ON A SEMITOPOLOGICAL POLYCYCLIC MONOID

SERHII BARDYLA AND OLEG GUTIK

Abstract. We study algebraic structure of the λ-polycyclic monoid P_λ and its topologizations. We show that the λ-polycyclic monoid for an infinite cardinal $\lambda \geqslant 2$ has similar algebraic properties so has the polycyclic monoid P_n with finitely many $n \geqslant 2$ generators. In particular we prove that for every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free combinatorial 0-bisimple 0-E-unitary inverse semigroup. Also we show that every non-zero element x is an isolated point in (P_λ, τ) for every Hausdorff topology τ on P_λ, such that (P_λ, τ) is a semitopological semigroup, and every locally compact Hausdorff semigroup topology on P_λ is discrete. The last statement extends results of the paper [33] obtaining for topological inverse graph semigroups. We describe all feebly compact topologies τ on P_λ such that (P_λ, τ) is a semitopological semigroup and its Bohr compactification as a topological semigroup. We prove that for every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating and there exists no a Hausdorff feebly compact topological semigroup which contains P_λ as a dense subsemigroup.

1. Introduction and preliminaries

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [8, 11, 14, 32]. If A is a subset of a topological space X, then we denote the closure of the set A in X by $\mathrm{cl}_X(A)$. By ω we denote the first infinite cardinal.

A semigroup S is called an *inverse semigroup* if every a in S possesses an unique inverse, i.e. if there exists an unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

A map which associates to any element of an inverse semigroup its inverse is called the *inversion*.

A *band* is a semigroup of idempotents. If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [36, Definition II.5.12] chain L is called ω-chain if L is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [16] or [11, Section 2.1]):

Date: January 7, 2016.

2010 *Mathematics Subject Classification.* Primary 22A15, 20M18. Secondary 20M05, 22A26, 54A10, 54D30, 54D35, 54D45, 54H11.

Key words and phrases. Inverse semigroup, bicyclic monoid, polycyclic monoid, free monoid, semigroup of matrix units, topological semigroup, semitopological semigroup, Bohr compactification, embedding, locally compact, countably compact, feebly compact.

$$\begin{array}{lcl} a\mathscr{R}b & \text{if and only if} & aS^1=bS^1; \\ a\mathscr{L}b & \text{if and only if} & S^1a=S^1b; \\ a\mathscr{J}b & \text{if and only if} & S^1aS^1=S^1bS^1; \end{array}$$
$$\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L};$$
$$\mathscr{H}=\mathscr{L}\cap\mathscr{R}.$$

A semigroup S is said to be:

- *simple* if S has no proper two-sided ideals, which is equivalent to $\mathscr{J}=S\times S$ in S;
- 0-*simple* if S has a zero and S contains no proper two-sided ideals distinct from the zero;
- *bisimple* if S contains a unique $\mathscr{D}$-class, i.e., $\mathscr{D}=S\times S$ in S;
- 0-*bisimple* if S has a zero and S contains two $\mathscr{D}$-classes: $\{0\}$ and $S\setminus\{0\}$;
- *congruence-free* if S has only identity and universal congruences.

An inverse semigroup S is said to be

- *combinatorial* if $\mathscr{H}$ is the equality relation on S;
- E-*unitary* if for any idempotents $e,f\in S$ the equality $ex=f$ implies that $x\in E(S)$;
- 0-E-*unitary* if S has a zero and for any non-zero idempotents $e,f\in S$ the equality $ex=f$ implies that $x\in E(S)$.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array

$$\begin{array}{ccccc} 1 & p & p^2 & p^3 & \cdots \\ q & qp & qp^2 & qp^3 & \cdots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

and the semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [11]. Also the nice Andersen Theorem states that *a simple semigroup S with an idempotent is completely simple if and only if S does not contains an isomorphic copy of the bicyclic semigroup* (see [1] and [11, Theorem 2.54]).

Let λ be a non-zero cardinal. On the set $B_\lambda=(\lambda\times\lambda)\cup\{0\}$, where $0\notin\lambda\times\lambda$, we define the semigroup operation "$\cdot$" as follows

$$(a,b)\cdot(c,d)=\left\{\begin{array}{cl} (a,d), & \text{if } b=c; \\ 0, & \text{if } b\neq c, \end{array}\right.$$

and $(a,b)\cdot 0=0\cdot(a,b)=0\cdot 0=0$ for $a,b,c,d\in\lambda$. The semigroup B_λ is called the *semigroup of $\lambda\times\lambda$-matrix units* (see [11]).

In 1970 Nivat and Perrot proposed the following generalization of the bicyclic monoid (see [35] and [32, Section 9.3]). For a non-zero cardinal λ, the polycyclic monoid P_λ on λ generators is the semigroup with zero given by the presentation:

$$P_\lambda=\left\langle \{p_i\}_{i\in\lambda},\{p_i^{-1}\}_{i\in\lambda}\mid p_ip_i^{-1}=1, p_ip_j^{-1}=0 \text{ for } i\neq j\right\rangle.$$

It is obvious that in the case when $\lambda=1$ the semigroup P_1 is isomorphic to the bicyclic semigroup with adjoined zero. For every finite non-zero cardinal $\lambda=n$ the polycyclic monoid P_n is a congruence free, combinatorial, 0-bisimple, 0-E-unitary inverse semigroup (see [32, Section 9.3]).

We recall that a topological space X is said to be:

- *compact* if each open cover of X has a finite subcover;
- *countably compact* if each open countable cover of X has a finite subcover;

- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;
- *feebly compact* if each locally finite open cover of X is finite.

According to Theorem 3.10.22 of [14], a Tychonoff topological space X is feebly compact if and only if each continuous real-valued function on X is bounded, i.e., X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space is countably compact, every countably compact space is countably pracompact, and every countably pracompact space is feebly compact (see [3] and [14]).

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S, τ) is a topological (inverse) semigroup, then we shall call τ a (*inverse*) *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{C}(p,q)$ is an open subset of S [13]. Bertman and West in [7] extended this result for the case of semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic semigroup [2, 30]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups discussed in [5, 6, 27]. In [13] Eberhart and Selden proved that if the bicyclic monoid $\mathscr{C}(p,q)$ is a dense subsemigroup of a topological monoid S and $I = S \setminus \mathscr{C}(p,q) \neq \varnothing$ then I is a two-sided ideal of the semigroup S. Also, there they described the closure the bicyclic monoid $\mathscr{C}(p,q)$ in a locally compact topological inverse semigroup. The closure of the bicyclic monoid in a countably compact (pseudocompact) topological semigroups was studied in [6].

In [15] Fihel and Gutik showed that any Hausdorff topology τ on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ such that $(\mathscr{C}_{\mathbb{Z}}, \tau)$ is a semitopological semigroup is discrete. Also in [15] studied a closure of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup.

For any Hausdorff topology τ on an infinite semigroup of $\lambda\times\lambda$-matrix units B_λ such that (B_λ, τ) is a semitopological semigroup every non-zero element of B_λ is an isolated point of (B_λ, τ) [22]. Also in [22] was proved that on any infinite semigroup of $\lambda\times\lambda$-matrix units B_λ there exists a unique feebly compact topology τ_A such that (B_λ, τ_A) is a semitopological semigroup and moroover this topology τ_A is compact. A closure of an infinite semigroup of $\lambda\times\lambda$-matrix units in semitopological and topological semigroups and its embeddings into compact-like semigroups were studied in [18, 22, 23].

Semigroup topologizations and closures of inverse semigroups of monotone co-finite partial bijections of some linearly ordered infinite sets, inverse semigroups of almost identity partial bijections and inverse semigroups of partial bijections of a bounded finite rank studied in [9, 10, 17, 20, 23, 24, 25, 28, 29].

To every directed graph E one can associate a graph inverse semigroup $G(E)$, where elements roughly correspond to possible paths in E. These semigroups generalize polycyclic monoids. In [33] the authors investigated topologies that turn $G(E)$ into a topological semigroup. For instance, they showed that in any such topology that is Hausdorff, $G(E) \setminus \{0\}$ must be discrete for any directed graph E. On the other hand, $G(E)$ need not be discrete in a Hausdorff semigroup topology, and for certain graphs E, $G(E)$ admits a T_1 semigroup topology in which $G(E) \setminus \{0\}$ is not discrete. In [33] the authors also described the algebraic structure and possible cardinality of the closure of $G(E)$ in larger topological semigroups.

In this paper we show that the λ-polycyclic monoid for in infinite cardinal $\lambda \geqslant 2$ has similar algebraic properties so has the polycyclic monoid P_n with finitely many $n \geqslant 2$ generators. In particular we prove that for every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free, combinatorial, 0-bisimple, 0-E-unitary inverse semigroup. Also we show that every non-zero element x is an isolated point in (P_λ, τ) for every Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup, and every locally compact Hausdorff semigroup topology on P_λ is discrete. The last statement extends results of the paper [33] obtaining for topological inverse graph semigroups. We describe all feebly compact topologies τ on P_λ such that (P_λ,τ) is a semitopological semigroup and its Bohr compactification as a topological semigroup. We prove that for every cardinal $\lambda\geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating and there exists no a Hausdorff feebly compact topological semigroup which contains P_λ as a dense subsemigroup.

2. Algebraic properties of the λ-polycyclic monoid for an infinite cardinal λ

In this section we assume that λ is an infinite cardinal.

We repeat the thinking and arguments from [32, Section 9.3].

We shall give a representation for the polycyclic monoid P_λ by means of partial bijections on the free monoid $\mathscr{M}_\lambda$ over the cardinal λ. Put $A=\{x_i\colon i\in\lambda\}$. Then the free monoid $\mathscr{M}_\lambda$ over the cardinal λ is isomorphic to the free monoid $\mathscr{M}_\lambda$ over the set A. Next we define for every $i\in\lambda$ the partial map $\alpha\colon \mathscr{M}_\lambda\to\mathscr{M}_\lambda$ by the formula $(u)\alpha_i=x_iu$ and put that $\mathscr{M}_\lambda$ is the domain and $x_i\mathscr{M}_\lambda$ is the range of α_i. Then for every $i\in\lambda$ we may regard so defined partial map as an element of the symmetric inverse monoid $\mathscr{I}(\mathscr{M}_\lambda)$ on the set $\mathscr{M}_\lambda$. Denote by I_λ the inverse submonoid of $\mathscr{I}(\mathscr{M}_\lambda)$ generated by the set $\{\alpha_i\colon i\in\lambda\}$. We observe that $\alpha_i\alpha_i^{-1}$ is the identity partial map on $\mathscr{M}_\lambda$ for each $i\in\lambda$ and whereas if $i\neq j$ then $\alpha_i\alpha_j^{-1}$ is the empty partial map on the set $\mathscr{M}_\lambda$, $i,j\in\lambda$. Define the map $h\colon P_\lambda\to I_\lambda$ by the formula $(p_i)h=\alpha_i$ and $(p_i^{-1})h=\alpha_i^{-1}$, $i\in\lambda$. Then by Proposition 2.3.5 of [32], I_λ is a homomorphic image of P_λ and by Proposition 9.3.1 from [32] the map $h\colon P_\lambda\to I_\lambda$ is an isomorphism. Since the band of the semigroup I_λ consists of partial identity maps, the identifying the semilattice of idempotents of I_λ with the free monoid $\mathscr{M}_\lambda^0$ with adjoined zero admits the following partial order on $\mathscr{M}_\lambda^0$:

(1) $\quad u\leqslant v$ if and only if v is a prefix of u for $u,v\in\mathscr{M}_\lambda^0$, and $0\leqslant u$ for every $u\in\mathscr{M}_\lambda^0$.

This partial order admits the following semilattice operation on $\mathscr{M}_\lambda^0$:

$$u*v=v*u=\begin{cases} u, & \text{if } v \text{ is a prefix of } u;\\ 0, & \text{otherwise,}\end{cases}$$

and $0*u=u*0=0*0=0$ for arbitrary words $u,v\in\mathscr{M}_\lambda^0$.

Remark 2.1. We observe that for an arbitrary non-zero cardinal λ the set $\mathscr{M}_\lambda^0\setminus\{0\}$ with the dual partial order to (1) is order isomorphic to the λ-ary tree T_λ with the countable height.

Hence, we proved the following proposition.

Proposition 2.2. *For every infinite cardinal λ the semigroup P_λ is isomorphic to the inverse semigroup I_λ and the semilattice $E(P_\lambda)$ is isomorphic to $(\mathscr{M}_\lambda^0,*)$.*

Let n be any positive integer and $i_1,\ldots,i_n\in\lambda$. We put

$$P_n^\lambda\left\langle i_1,\ldots,i_n\right\rangle=\left\langle p_{i_1},\ldots,p_{i_n},p_{i_1}^{-1},\ldots,p_{i_n}^{-1}\mid p_{i_k}p_{i_k}^{-1}=1, p_{i_k}p_{i_l}^{-1}=0 \text{ for } i_k\neq i_l\right\rangle.$$

The statement of the following lemma is trivial.

Lemma 2.3. *Let λ be an infinite cardinal and n be an arbitrary positive integer. Then $P_n^\lambda\left\langle i_1,\ldots,i_n\right\rangle$ is a submonoid of the polycyclic monoid P_λ such that $P_n^\lambda\left\langle i_1,\ldots,i_n\right\rangle$ is isomorphic to P_n for arbitrary $i_1,\ldots,i_n\in\lambda$.*

Our above representation of the polycyclic monoid P_λ by means of partial bijections on the free monoid $\mathscr{M}_\lambda$ over the cardinal λ implies the following lemma.

Lemma 2.4. *Let λ be an infinite cardinal. Then for any elements $x_1,\ldots,x_k\in P_\lambda$ there exist $i_1,\ldots,i_n\in\lambda$ such that $x_1,\ldots,x_k\in P_n^\lambda\left\langle i_1,\ldots,i_n\right\rangle$.*

Theorem 2.5. *For every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free combinatorial 0-bisimple 0-E-unitary inverse semigroup.*

Proof. By Proposition 2.2 the semigroup P_λ is inverse.

First we show that the semigroup P_λ is 0-bisimple. Then by the Munn Lemma (see [34, Lemma 1.1] and [32, Proposition 3.2.5]) it is sufficient to show that for any two non-zero idempotents $e,f\in P_\lambda$ there exists $x\in P_\lambda$ such that $xx^{-1}=e$ and $x^{-1}x=f$. Fix arbitrary two non-zero idempotents $e,f\in P_\lambda$. By Lemma 2.4 there exist $i_1,\ldots,i_n\in\lambda$ such that $e,f\in P_n^\lambda\langle i_1,\ldots,i_n\rangle$. Lemma 2.3, Theorem 9.3.4 of [32] and Proposition 3.2.5 of [32] imply that there exists $x\in P_n^\lambda\langle i_1,\ldots,i_n\rangle\subset P_\lambda$ such that $xx^{-1}=e$ and $x^{-1}x=f$. Hence the semigroup P_λ is 0-bisimple.

The above representation of the polycyclic monoid P_λ by means of partial bijections on the free monoid $\mathscr{M}_\lambda$ over the cardinal λ implies that the $\mathscr{H}$-class in P_λ which contains the unity is a singleton. Then since the polycyclic monoid P_λ is 0-bisimple Theorem 2.20 of [11] implies that every non-zero $\mathscr{H}$-class in P_λ is a singleton. It is obvious that $\mathscr{H}$-class in P_λ which contains zero is a singleton. This implies that the polycyclic monoid P_λ is combinatorial.

Suppose to the contrary that the monoid P_λ is not 0-E-unitary. Then there exist a non-idempotent element $x\in P_\lambda$ and non-zero idempotents $e,f\in P_\lambda$ such that $xe=f$. By Lemma 2.4 there exist $i_1,\ldots,i_n\in\lambda$ such that $x,e,f\in P_n^\lambda\langle i_1,\ldots,i_n\rangle$. Hence the monoid $P_n^\lambda\langle i_1,\ldots,i_n\rangle$ is not 0-E-unitary, which contradicts Lemma 2.3 and Theorem 9.3.4 of [32]. The obtained contradiction implies that the polycyclic monoid P_λ is a 0-E-unitary inverse semigroup.

Suppose the contrary that there exists a congruence $\mathfrak{C}$ on the polycyclic monoid P_λ which is distinct from the identity and the universal congruence on P_λ. Then there exist distinct $x,y\in P_\lambda$ such that $x\mathfrak{C}y$. By Lemma 2.4 there exist $i_1,\ldots,i_n\in\lambda$ such that $x,y\in P_n^\lambda\langle i_1,\ldots,i_n\rangle$. By Lemma 2.3 and Theorem 9.3.4 of [32], since the polycyclic monoid P_n is congruence-free we have that the unity and zero of the polycyclic monoid P_λ are $\mathfrak{C}$-equivalent and hence all elements of P_λ are $\mathfrak{C}$-equivalent. This contradicts our assumption. The obtained contradiction implies that the polycyclic monoid P_λ is a congruence-free semigroup. □

From now for an arbitrary cardinal $\lambda\geqslant 2$ we shall call the semigroup P_λ the λ-*polycyclic monoid*.

Fix an arbitrary cardinal $\lambda\geqslant 2$ and two distinct elements $a,b\in\lambda$. We consider the following subset $A=\{b^ia\colon i=0,1,2,3,\ldots\}$ of the free monoid $\mathscr{M}_\lambda$. The definition of the above defined partial order $\leqslant$ on $\mathscr{M}_\lambda^0$ implies that two arbitrary distinct elements of the set A are incomparable in $(\mathscr{M}_\lambda^0,\leqslant)$. Let $B(b^ia)$ be a subsemigroup of I_λ generated by the subset

$$\left\{\alpha\in I_\lambda\colon\ \operatorname{dom}\alpha=b^ia\mathscr{M}_\lambda \text{ and } \operatorname{ran}\alpha=b^ja\mathscr{M}_\lambda \text{ for some } i,j\in\omega\right\}$$

of the semigroup I_λ. Since two arbitrary distinct elements of the set A are incomparable in the partially ordered set $(\mathscr{M}_\lambda^0,\leqslant)$ the semigroup operation of I_λ implies that the following conditions hold:

(i) $\alpha\beta$ is a non-zero element of the semigroup I_λ if and only if $\operatorname{ran}\alpha=\operatorname{dom}\beta$;
(ii) $\alpha\beta=0$ in I_λ if and only if $\operatorname{ran}\alpha\neq\operatorname{dom}\beta$;
(iii) if $\alpha\beta\neq 0$ in I_λ then $\operatorname{dom}(\alpha\beta)=\operatorname{dom}\alpha$ and $\operatorname{ran}(\alpha\beta)=\operatorname{ran}\beta$;
(iv) $B(b^ia)$ is an inverse subsemigroup of I_λ,

for arbitrary $\alpha,\beta\in B(b^ia)$.

Now, if we identify ω with the set of all non-negative integers $\{0,1,2,3,4,\ldots\}$, then simple verifications show that the map $\mathfrak{h}\colon B(b^ia)\to B_\omega$ defined in the following way:

(a) if $\alpha\neq 0$, $\operatorname{dom}\alpha=b^ia\mathscr{M}_\lambda$ and $\operatorname{ran}\alpha=b^ja\mathscr{M}_\lambda$, then $(\alpha)\mathfrak{h}=(i,j)$, for $i,j\in\{0,1,2,3,4,\ldots\}$;
(b) $(0)\mathfrak{h}=0$,

is a semigroup isomorphism.

Hence we proved the following proposition.

Proposition 2.6. *For every cardinal $\lambda\geqslant 2$ the λ-polycyclic monoid P_λ contains an isomorphic copy of the semigroup of $\omega\times\omega$-matrix units B_ω.*

Proposition 2.7. *For every non-zero cardinal λ and any $\alpha,\beta\in P_\lambda\setminus\{0\}$, both sets $\{\chi\in P_\lambda\colon \alpha\cdot\chi=\beta\}$ and $\{\chi\in P_\lambda\colon \chi\cdot\alpha=\beta\}$ are finite.*

Proof. We show that the set $\{\chi \in P_\lambda \colon \alpha \cdot \chi = \beta\}$ is finite. The proof in other case is similar.

It is obvious that

$$\{\chi \in P_\lambda \colon \alpha \cdot \chi = \beta\} \subseteq \left\{\chi \in P_\lambda \colon \alpha^{-1} \cdot \alpha \cdot \chi = \alpha^{-1} \cdot \beta\right\}.$$

Then the definition of the semigroup I_λ implies there exist words $u, v \in \mathscr{M}_\lambda$ such that the partial map $\alpha^{-1} \cdot \beta$ is the map from $u\mathscr{M}_\lambda$ onto $v\mathscr{M}_\lambda$ defined by the formula $(ux)(\alpha^{-1} \cdot \beta) = vx$ for any $x \in \mathscr{M}_\lambda$. Since $\alpha^{-1} \cdot \alpha$ is an identity partial map of $\mathscr{M}_\lambda$ we get that the partial map $\alpha^{-1} \cdot \beta$ is a restriction of the partial map χ on the set $\operatorname{dom}(\alpha^{-1} \cdot \alpha)$. Hence by the definition of the semigroup I_λ there exists words $u_1, v_1 \in \mathscr{M}_\lambda$ such that u_1 is a prefix of u, v_1 is a prefix of v and χ is the map from $u_1\mathscr{M}_\lambda$ onto $v_1\mathscr{M}_\lambda$ defined by the formula $(u_1x)(\alpha^{-1} \cdot \beta) = v_1x$ for any $x \in \mathscr{M}_\lambda$. Now, since every word of free monoid $\mathscr{M}_\lambda$ has finitely many prefixes we conclude that the set $\{\chi \in P_\lambda \colon \alpha^{-1} \cdot \alpha \cdot \chi = \alpha^{-1} \cdot \beta\}$ is finite, and hence so is $\{\chi \in P_\lambda \colon \alpha \cdot \chi = \beta\}$. □

Later we need the following lemma.

Lemma 2.8. *Let λ be any cardinal $\geqslant 2$. Then an element x of the λ-polycyclic monoid P_λ is $\mathscr{R}$-equivalent to the identity 1 of P_λ if and only if $x = p_{i_1} \dots p_{i_n}$ for some generators $p_{i_1}, \dots, p_{i_n} \in \{p_i\}_{i \in \lambda}$.*

Proof. We observe that the definition of the $\mathscr{R}$-relation implies that $x\mathscr{R}1$ if and only if $xx^{-1} = 1$ (see [32, Section 3.2]).

($\Rightarrow$) Suppose that an element x of P_λ has a form $x = p_{i_1} \dots p_{i_n}$. Then the definition of the λ-polycyclic monoid P_λ implies that

$$xx^{-1} = (p_{i_1} \dots p_{i_n})\,(p_{i_1} \dots p_{i_n})^{-1} = p_{i_1} \dots p_{i_n} p_{i_n}^{-1} \dots p_{i_1}^{-1} = 1,$$

and hence $x\mathscr{R}1$.

($\Leftarrow$) Suppose that some element x of the λ-polycyclic monoid P_λ is $\mathscr{R}$-equivalent to the identity 1 of P_λ. Then the definition of the semigroup P_λ implies that there exist finitely many $p_{i_1}, \dots, p_{i_n} \in \{p_i\}_{i \in \lambda}$ such that x is an element of the submonoid $P_n^\lambda \langle i_1, \dots, i_n \rangle$ of P_λ, which is generated by elements $p_{i_1}, \dots, p_{i_n}$, i.e.,

$$P_n^\lambda \langle i_1, \dots, i_n \rangle = \left\langle p_{i_1}, \dots, p_{i_n}, p_{i_1}^{-1}, \dots, p_{i_n}^{-1} \colon p_{i_k} p_{i_k}^{-1} = 1,\ p_{i_k} p_{i_l}^{-1} = 0 \ \text{ for } \ i_k \neq i_l \right\rangle.$$

Proposition 9.3.1 of [32] implies that the element x is equal to the unique string of the form $u^{-1}v$, where u and v are strings of the free monoid $\mathscr{M}_{\{p_{i_1}, \dots, p_{i_n}\}}$ over the set $\{p_{i_1}, \dots, p_{i_n}\}$. Next we shall show that u is the empty string of $\mathscr{M}_{\{p_{i_1}, \dots, p_{i_n}\}}$. Suppose that $u = a_1 \dots a_k$ and $v = b_1 \dots b_l$, for some $a_1, \dots, a_k, b_1, \dots, b_l \in \{p_{i_1}, \dots, p_{i_n}\}$ and u is not the empty-string of $\mathscr{M}_{\{p_{i_1}, \dots, p_{i_n}\}}$. Then the definition of the λ-polycyclic monoid P_λ implies that

$$\begin{aligned}
xx^{-1} &= \left(u^{-1}v\right)\left(u^{-1}v\right)^{-1} = u^{-1}vv^{-1}u = \\
&= (a_1 \dots a_k)^{-1}\,(b_1 \dots b_l)\,(b_1 \dots b_l)^{-1}\,(a_1 \dots a_k) = \\
&= a_k^{-1} \dots a_1^{-1} b_1 \dots b_l b_l^{-1} \dots b_1^{-1} a_1 \dots a_k = \\
&= \dots = \\
&= a_k^{-1} \dots a_1^{-1} 1 a_1 \dots a_k = \\
&= a_k^{-1} \dots a_1^{-1} a_1 \dots a_k \neq 1,
\end{aligned}$$

which contradicts the assumption that $x\mathscr{R}1$. The obtained contradiction implies that the element x has the form $x = p_{i_1} \dots p_{i_n}$ for some generators $p_{i_1}, \dots, p_{i_n}$ from the set $\{p_i\}_{i \in \lambda}$. □

3. On semigroup topologizations of the λ-polycyclic monoid

In [13] Eberhart and Selden proved that if τ is a Hausdorff topology on the bicyclic monoid $\mathscr{C}(p,q)$ such that $(\mathscr{C}(p,q), \tau)$ is a topological semigroup then τ is discrete. In [7] Bertman and West extended this results for the case when $(\mathscr{C}(p,q), \tau)$ is a Hausdorff semitopological semigroup. In [33] there proved that for any positive integer $n > 1$ every non-zero element in a Hausdorff topological n-polycyclic monoid P_n is an isolated point. The following proposition generalizes the above results.

Proposition 3.1. *Let λ be any cardinal $\geqslant 2$ and τ be any Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup. Then every non-zero element x is an isolated point in (P_λ, τ).*

Proof. We observe that the λ-polycyclic monoid P_λ is a 0-bisimple semigroup, and hence is a 0-simple semigroup. Then the continuity of right and left translations in (P_λ, τ) and Proposition 2.7 imply that it is complete to show that there exists an non-zero element x of P_λ such that x is an isolated point in the topological space (P_λ, τ).

Suppose to the contrary that the unit 1 of the λ-polycyclic monoid P_λ is a non-isolated point of the topological space (P_λ, τ). Then every open neighbourhood $U(1)$ of 1 in (P_λ, τ) is infinite subset.

Fix a singleton word x in the free monoid $\mathscr{M}_\lambda$. Let ε be an idempotent of the λ-polycyclic monoid P_λ which corresponds to the identity partial map of $x\mathscr{M}_\lambda$. Since left and right translation on the idempotent ε are retractions of the topological space (P_λ, τ) the Hausdorffness of (P_λ, τ) implies that εP_λ and $P_\lambda \varepsilon$ are closed subsets of the topological space (P_λ, τ), and hence so is the set $\varepsilon P_\lambda \cup P_\lambda \varepsilon$. The separate continuity of the semigroup operation and Hausdorffness of (P_λ, τ) imply that for every open neighbourhood $U(\varepsilon) \not\ni 0$ of the point ε in (P_λ, τ) there exists an open neighbourhood $U(1)$ of the unit 1 in (P_λ, τ) such that

$$U(1) \subseteq P_\lambda \setminus (\varepsilon P_\lambda \cup P_\lambda \varepsilon), \qquad \varepsilon \cdot U(1) \subseteq U(\varepsilon) \qquad \text{and} \qquad U(1) \cdot \varepsilon \subseteq U(\varepsilon).$$

We observe that the idempotent ε is maximal in $P_\lambda \setminus \{1\}$. Hence any other idempotent $\iota \in P_\lambda \setminus (\varepsilon P_\lambda \cup P_\lambda \varepsilon)$ is incomparable with ε. Since the set $U(1)$ is infinite there exists an element $\alpha \in U(1)$ such that either $\alpha \cdot \alpha^{-1}$ or $\alpha^{-1} \cdot \alpha$ is an incomparable idempotent with ε. Then we get that either

$$\varepsilon \cdot \alpha = \varepsilon \cdot (\alpha \cdot \alpha^{-1} \cdot \alpha) = (\varepsilon \cdot \alpha \cdot \alpha^{-1}) \cdot \alpha = 0 \cdot \alpha = 0 \in U(\varepsilon)$$

or

$$\alpha \cdot \varepsilon = (\alpha \cdot \alpha^{-1} \cdot \alpha) \cdot \varepsilon = \alpha \cdot (\alpha^{-1} \cdot \alpha \cdot \varepsilon) = \alpha \cdot 0 = 0 \in U(\varepsilon).$$

The obtained contradiction implies that the unit 1 is an isolated point of the topological space (P_λ, τ), which completes the proof of our proposition. □

A topological space X is called *collectionwise normal* if X is T_1-space and for every discrete family $\{F_\alpha\}_{\alpha \in \mathscr{J}}$ of closed subsets of X there exists a discrete family $\{S_\alpha\}_{\alpha \in \mathscr{J}}$ of open subsets of X such that $F_\alpha \subseteq S_\alpha$ for every $\alpha \in \mathscr{J}$ [14].

Proposition 3.2. *Every Hausdorff topological space X with a unique non-isoloated point is collectionwise normal.*

Proof. Suppose that a is a non-isolated point of X. Fix an arbitrary discrete family $\{F_\alpha\}_{\alpha \in \mathscr{J}}$ of closed subsets of the topological space X. Then there exists an open neighbourhood $U(a)$ of the point a in X which intersects at most one element of the family $\{F_\alpha\}_{\alpha \in \mathscr{J}}$. In the case when $U(a) \cap F_\alpha = \varnothing$ for every $\alpha \in \mathscr{J}$ we put $S_\alpha = F_\alpha$ for all $\alpha \in \mathscr{J}$. If $U(a) \cap F_{\alpha_0} \neq \varnothing$ for some $\alpha_0 \in \mathscr{J}$ we put $S_{\alpha_0} = U(a) \cup F_{\alpha_0}$ and $S_\alpha = F_\alpha$ for all $\alpha \in \mathscr{J} \setminus \{\alpha_0\}$. Then $\{S_\alpha\}_{\alpha \in \mathscr{J}}$ is a discrete family of open subsets of X such that $F_\alpha \subseteq S_\alpha$ for every $\alpha \in \mathscr{J}$. □

Propositions 3.1 and 3.2 imply the following corollary.

Corollary 3.3. *Let λ be any cardinal $\geqslant 2$ and τ be any Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup. Then the topological space (P_λ, τ) is collectionwise normal.*

In [33] there proved that for arbitrary finite cardinal $\geqslant 2$ every Hausdorff locally compact topology τ on P_λ such that (P_λ, τ) is a topological semigroup, is discrete. The following proposition extends this result for any infinite cardinal λ.

Proposition 3.4. *Let λ be an infinite cardinal and τ be a locally compact Hausdorff topology on P_λ such that (P_λ, τ) is a topological semigroup. Then τ is discrete.*

Proof. Suppose to the contrary that there exist a Hausdorff locally compact non-discrete semigroup topology τ on P_λ. Then by Proposition 3.1 every non-zero element the semigroup P_λ is an isolated point in (P_λ,τ). This implies that for any compact open neighbourhoods $U(0)$ and $V(0)$ of zero 0 in (P_λ,τ) the set $U(0)\setminus V(0)$ is finite. Hence zero 0 of P_λ is an accumulation point of any infinite subset of an arbitrary open compact neighbourhood $U(0)$ of zero in (P_λ,τ).

Put R_1 is the $\mathscr{R}$-class of the semigroup P_λ which contains the identity 1 of P_λ. Then only one of the following conditions holds:

(1) there exists a compact open neighbourhood $U(0)$ of zero 0 in (P_λ,τ) such that $U(0)\cap R_1=\varnothing$;
(2) $U(0)\cap R_1$ is an infinite set for every compact open neighbourhood $U(0)$ of zero 0 in (P_λ,τ).

Suppose that case (1) holds. For arbitrary $x\in R_1$ we put

$$R[x]=\left\{a\in R_1\colon x^{-1}a\in U(0)\right\}.$$

Next we shall show that the set $R[x]$ is finite for any $x\in R_1$. Suppose to the contrary that $R[x]$ is infinite for some $x\in R_1$. Then Lemma 2.8 implies that $x^{-1}a$ is non-zero element of P_λ for every $a\in R[x]$, and hence by Proposition 2.7,

$$B=\left\{x^{-1}a\colon a\in R[x]\right\}$$

is an infinite subset of the neighbourhood $U(0)$. Therefore, the above arguments imply that $0\in\operatorname{cl}_{P_\lambda}(B)$. Now, the continuity of the semigroup operation in (P_λ,τ) implies that

$$0=x\cdot 0\in x\cdot\operatorname{cl}_{P_\lambda}(B)\subseteq\operatorname{cl}_{P_\lambda}(x\cdot B).$$

Then Lemma 2.8 implies that $xx^{-1}=1$ for any $x\in R_1$ and hence we have that

$$x\cdot B=\left\{xx^{-1}a\colon a\in R[x]\right\}=\{a\colon a\in R[x]\}=R[x]\subseteq R_1.$$

This implies that every open neighbourhood $U(0)$ of zero 0 in (P_λ,τ) contains infinitely many elements from the class R_1, which contradicts our assumption.

Suppose that case (2) holds. Then the set $\{0\}$ is a compact minimal ideal of the topological semigroup (P_λ,τ). Now, by Lemma 1 of [31] (also see [8, Vol. 1, Lemma 3,12]) for every open neighbourhood $W(0)$ of zero 0 in (P_λ,τ) there exists an open neighbourhood $O(0)$ of zero 0 in (P_λ,τ) such that $O(0)\subseteq W(0)$ and $O(0)$ is an ideal of $\operatorname{cl}_{P_\lambda}(O(0))$, i.e., $O(0)\cdot\operatorname{cl}_{P_\lambda}(O(0))\cup\operatorname{cl}_{P_\lambda}(O(0))\cdot O(0)\subseteq O(0)$. But by Proposition 3.1 all non-zero elements of P_λ are isolated points in (P_λ,τ), and hence we have that $\operatorname{cl}_{P_\lambda}(O(0))=O(0)$. This implies that $O(0)$ is an open-and-closed subsemigroup of the topological semigroup (P_λ,τ). Therefore, the topological λ-polycyclic monoid (P_λ,τ) has a base $\mathscr{B}(0)$ at zero 0 which consists of open-and-closed subsemigroups of (P_λ,τ). Fix an arbitrary $S\in\mathscr{B}(0)$. Then our assumption implies that there exists $x\in S\cap R_1$. Since $x\in R_1$, Lemma 2.8 implies that $xx^{-1}=1$. Without loss of generality we may assume that $x^{-1}x\neq 1$, because S is a proper ideal of P_λ. Put $\mathbb{B}(x)=\langle x,x^{-1}\rangle$. Then Lemma 1.31 of [11] implies that $\mathbb{B}(x)$ is isomorphic to the bicyclic monoid, and since by Proposition 3.1 all non-zero elements of P_λ are isolated points in (P_λ,τ), $\mathbb{B}^0(x)=\mathbb{B}(x)\sqcup\{0\}$ is a closed subsemigroup of the topological semigroup (P_λ,τ), and hence by Corollary 3.3.10 of [14], $\mathbb{B}^0(x)$ with the induced topology $\tau_{\mathbb{B}}$ from (P_λ,τ) is a Hausdorff locally compact topological semigroup. Also, the above presented arguments imply that $\langle x\rangle\cup\{0\}$ with the induced topology from (P_λ,τ) is a compact topological semigroup, which is contained in $\mathbb{B}^0(x)$ as a subsemigroup. But by Corollary 1 from [19], $(\mathbb{B}^0(x),\tau_{\mathbb{B}})$ is the discrete space, which contains a compact infinite subspace $\langle x\rangle\cup\{0\}$. Hence case (2) does not hold.

The presented above arguments imply that there exists no non-discrete Hausdorff locally compact semigroup topology on the λ-polycyclic monoid P_λ. □

The following example shows that the statements of Proposition 3.4 does not extend in the case when (P_λ,τ) is a semitopological semigroup with continuous inversion. Moreover there exists a compact Hausdorff topology $\tau_{\mathsf{A\text{-}c}}$ on P_λ such that $(P_\lambda,\tau_{\mathsf{A\text{-}c}})$ is semitopological inverse semigroup with continuous inversion.

Example 3.5. Let λ is any cardinal $\geqslant 2$. Put $\tau_{\mathsf{A\text{-}c}}$ is the topology of the one-point Alexandroff compactification of the discrete space $P_\lambda \setminus \{0\}$ with the narrow $\{0\}$, where 0 is the zero of the λ-polycyclic monoid P_λ. Since $P_\lambda \setminus \{0\}$ is a discrete open subspace of $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$, it is complete to show that the semigroup operation is separately continuous in $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$ in the following two cases:

$$x \cdot 0 \qquad \text{and} \qquad 0 \cdot x,$$

where x is an arbitrary non-zero element of the semigroup P_λ. Fix an arbitrary open neighbourhood $U_A(0)$ of the zero in $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$ such that $A = P_\lambda \setminus U_A(0)$ is a finite subset of P_λ. By Proposition 2.7,

$$R_x^A = \{a \in P_\lambda \colon x \cdot a \in A\} \qquad \text{and} \qquad L_x^A = \{a \in P_\lambda \colon a \cdot x \in A\}$$

are finite not necessary non-empty subsets of the semigroup P_λ. Put $U_{R_x^A}(0) = P_\lambda \setminus R_x^A$, $U_{L_x^A}(0) = P_\lambda \setminus L_x^A$ and $U_{A^{-1}} = P_\lambda \setminus \{a \colon a^{-1} \in A\}$. Then we get that

$$x \cdot U_{R_x^A}(0) \subseteq U_A(0), \qquad U_{L_x^A}(0) \cdot x \subseteq U_A(0) \qquad \text{and} \qquad (U_{A^{-1}})^{-1} \subseteq U_A(0),$$

and hence the semigroup operation is separately continuous and the inversion is continuous in $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$.

Proposition 3.6. *Let λ is any cardinal $\geqslant 2$ and τ be a Hausdorff topology on P_λ such that (P_λ, τ) is a semitopological semigroup. Then the following conditions are equivalent:*

(i) $\tau = \tau_{\mathsf{A\text{-}c}}$;
(ii) (P_λ, τ) *is a compact semitopological semigroup;*
(iii) (P_λ, τ) *is a feebly compact semitopological semigroup.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial and implication $(ii) \Rightarrow (i)$ follows from Proposition 3.1.

$(iii) \Rightarrow (ii)$ Suppose there exists a feebly compact Hausdorff topology τ on P_λ such that (P_λ, τ) is a non-compact semitopological semigroup. Then there exists an open cover $\{U_\alpha\}_{\alpha \in \mathscr{J}}$ which does not contain a finite subcover. Let U_{α_0} be an arbitrary element of the family $\{U_\alpha\}_{\alpha \in \mathscr{J}}$ which contains zero 0 of the semigroup P_λ. Then $P_\lambda \setminus U_{\alpha_0} = A_{U_{\alpha_0}}$ is an infinite subset of P_λ. By Proposition 3.1, $\{U_{\alpha_0}\} \cup \{\{x\} \colon x \in A_{U_{\alpha_0}}\}$ is an infinite locally finite family of open subset of the topological space (P_λ, τ), which contradicts that the space (P_λ, τ) is feebly compact. The obtained contradiction implies the requested implication. □

It is well known that the closure $\operatorname{cl}_S(T)$ of an arbitrary subsemigroup T in a semitopological semigroup S again is a subsemigroup of S (see [37, Proposition I.1.8(ii)]). The following proposition describes the structure of a narrow of the λ-polycyclic monoid P_λ in a semitopological semigroup.

Proposition 3.7. *Let λ is any cardinal $\geqslant 2$, S be a Hausdorff semitopological semigroup and P_λ is a dense subsemigroup of S. Then $S \setminus P_\lambda \cup \{0\}$ is a closed ideal of S.*

Proof. First we observe by Proposition I.1.8(iii) from [37] the zero 0 of the λ-polycyclic monoid P_λ is a zero of the semitopological semigroup S. Hence the statement of the proposition is trivial when $S \setminus P_\lambda = \varnothing$.

Assume that $S \setminus P_\lambda \neq \varnothing$. Put $I = S \setminus P_\lambda \cup \{0\}$. By Theorem 3.3.9 of [14], I is a closed subspace of S. Suppose to the contrary that I is not an ideal of S. If $I \cdot S \nsubseteq I$ then there exist $x \in I \setminus \{0\}$ and $y \in P_\lambda \setminus \{0\}$ such that $x \cdot y = z \in P_\lambda \setminus \{0\}$. By Theorem 3.3.9 of [14], y and z are isolated points of the topological space S. Then the separate continuity of the semigroup operation in S implies that there exists an open neighbourhood $U(x)$ of the point x in S such that $U(x) \cdot \{y\} = \{z\}$. Then we get that $|U(x) \cap P_\lambda| \geqslant \omega$ which contradicts Proposition 2.7. The obtained contradiction implies the inclusion $I \cdot S \subseteq I$. The proof of the inclusion $S \cdot I \subseteq I$ is similar.

Now we shall show that $I \cdot I \subseteq I$. Suppose to the contrary that there exist $x, y \in I \setminus \{0\}$ such that $x \cdot y = z \in P_\lambda \setminus \{0\}$. By Theorem 3.3.9 of [14], z is an isolated point of the topological space S. Then the separate continuity of the semigroup operation in S implies that there exists an open neighbourhood $U(x)$ of the point x in S such that $U(x) \cdot \{y\} = \{z\}$. Since $|U(x) \cap P_\lambda| \geqslant \omega$ there exists $a \in P_\lambda \setminus \{0\}$

such that $a \cdot y \in a \cdot I \nsubseteq I$ which contradicts the above part of our proof. The obtained contradiction implies the statement of the proposition. □

4. Embeddings of the λ-polycyclic monoid into compact-like topological semigroups

By Theorem 5 of [23] the semigroup of $\omega\times\omega$-matrix units does not embed into any countably compact topological semigroup. Then by Proposition 2.6 we have that for every cardinal $\lambda \geqslant 2$ the λ-polycyclic monoid P_λ does not embed into any countably compact topological semigroup too.

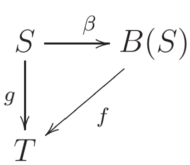
A homomorphism $\mathfrak{h}$ from a semigroup S into a semigroup T is called *annihilating* if there exists $c \in T$ such that $(s)\mathfrak{h} = c$ for all $s \in S$. By Theorem 6 of [23] every continuous homomorphism from the semigroup of $\omega\times\omega$-matrix units into an arbitrary countably compact topological semigroup is annihilating. Then since by Theorem 2.5 the semigroup P_λ is congruence-free Theorem 6 of [23] and Theorem 2.5 imply the following corollary.

Corollary 4.1. *For every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating.*

Proposition 4.2. *For every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into a topological semigroup S such that $S \times S$ is a Tychonoff pseudocompact space is annihilating, and hence S does not contain the λ-polycyclic monoid P_λ.*

Proof. First we shall show that S does not contain the λ-polycyclic monoid P_λ. By [4, Theorem 1.3] for any topological semigroup S with the pseudocompact square $S\times S$ the semigroup operation $\mu\colon S\times S \to S$ extends to a continuous semigroup operation $\beta\mu\colon \beta S \times \beta S \to \beta S$, so S is a subsemigroup of the compact topological semigroup βS. Therefore the λ-polycyclic monoid P_λ is a subsemigroup of compact topological semigroup βS which contradicts Corollary 4.1. The first statement of the proposition implies from the statement that P_λ is a congruence-free semigroup. □

Recall [12] that a *Bohr compactification of a topological semigroup* S is a pair $(\beta, B(S))$ such that $B(S)$ is a compact topological semigroup, $\beta\colon S \to B(S)$ is a continuous homomorphism, and if $g\colon S \to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f\colon B(S) \to T$ such that the diagram



commutes.

By Theorem 2.5 for every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free inverse semigroup and hence Corollary 4.1 implies the following corollary.

Corollary 4.3. *For every cardinal $\lambda \geqslant 2$ the Bohr compactification of a topological λ-polycyclic monoid P_λ is a trivial semigroup.*

The following theorem generalized Theorem 5 from [23].

Theorem 4.4. *For every infinite cardinal λ the semigroup of $\lambda\times\lambda$-matrix units B_λ does not densely embed into a Hausdorff feebly compact topological semigroup.*

Proof. Suppose to the contrary that there exists a Hausdorff feebly compact topological semigroup S which contains the semigroup of $\lambda\times\lambda$-matrix units B_λ as a dense subsemigroup.

First we shall show that the subsemigroup of idempotents $E(B_\lambda)$ of the semigroup $\lambda\times\lambda$-matrix units B_λ with the induced topology from S is compact. Suppose to the contrary that $E(B_\lambda)$ is not a compact subspace of S. Then there exists an open neighbourhood $U(0)$ of the zero 0 of S such that $E(B_\lambda)\setminus U(0)$ is an infinite subset of $E(B_\lambda)$. Since the closure of semilattice in a topological semigroup is subsemilattice (see [21, Corollary 19]) and every maximal chain of $E(B_\lambda)$ is finite, Theorem 9 of [38] implies that the band $E(B_\lambda)$ is a closed subsemigroup of S. Now, by Lemma 2 from [22] every non-zero element of the semigroup B_λ is an isolated point in the space S, and hence by Theorem 3.3.9 of [14], $B_\lambda \setminus \{0\}$ is an open discrete subspace of the topological space S. Therefore we get that $E(B_\lambda) \setminus U(0)$ is an infinite open-and-closed discrete subspace of S. This contradicts the condition that S is a feebly compact space.

If the subsemigroup of idempotents $E(B_\lambda)$ is compact then by Theorem 1 from [23] the semigroup of $\lambda\times\lambda$-matrix units B_λ is closed subsemigroup of S and since B_λ is dense in S, the semigroup B_λ coincides with the topological semigroup S. This contradicts Theorem 2 of [22] which states that there exists no a feebly compact Hausdorff topology τ on the semigroup of $\lambda\times\lambda$-matrix units B_λ such that (B_λ, τ) is a topological semigroup. The obtained contradiction implies the statement of the theorem. □

Lemma 4.5. *Every Hausdorff feebly compact topological space with a dense discrete subspace is countably pracompact.*

Proof. Suppose to the contrary that there exists a feebly compact topological space X with a dense discrete subspace D such that X is not countably pracompact. Then every dense subset A in the topological space X contains an infinite subset B_A such that B_A hasn't an accumulation point in X. Hence the dense discrete subspace D of X contains an infinite subset B_D such that B_D hasn't an accumulation point in the topological space X. Then B_D is a closed subset of X. By Theorem 3.3.9 of [14], D is an open subspace of X, and hence we have that B_D is a closed-and-open discrete subspace of the space X, which contradicts the feeble compactness of the space S. The obtained contradiction implies the statement of the lemma. □

Theorem 4.6. *For arbitrary cardinal $\lambda \geqslant 2$ there exists no Hausdorff feebly compact topological semigroup which contains the λ-polycyclic monoid P_λ as a dense subsemigroup.*

Proof. By Proposition 3.1 and Lemma 4.5 it is suffices to show that there does not exist a Hausdorff countably pracompact topological semigroup which contains the λ-polycyclic monoid P_λ as a dense subsemigroup.

Suppose to the contrary that there exists a Hausdorff countably pracompact topological semigroup S which contains the λ-polycyclic monoid P_λ as a dense subsemigroup. Then there exists a dense subset A in S such that every infinite subset $B \subseteq A$ has an accumulation point in the topological space S. By Proposition 3.1, $P_\lambda \setminus \{0\}$ is a discrete dense subspace of S and hence Theorem 3.3.9 of [14] implies that $P_\lambda \setminus \{0\}$ is an open subspace of S. Therefore we have that $P_\lambda \setminus \{0\} \subseteq A$. Now, by Proposition 2.6 the λ-polycyclic monoid P_λ contains an isomorphic copy of the semigroup of $\omega\times\omega$-matrix units B_ω. Then the countable pracompactness of the space S implies that every infinite subset C of the set $B_\omega\{0\}$ has an accumulating point in X, and hence the closure $\operatorname{cl}_S(B_\omega)$ is a countably pracompact subsemigroup of the topological semigroup S. This contradicts Theorem 4.4. The obtained contradiction implies the statement of the theorem. □

Acknowledgements

We acknowledge Alex Ravsky for his comments and suggestions.

References

[1] O. Andersen, *Ein Bericht über die Struktur abstrakter Halbgruppen*, PhD Thesis, Hamburg, 1952.
[2] L.W. Anderson, R.P. Hunter, R.J. Koch, *Some results on stability in semigroups*, Trans. Amer. Math. Soc. **117** (1965), 521–529.
[3] A. V. Arkhangel'skii, *Topological Function Spaces*, Kluwer Publ., Dordrecht, 1992.
[4] T. O. Banakh and S. Dimitrova *Openly factorizable spaces and compact extensions of topological semigroups*, Commentat. Math. Univ. Carol. **51**:1 (2010), 113–131.
[5] T. Banakh, S. Dimitrova, and O. Gutik, *The Rees-Suschkiewitsch Theorem for simple topological semigroups*, Mat. Stud. **31**:2 (2009), 211–218.
[6] T. Banakh, S. Dimitrova, and O. Gutik, *Embedding the bicyclic semigroup into countably compact topological semigroups*, Topology Appl. **157**:18 (2010), 2803–2814.

[7] M. O. Bertman and T. T. West, *Conditionally compact bicyclic semitopological semigroups*, Proc. Roy. Irish Acad. **A76**:21–23 (1976), 219–226.

[8] J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The Theory of Topological Semigroups*, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.

[9] I. Chuchman and O. Gutik, *Topological monoids of almost monotone injective co-finite partial selfmaps of the set of positive integers*, Carpathian Math. Publ. **2**:1 (2010), 119–132.

[10] I. Chuchman and O. Gutik, *On monoids of injective partial selfmaps almost everywhere the identity*, Demonstr. Math. **44**:4 (2011), 699–722.

[11] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vols. I and II, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961 and 1967.

[12] K. DeLeeuw and I. Glicksberg, *Almost-periodic functions on semigroups*, Acta Math. **105** (1961), 99–140.

[13] C. Eberhart and J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. **144** (1969), 115–126.

[14] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.

[15] I. R. Fihel and O. V. Gutik, *On the closure of the extended bicyclic semigroup*, Carpathian Math. Publ. **3**:2, (2011) 131–157.

[16] J. A. Green, *On the structure of semigroups*, Ann. Math. (2) **54** (1951), 163-172.

[17] I. Guran, O. Gutik, O. Ravskyj, and I. Chuchman, *Symmetric topological groups and semigroups*, Visn. Lviv. Univ., Ser. Mekh.-Mat. **74** (2011), 61–73.

[18] O. Gutik, *On closures in semitopological inverse semigroups with continuous inversion*, Algebra Discr. Math. **18**:1 (2014), 59-85.

[19] O. Gutik, *On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero*, Visn. Lviv. Univ. Ser. Mech. Math. (2015), (to appear) (arXiv:1509.02148).

[20] O. Gutik, J. Lawson, and D. Repovs, *Semigroup closures of finite rank symmetric inverse semigroups*, Semigroup Forum **78**:2 (2009), 326–336.

[21] O. V. Gutik and K. P. Pavlyk, *Topological Brandt λ-extensions of absolutely H-closed topological inverse semigroups*, Visnyk Lviv Univ., Ser. Mekh.-Math. **61** (2003), 98–105.

[22] O. V. Gutik and K. P. Pavlyk, *Topological semigroups of matrix units*, Algebra Discrete Math. no. **3** (2005), 1–17.

[23] O. Gutik, K. Pavlyk, and A. Reiter, Topological semigroups of matrix units and countably compact Brandt λ^0-extensions, Mat. Stud. **32**:2 (2009), 115–131.

[24] O. Gutik and I. Pozdnyakova, *On monoids of monotone injective partial selfmaps of $L_n \times_{\mathrm{lex}} \mathbb{Z}$ with co-finite domains and images*, Algebra Discrete Math. **17**:2 (2014), 256–279.

[25] O. V. Gutik and A. R. Reiter, *Symmetric inverse topological semigroups of finite* rank $\leqslant n$, Math. Methods and Phys.-Mech. Fields **52**:3 (2009), 7–14; reprinted version: J. Math. Sc. **171**:4 (2010), 425–432.

[26] O. Gutik and A. Reiter, *On semitopological symmetric inverse semigroups of a bounded finite rank*, Visnyk Lviv Univ. Ser. Mech. Math. **72** (2010), 94–106 (in Ukrainian).

[27] O. Gutik and D. Repovš, *On countably compact* 0*-simple topological inverse semigroups*, Semigroup Forum **75**:2 (2007), 464–469.

[28] O. Gutik and D. Repovš, *Topological monoids of monotone injective partial selfmaps of $\mathbb{N}$ with cofinite domain and image*, Stud. Sci. Math. Hung. **48**:3 (2011), 342–353.

[29] O. Gutik and D. Repovš, *On monoids of monotone injective partial selfmaps of integers with cofinite domains and images*, Georgian Math. J. **19**:3 (2012), 511–532.

[30] J. A. Hildebrant and R. J. Koch, *Swelling actions of Γ-compact semigroups*, Semigroup Forum **33**:1 (1986), 65–85.

[31] R. J. Koch, *On monothetic semigroups*, Proc. Amer. Mathe Soc. **8** (1957), 397–401.

[32] M. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries*, Singapore: World Scientific, 1998.

[33] Z. Mesyan, J. D. Mitchell, M. Morayne, and Y. H. Péresse, *Topological graph inverse semigroups*, Preprint (arXiv:1306.5388).

[34] W. D. Munn, *Uniform semilattices and bisimple inverse semigroups*, Quart. J. Math. **17**:1 (1966), 151–159.

[35] M. Nivat and J.-F. Perrot, *Une généralisation du monoide bicyclique*, C. R. Acad. Sci., Paris, Sér. A **271** (1970), 824–827.

[36] M. Petrich, *Inverse Semigroups*, John Wiley & Sons, New York, 1984.

[37] W. Ruppert, *Compact Semitopological Semigroups: An Intrinsic Theory*, Lect. Notes Math., **1079**, Springer, Berlin, 1984.

[38] J. W. Stepp, *Algebraic maximal semilattices.* Pacific J. Math. **58**:1 (1975), 243–248.

Faculty of Mathematics, National University of Lviv, Universytetska 1, Lviv, 79000, Ukraine
E-mail address: sbardyla@yahoo.com, o_gutik@franko.lviv.ua, ovgutik@yahoo.com