properties.

Lemma 4.3 *If $\mathbf{u}_1 \preceq \mathbf{v}_1, \mathbf{u}_2 \preceq \mathbf{v}_2$,then $x\mathbf{u}_1 \preceq x\mathbf{v}_1, \mathbf{u}_1 + \mathbf{u}_2 \preceq \mathbf{v}_1 + \mathbf{v}_2$.*

Proof: It is obvious. □

For $f = \sum_{i=0}^l f_ix^i \in \mathbb{N}[x]$, denote $|f| = \sum_{i=0}^l f_i$.

Lemma 4.4 *Let $S \subseteq \mathbb{N}[x]$ such that $|f| = a$ is a constant for all $f \in S$. Then the set of minimal elements of S under the partial order $\preceq$ is finite.*

Proof: Do induction on a. The case $a = 1$ is clear. Assume the conclusion is correct for the case $< a$. Choose an $f = \sum_{i=0}^l f_ix^i \in S$. The set $G = \{g = \sum_{i=0}^l g_ix^i \mid |g| \leqslant a\}$ is finite. For each $g = \sum_{i=0}^l g_ix^i \in G$, define

$$S_g = \{\sum_{i=0}^m h_ix^i \in S \mid m > l, \sum_{i=0}^l h_ix^i = g\}$$

and

$$S'_g = \{ \sum_{i=l+1}^m h_ix^i \mid \sum_{i=0}^m h_ix^i \in S_g\}.$$

It follows $S = \cup_{g\in G} S_g$. For each $g \in G$, if $g \neq 0$, then $\forall h \in S'_g$, $|h|$ is a constant which is lower than a. So by the induction hypothesis, the set of minimal elements of S'_g under the partial order $\preceq$ is finite and hence so is S_g. Note that $\forall h \in S_0$, $f \preceq h$. Therefore, the set of minimal elements of S is contained in the union of the set of minimal elements of S_g, where $g \in G$. Since the latter is a finite set, the former must be finite. □

Lemma 4.5 *Let $S \subseteq \mathbb{N}[x]$ such that $\forall f \in S, |f| \leqslant a$ for some $a \in \mathbb{N}$. Then the set of minimal elements of S under the partial order $\preceq$ is finite.*

Proof: It is an immediate corollary from the above lemma. □

Lemma 4.6 *Let $S \subseteq \mathbb{N}[x]$ such that $\forall f \in S, \deg(f) \leqslant k$ for some $k \in \mathbb{N}$. Then the set of minimal elements of S under the partial order $\preceq$ is finite.*

Proof: Do induction on k. The case $k = 0$ is clear. Assume the conclusion is correct for the case $< k$. Choose an $f = \sum_{i=0}^{k} f_i x^i \in S$ and denote $c = \sum_{i=0}^{k} f_i$. For any $j < c, j \in \mathbb{N}$, suppose s, $0 \leqslant s \leqslant k$, such that $\sum_{i=s+1}^{k} f_i \leqslant j < \sum_{i=s}^{k} f_i$, and define

$$U_j = \{\sum_{i=s}^{k} g_i x^i \mid \sum_{i=s}^{k} g_i = j\}$$

which is obviously a finite set. For each $g \in U_j$, define

$$S_g = \{\sum_{i=0}^{k} g_i x^i \in S \mid \sum_{i=s}^{k} g_i x^i = g\}$$

and

$$S'_g = \{\sum_{i=0}^{s-1} g_i x^i \mid \sum_{i=0}^{k} g_i x^i \in S_g\}.$$

In addition, we define

$$S_f = \{g \in S \mid f \preceq g\}.$$

Then we have

$$S = (\cup_{j=0}^{c-1} \cup_{g\in U_j} S_g) \cup S_f. \tag{4.1}$$

Note for each $g \in U_j$, $\forall h \in S'_g$, $\deg(h) < k$, so by the induction hypothesis, the set of minimal elements of S'_g under the partial order $\preceq$ is finite and hence so is S_g. Because of (4.1), we have the set of minimal elements of S is contained in the union of the set of minimal elements of S_g and $\{f\}$, where $g \in U_j$ and $0 \leqslant j < c$. Since the latter is a finite set, the former must be finite. □

Lemma 4.7 *Let $S \subseteq \mathbb{N}[x]$. Then the set of minimal elements of S under the partial order $\preceq$ is finite.*

Proof: Choose an $f = \sum_{i=0}^{k} f_i x^i \in S$ and denote $c = \sum_{i=0}^{k} f_i$. For any $j < c - f_0, j \in \mathbb{N}$, suppose s, $1 \leqslant s \leqslant k$, such that $\sum_{i=s+1}^{k} f_i \leqslant j < \sum_{i=s}^{k} f_i$, and define

$$U_j = \{\sum_{i=s}^{l} g_i x^i \mid \sum_{i=0}^{l} g_i x^i \in S, \sum_{i=s}^{l} g_i = j\},$$
$$S_j = \{\sum_{i=0}^{l} g_i x^i \in S \mid \sum_{i=s}^{l} g_i = j\}$$

and

$$S'_j = \{\sum_{i=0}^{s-1} g_i x^i \mid \sum_{i=0}^{l} g_i x^i \in S_j\}.$$

By Lemma 4.4, for each j, the set of minimal elements of U_j is finite, which is denoted by V_j. By Lemma 4.6, for each j, the set of minimal elements of S'_j is finite, which is denoted by W_j.

In addition, we define

$$S_c = \{\sum_{i=0}^{l} g_i x^i \in S \mid \sum_{i=0}^{l} g_i < c\},$$
$$S_f = \{g \in S \mid f \preceq g\}.$$

Then we have

$$S = (\cup_{j=0}^{c-f_0-1} S_j) \cup S_c \cup S_f. \tag{4.2}$$

By Lemma 4.5, the set of minimal elements of S_c is finite, which is denoted by C. Claim that the set of minimal elements of S is contained in $(\cup_{j=0}^{c-f_0-1}(V_j + W_j)) \cup C \cup \{f\}$, where $V_j + W_j$ means $\{g + h \mid g \in V_j, h \in W_j\}$. To prove this, $\forall g = \sum_{i=0}^{l} g_i x^i \in S$. By (4.2), if $g \notin S_c$ and $g \notin S_f$, then there exists j such that $g \in S_j$. By definition, $\sum_{i=s}^{l} g_i x^i \in U_j$ and $\sum_{i=0}^{s-1} g_i x^i \in S'_j$. So $\exists h \in V_j$ and $\exists h' \in W_j$ such that $h \preceq \sum_{i=s}^{l} g_i x^i$ and $h' \preceq \sum_{i=0}^{s-1} g_i x^i$. Therefore, $\exists h + h' \in V_j + W_j$ such that $h + h' \preceq \sum_{i=s}^{l} g_i x^i + \sum_{i=0}^{s-1} g_i x^i = g$, which proves the claim.

Since $(\cup_{j=0}^{c-f_0-1}(V_j + W_j)) \cup C \cup \{f\}$ is a finite set, it follows the set of minimal elements of S is finite. □

Lemma 4.8 *Let $S \subseteq \mathbb{N}[x]^n$. Then the set of minimal elements of S under the partial order $\preceq$ is finite.*

Proof: Do induction on n. The case $n = 1$ is proved by Lemma 4.7. Assume the conclusion is correct for the case $= n - 1$. Define

$$S_1 = \{u_1 \mid (u_1, \ldots, u_n) \in S\}.$$

By Lemma 4.7, the set of minimal elements of S_1 is finite, which is denoted by U. For each $u \in U$, define

$$S_u = \{(u_1, u_2, \ldots, u_n) \in S \mid u \preceq u_1\}$$

and

$$S'_u = \{(u_2, \ldots, u_n) \mid (u_1, u_2, \ldots, u_n) \in S_u\}.$$

Obviously, $S = \cup_{u \in U} S_u$.

By the induction hypothesis, for each $u \in U$, the set of minimal elements of S'_u under the partial order $\preceq$ is finite, which is denoted by V_u. Let $u \times V_u = \{(u, \mathbf{v}) \mid \mathbf{v} \in V_u\}$. Claim that the set of minimal elements of S is contained in $\cup_{u \in U} u \times V_u$. To prove this, $\forall \mathbf{u} = (u_1, u_2, \ldots, u_n) \in S$, there exists $u \in U$ such that $\mathbf{u} \in S_u$. By definition, $(u_2, \ldots, u_n) \in S'_u$. So $\exists \mathbf{v} \in V_u$ such that $\mathbf{v} \preceq (u_2, \ldots, u_n)$. Therefore, $(u, \mathbf{v}) \preceq (u_1, u_2, \ldots, u_n) = \mathbf{u}$ and $(u, \mathbf{v}) \in u \times V_u$ which proves the claim.

Since $\cup_{u \in U} u \times V_u$ is a finite set, the set of minimal elements of S is finite. □

Theorem 4.9 *Let $I = \langle \mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U \rangle$ for some $U \subseteq \mathbb{N}[x]^n$. Then I is generated by a finite set of monomials as a well-mixed σ-ideal.*

Proof: If $\mathbf{u} \preceq \mathbf{v}$, since $\mathbb{Y}^{\mathbf{v}} \in \langle \mathbb{Y}^{\mathbf{u}} \rangle$, we can delete $\mathbf{v}$ from the generating set U to get the same well-mixed σ-ideal. So we just need to show that the set of minimal elements of U under the partial order $\preceq$ is finite, which follows from Lemma 4.8. □

Corollary 4.10 *Any strictly ascending chain of well-mixed σ-ideals generated by monomials in R is finite.*

Proof: Assume that $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k \ldots$ is an ascending chain of well-mixed σ-ideals generated by monomials. Then $\cup_{i=1}^{\infty} I_i$ is still a well-mixed σ-ideal generated by monomials. By Theorem 4.9, $\cup_{i=1}^{\infty} I_i$ is finitely generated, say by $\{a_1, \ldots, a_m\}$. Then there exists $k \in \mathbb{N}$ large enough such that $\{a_1, \ldots, a_m\} \subset I_k$. So $I_k = I_{k+1} = \ldots = \cup_{i=1}^{\infty} I_i$. □

Remark 4.11 *It should be pointed out that a counter example due to Levin in [5] shows Corollary 4.10 does not hold even for well-mixed σ-ideals generated by binomials in R.*

5 Prime Decomposition of Radical Well-Mixed Monomial σ-Ideals

In this section, we will give a finite prime decomposition of radical well-mixed monomial σ-ideals. First prove some lemmas. Notations follow as Section 4.

Lemma 5.1 *Let F and G be subsets of any σ-ring R. Then*

(1) $F^{[1]} G^{[1]} \subset (FG)^{[1]}$;

(2) $F^{[i]} G^{[i]} \subset (FG)^{[i]}$ *for* $i = 1, 2, \ldots$;

(3) $F^{[i]} \cup G^{[i]} \subset \sqrt{(FG)^{[i]}}$ *for* $i = 1, 2, \ldots$.

Proof:

(1) $\forall a\sigma(b) \in F^{[1]}, \forall c\sigma(d) \in G^{[1]}$ such that $ab \in (F)$ and $cd \in (G)$. Then $abcd \in (FG)$ and it follows $ac\sigma(bd) = a\sigma(b)c\sigma(d) \in (FG)^{[1]}$. So $F^{[1]}G^{[1]} \subset (FG)^{[1]}$.

(2) Do induction on i.

$$\begin{aligned} F^{[i]}G^{[i]} &= (F^{[i-1]})^{[1]}(G^{[i-1]})^{[1]} \subset (F^{[i-1]}G^{[i-1]})^{[1]} \\ &\subseteq ((FG)^{[i-1]})^{[1]} = (FG)^{[i]}. \end{aligned}$$

(3) $\forall a \in F^{[i]} \cup G^{[i]}$, then $a^2 \in F^{[i]}G^{[i]}$. It follows $a \in \sqrt{F^{[i]}G^{[i]}} \subset \sqrt{(FG)^{[i]}}$.

□

Proposition 5.2 *Let F and G be subsets of any σ-ring R. Then*

$$\langle F\rangle_r \cap \langle G\rangle_r = \langle FG\rangle_r.$$

As a corollary, if I and J are two σ-ideals of R, then

$$\langle I\rangle_r \cap \langle J\rangle_r = \langle I \cap J\rangle_r = \langle IJ\rangle_r.$$

Proof: $\langle F\rangle_r \cap \langle G\rangle_r \supseteq \langle FG\rangle_r$ is clear. It is enough to show the converse.

$$\begin{aligned} \langle F\rangle_r \cap \langle G\rangle_r &= \sqrt{\langle F\rangle} \cap \sqrt{\langle G\rangle} = \sqrt{\cup_{i=0}^{\infty} F^{[i]})} \cap \sqrt{\cup_{i=0}^{\infty} G^{[i]})} \\ &= \sqrt{\cup_{i=0}^{\infty}(F^{[i]} \cap G^{[i]})} \subseteq \sqrt{\cup_{i=0}^{\infty}\sqrt{(FG)^{[i]}}} = \langle FG\rangle_r, \end{aligned}$$

where the inclusion follows from Lemma 5.1(3). □

Lemma 5.3 *Let I be a radical well-mixed σ-ideal of R. Suppose $\mathbb{Y}^{\mathbf{u}_1}, \mathbb{Y}^{\mathbf{u}_2}$ are two monomials in R such that $\mathbb{Y}^{\mathbf{u}_1+\mathbf{u}_2} \in I$. Then*

$$I = \langle I, \mathbb{Y}^{\mathbf{u}_1}\rangle_r \cap \langle I, \mathbb{Y}^{\mathbf{u}_2}\rangle_r.$$

Proof: By Proposition 5.2,

$$\langle I, \mathbb{Y}^{\mathbf{u}_1}\rangle_r \cap \langle I, \mathbb{Y}^{\mathbf{u}_2}\rangle_r = \langle I, \mathbb{Y}^{\mathbf{u}_1+\mathbf{u}_2}\rangle_r = I.$$

□

For $\mathbf{b} = (b_1, \ldots, b_n) \in (\mathbb{N} \cup \{-1\})^n$, define

$$\mathfrak{m}^{\mathbf{b}} := [y_i^{x^{b_i}} \mid b_i \neq -1]$$

which is a prime monomial σ-ideal.

For $m \in \mathbb{N}^*$, denote $[m] = \{1, \ldots, m\}$.

Theorem 5.4 *Let $I = \langle \mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U \rangle_r$ where $U \subset \mathbb{N}[x]^n$. Then I can be written as a finite intersection of prime monomial σ-ideals of forms $\mathfrak{m}^{\mathbf{b}}$. That is, $\exists \mathbf{b}_1, \ldots, \mathbf{b}_s \in (\mathbb{N} \cup \{-1\})^n$ such that*

$$I = \mathfrak{m}^{\mathbf{b}_1} \cap \ldots \cap \mathfrak{m}^{\mathbf{b}_s}.$$

Moreover, if the decomposition is irredundant, then it is unique.

Proof: By Lemma 5.3, if a monomial $\mathbb{Y}^{\mathbf{u}} \in I$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, then $I = \langle I, \mathbb{Y}^{\mathbf{u}_1} \rangle_r \cap \langle I, \mathbb{Y}^{\mathbf{u}_2} \rangle_r$. Iterating this process eventually write I as follows:

$$I = \cap \langle y_{i_1}^{x^{b_{i_1}}}, \ldots, y_{i_k}^{x^{b_{i_k}}} \rangle_r.$$

Note $[y_{i_1}^{x^{b_{i_1}}}, \ldots, y_{i_k}^{x^{b_{i_k}}}]$ is a prime σ-ideal, therefore

$$\langle y_{i_1}^{x^{b_{i_1}}}, \ldots, y_{i_k}^{x^{b_{i_k}}} \rangle_r = [y_{i_1}^{x^{b_{i_1}}}, \ldots, y_{i_k}^{x^{b_{i_k}}}]$$

and

$$I = \cap [y_{i_1}^{x^{b_{i_1}}}, \ldots, y_{i_k}^{x^{b_{i_k}}}].$$

After deleting unnecessary intersectands, we can assume that the intersection is irredundant. Using an argument similar to the proof of Dickson's lemma, we see this irredundant intersection must be finite. So $\exists \mathbf{b}_1, \ldots, \mathbf{b}_s \in (\mathbb{N} \cup \{-1\})^n$ such that

$$I = \mathfrak{m}^{\mathbf{b}_1} \cap \ldots \cap \mathfrak{m}^{\mathbf{b}_s}.$$

Let $\mathfrak{m}^{\mathbf{b}_1} \cap \ldots \cap \mathfrak{m}^{\mathbf{b}_s} = \mathfrak{m}^{\mathbf{a}_1} \cap \ldots \cap \mathfrak{m}^{\mathbf{a}_t}$ be two irredundant decompositions of I. We will show that for each $i \in [s]$, there exists a $j \in [t]$ such that $\mathfrak{m}^{\mathbf{a}_j} \subseteq \mathfrak{m}^{\mathbf{b}_i}$. By symmetry, we then also have that for each $k \in [t]$, there exists an $l \in [s]$ such that $\mathfrak{m}^{\mathbf{b}_l} \subseteq \mathfrak{m}^{\mathbf{a}_k}$. This implies that $s = t$ and $\{\mathfrak{m}^{\mathbf{b}_1}, \ldots, \mathfrak{m}^{\mathbf{b}_s}\} = \{\mathfrak{m}^{\mathbf{a}_1}, \ldots, \mathfrak{m}^{\mathbf{a}_t}\}$.

In fact, let $i \in [s]$. We may assume that $\mathfrak{m}^{\mathbf{b}_i} = [y_1^{x^{b_{i1}}}, \ldots, y_r^{x^{b_{ir}}}]$. Suppose that $\mathfrak{m}^{\mathbf{a}_j} \not\subseteq \mathfrak{m}^{\mathbf{b}_i}$ for all $j \in [t]$. Then for each j there exists $y_{l_j}^{x^{c_j}} \in \mathfrak{m}^{\mathbf{a}_j} \backslash \mathfrak{m}^{\mathbf{b}_i}$. It follows that either $l_j \notin [r]$ or $c_j < b_{il_j}$. Let

$$a = \prod_{j=1}^{t} y_{l_j}^{x^{c_j}}.$$

We have $a \in \cap_{j=1}^{t} \mathfrak{m}^{\mathbf{a}_j} \subseteq \mathfrak{m}^{\mathbf{b}_i}$. Therefore, there exists $h \in [r]$ such that $b_{ih} \leqslant \deg(C(a)_h)$. This is impossible. □

Corollary 5.5 *The radical well-mixed closure of a monomial σ-ideal is still a monomial σ-ideal.*

Proof: Suppose I is a monomial σ-ideal. By Theorem 5.4, $\langle I \rangle_r = \cap_{i=1}^{s} \mathfrak{m}^{\mathbf{b}_i}$. Since $\mathfrak{m}^{\mathbf{b}_i}$ are monomial σ-ideals and the intersection of monomial σ-ideals is a monomial σ-ideal, it follows $\langle I \rangle_r$ is a monomial σ-ideal. □

If I is a radical well-mixed monomial σ-ideal, then the irredundant prime decomposition of I obtained in Theorem 5.4 is called the standard prime decomposition of I and each $\mathfrak{m}^{\mathbf{b}_i}$ is called an irreducible component of I.

Corollary 5.6 *Every radical well-mixed monomial σ-ideal in R could be generated by finitely many monomials as a radical well-mixed σ-ideal.*

Proof: Suppose I is a radical well-mixed monomial σ-ideal. Let $I = \cap_{i=1}^{s} \mathfrak{m}^{\mathbf{b}_i}$ be the standard prime decomposition of I. By Proposition 5.2, $\cap_{i=1}^{s} \mathfrak{m}^{\mathbf{b}_i}$ equals to a radical well-mixed σ-ideal which is generated by finitely many monomials, so I is finitely generated as a radical well-mixed σ-ideal. □

From the above corollary, for any radical well-mixed monomial σ-ideal I, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_m \in (\mathbb{N} \cap \{-1\})^n$ with $\mathbf{a}_j = (a_{ji})_{i=1}^n, j = 1, \ldots, m$ such that

$$I = \langle \prod_{i=1}^{n} y_i^{x^{a_{1i}}}, \ldots, \prod_{i=1}^{n} y_i^{x^{a_{mi}}} \rangle_r,$$

where we set that $x^{-1} = 0$. We call $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ the *character vectors* of I and call $\{\prod_{i=1}^{n} y_i^{x^{\mathbf{a}_{1i}}}, \ldots, \prod_{i=1}^{n} y_i^{x^{\mathbf{a}_{mi}}}\}$ the *set of minimal generators* of I, denoted by $G(I)$.

Corollary 5.7 *Any strictly ascending chain of radical well-mixed monomial σ-ideals in R is finite.*

Proof: Assume that $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k \ldots$ is an ascending chain of radical well-mixed monomial σ-ideals. Then $\cup_{i=1}^{\infty} I_i$ is still a radical well-mixed monomial σ-ideal. By Corollary 5.6, $\cup_{i=1}^{\infty} I_i$ is finitely generated, say by $\{a_1, \ldots, a_m\}$. Then there exists $k \in \mathbb{N}$ large enough such that $\{a_1, \ldots, a_m\} \subset I_k$. So $I_k = I_{k+1} = \ldots = \cup_{i=1}^{\infty} I_i$. □

Remark 5.8 *By Corollary 5.7, Conjecture 1.1 is valid for radical well-mixed monomial σ-ideals.*

In the following, we give a criterion to check if a monomial σ-ideal is radical well-mixed using its support set.

Lemma 5.9 *An intersection of prime σ-ideals is radical well-mixed.*

Proof: A prime σ-ideal is radical well-mixed and an intersection of radical well-mixed σ-ideals is radical well-mixed. □

Corollary 5.10 *Let $I = k[S]$ be a monomial σ-ideal of R. Then I is radical well-mixed if and only if the following conditions are satisfied:*

(a) $\forall \mathbf{u} \in \mathbb{N}[x]^n, \forall m \in \mathbb{N}^*$, $m\mathbf{u} \in S \Rightarrow \mathbf{u} \in S$;

(b) $\forall \mathbf{u}, \mathbf{v} \in \mathbb{N}[x]^n$, $\mathbf{u} + \mathbf{v} \in S \Rightarrow \mathbf{u} + x\mathbf{v} \in S$.

Proof: "⇒" is clear.

"⇐". For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}[x]^n$, define $\deg(\mathbf{u}) = (\deg(u_1), \ldots, \deg(u_n))$ and set that $\deg(0) = -1$. If $\mathbf{b} = (b_1, \ldots, b_n) \in (\mathbb{N} \cup \{-1\})^n$, then let $x^{\mathbf{b}} = (x^{b_1}, \ldots, x^{b_n})$ and set $x^{-1} = 0$. So from (a) and (b), we obtain

$$\forall \mathbf{u} \in S \Rightarrow x^{\deg(\mathbf{u})} \in S.$$

Let U be the subset of S which is the set of minimal elements in $\{x^{\deg(\mathbf{u})} \mid \mathbf{u} \in S\}$ under the order $\leqslant(\mathbf{u} = (u_i)_{i=1}^n \leqslant \mathbf{v} = (v_i)_{i=1}^n \Leftrightarrow \deg(u_i) \leqslant \deg(v_i)$ for all i). Then using an argument similar to the proof of Dickson's lemma, we see that U is a finite set. Moreover,

$$S = \{\mathbf{v} \in \mathbb{N}[x]^n \mid \mathbf{u} \leqslant \mathbf{v} \text{ for some } \mathbf{u} \in U\}.$$

Or equivalently,

$$S = \cup_{\mathbf{u}\in U} \cap_{i=1}^n \{\mathbf{v} \in \mathbb{N}[x]^n \mid \deg(u_i) \leqslant \deg(v_i)\}.$$

Exchange $\cup$ and $\cap$, we get $\exists \mathbf{b}_1, \ldots, \mathbf{b}_s \in (\mathbb{N} \cup \{-1\})^n$ such that

$$S = \cap_{i=1}^s \cup_{j=1, b_{ij}\neq -1}^n \{\mathbf{v} \in \mathbb{N}[x]^n \mid b_{ij} \leqslant \deg(v_i)\}.$$

It follows

$$I = k[S] = \cap_{i=1}^s \mathfrak{m}^{\mathbf{b}_i}.$$

Since $\mathfrak{m}^{\mathbf{b}_i}$ are prime σ-ideals, I is radical well-mixed. □

Suppose S is a subset of $\mathbb{N}[x]^n$. Let

$$S' = \{\mathbf{u} + x\mathbf{v} \mid \mathbf{u} + \mathbf{v} \in S, \mathbf{u}, \mathbf{v} \in \mathbb{N}[x]^n\}.$$

Let $S^{[0]} = S$ and recursively define $S^{[k]} = [S^{[k-1]}]'(k = 1, 2, \ldots)$. Denote

$$\langle S\rangle = \cup_{k=0}^{\infty} S^{[k]}.$$

Corollary 5.11 *Let $I = k[S]$ be a monomial σ-ideal of R. Then $\langle I\rangle_r = k[\sqrt{\langle S\rangle}]$.*

Proof: Clearly, $\langle I\rangle_r \supset k[\sqrt{\langle S\rangle}]$. We just need to show $k[\sqrt{\langle S\rangle}]$ is already a radical well-mixed σ-ideal. It is easy to see that $k[\sqrt{\langle S\rangle}]$ is a σ-ideal. To show it is radical well-mixed, we need to check $\sqrt{\langle S\rangle}$ satisfies conditions (a) and (b) of Corollary 5.10. (a) is obvious. For (b), $\forall \mathbf{u} + \mathbf{v} \in \sqrt{\langle S\rangle}$, then $\exists m \in \mathbb{N}^*$ such that $m(\mathbf{u} + \mathbf{v}) \in \sqrt{\langle S\rangle} = \cup_{k=0}^{\infty} S^{[k]}$. So $\exists k \in \mathbb{N}$ such that $m(\mathbf{u} + \mathbf{v}) \in S^{[k]}$. Hence $m(\mathbf{u} + x\mathbf{v}) \in S^{[k+1]} \subseteq \langle S\rangle$. Therefore, $\mathbf{u} + x\mathbf{v} \in \sqrt{\langle S\rangle}$. □

Corollary 5.12 *Suppose $U_1, U_2 \subseteq \mathbb{N}[x]^n$. Then $\sqrt{\langle U_1 \cup U_2\rangle} = \sqrt{\langle U_1\rangle} \cup \sqrt{\langle U_2\rangle}$.*

Proof: Clearly $\sqrt{\langle U_1 \cup U_2\rangle} \supseteq \sqrt{\langle U_1\rangle} \cup \sqrt{\langle U_2\rangle}$, we only need to show that $k[\sqrt{\langle U_1\rangle} \cup \sqrt{\langle U_2\rangle}]$ is already a radical well-mixed σ-ideal.

Obviously, $\sqrt{\langle U_1\rangle} \cup \sqrt{\langle U_2\rangle}$ is a character set. Let $\mathbf{u} + \mathbf{v} \in \sqrt{\langle U_1\rangle} \cup \sqrt{\langle U_2\rangle}$, then $\mathbf{u} + \mathbf{v} \in \sqrt{\langle U_1\rangle}$ or $\sqrt{\langle U_2\rangle}$ and hence $\mathbf{u} + x\mathbf{v} \in \sqrt{\langle U_1\rangle}$ or $\sqrt{\langle U_2\rangle}$. So $\mathbf{u} + x\mathbf{v} \in \sqrt{\langle U_1\rangle} \cup \sqrt{\langle U_2\rangle}$ which proves conditions (b) of Corollary 5.10. Similarly for conditions (a) of Corollary 5.10. □

Corollary 5.13 *Let I, J be two monomial σ-ideals. Then $\langle I + J\rangle_r = \langle I\rangle_r + \langle J\rangle_r$.*

Proof: Suppose $I = k[S_1], J = k[S_2]$. Then $\langle I + J\rangle_r = k[\sqrt{\langle S_1 \cup S_2\rangle}]$ and $\langle I\rangle_r + \langle J\rangle_r = k[\sqrt{\langle S_1\rangle} \cup \sqrt{\langle S_2\rangle}]$. So the equality follows from Corollary 5.12. □

6 σ-Prime Decomposition of Perfect Monomial σ-Ideals

Similarly to Proposition 5.2 and Lemma 5.3, we have

Proposition 6.1 *Let F and G be subsets of any σ-ring R. Then*

$$\{F\} \cap \{G\} = \{FG\}.$$

Lemma 6.2 *Let I be a perfect σ-ideal of R. Suppose $\mathbb{Y}^{\mathbf{u}_1}, \mathbb{Y}^{\mathbf{u}_2}$ are two monomials in R such that $\mathbb{Y}^{\mathbf{u}_1+\mathbf{u}_2} \in I$. Then*

$$I = \{I, \mathbb{Y}^{\mathbf{u}_1}\} \cap \{I, \mathbb{Y}^{\mathbf{u}_2}\}.$$

For $\mathbf{b} = (b_1, \ldots, b_n) \in \{0,1\}^n$, define

$$\mathfrak{p}^{\mathbf{b}} := [y_i \mid b_i \neq 0]$$

which is a σ-prime ideal.

Theorem 6.3 *Let $I = \{\mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U\}$ where $U \subset \mathbb{N}[x]^n$. Then I can be written as a finite intersection of σ-prime ideals of forms $\mathfrak{p}^{\mathbf{b}}$. That is, $\exists \mathbf{b}_1, \ldots, \mathbf{b}_s \in \{0,1\}^n$ such that*

$$I = \mathfrak{p}^{\mathbf{b}_1} \cap \ldots \cap \mathfrak{p}^{\mathbf{b}_s}.$$

Moreover, if the decomposition is irredundant, then it is unique.

Proof: Without loss of generality, we can assume that $U \subset \{0,1\}^n$. By Lemma 6.2, if a monomial $\mathbb{Y}^{\mathbf{u}} \in I$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, then $I = \{I, \mathbb{Y}^{\mathbf{u}_1}\} \cap \{I, \mathbb{Y}^{\mathbf{u}_2}\}$. Iterating this process eventually write I as follows:

$$I = \cap\{y_{i_1}, \ldots, y_{i_k}\} = \cap[y_{i_1}, \ldots, y_{i_k}].$$

After deleting unnecessary intersectands, we can assume that the intersection is irredundant. It is easy to see this irredundant intersection is finite. Thus $\exists \mathbf{b}_1, \ldots, \mathbf{b}_s \in \{0,1\}^n$ such that

$$I = \mathfrak{p}^{\mathbf{b}_1} \cap \ldots \cap \mathfrak{p}^{\mathbf{b}_s}.$$

The uniqueness is similar to Theorem 5.4. □

Remark 6.4 *In fact, it is more straightforward to get the σ-prime decomposition of perfect monomial σ-ideals by using Theorem 5.4. Assume that $U \subset \{0,1\}^n$. Then by Theorem 5.4, $\langle \mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U\rangle_r = \cap\langle y_{i_1}, \ldots, y_{i_k}\rangle_r = \cap[y_{i_1}, \ldots, y_{i_k}]$. Since $[y_{i_1}, \ldots, y_{i_k}]$ are σ-prime ideals, it follows $\langle \mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U\rangle_r$ is a perfect σ-ideal. Thus*

$$I = \{\mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U\} = \langle \mathbb{Y}^{\mathbf{u}} : \mathbf{u} \in U\rangle_r = \cap[y_{i_1}, \ldots, y_{i_k}].$$

7 Alexander Duality of Monomial σ-Ideals

Definition 7.1 *Given two vectors* $\mathbf{a}, \mathbf{b} \in (\mathbb{N} \cup \{-1\})^n$ *with* $\mathbf{b} \leqslant \mathbf{a}$*(that is,* $b_i \leqslant a_i$ *for* $i = 1, \ldots, n$*), let* $\mathbf{a} \backslash \mathbf{b}$ *denote the vector whose* i^{th} *coordinate is*

$$a_i \backslash b_i = \begin{cases} a_i + 1 - b_i, & \text{if } b_i \geqslant 0; \\ -1, & \text{if } b_i = -1. \end{cases}$$

Suppose I *is a radical well-mixed monomial* σ*-ideal. If* $\mathbf{a}$ *is a vector in* $(\mathbb{N} \cup \{-1\})^n$ *satisfying* $\mathbf{a} \geqslant \mathbf{b}$ *for any character vector* $\mathbf{b}$ *of* I*, then the* **Alexander dual** *of* I *with respect to* $\mathbf{a}$ *is*

$$I^{[\mathbf{a}]} = \cap \{\mathfrak{m}^{\mathbf{a} \backslash \mathbf{b}} \mid \mathbf{b} \text{ is a character vector of } I\}.$$

Note that for vectors $\mathbf{b} \leqslant \mathbf{a}$ in $(\mathbb{N} \cup \{-1\})^n$, $\mathbf{a} \backslash (\mathbf{a} \backslash \mathbf{b}) = \mathbf{b}$.

As in Corollary 5.10, for $\mathbf{b} = (b_1, \ldots, b_n) \in (\mathbb{N} \cup \{-1\})^n$, let $x^{\mathbf{b}} = (x^{b_1}, \ldots, x^{b_n})$ and set $x^{-1} = 0$.

Proposition 7.2 *Suppose* I *is a radical well-mixed monomial* σ*-ideal and* $\mathbf{a}$ *is a vector in* $(\mathbb{N} \cup \{-1\})^n$ *satisfying* $\mathbf{a} \geqslant \mathbf{c}$ *for any character vector* $\mathbf{c}$ *of* I*. If* $\mathbf{b} \leqslant \mathbf{a}$*, then* $\mathbb{Y}^{x^{\mathbf{b}}}$ *lies outside* I *if and only if* $\mathbb{Y}^{x^{\mathbf{a}-\mathbf{b}}}$ *lies inside* $I^{[\mathbf{a}]}$*.*

Proof: Suppose $\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$is the set of character vectors of I. Then $\mathbb{Y}^{x^{\mathbf{b}}} \notin I$ if and only if $\mathbf{b} \ngeqslant \mathbf{c}_i$, or equivalently, $\mathbf{a} - \mathbf{b} \nleqslant \mathbf{a} - \mathbf{c}_i$ for all i. This means that for each i, some coordinate of $\mathbf{a} - \mathbf{b}$ equals at least the corresponding coordinate of $\mathbf{a} + 1 - \mathbf{c}_i$. That is $\mathbb{Y}^{x^{\mathbf{a}-\mathbf{b}}} \in \mathfrak{m}^{\mathbf{a}+1-\mathbf{c}_i}$ for all i, i.e. $\mathbb{Y}^{x^{\mathbf{a}-\mathbf{b}}} \in \cap_{i=1}^m \mathfrak{m}^{\mathbf{a}+1-\mathbf{c}_i}$. Next, let us show that

$$\cap_{i=1}^m \mathfrak{m}^{\mathbf{a}+1-\mathbf{c}_i} = \cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2} = I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+2}. \tag{7.1}$$

It is obvious that $\cap_{i=1}^m (\mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}) \supseteq \cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}$. For the converse, choose $f \in \cap_{i=1}^m (\mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2})$, then for each i, we could write $f = f_i + g_i$, where $f_i \in \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i}, g_i \in \mathfrak{m}^{\mathbf{a}+2}$. Thus $f^m = (f_1 + g_1) \ldots (f_m + g_m) \in \cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}$. By Corollary 5.13, $\cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}$ is radical, so $f \in \cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}$. Hence we have

$$\cap_{i=1}^m (\mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}) = \cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2}.$$

By Corollary 5.13 again,

$$\begin{aligned} \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2} &= \langle y_j^{x^{a_j+1-c_{ij}}} : c_{ij} \neq -1, j = 1, \ldots, n \rangle_r + \langle y_j^{x^{a_j+2}} : j = 1, \ldots, n \rangle_r \\ &= \langle y_j^{x^{a_j+1-c_{ij}}}, y_j^{x^{a_j+2}} : c_{ij} \neq -1, j = 1, \ldots, n \rangle_r \\ &= \langle y_j^{x^{a_j+1-c_{ij}}}, j = 1, \ldots, n \rangle_r = \mathfrak{m}^{\mathbf{a}+1-\mathbf{c}_i}. \end{aligned}$$

Thus (7.1) holds. Since $\mathbf{a} - \mathbf{b} \leqslant \mathbf{a} + 1$, $\mathbb{Y}^{x^{\mathbf{a}-\mathbf{b}}} \in \cap_{i=1}^m \mathfrak{m}^{\mathbf{a}+1-\mathbf{c}_i} = \cap_{i=1}^m \mathfrak{m}^{\mathbf{a} \backslash \mathbf{c}_i} + \mathfrak{m}^{\mathbf{a}+2} = I^{[\mathbf{a}]} + \mathfrak{m}^{\mathbf{a}+2}$ exactly when $\mathbb{Y}^{x^{\mathbf{a}-\mathbf{b}}} \in I^{[\mathbf{a}]}$. □

Theorem 7.3 *Suppose* I *is a radical well-mixed monomial* σ*-ideal and* $\mathbf{a}$ *is a vector in* $(\mathbb{N} \cup \{-1\})^n$ *satisfying* $\mathbf{a} \geqslant \mathbf{b}$ *for any character vector* $\mathbf{b}$ *of* I*. Then* $\mathbf{a} \geqslant \mathbf{c}$ *for any character vector* $\mathbf{c}$ *of* $I^{[\mathbf{a}]}$*, and* $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$*.*

Proof: The proof is similar to the proof of Theorem 5.24 in p.90 in [1]. □

Theorem 7.4 *Suppose I is a radical well-mixed monomial σ-ideal and $\mathbf{a}$ is a vector in $(\mathbb{N} \cup \{-1\})^n$ satisfying $\mathbf{a} \geqslant \mathbf{b}$ for any character vector $\mathbf{b}$ of I. Then*

$$I = \cap\{\mathfrak{m}^{\mathbf{a}\setminus\mathbf{b}} \mid \mathbf{b} \textit{ is a character vector of } I^{[\mathbf{a}]}\},$$

and

$$I^{[\mathbf{a}]} = \langle \mathbb{Y}^{x^{\mathbf{a}\setminus\mathbf{b}}} \mid \mathfrak{m}^{\mathbf{b}} \textit{ is an irreducilbe component of } I\rangle_r.$$

Proof: The proof is similar to the proof of Theorem 5.27 in p.90 in [1]. □

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