ALGEBRAIC RELATIONS, TAYLOR COEFFICIENTS OF HYPERLOGARITHMS AND IMAGES BY FROBENIUS - II

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Abstract. In part I, we defined and studied the algebraic properties of a "prime multiple harmonic sum motive" $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}$ and its periods.
Here, we study their relationships with the usual hyperlogarithm motives and periods, and their "finite" variant.
One of the results provides a p-adic lift of the congruence $\sum_{0<n<p} n^s \equiv 0 \mod p$ if $p-1 \nmid s$. Another one concerns a question of Deligne and Goncharov on how to read explicitly the series shuffle relation on p-adic multiple zeta values.
On the other hand, we interpret some of the information on the valuation on multiple harmonic sums in terms of these objects.
The last generic subject of this paper is the definition of the "Taylor period map", which we have delayed in part I. We state it and we see that it englobes questions on lifts of congruences and of the question to find a motivic analogue to some of the information on the valuation of multiple harmonic sums.

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1. Introduction

We give a short introduction to this paper in §1.1 ; we explain the heuristical situation in §1.2 ; we recall some notations in §1.3 ; we give an outline in §1.4.

1.1. **Context.**

1.1.1. *General framework.* Let Z be a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$, containing $\{0,1,\infty\}$; we denote the elements of $Z-\{0,1,\infty\}$ by $z_1,\ldots,z_r$, where $r\in\mathbb{N}$, and $z_0=0$, $z_{r+1}=1$. Our framework is mostly the pro-unipotent fundamental group of the variety $\mathbb{P}^1-Z$ over the number field $\mathbb{Q}(z_1,\ldots,z_r)$. We are interested in its periods. The most usual ones are hyperlogarithms, and, in the case of $\mathbb{P}^1-\{0,1,\infty\}$, multiple zeta values. Hyperlogarithms are iterated integrals of the form $\mathrm{Li}\left(\begin{array}{c} z_{i_d},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)(z) = \int_0^z \omega_{y_n}(t_n)\int_0^{t_n}\ldots\int_0^{t_3}\omega_{y_2}(t_2)\int_0^{t_2}\omega_{y_1}(t_1)$ where $\omega_{y_i}=\frac{dz}{z-y_i}$ and $(y_n,\ldots,y_1) = (\overbrace{0,\ldots,0}^{s_d-1},z_{i_d},\ldots\overbrace{0,\ldots,0}^{s_1-1},z_{i_1})$, with $i_1,\ldots,i_d\in\{1,\ldots,r\}$. We will consider their values at points $z\in Z-\{0,\infty\}$ - but note that we have, for compatible choices of paths, $\mathrm{Li}\left(\begin{array}{c} z_{i_d},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)(z) = \mathrm{Li}\left(\begin{array}{c} z_{i_d}/z,\ldots,z_{i_1}/z \\ s_d,\ldots,s_1 \end{array}\right)(1)$. In the complex case, we choose the straight path from 0 to 1. When $z_{i_1}=\ldots=z_{i_d}=z=1$, the numbers obtained are multiple zeta values $\zeta(s_d,\ldots,s_1)\in\mathbb{R}$.
The common way to formulate their algebraic theory is via :
i) the existence of motives called "motivic hyperlogarithms", "motivic multiple zeta values", forming $\mathbb{Q}$-algebras equipped with period morphisms onto the corresponding $\mathbb{Q}$-algebras of periods, which are conjecturally isomorphisms. This is a powerful tool having striking arithmetic applications, given the rich structure that motives have.
ii) some standard families of algebraic relations, between both the periods and their motivic lifts, which conjecturally generate all the existing algebraic relations among them.

1.1.2. *Review of part I.* In part I, we built a variant of this setting. Let multiple harmonic sums be the following numbers :

$$H_n\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right) = \sum_{0<n_1<\ldots<n_r<n} \frac{\left(z_{i_2}/z_{i_1}\right)^{n_1}\ldots\left(z_{i_{d+1}}/z_{i_d}\right)^{n_d}\left(1/z_{i_{d+1}}\right)^{n}}{n_1^{s_1}\ldots n_d^{s_d}} \in \overline{\mathbb{Q}}$$

where $d\in\mathbb{N}^*$, $i_1,\ldots,i_{d+1}\in\{1,\ldots,r\}$, $s_d,\ldots,s_1\in\mathbb{N}^*$, and $n\in\mathbb{N}^*$.
We call "prime" multiple harmonic sums those whose upper bound n is a power of a prime number.
In part I, we defined, and studied as a main topic a "prime multiple harmonic sum motive". For each index $\tilde{w}=\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)$, it is an element

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}]$$

of the weight-adic completion of the Hopf algebra of motivic hyperlogarithms.
It has, as what we call a "Taylor period" (we will study intrinsically this notion later),

the sequence :

$$(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}] = \Big((p^k)^{\mathrm{weight}} H_{p^k}(\tilde{w}) \Big)_p \in \Big(\prod_{p \text{ prime}} \overline{\mathbb{Q}_p} \Big)$$

where k is any element of $\mathbb{N}^*$, arising as the power of Frobenius. This motive also has "formal complex" and "formal p-adic" periods $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{(\mathcal{Z}/\zeta(2)\mathcal{Z})[[\Lambda]]}[\tilde{w}]$, respectively $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{Z}_p[[\Lambda]]}[\tilde{w}]$ in formal power series rings

$$(\mathcal{Z}/\zeta(2)\mathcal{Z})[[\Lambda]] \ , \ \text{resp.} \quad \mathcal{Z}_p[[\Lambda]]$$

where the formal variable Λ is the dual of the weight, and $\mathcal{Z}$, resp. $\mathcal{Z}_p$ is the $\mathbb{Q}$-algebra of complex, resp. p-adic hyperlogarithms. The Taylor period is equal to the sequence, indexed by primes p, of values at $\Lambda = 1$ of the formal p-adic periods. In more concrete words, prime multiple harmonic sums can be expressed as infinite sums of p-adic hyperlogarithms. We proved this in an earlier work by a p-adic analytic method.

We established families of algebraic relations for this motive and its periods, which are variants of the standard families of algebraic relations between multiple zeta values, and conjecturally generate all their relations. A particularity of the context of the weight-adic completion of motivic hyperlogarithms, is that it makes sense define period maps and state period conjectures in the context of complete topological algebras, and not just algebras. Such maps and conjecture contain more information, can be sometimes more subtle to define ; for this reason, we have delayed to part II the definition of the Taylor period map, the existence of such a map being suggested by part I.

1.2. **Heuristics for part II.** Let us now discuss how to define a priori a "Taylor period map" given the amount of information provided by part I. The only way to obtain a well-defined map is to use the post-composition of the sequence of formal p-adic period maps for all p, evoked above, by the quotient of $\prod_p \mathcal{Z}_p[[\Lambda]]$ by $(\Lambda - 1)$; by our theorem expressing prime multiple harmonic sums as infinite sums of p-adic hyperlogarithms, we obtain at first sight a well-defined map that sends each prime multiple harmonic sum motive $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}[\tilde{w}]$ to the corresponding sequence of prime multiple harmonic sums $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}]$. However, this simple idea is not sufficient to obtain a good candidate for the desired map.

First, we have to take into account the issues of convergence of p-adic series. This convergence is saved and we have a well-defined map if, for example, we restrict to $\mathbb{Z}$-algebras ; but the composed map evoked above is not well defined as, for example, a map between $\mathbb{Q}$-algebras. It is not clear a priori which rings of coefficients we can choose.

According to our principle of placing ourselves in the context of topologically complete algebras, we also have to choose a topology at the target. If we follow moreover the principle of considering prime multiple harmonic sums as a period by themselves, and not just a byproduct of the p-adic period, the complete topological algebra at the target has to be defined purely in terms of the ring $\prod_p \overline{\mathbb{Q}_p}$ and multiple harmonic sums.

The continuity of the map is also put into question ; it should be related to the fact that the valuation of a prime multiple harmonic sum $(p^k)^{\mathrm{weight}(\tilde{w})} H_{p^k}[\tilde{w}]$ is lower bounded by its weight.

A primary guess for the topology on the target would be, for example, to consider the uniform topology on $\prod_p \overline{\mathbb{Q}_p}$ relative to the p-adic topologies. Actually, with this choice, the conjecture of periods for the given map would be an extremely strong statement ; and there would be no reason - nor conceptual, nor experimental - to believe in it. It would imply, essentially, that the filtration on the target defined through the p-adic valuation of multiple harmonic sums is very close to the weight filtration on the completed algebra of motivic hyperlogarithms. It would also imply a statement on lift of congruences between finite multiple zeta values, and on the rational coefficients of the lifts.
Finally, note that given the relationship between prime multiple harmonic sums and p-adic hyperlogarithms, any definition of a Taylor period map as above would implicitly relate conjecturally, more than only comparing, the slopes of Frobenius and the Hodge filtration on the log-crystalline cohomology groups associated to the pro-unipotent fundamental groupoid.

For all these reasons, it seems that to attain a reasonable definition of this period map requires some additional experimentation. Most results of this paper - which also can be thought of independently of the question of the Taylor period - can be viewed as some evidence for the existence of a Taylor period map, of a nature different from the one of part I, and also as indication on what precise form the map should have. We tackle the following two themes :

A) The interplay between the algebraic properties of the prime multiple harmonic sum motive and periods, and the algebraic properties of the following motives and periods :
1) the usual hyperlogarithm motive $\mathrm{Li}^{\mathcal{M}}$, and its periods. We will discuss especially p-adic hyperlogarithms.
2) the "finite multiple zeta motive" $\zeta_{\mathcal{A}}^{\mathcal{M}}$, and its generalization from $\mathbb{P}^1 - \{0,1,\infty\}$ to $\mathbb{P}^1 - Z$. It has, as a period, the finite multiple zeta values of Kaneko-Zagier :

$$\zeta_{\mathcal{A}}(s_d,\ldots,s_1) = \left(H_p(s_d,\ldots,s_1) \mod p\right) \in \mathcal{A} = \big(\prod_p \mathbb{F}_p\big)/\big(\oplus_p \mathbb{F}_p\big)$$

(here $z_{i_{d+1}} = \ldots = z_{i_1} = 1$) which could be referred to as a "finite period".

There are several aspects to the theme A : reduction modulo primes, or modulo Λ of prime multiple harmonic sums ; in the converse way, lift of congruences ; and the question to derive an explicit algebraic theory of p-adic multiple zeta values, and more generally, of p-adic hyperlogarithms. This last issue has been first raised by Deligne and Goncharov in [DG], §5.28.

B) Arguments for the existence of a "motivic origin" of a significant part of the valuation of multiple harmonic sums. Let us give here the first example of such a phenomenon ; it is implicitly provided by Kaneko-Zagier's conjecture on finite multiple zeta values. Indeed, this conjecture, combined to the usual period conjecture for multiple zeta values, implies, for all indices $w=(s_d,\dots,s_1)$, the following equivalence :

$$\zeta_{\mathcal{A}}^{\mathcal{M}}(w)\neq 0 \Leftrightarrow v_p(H_p(w))=0 \text{ for infinitely many } p$$

1.3. **Notations for the fundamental groupoid.** Here, let us take again a curve $\mathbb{P}^1-Z$ with the same notations : $Z\subset\mathbb{P}^1(\overline{\mathbb{Q}})$ contains $\{0,1,\infty\}$ and $Z-\{0,1,\infty\}=\{z_1,\dots,z_r\}$, where $r\in\mathbb{N}$, and $z_0=0$, $z_{r+1}=1$. We reviewed definitions and facts on the pro-unipotent fundamental group of $\mathbb{P}^1-Z$ in part I, §2. Let us recall here a few notations. A few other definitions will be recalled throughout the paper when necessary.

1.3.1. *Notation for* $\pi_1^{un,dR}(\mathbb{P}^1-Z,\text{can})$. Let e_Z be the alphabet $\{e_0,e_{z_1}\dots,e_{z_r},e_1\}$. Let the non-commutative algebra of formal power series with variables in e_Z and coefficients in an algebra R be denoted by

$$R\langle\langle e_Z\rangle\rangle=R\langle\langle e_0,e_{z_1}\dots,e_{z_r},e_1\rangle\rangle$$

An element of it can be written uniquely as

$$f=f[\emptyset]+\sum_{\substack{s_d,\dots,s_0\in\mathbb{N}^*\\ i_d,\dots,i_1\in Z-\{0,\infty\}}} f[e_0^{s_d-1}e_{z_{i_d}}\dots e_0^{s_1-1}e_{z_{i_1}}e_0^{s_0-1}]e_0^{s_d-1}e_{z_{i_d}}\dots e_0^{s_1-1}e_{z_{i_1}}e_0^{s_0-1}$$

1.3.2. *Generating series of hyperlogarithms and multiple zeta values.* Generating series of (complex, p-adic, motivic) hyperlogarithms, resp. multiple zeta values are elements

$$\Phi_{0z}\in\mathbb{C}\langle\langle e_0,e_1\rangle\rangle,\quad (\Phi_{0z})_{p,-k}\in\mathbb{C}_p\langle\langle e_0,e_1\rangle\rangle,\quad \Phi_{0z}^{\mathcal{M}}\in\mathcal{Z}_{\mathcal{M}}\langle\langle e_0,e_1\rangle\rangle$$

resp.

$$\Phi\in\mathbb{R}\langle\langle e_0,e_1\rangle\rangle,\quad \Phi_{p,-k}\in\mathbb{Q}_p\langle\langle e_0,e_1\rangle\rangle,\quad \Phi^{\mathcal{M}}\in\mathbb{Q}_p\langle\langle e_0,e_1\rangle\rangle,$$

where $k\in\mathbb{N}^*$ is the number of iterations of Frobenius, and z is the extremity of the path of integration appearing in the definition of §1.1.

Take the following correspondence between words on e_Z and indices that appeared in §1.1 :

$$\left(\begin{array}{c} z_{i_d},\dots,z_{i_1}\\ s_d,\dots,s_1\end{array}\right)\leftrightarrow(-1)^d e_0^{s_d-1}e_{z_{i_d}}\dots e_0^{s_1-1}e_{z_{i_1}}$$

Then, we have, for all indices :

$$\zeta(s_d,\dots,s_1)=(-1)^d\Phi[e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1]\quad\text{etc.}$$

The factor $(-1)^d$ comes from that the series expansion at 0 with respect to z of a differential form $\frac{dz}{z-z_i}$, which occurs in the iterated integral representation of multiple zeta values, has a negative sign.

Finally, the p-adic, resp. motivic analogues of multiple zeta values are denoted by $\zeta_{p,-k}$, resp. $\zeta_{\mathcal{M}}$. The p-adic, resp. motivic analogues of hyperlogarithms are denoted by $\mathrm{Li}_{p,-k}$, $\mathrm{Li}^{\mathcal{M}}$.

We denote by $\zeta_{p,-\infty}$, resp. $\mathrm{Li}_{p,-\infty}$ the numbers obtained from $\zeta_{p,-k}$, $\mathrm{Li}_{p,-k}$ by taking limits $k\to\infty$. These are the inverse for the Ihara action of the numbers reflecting the Frobenius-invariant path in the fundamental group.

We will denote Kaneko-Zagier's finite multiple zeta values by $\zeta_{\mathcal{A}}$; their motivic versions by $\zeta_{\mathcal{A}}^{\mathcal{M}}$. They have complex and p-adic analogues, which we will denote by $\zeta_{\mathcal{A},\mathcal{Z}/\zeta(2)\mathcal{Z}}$, and $\zeta_{\mathcal{A},\mathcal{Z}_p}$.

1.3.3. *The prime multiple harmonic sum motive.* For each index $\tilde{w} = \begin{pmatrix} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{pmatrix}$, the associated prime multiple harmonic sum motive is :

(1) $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{\mathcal{O},\mathrm{prime}}[\tilde{w}] =$

$$z_{i_{d+1}}^{-p^k}(-1)^d \sum_{\substack{0\leq d'\leq d \mid z_{i_{d'}}=z_{i_{d+1}} \\ l_{d'+1},\ldots,l_d\geq 0}} \prod_{i=d'}^{d} \binom{-s_i}{l_i}(-1)^{s_i}(\Phi_{0z})^{\mathcal{M}}\begin{pmatrix} z_{i_{d'+1}}\ldots\ldots z_{i_{d+1}} \\ s_{d'+1}+l_{d'+1},\ldots,s_d+l_d \end{pmatrix}$$

$$\times (\Phi_{0z})^{\mathcal{M}}\begin{pmatrix} z_{i_{d'}},\ldots,z_{i_1} \\ s_{d'},\ldots,s_1 \end{pmatrix}$$

1.4. **Outline.** The §2 is devoted to the simplest transfer of algebraic properties :

$$\left\{ \begin{array}{c} \text{prime multiple harmonic sums} \\ (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}} \\ (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}} \end{array} \right\} \rightarrow \begin{array}{c} {}^{\text{red. mod. }\Lambda} \\ {}^{\text{red. mod. }p} \end{array} \left\{ \begin{array}{c} \text{finite multiple zetas} \\ \zeta_{\mathcal{A}}^{\mathcal{M}} \\ \zeta_{\mathcal{A}} \end{array} \right\}$$

Proposition 1. The relations of theorem 1 and 3 of part I imply, by taking reduction modulo Λ, resp. modulo primes, relations between the finite multiple zeta motive and its periods, and similarly for their generalizations to curves $\mathbb{P}^1 - Z$.

We conjecture that they generate all their possible relations.

The §3 is devoted to the question of going backwards from finite multiple zeta and usual multiple zeta motives, to the prime multiple harmonic sums motive :

$$\left\{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}},\, (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}\right\} \leftarrow^{\text{lifts}} \left\{\zeta^{\mathcal{M}}, \zeta, \text{ and } \zeta_{\mathcal{A}}^{\mathcal{M}},\, \zeta_{\mathcal{A}}\right\}$$

We address this question in the case of $\mathbb{P}^1 - \{0,1,\infty\}$, considering both the lift of congruences and the lift of the motives themselves.

In §3.1, we show that there exists a lift to a well-known family of congruences between harmonic sums of $\mathbb{P}^1 - \{0,1,\infty\}$: for all $s \in \mathbb{Z}$ and p prime, we have :

(2) $$\sum_{0<n<p} n^s \equiv \left\{ \begin{array}{l} 0 \mod p \text{ if } p-1|s \\ -1 \mod p \text{ otherwise} \end{array} \right.$$

Several people, in particular Rosen [Ro], have proved the existence of some lifts to higher powers of p of certain cases of this congruence, involving prime multiple harmonic sums $p^{\mathrm{weight}}H_p$.

Theorem 2. There is an relation in $\widehat{\mathcal{Z}_{\mathcal{M}}}$ written as the vanishing of an infinite sum of elements $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}]$'s, which implies the vanishing of an absolutely convergent p-adic sum of prime multiple harmonic sums, lifting p-adically the congruence (2).

In §3.2, using a result of Yasuda [Y1] on $\zeta_{\mathcal{A}}^{\mathcal{M}}$, we prove that $\zeta^{\mathcal{M}}$ and $\zeta_{\mathcal{A}}^{\mathcal{M}}$ admit expansions in terms of of $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}$. Precisely, given the natural map $\Sigma_{(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}} : Z_{\mathcal{M}} \to \widehat{Z_{\mathcal{M}}}$ satisfying $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}} = \Sigma_{(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}} \zeta^{\mathcal{M}}$, we have :

Proposition 3. There is a converse expansion :

$$\zeta^{\mathcal{M}} = \Sigma^{-1}_{(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}} (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}$$

In particular, $\zeta_{\mathcal{A}}^{\mathcal{M}}$ can also be expanded in terms of $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}$.
This can also be formulated by saying that the motives $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}[\tilde{w}]$ generate topologically the weight-adic completion of the $\mathbb{Q}$-algebra of motivic multiple zeta values.

It implies also a kind of abstract general lift of congruences between finite multiple zeta values, related to Rosen's conjecture of the existence of lift of congruences between finite multiple zeta values into equalities involving infinite sums of prime multiple harmonic sums $p^{\mathrm{weight}} H_p$ in the ring $\varprojlim \big(\prod_p \mathbb{Z}/p^n\mathbb{Z}\big) / \big(\oplus_p \mathbb{Z}/p^n\mathbb{Z}\big)$.

In §4, we discuss how to obtain relations between p-adic hyperlogarithms starting with relations between prime multiple harmonic sums. This question is close to a problem raised by Deligne and Goncharov ([DG] §5.28) who asked how to obtain, for p-adic multiple zeta values, explicit formulas on which the series shuffle relation can be visualized explicitly.

The method that we describe combines the relations between prime multiple harmonic sums of part I with some of our work p-adic analytic work, where we have proved that prime multiple harmonic sums admit a p-adic analytic expansion in terms of the upper bound p^k, with coefficients equal to certain infinite sums of products of p-adic hyperlogarithms, and that this characterizes p-adic hyperlogarithms entirely.

Theorem 4. There is a general implication

$$\left\{ \begin{array}{c} \text{relations between prime} \\ \text{multiple harmonic sums} \end{array} \right\} \underset{\substack{\text{coeff w.r.t number of} \\ \text{iterations of Frobenius}}}{\rightarrow} \left\{ \begin{array}{c} \text{relations between p-adic} \\ \text{hyperlogarithms} \end{array} \right\}$$

In part I, by our proofs of theorems 1 and 3 we had constructed implicitly a passage in the opposite direction. Note also that, in §2, a variant of the operation of reduction modulo Λ was to take the coefficients of Λ^n, for all values of $n \in \mathbb{N}$. The process of §4 could be seen as an analogue with p^k instead of Λ ; in some sense, we have made p into a variable.

Since the expansion of multiple harmonic sums entirely characterizes p-adic multiple zeta values, we have an answer to a sort of indirect version of Deligne-Goncharov's question ; separately, since the method yields a variant of the series shuffle relation on p-adic multiple zeta values, we can also say that we have a partial answer to this question.

In §5, we make a remark on the arithmetic nature of the sequences of prime multiple harmonic sums, $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}] \in \prod_p \overline{\mathbb{Q}_p}$: we explain that, in certain cases, the question of their transcendence can be settled. This will be used in the next part.

In §6, we explain that certain parts of the information on the valuation of multiple harmonic sums should have a particular algebraic meaning : these are, respectively, some information on the complex valuation of $H_n(\tilde{w})$ when $n \to \infty$, and some information on the p-adic valuation $v_p(H_{p^k}(\tilde{w}))$ when $p \to \infty$. The main results of this paragraph are implications relating such type of information with algebraic properties.

In §7, we address finally the question of the Taylor period map, explaining how it is connected to the previous topics. Then, the main result of this part is the statement of a version of this conjecture.

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2. Reductions of $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}$ modulo Λ

2.1. Kaneko-Zagier's finite multiple zeta values and their generalization.

2.1.1. *Definition.* In the 2000's, several people, in particular Hoffman and Zhao, have shown in several papers that the quantities

$$H_p(s_d,\ldots,s_1) \mod p, \text{ especially for } p \text{ large}$$

admit significative analogies with multiple zeta values - where $H_p(w) \mod p$ is well defined because the denominators of $H_p(w)$ are products of integers in $\{1,\ldots,p-1\}$. More recently, Zagier and Kaneko have given the following notion and a more explicit, striking conjecture making this analogy precise :

Definition 2.1. (Zagier, 2011) Let finite multiple zeta values be

$$\zeta_{\mathcal{A}}(w) = \Big(H_p(w) \mod p\Big) \in \mathcal{A} := \big(\prod_p \mathbb{F}_p\big)/\big(\oplus_p \mathbb{F}_p\big) \tag{3}$$

The ring $\mathcal{A} = \big(\prod_p \mathbb{F}_p\big)/\big(\oplus_p \mathbb{F}_p\big)$ is the ring of "integers modulo infinitely large primes". It is the canonical $\mathbb{Q}$-algebra attached to the $\mathbb{Z}$-module $\prod_p \mathbb{F}_p$, i.e. we have

$$\mathcal{A} = \big(\prod_p \mathbb{F}_p\big) \otimes_{\mathbb{Z}} \mathbb{Q} = \big(\prod_p \mathbb{F}_p\big)/\big(\prod_p \mathbb{F}_p\big)_{tors}$$

The terminology "finite multiple zeta values" is explained by the following precise conjecture, which means that the $\mathbb{Q}$-vector spaces generated by finite multiple zeta values of a given weight, has some identical conjectural properties with the analogous $\mathbb{Q}$-vector spaces of, respectively, multiple zeta values modulo $\zeta(2)$ and p-adic multiple zeta values.

Conjecture 2.2. (Zagier, 2011) For $s \in \mathbb{N}$ let V_s be the $\mathbb{Q}$-vector space of finite multiple zeta values of weight s. (By convention $V_0 = \mathbb{Q}$ and $\zeta_{\mathcal{A}}(\emptyset) = 1$). Then

i) The sum of the V_s's is direct
ii) Let, for all $s \in \mathbb{N}$, $d_s = \dim V_s$. Then we have :

$$\sum_{s \geq 0} d_s x^s = \frac{1-x^2}{1-x^2-x^3}$$

Later, Kaneko and Zagier made a more precise conjecture :

Conjecture 2.3. (Kaneko-Zagier, 2013) There is a well-defined map from the $\mathbb{Q}$-algebra of finite multiple zeta values to the $\mathbb{Q}$-algebra of multiple zeta values modulo $(\zeta(2))$:

$$\zeta_{\mathcal{A}}(s_d, \ldots, s_1) \mapsto \sum_{k=0}^{d} (-1)^{s_{k+1}+\ldots+s_d} \zeta(s_{k+1}, \ldots, s_d)\zeta(s_k, \ldots, s_1)$$

which is an isomorphism.

A good way to formulate such conjectural isomorphisms is via motives and periods. An essential message of Kaneko-Zagier's conjecture can be stated as :

Message. The motive

$$\sum_{k=0}^{d} (-1)^{s_{k+1}+\ldots+s_d} \zeta^{\mathcal{M}}(s_{k+1}, \ldots, s_d)\zeta^{\mathcal{M}}(s_k, \ldots, s_1) \mod \zeta^{\mathcal{M}}(2)$$

has a "finite period", namely a "period" in the ring $\mathcal{A}$.

This is the first occurence of such a phenomenon in the theory of multiple zeta values.

Although most of Kaneko-Zagier's conjecture is an inacessible transcendence conjecture, an interesting part of it is accessible : recall our theorem expressing prime multiple harmonic sums as infinite sums of p-adic multiple zeta values - in the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, and of the first power of Frobenius :

$$p^{s_d+\ldots+s_1} H_p(s_d, \ldots, s_1) =$$

$$\sum_{d'=0}^{d} \sum_{l_{k+1},\ldots,l_d \in \mathbb{N}} (-1)^{s_{d'+1}+\ldots+s_d} \prod_{i=d'+1}^{d} \binom{-s_i}{l_i} \zeta_{p,-1}(s_{d'+1}+l_{d'+1}, \ldots, s_d+l_d)\zeta_{p,-1}(s_{d'}, \ldots, s_1)$$

By work of Yasuda, we have, for all indices w, and all primes p such that $p > \text{weight}(w)$, that $\zeta_p(w) \in p^{\text{weight}(w)}\mathbb{Z}_p$, whence, for $p > s_d + \ldots + s_1$,

$$(4) \quad H_p(s_d, \ldots, s_1)$$

$$\equiv p^{-(s_d+\ldots+s_1)} \sum_{d'=0}^{d} (-1)^{s_{d'+1}+\ldots+s_d} \zeta_{p,-1}(s_{d'+1}, \ldots, s_d)\zeta_{p,-1}(s_{d'}, \ldots, s_1) \mod p$$

For each $(s_d, \ldots, s_1)$, this equality being true for all p large enough, it descends to an expression of $\zeta_{\mathcal{A}}(s_d, \ldots, s_1)$ in terms of p-adic multiple zeta values, which explains the formula in the conjecture 2.3.

As Yasuda pointed out, this also enables to prove that the dimension of the $\mathbb{Q}$-vector spaces V_s in conjecture 2.2 are upper bounded by those appearing in the conjecture ; this relies the same upper bound for the dimension of $\mathbb{Q}$-vector spaces of p-adic multiple zeta values, via unpublished work of Yamashita on the crystalline realization of mixed Tate motives.

2.1.2. *Generalization to* $\mathbb{P}^1 - Z$. We take the same notation as in the introduction : Z a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$, containing $\{0,1,\infty\}$; $Z - \{0,1,\infty\}$ as $z_1,\ldots,z_r$, where $r \in \mathbb{N}$, and $z_0 = 0$, $z_{r+1} = 1$. We generalize Kaneko-Zagier's finite multiple zeta values to this case. For each prime number p, $\overline{\mathbb{Q}}$ is embedded into $\overline{\mathbb{Q}_p}$; given a $z \in \overline{\mathbb{Q}} - \{0\}$, we have, for all p large enough $|z|_p = 1$; in particular, z is in $\mathcal{O}_{\mathbb{C}_p}$ and we can consider its reduction modulo $\mathfrak{m}_{\mathbb{C}_p}$ which (is non zero and) lies in $\overline{\mathbb{F}_p}$.

Definition 2.4. For all indices $\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)$, let finite hyperlogarithms be :

$$\mathrm{Li}^{\mathcal{A}}(\tilde{w}) = \big(H_p(\tilde{w}) \mod p\big) \in \big(\prod_p \overline{\mathbb{F}_p}\big)/\big(\oplus_p \overline{\mathbb{F}_p}\big)$$

This contains in particular a definition of cyclotomic analogues of finite multiple zeta values, corresponding to the case where $Z - \{0,\infty\} = \mu_N$ with $N \in \mathbb{N}^*$.

Relying on our expression of prime multiple harmonic sums in terms of p-adic hyperlogarithms, we extend the philosophy of Kaneko-Zagier to this case, in the following language.

Definition 2.5. Let the motivic finite hyperlogarithms be :

$$\mathrm{Li}_{\mathcal{A}}^{\mathcal{M}}[\tilde{w}] = (-1)^d \big(\Phi_{0z_{i_{d+1}}}^{-1} e_{z_{i_{d+1}}} \Phi_{0z_{i_{d+1}}}\big)^{\mathcal{M}} \big[e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}}\big] \tag{5}$$

$$(-1)^d \sum_{0 \leq d' \leq d \mid z_{i_{d'}} = z_{i_{d+1}}} (-1)^{s_{d'+1}+\ldots+s_d} \Phi_{0z}^{\mathcal{M}}\left(\begin{array}{c} z_{i_{d'+1}} \ldots\ldots z_{i_{d+1}} \\ s_{d'+1},\ldots,s_d \end{array}\right) \Phi_{0z}^{\mathcal{M}}\left(\begin{array}{c} z_{i_{d'}},\ldots,z_{i_1} \\ s_{d'},\ldots,s_1 \end{array}\right)$$

2.2. **The reduction modulo Λ and reduction modulo primes of prime multiple harmonic sums.** Recall that the prime multiple harmonic sum motive is given by the formulas :

(6)

$$(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}^{\mathcal{M}}[\tilde{w}] = (-1)^d \big(\Phi_{0z_{i_{d+1}}}^{-1} e_{z_{i_{d+1}}} \Phi_{0z_{i_{d+1}}}\big)^{\mathcal{M}} \Big[\frac{1}{1-e_0} e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}}\Big]$$

$$= \sum_{\substack{0 \leq d' \leq d \mid z_{i_{d'}} = z_{i_{d+1}} \\ l_{d'+1},\ldots,l_d \geq 0}} \prod_{i=d'}^{d} \binom{-s_i}{l_i} (-1)^{s_i} \Phi_{0z_{i_{d+1}}}^{\mathcal{M}}\left(\begin{array}{c} z_{i_{d'+1}} \ldots\ldots z_{i_{d+1}} \\ s_{d'+1}+l_{d'+1},\ldots,s_d+l_d \end{array}\right)$$

$$\times \Phi_{0z_{i_{d+1}}}^{\mathcal{M}}\left(\begin{array}{c} z_{i_{d'}},\ldots,z_{i_1} \\ s_{d'},\ldots,s_1 \end{array}\right)$$

We view it as an element of $\mathcal{Z}_{\mathcal{M}}[[\Lambda]]$, where Λ is a formal variable, via the weight homogeneity ; namely

$$(7)\quad (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}] = \big(\Phi^{-1}_{0z_{i_{d+1}}} e_{z_{i_{d+1}}} \Phi_{0z_{i_{d+1}}}\big)^{\mathcal{M}} \left[\frac{\Lambda^{\sum_{i=1}^{d} s_i}}{1-\Lambda e_0} e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \ldots e_0^{s_1-1} e_{z_{i_1}}\right]$$

$$= \sum_{\substack{0\leq d'\leq d \mid z_{i_{d'}}=z_{i_{d+1}} \\ l_{d'+1},\ldots,l_d\geq 0}} \Lambda^{\sum_{i=1}^{d} s_i+l_i} \prod_{i=d'}^{d} \binom{-s_i}{l_i} (-1)^{s_i} \Phi^{\mathcal{M}}_{0z_{i_{d+1}}} \left(\begin{array}{c} z_{i_{d'+1}} \ldots\ldots z_{i_{d+1}} \\ s_{d'+1}+l_{d'+1},\ldots,s_d+l_d \end{array} \right)$$

$$\times\, \Phi^{\mathcal{M}}_{0z_{i_{d+1}}} \left(\begin{array}{c} z_{i_{d'}},\ldots,z_{i_1} \\ s_{d'},\ldots,s_1 \end{array} \right)$$

Proposition 2.6. Let $\tilde{w} = \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array} \right)$ be an index of prime multiple harmonic sums.
i) We have

$$\mathrm{Li}^{\mathcal{M}}_{\mathcal{A}}[\tilde{w}] \equiv \Lambda^{-\,\mathrm{weight}(\tilde{w})} (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}] \mod \Lambda$$

Thus, we also have the same congruence on the level of the complex and p-adic periods.
ii) Let $k \in \mathbb{N}^*$ be the chosen power of Frobenius. Let $\tilde{w}_k = \left(\begin{array}{c} z^{p^{k-1}}_{i_{d+1}},\ldots,z^{p^{k-1}}_{i_1} \\ s_d,\ldots,s_1 \end{array} \right)$ (note that $\tilde{w}_1 = \tilde{w}$. The right-hand side below is well-defined and we have :

$$\mathrm{Li}_{\mathcal{A}}[\tilde{w}_k] \equiv \big(p^{-\,\mathrm{weight}\,\tilde{w}} (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}] \mod \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}\big)_{p \text{ large enough}} \in \big(\prod_p \overline{\mathbb{F}_p}\big)/\big(\oplus_p \overline{\mathbb{F}_p}\big)$$

Proof. i) is clear. To prove ii), let p a prime number such that $|z_{i_1}|_p = \ldots = |z_{i_d}|_p = 1$ (this assumption is satisfied if p is large enough). We have :

$$v_p\Big((p^k)^{s_d+\ldots+s_1} H_{p^k}\left(\begin{array}{c} z_{i_{d+1}} \ldots z_{i_1} \\ s_d \ldots s_1 \end{array} \right) - p^{s_d+\ldots+s_1} H_p\left(\begin{array}{c} z^{p^{k-1}}_{i_{d+1}} \ldots z^{p^{k-1}}_{i_1} \\ s_d \ldots s_1 \end{array} \right)\Big) \geq s_d+\ldots+s_1+1$$

Indeed, the subsum of $(p^k)^{s_d+\ldots+s_1} \sum_{0<n_1<\ldots<n_d<p^k}$ made of indices $(n_1,\ldots,n_d) \in p^{k-1}\mathbb{N}^* \times \ldots \times p^{k-1}\mathbb{N}^*$ is equal to $p^{s_d+\ldots+s_1} H_p\left(\begin{array}{c} z^{p^{k-1}}_{i_{d+1}} \ldots z^{p^{k-1}}_{i_1} \\ s_d \ldots s_1 \end{array} \right)$, as we can see by making a change of variable $n_i = p^{k-1}n'_i$. The complementary subsum has higher p-adic valuation. $\square$

Remark 2.7. Assume that $z_{i_{d+1}},\ldots,z_{i_1} \in \mathbb{Q}$. Then the reduction modulo large primes of $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}]$ defines an element of $\big(\prod_p \mathbb{F}_p\big)/\big(\oplus_p \mathbb{F}_p\big)$, which is independent of the chosen power of Frobenius, since we have, for all primes p and all elements $x \in \mathbb{F}_p$, that $x^p = x$.

We can now state the application of our results of part I to the finite setting.

Application 2.8. The reduction modulo primes and modulo Λ of theorems 1 and theorem 3 of I give analogues of double shuffle and Kashiwara-Vergne equations for finite hyperlogarithms and their motivic, complex, p-adic analogues.

Conjecture 2.9. Those relations generate all the algebraic relations among motivic finite hyperlogarithms and their periods.

Remark 2.10. We can write not only the reduction modulo Λ of each relation between $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}]$'s, but also the coefficient of each Λ^n, $n \in \mathbb{N}$, of such relations : we will study this aspect separately in part III. On the other hand, in §4 we will consider "coefficients with respects to powers of p^k".

2.3. **Related work.** What follows is, to our knowledge, the work related to the reductions of our theorems 1 and 3 of part I.

We start by the known work on finite multiple zeta values.
The theorem 1 of part I concerns double shuffle relations. Its reduction modulo Λ and modulo primes has been proved by Kaneko and Zagier. To our knowledge, the method of proof of the shuffle relation is different from ours ; it does not involve the generating series $\Phi^{-1}e_1\Phi$.
A particular consequence of the integral shuffle is a "reversal" relation on prime multiple harmonic sums stated in I, §4. Its reduction modulo primes, in the case of $\mathbb{P}^1 - \{0,1,\infty\}$, is the formula $H_p(s_1,\ldots,s_d) = (-1)^{s_1+\ldots+s_d} H_p(s_1,\ldots,s_d) \mod p$. This result had been proved by Zhao ([Z], lemma 3.3) and Hoffman ([H], theorem 4.5).

The theorem 3 of part I concerns Kashiwara-Vergne relations. The reduction modulo primes of the one-dimensional part of the equations has been proved by Hoffman, using Newton series [H], and named the "duality theorem".

There are also notions of finite multiple polylogarithms.

Sakugawa and Seki have defined and studied the following notion [SaSe] :

Definition 2.11. (Sakugawa, Seki) Let finite multiple polylogarithms be :

$$L_{\mathcal{A},(s_d,\ldots,s_1)}(t) = \sum_{0<n_1<\ldots<n_d<p} \frac{t^{n_d}}{n_1^{s_1}\ldots n_d^{s_d}} \in \Big(\prod_p (\mathbb{Z}/p\mathbb{Z})[t]\Big) / \Big(\oplus_p\,(\mathbb{Z}/p\mathbb{Z})[t]\Big)$$

They show in [SaSe] several functional equations among those finite multiple polylogarithms.

Remark 2.12. Note that our results imply more generally algebraic relations among the following more general object :

$$L_{\mathcal{A},(s_d,\ldots,s_1)}(t_1,\ldots,t_d) = \sum_{0<n_1<\ldots<n_d<p} \frac{t_1^{n_1}\ldots t_d^{n_d}}{n_1^{s_1}\ldots n_d^{s_d}}$$
$$\in \Big(\prod_p (\mathbb{Z}/p\mathbb{Z})[t_1,\ldots,t_d]\Big) / \Big(\oplus_p\,(\mathbb{Z}/p\mathbb{Z})[t_1,\ldots,t_d]\Big)$$

Ono and Yamamoto have defined another notion of finite multiple polylogarithms and proved that it satisfies an integral shuffle relation [OY] :

Definition 2.13. (Ono-Yamamoto) Let finite multiple polylogarithms be :

$$li_{(s_d,\ldots,s_1)}(t) = \sum_{\substack{0<l_1,\ldots,l_d<p \\ p\nmid l_1,\ldots,p\nmid l_1+\ldots+l_d}} \frac{t^{l_1+\ldots+l_d}}{l_1^{s_1}\ldots(l_1+\ldots+l_d)^{s_d}} \in \Big(\prod_p (\mathbb{Z}/p\mathbb{Z})[t]\Big) / \Big(\oplus_p(\mathbb{Z}/p\mathbb{Z})[t]\Big)$$

The relationship between the notions of Sakugawa-Seki and Ono-Yamamoto is established in [SaSe], §4.2. Again, we can consider the variant with several variables.

Remark 2.14. Again, we can consider the generalization to $\mathbb{P}^1 - \{0,1,\infty\}$ to $\mathbb{P}^1 - Z$: replace $\mathbb{Z}_p$ by $\mathcal{O}_{\overline{\mathbb{Q}_p}}$, and replace $H_p(s_d,\ldots,s_1)$ by $H_p\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array}\right)$.

Last related work : we stated as a remark the existence of notions of cyclotomic finite multiple zeta values, their variants defined through the series $\Phi_{0z}^{-1}e_z\Phi_{0z}$, and their algebraic properties, in the first version of part I.

3. Lifts of $\zeta^{\mathcal{M}}$ and $\zeta^{\mathcal{M}}_{\mathcal{A}}$ and their congruences in the case of $\mathbb{P}^1 - \{0,1,\infty\}$

3.1. **The problem of lift of congruences and related questions.** The finding of congruences between prime multiple harmonic sums H_p modulo p in the 2000's have raised the question to find lifts of them modulo higher powers of p.

Let us give an example : for all primes p and all all $s \in \mathbb{Z}$ such that $p-1$ does not divide s, we have

$$H_p(s) \equiv 0 \mod p$$

Here is a lift of this congruence to a congruence modulo p^2 [Z], [**?**] : assume that $p > s+1$; we have

$$H_p(s) \equiv p\frac{s}{s+1}H_p(1,s) \mod p^2 \tag{8}$$

Indeed, we have first of all $H_p(s) \equiv \sum_{n=1}^{p-1} n^{p(p-1)-s} \mod p^2$. This reduces us to positive powers of n ; the sums of positive powers of integers from 1 to $p-1$ are given by, a standard fact, by a polynomial of p : precisely,

$$H_p(s) \equiv \sum_{l=1}^{p(p-1)-s+1} \frac{1}{p(p-1)-s+1}\binom{p(p-1)-s+1}{l} B_{(p-1)p-s+1-l}\,p^l$$

Because of the hypothesis on p and s, only the $l=1$ term is non zero; finally, by Kummer congruences, we have $B_{p(p-1)-s} \equiv \frac{(p-1)p-s}{p-1-s}B_{p-1-s} \equiv \frac{s}{s+1}B_{p-1-s} \mod p$.

The problem of lift of congruences has been made systematic by Rosen in [Ro], who defined a lift of Zagier's finite multiple zeta values involving p-adic numbers for "p infinitely large"

Definition 3.1. (Rosen) Let the weighted finite multiple zeta values be :

$$\zeta_{\hat{\mathcal{A}}}(s_d,\ldots,s_1) = (H_p(s_d,\ldots,s_1))_{p\text{ prime}} \in \hat{\mathcal{A}} = \varprojlim \left(\prod_p \mathbb{Z}/p^n\mathbb{Z}\right) \Big/ \left(\oplus\,\mathbb{Z}/p^n\mathbb{Z}\right)$$

The ring $\hat{\mathcal{A}}$ defined in [Ro] is the quotient of $\prod_p \mathbb{Z}_p$ by the closure of $\oplus_p\mathbb{Z}_p$ relatively to the uniform topology on $\prod_p \mathbb{Z}_p$ (where the uniform topology is relative to the p-adic topologies on each $\mathbb{Z}_p$). It is a complete topological ring.
We have reviewed in I, §8, how the two algebraic relations proved in [Ro] are particular cases of the theorems 1 and 3 of part I. Another aspect of Rosen's work is a philosophy of lift of congruences between finite multiple zeta values. Rosen conjectures that all

equalities between finite multiple zeta values admit lift to equalities between weighted finite multiple zeta values in the ring $\hat{\mathcal{A}}$. For $s \in \{1,2\}$, the result of existence of such a lift is stated in (the first version only of) [Ro], §5, and proven up to $\mod p^7$, with a reference to algorithms for higher congruences.

We are going to see that we can find p-adic lifts of certain congruences, using our theorems 1 and 3 of part I. Another issue is at stake : our theorems of part I give equalities on prime multiple harmonic sums valid for all primes p and having coefficients in $\mathbb{Z}$; on the other hand, Kaneko-Zagier's finite multiple zeta values are defined "modulo large primes", and their structure comes from congruences valid only for p large. The proof of theorem 2 shows how, by rearranging equalities with coefficients in $\mathbb{Z}$, we arrive indeed at certain non trivial rational coefficients.

On the other hand, we are going to see that we can lift Λ-adically the finite multiple zeta motives themselves, by expressing them as formal infinite sums of prime multiple harmonic sum motives. This implies a general lift of congruences between prime multiple harmonic sums, up to the determination of the convergence of certain p-adic series.

3.2. **A lift of congruences between the harmonic sums of $\mathbb{P}^1 - \{0,1,\infty\}$.**

3.2.1. *Introduction.* We take the case of $\mathbb{P}^1 - \{0,1,\infty\}$. Let us recall the expression of certain parts of the theorem 1 and 3 of part I in this case. Some avatars of these equations appear in [Ro].

i) the series shuffle equation (particular case of theorem 1) : for all words w, w' :

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(w \ * \ w') = (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(w)(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(w') \tag{9}$$

ii) a particular case of a "symmetry equation" that is a particular case of the shuffle equation (particular case of theorem 1)

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(w) = (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}((\Sigma_{\omega})_{*}\,\mathrm{inv}(w)) \tag{10}$$

ii) the one-dimensional part of Kashiwara-Vergne equations (particular case of theorem 3) :

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(w(e_0+e_1,-e_1)) = -\sum_{z\in\ker\tilde{\partial}_{e_0}} (-1)^{\mathrm{depth}(z)}(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(zw) \tag{11}$$

For the simplicity of the notation we will rename $h(w) = (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}(w)$. For R a ring, and $s \in \mathbb{N}^{*}$, denote by $R.h(_{\mathrm{weight}\geq s})$ the weight-adic completion of the R-module generated by polynomials, with constant coefficient 0, of numbers $h(w_n)$ with for all n, $\mathrm{weight}(w_n) \geq s$.

Theorem. We have, for all $s \in \mathbb{N}^{*}$:

$$h(s) \equiv 0 \mod \mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s+1}].h(_{\mathrm{weight}\geq s+1})$$

and this can be lifted adically as follows.

By induction on N, let us assume that, for $N \in \mathbb{N}^{*}$, $n \in \{1,\ldots,N-1\}$ and $s \in \mathbb{N}^{*}$, there exists $w_n^N(s) \in \mathcal{H}_*$ of weight $s+n$ such that

$$h(s) \equiv \sum_{n=1}^{N-1} h(w_n^N(s)) \mod \mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s+1}]h(_{\text{weight}\geq s+N})$$

Then, for $N \in \mathbb{N}^*$, $n \in \{1,\ldots,N\}$, and $s \in \mathbb{N}^*$, denoting by

$$(12)\quad w_n^{N+1}(s) = -\frac{s}{s+1}\sum_{z,\text{word of weight } n}(-1)^{\text{depth}(z)}h(z.y_s)$$
$$-\frac{s}{s+1}\sum_{\substack{2\leq l\leq s \\ \{(k_1,\ldots,k_l)\in(\mathbb{N}^*)^l| \\ k_1+\ldots+k_l=s\}/S_l}}\sum_{\substack{n_1,\ldots,n_l\in\mathbb{N}^* \\ n_1+\ldots+n_l=n}}\frac{1}{\prod_{i=1}^l k_i}w_{n_1}^N(k_1)*\ldots*w_{n_1}^N(k_l)$$

we have

$$h(s) \equiv \sum_{n=1}^{N} h(w_n^{N+1}(s)) \mod \mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s'+1}]h(_{\text{weight}\geq s+N+1})$$

3.2.2. *Proof.*

Remark 3.2. The formulas of [J6], §3.4, which are true in the motivic case, imply that :

$$(\Phi^{-1}e_1\Phi)^{\mathcal{M}}[\frac{1}{1-\Lambda e_0}e_1e_0^{s-1}e_1] = \Lambda\frac{s}{s+1}(\Phi^{-1}e_1\Phi)^{\mathcal{M}}[\frac{1}{1-\Lambda e_0}e_1^2e_0^{s-1}e_1)] \mod \Lambda^2$$

This is the motivic analogue of the lift (8)

Lemma 3.3. It follows from the series shuffle equation that

$$\sum_{w \text{ of weight } s} h(w) = \sum_{l=1}^{s}\sum_{\substack{\{(k_1,\ldots,k_l)\in(\mathbb{N}^*)^l| \\ k_1+\ldots+k_l=s\}/S_l}}\prod_{i=1}^{l}\frac{h(k_i)}{k_i}$$

Proof. In [H], theorem 2.2, Hoffman proves that the series shuffle equation implies, for all $s_d,\ldots,s_1 \in \mathbb{N}^*$,

(13)

$$\sum_{\substack{\phi \text{ permutation} \\ \text{of } \{1,\ldots,d\}}} h(s_{\phi(d)},\ldots,s_{\phi(1)}) = \sum_{\substack{\{B_1,\ldots,B_l\} \text{ partitions} \\ \text{of } \{1,\ldots,d\}}}(-1)^{d-l}\prod_{i=1}^{l}\Big((\sharp B_i-1)!h(\sum_{u\in B_i}s_u)\Big)$$

This implies that $\sum_{w \text{ of weight s}} h(w)$ is equal to

$$\sum_{l=1}^{s}\sum_{\substack{(k_1,\ldots,k_l)\in(\mathbb{N}^*)^l| \\ k_1+\ldots+k_l=s\}/S_l}}\prod_{i=1}^{l}h(k_i)$$

$$\sum_{d\geq l}(-1)^{d-l}\sum_{\substack{n_1,\ldots,n_l\geq 1 \\ n_1+\ldots+n_l=d}}\prod_{i=1}^{l}\frac{(n_i-1)!}{n_i!}|\{(m_1,\ldots,m_{n_i})\in(\mathbb{N}^*)^{n_i}|\sum m_j = k_i\}$$

$$= \sum_{l=1}^{s}\sum_{\substack{(k_1,\ldots,k_l)\in(\mathbb{N}^*)^l| \\ k_1+\ldots+k_l=s\}/S_l}}\prod_{i=1}^{l}h(k_i)(-1)^l\sum_{d\geq l}\sum_{\substack{n_1,\ldots,n_l\geq 1 \\ n_1+\ldots+n_l=d}}\frac{1}{k_i}\binom{k_i}{n_i}(-1)^{n_i}$$

hence the result. $\square$

Corollary 3.4. With the assumption of theorem 4, denoting by S_l is the group of permutations of $\{1,\ldots,l\}$ we have :

$$(14)\qquad -\sum_{\substack{2\leq l\leq s\\ \{(k_1,\ldots,k_l)\in(\mathbb{N}^*)^l\,|\\ k_1+\ldots+k_l=s\}/S_l}}\prod_{i=1}^{l}\frac{h(k_i)}{k_i}=\frac{s+1}{s}h(s)+\sum_{z\in W'\setminus\{*\}}(-1)^{\mathrm{depth}(z)}h(z.y_s)$$

Proof. This combines the equation (11) in depth one with the previous lemma. $\square$

Lemma 3.5. We have, for all $s\in\mathbb{N}^*$,

$$h(s)\equiv 0 \mod \mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s+1}].h(_{\mathrm{weight}\geq s+1})$$

Proof. Let us take $s=1$ in the symmetry relation (10). This gives

$$2h(1)\equiv 0 \mod \mathbb{Z}[h_{\mathrm{weight}\geq 2}]$$

i.e. the result for $s=1$. On the other hand, the previous corollary implies

$$\frac{-1}{s}h(s)+\big(\mathbb{Z}\text{-linear combinations of products of }\frac{h(s')}{s'}\text{ with }s'<s\big)=h(s)\mod \mathbb{Z}.h_{(\mathrm{weight}\geq s+1)}$$

Hence the result by induction on s, by regrouping the two terms involving $h(s)$ and obtaining a term $\frac{s+1}{s}h(s)$. $\square$

Lemma 3.6. Let us assume that, for a given $N\in\mathbb{N}^*$, for all $s'\in\mathbb{N}^*$, we have :

$$h(s')\equiv\sum_{n=1}^{N-1}h(w_n^N(s'))\mod \mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s'+1}]h(_{\mathrm{weight}\geq s'+N})$$

with, for all $n=1,\ldots,N-1$, $\mathrm{weight}(w_n^N(s'))=s'+n$.
Let us take $s\in\mathbb{N}^*$, $(k_1,\ldots,k_l)\in(\mathbb{N}^*)^l$ with $l\geq 2$ and $k_1+\ldots+k_l=s$. Then :

$$\prod_{i=1}^{l}\frac{h(k_i)}{k_i}=\prod_{i=1}^{l}\sum_{n_i=1}^{N-1}h(w_n^N(k_i))\mod \mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s'+1}]h(_{\mathrm{weight}\geq s+N+1})$$

Proof. We apply the hypothesis to $s'=k_1,\ldots,s=k_l$. First of all, for all i, we have $\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{k_i+1}]\subset\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s+1}]$. The numbers $\frac{1}{k_i}h(w_n^N(k_i))$ are of weight $\geq k_i+1$, and the elements of
$\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s'+1}]h(_{\mathrm{weight}\geq k_i+N})$ are of weight $\geq k_i+N$. Since we have

$$\prod_{i=1}^{l}\frac{h(k_i)}{k_i}\in\prod_{i=1}^{l}\Big(\sum_{n=1}^{N-1}h(w_n^N(s'))+\mathbb{Z}[\frac{1}{2},\ldots,\frac{1}{s'+1}]h(_{\mathrm{weight}\geq s'+N})\Big)$$

then, the difference

$$\prod_{i=1}^{l}\frac{h(k_i)}{k_i}-\prod_{i=1}^{l}\sum_{n_i=1}^{N-1}h(w_n^N(s'))$$

is made of terms whose weight is superior or equal to $\sum_{\substack{1\leq i\leq l\\ i\neq i_0}}(k_i+1)+k_{i_0}+N=(\sum_{i=1}^{l}k_i)+N+l-1$ for all $i_0\in\{1,\ldots,l\}$, which is itself superior or equal to $\sum_{i=1}^{l}k_i+N+1$ since $l\geq 2$. $\square$

Finally, the theorem follows from this last lemma and the corollary 3.4.

3.2.3. *Remarks and corollaries.* We derive four particular consequences of the theorem.

Remark 3.7. Variant for the multiple harmonic sums with large inequalities One has also in [H], theorem 2.2, a variant of equation (13). It applies to the variant of multiple harmonic sums defined by large inequalities

$$\sum_{0<n_1\leq \ldots \leq n_d<N} \frac{1}{n_1^{s_1}\ldots n_d^{s_d}}$$

It is the same equation with (13) without the sign $(-1)^d$. It enables to show an analogue of the theorem for the associated variant of prime multiple harmonic sums. In the proof, the sums of the type $\sum_{v=1}^{u}(-1)^v\binom{u}{v} = -1$, that appear in the rational coefficients, are replaced by sums of the form $\sum_{v=1}^{u}\binom{u}{v} = 2^u - 1$. This changes slightly the computation. The variants of multiple zeta values involving sums with large inequalities are usually refered to as "multiple zeta star values".

Let $s_1, s_2 \in \mathbb{N}^*$ such that $s_1 + s_2$ is even. The series shuffle relation (9) $h(s_1)h(s_2) = h(s_1+s_2) + h(s_1,s_2) + h(s_2,s_1)$ combined to the symmetry equation (10) which gives, by the parity assumption, that $h(s_1,s_2) \equiv h(s_2,s_1) \mod h(_{\text{weight}\geq s_1+s_2+1})$, implies an expression of the lowest weight term of $h(s_2,s_1)$ in depth one :

$$h(s_2,s_1) = \frac{1}{2}\big(h(s_2)h(s_1) - h(s_1+s_2)\big) \mod h(_{\text{weight}\geq s_1+s_2+1})$$

Combining the theorem and this equality gives :

Corollary 3.8. $h(s_2,s_1)$ admits a weight-adic expression in $h(_{\text{weight}\geq s_1+s_2+1})$. It gives a p-adic absolutely convergent equality on prime multiple harmonic sums for $p > s_1+s_2+1$.

This is a lift of the vanishing of $\zeta_{\mathcal{A}}(s_2,s_1)$ when the weight s_1+s_2 is even. Recall that this vanishing is the finite counterpart of the vanishing of even p-adic zeta values, or, equivalently, of the fact that we have $\zeta(2n) \in \mathbb{Q}\pi^{2n}$ for all $n \in \mathbb{N}^*$.

Remark 3.9. By considering the coefficient of each Λ^l in the theorem, we obtain an infinite family of relations among multiple zeta values, which is a consequence of the double shuffle and Kashiwara-Vergne equations.

The p-adic multiple zeta values of depth one admit the following expression as sums of series in terms of prime multiple harmonic sums : for all $s \in \mathbb{N}^*$, for all primes p, and for all $k \in \mathbb{N}^*$:

$$\zeta_{p,-k}(s) = \frac{p^s}{s-1}\sum_{n\geq -1} B_{n+1}(-1)^s\binom{n+s}{s-1}(p^k)^n H_{p^k}(s+n) \tag{15}$$

Recall that those numbers are also values of the Kubota-Leopoldt p-adic zeta function : $\zeta_{p,-k}(s) = p^s L_p(s,\omega^{1-s})$.

Let us apply the previous theorem to the term $n=-1$ of the series, i.e. to the factor $(p^k)^{s-1}H_{p^k}(s-1)$.

Corollary 3.10. For $p > s$, there is another series expansion of the p-adic zeta value $\zeta_{p,-k}(s)$ in terms of multiple harmonic sums, that reflect the integrality property coming from cohomology

$$\zeta_{p,-k}(s) \in \sum_{s' \geq s} \frac{p^{s'}}{s'!} \mathbb{Z}_p$$

3.3. **Lift of $\zeta^{\mathcal{M}}$ and $\zeta^{\mathcal{M}}_{\mathcal{A}}$ in terms of $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathbf{prime}}$ for $\mathbb{P}^1 - \{0,1,\infty\}$.**

3.3.1. *Introduction.* Among all the information implied by the conjecture of Kaneko and Zagier, combined to the usual conjecture of periods for multiple zeta values, one has the following assertion : the numbers

$$\sum_{n=0}^{d} (-1)^{s_{n+1}+\ldots+s_d} \zeta^{\mathcal{M}}(s_{n+1},\ldots,s_d)\zeta^{\mathcal{M}}(s_n,\ldots,s_1) \mod \zeta(2)$$

generate the $\mathbb{Q}$-vector space of motivic multiple zeta values modulo $\zeta(2)$; and thus similarly for all versions of multiple zeta values. (It is indeed implied by the conjectural equality of dimensions of the $\mathbb{Q}$-vector spaces generated by, respectively, these numbers and the usual motivic multiple zeta values modulo $\zeta(2)$).
This assertion has been proven true by Yasuda, as a consequence of the double shuffle relations [Y1].

Remark 3.11. The main technical step of Yasuda's proof is the corollary 3.3 of [Y1] : it is an expression of $\sum_{n=0}^{d}(-1)^{s_{n+1}+\ldots+s_d}\zeta^{\mathcal{M}}(s_{n+1},\ldots,s_d)\zeta^{\mathcal{M}}(s_n,\ldots,s_1)$ in as a polynomial expression over $\mathbb{Q}$ of motivic multiple zeta values of depth $\leq d-1$ and $2i\pi$. We have given a different proof of it via associator relations in [J6], where we study more generally the phenomena of depth reduction via associator relations.

3.3.2. *Statement and proof.* Using Yasuda's result, we can obtain :

Theorem 3.12. Each coefficient $\Phi^{\mathcal{M}}[w]$ of the motivic Drinfeld associator $\Phi^{\mathcal{M}}$ admits the following expression, for a formal variable Λ :

$$\Phi[w] = \sum_{n \geq 0} (\Phi^{-1}e_1\Phi)^{\mathcal{M}}[\frac{1}{1-\Lambda e_0} e_1 w_n]\Lambda^n$$

where $w_n \in \mathcal{H}_{ш}(e_0,e_1)$ is of weight equal to weight$(w)+n$.
In other words, each motivic multiple zeta value can be written as an infinite sum of the prime multiple harmonic sum motives $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}]$

This has the following consequences :

Corollary 3.13. The weight-adic completion of the algebra of motivic multiple zeta values is topologically generated by the prime multiple harmonic sum motives $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}]$.

Corollary 3.14. Every algebraic relation between motivic multiple zeta values implies a relation between prime multiple harmonic sum motives.

Now let us give the proof of the statement.

Proof. By Yasuda's theorem, the $\mathbb{Q}$-vector spaces generated by the coefficients of Φ of a given weight are generated by the numbers of the form $(\Phi^{-1}e_1\Phi)[e_1\ldots e_1]$. Thus, given a word w, there exists $z_0 \in \mathcal{H}_{ш}(e_0,e_1)$ such that

$$\Phi[w] = (\Phi^{-1}e_1\Phi)[e_1 z_0 e_1]$$

By induction on $l \in \mathbb{N}$, let $z_l \in \mathcal{H}_{ш}(e_0,e_1)$ such that

$$(\Phi^{-1}e_1\Phi)[e_1 z_l e_1] = \sum_{m=0}^{l-1} (\Phi^{-1}e_1\Phi)[e_0^{l-m} e_1 z_m e_1]$$

By the definition of the sequence (z_l) we have, for all l,

$$\Phi^{\mathcal{M}}[w] = \sum_{m=0}^{l} (\Phi^{-1}e_1\Phi)^{\mathcal{M}}[\frac{\Lambda^m}{1-\Lambda e_0} e_1 z_m e_1] - \sum_{m=0}^{l} \sum_{u \geq l-m} (\Phi^{-1}e_1\Phi)[e_0^u e_1 z_m e_1]\Lambda^{u+m}$$

The assertion is obtained by taking the limit $l \to \infty$. □

4. Transfer of algebraic relations from $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}$ to p-adic hyperlogarithms

4.1. **The problem of a theory of series of p-adic multiple zeta values.** The p-adic multiple zeta values have been first defined in [DG], §5.28. There, it is conjectured that they satisfy the same relations with complex multiple zeta values, plus the analogue of "$2i\pi = 0$". Later, it has been proved that they satisfy double shuffle relations [BF], [FJ], and associator relations with parameter zero [U2]. The analogue of the relation "$2i\pi = 0$" is reflected, among others, on the fact that the parameter of associator relations is zero, and on the fact that p-adic multiple zeta values of depth one and even weight vanish : this recasts a classical statement on values of the Kubota-Leopoldt zeta function.

In [DG] §5.28, Deligne and Goncharov also ask the following question : "*il serait intéressant aussi de disposer pour ces coefficients* [p-adic multiple zeta values] *d'expressions p-adiques qui rendent clair qu'ils vérifient des identités du type* [series shuffle relation]". Separately, the question to compute p-adic multiple zeta values in depth two, or to make an "educated guess", has also been raised by Deligne (Arizona Winter School, 2002, unpublished). This last question has been solved by Unver in [U1].

Our formula expressing prime multiple harmonic sums in terms of p-adic multiple zeta values is the first close formula, valid for every value of the depth and relating p-adic multiple zeta values and explicit functions :

$$(p^k)^{s_d+\ldots+s_1} H_{p^k}(s_d,\ldots,s_1) =$$
$$(-1)^d \sum_{d'=0}^{d} \sum_{l_{d'+1},\ldots,l_d \geq 0} \prod_{i=d'}^{d} \binom{-s_i}{l_i} \zeta_{p,-k}(s_{d'+1}+l_{d'+1},\ldots,s_d+l_d)\zeta_{p,-k}(s_{d'},\ldots,s_1)$$

On the level of prime multiple harmonic sums, the series shuffle relation is clear. More generally, as we will explain in III, all the algebraic relations of prime multiple harmonic sums can be read explicitly via elementary operations on series.

This formula suggests that the present work could be seen as an indirect algbraic theory of series for p-adic multiple zeta values, related to Deligne-Goncharov's question, and more generally of p-adic hyperlogarithms. Here, we sketch how, combining part I and some of our p-adic analytic results, to make this idea more precise and explain how to turn it into a less indirect theory. This is only one point of view among other possible points of view : we will provide a different one in part III.

4.2. **Sketch of the statements.** For simplicity, we restrict to the case of $\mathbb{P}^1 - \{0,1,\infty\}$. We showed in [J2] the following result, which shows the importance of considering all powers of Frobenius at the same time :

Proposition 4.1. The expression of prime multiple harmonic sums in terms of p-adic multiple zeta values and the integral shuffle relation of p-adic multiple zeta values characterize entirely p-adic multiple zeta values.

Precisely, we have showed that p-adic iterated integrals expressing the action of the Frobenius map F_* (of [D] §11) on the canonical paths ${}_y1_x$ of the fundamental groupoid can be expanded analytically in terms of the power of Frobenius, and that the coefficients of the expansion have a natural expression in terms of the iterated integrals associated with the Frobenius-invariant path.

Theorem 4.2. Each algebraic property among prime multiple harmonic sums, valid for all possible powers of p, is equivalent with a family of relations involving infinite sums of polynomials over $\mathbb{Q}$ of p-adic multiple zeta values.

Corollary 4.3. The explicit double shuffle and Kashiwara-Vergne relations of part I, theorems 1 and 3, for prime multiple harmonic sums are equivalent to variants of the double shuffle and Kashiwara-Vergne relations for p-adic multiple zeta values.

This gives an answer to a variant of the question of Deligne-Goncharov. It would remain to characterize more precisely these variants.

5. Interlude on the transcendence of $(\mathrm{Li}\,\mathcal{T})_{O,\text{prime}}$

We are going to see that the question of the transcendence of the sequences of algebraic numbers $(\mathrm{Li}\,\mathcal{T})_{O,\text{prime}}[w]$ is sometimes accessible. This means, heuristically, that their nature may be closer to the one of hyperlogarithms, analytic functions on $\mathbb{P}^1 - Z$, rather than their special values at tangential base-points such as multiple zeta values. We will use several times the following fact.

Fact 5.1. Let an infinite countable product $\prod_{n\in\mathbb{N}} A_n$ of $\mathbb{Q}$-algebras, such that either for all $n \in \mathbb{N}$, $A_n \subset \mathbb{C}$, or for all $n \in \mathbb{N}$, $A_n \in \mathbb{C}_p$. An element $a \in \prod_{n\in\mathbb{N}} A_n$ is transcendental as soon as it has infinitely many components that are pairwise distinct. Indeed, a polynomial $P \in \mathbb{Q}[T]$ such that $P(a) = 0$ has then infinitely many roots, and is thus equal to 0.

5.1. **The case of $\mathbb{P}^1 - \{0,1,\infty\}$.**

Proposition 5.2. Assume that $Z = \{0,1,\infty\}$. Then, all the sequences $(\mathrm{Li}\,\mathcal{T})_{O,\text{prime}}[w]$, elements of $\prod_p \mathbb{Q} \subset \prod_p \mathbb{Q}_p$, are transcendental.

Proof. We will view $\prod_p \mathbb{Q} \subset \prod_p \mathbb{C}$. Let us fix an index $(s_d,\ldots,s_1)$. The maps $n \mapsto H_n(s_d,\ldots,s_1)$ and $n \mapsto n^{s_d+\cdots+s_1}$ are strictly increasing functions $\mathbb{N}^* \to \mathbb{R}$; thus so is their product

$n \mapsto n^{s_d+\cdots+s_1} H_n(s_d, \ldots, s_1)$. This last map is in particular injective. A polynomial, not necessarily with rational coefficients, mapping infinitely many values $n^{s_d+\cdots+s_1} H_n(s_d, \ldots, s_1)$ to 0, has infinitely many roots in $\mathbb{Q}$, and is equal to 0. □

Remark 5.3. Kaneko-Zagier's finite multiple zeta values are obtained by reduction modulo large primes of these sequence of prime multiple harmonic sums. The study of their transcendence may use one day this simple fact.

5.2. **On the totally real case.** Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and let $z_{i_{d+1}}, \ldots, z_{i_1}$ be non-zero algebraic numbers whose images by this embedding are in $\mathbb{R}^{+*}$. Let w an index of the form $\left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \right)$, with $s_d, \ldots, s_1 \in \mathbb{N}^*$.

Proposition 5.4. The sequence $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[w]$, in $\prod_p \overline{\mathbb{Q}} \subset \prod_p \overline{\mathbb{Q}_p}$ is transcendental.

Proof. Viewing $\prod_p \overline{\mathbb{Q}} \subset \prod_p \mathbb{C}$ by the embedding above, the map $n \in \mathbb{N}^* \mapsto n^{\mathrm{weight}(w)} H_n[w]$ is again injective as in the previous proposition. □

5.3. **The "universal" case.** The results of part I have been stated as being valid for all curves $\mathbb{P}^1 - Z$ over a number field. Except for the motivic results, they can be restated as concerning by replacing elements of Z by formal variables. Then, the results concern the following polynomials :

Definition 5.5. Let, for $d \in \mathbb{N}^*$, and $(s_d, \ldots, s_1) \in (\mathbb{N}^*)^d$, and $n \in \mathbb{N}^*$,

$$P_{s_1,\ldots,s_d,n}(T_1, \ldots, T_d, T_{d+1}) = \sum_{0<n_1<\ldots<n_d<n} \frac{T_1^{n_1} \ldots T_d^{n_d}}{n_1^{s_1} \ldots n_d^{s_d}} T_{d+1}^n \in \mathbb{Q}[T_1, \ldots, T_d, T_{d+1}]$$

The very simple statement below can be viewed as a "universal" analogue of the previous propositions, involving implicitly all curves $\mathbb{P}^1 - Z$. We fix $d \in \mathbb{N}^*$ and $(s_1, \ldots, s_d) \in (\mathbb{N}^*)^d$.

Proposition 5.6. The polynomials $P_{s_1,\ldots,s_d,n}(T_1, \ldots, T_d, T_{d+1})$ for $n \in \mathbb{N}^*$ are pairwise distinct.

Proof. For each $n \in \mathbb{N}^*$, the degree of $P_{s_1,\ldots,s_d,n}(T_1, \ldots, T_d, T_{d+1})$ with respect to T_{d+1} is equal to n. □

5.4. **On the general case.** For a general curve $\mathbb{P}^1 - Z$, we make the following conjecture.

Conjecture 5.7. Let an index $w = \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \right)$ with $z_{i_{d+1}}, \ldots, z_{i_1} \in \overline{\mathbb{Q}} - \{0\}$. Then, the subsets I of $\mathbb{N}$ such that $H_n[\tilde{w}]$ takes the same value for all $n \in I$ are finite. In particular, there are infinitely many distinct values of $H_n[\tilde{w}]$ when n varies in $\mathbb{N}$. Then $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[w]$ is transcendental, by fact 5.1.

The reason for this conjecture is the following.

Remark 5.8. Assume first $s_d \geq 2$. If there are infinitely many n such that $H_n[w]$ takes the same value, then the limit $\lim_{n\to\infty} H_n[w]$, which is a value of an hyperlogarithm at a tangential base-point, is an algebraic number. This contradicts the usual transcendence conjectures on values of hyperlogarithms at tangential base-points. In the case where $s_d = 1$, we can make a similar reasoning using the asymptotic expansion of multiple harmonic sums, which we will recall in the next paragraph.

5.5. **The case of any complex or p-adic values of $z_{i_{d+1}},\ldots,z_{i_1}$.** The definitions of multiple harmonic sums $H_n[w]$, $n \in \mathbb{N}^*$ and of $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[w]$, extend to all indices $w = \left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{array} \right)$ with $(z_{i_{d+1}},\ldots,z_{i_1}) \in \mathbb{C}^{d+1}$ or $(z_{i_{d+1}},\ldots,z_{i_1}) \in \mathbb{C}_p^{d+1}$, i.e. where z_{i_j} are not necessarily algebraic, by the same formulas with the usual ones. As mentioned in the paragraph on the universal case, the algebraic relations of part I remain true. We are going see below that, not surprisingly, for most values of $(z_{i_{d+1}},\ldots,z_{i_1})$, the obtained sequence of prime multiple harmonic sums is trasncendental.

First, let $Q \in \mathbb{Q}[T_1,\ldots,T_d,T_{d+1}]$ be a polynomial such that its set of roots inside $\mathbb{C}^{d+1}$ or $\mathbb{C}_p^{d+1}$ satisfies that, for each $i \in \{1,\ldots,d+1\}$, its image by π_i the projection $\mathbb{C}^{d+1} \to \mathbb{C}$, resp. $\mathbb{C}_p^{d+1} \to \mathbb{C}_p$, onto the ith coordinate, is infinite. Then $Q = 0$. Indeed, this follows by induction on d using that a polynomial in one variable having infinitely many roots is equal to zero.
This, together with proposition 5.6, implies that, for $n, n' \in \mathbb{N}^*$ with $n \neq n'$, the subset of $\mathbb{C}^{d+1}$ characterized by $P_{s_1,\ldots,s_d,n} - P_{s_1,\ldots,s_d,n'} \neq 0$ is a dense open subset for the complex resp. p-adic topology. In particular

Remark 5.9. The subset of $\mathbb{C}^{d+1}$, resp. $\mathbb{C}_p^{d+1}$ such that $P_{s_1,\ldots,s_d,n} - P_{s_1,\ldots,s_d,n} \neq 0$ for all $n, n' \in \mathbb{N}^*$ such that $n \neq n'$ is dense inside $\mathbb{C}^{d+1}$, resp. $\mathbb{C}_p^{d+1}$. This follows from the facts above and Baire's theorem that a countable intersection of dense open subsets of a complete metric space is dense in the complete metric space. This very simple remark is of course not sufficient to deal with the case that we want to study, where $z_{i_{d+1}},\ldots,z_{i_1}$ are algebraic.

6. On the valuation of multiple harmonic sums and algebraic relations

We know that some of the information on the p-adic valuation of multiple harmonic sums $H_{p^k}[\tilde{w}]$ is equivalent to relations between motives, up to two conjectures : the usual period conjecture for multiple zeta values, and Kaneko-Zagier's conjecture on finite multiple zeta values.
Here, we sketch how this principle can be extended in three different directions.

6.1. **Constraints relating the p-adic valuation of H_n and $H_{p^k n}$.** Recall from our p-adic analytic work [J2], that the multiplication by a power of a prime number of the upper bound of multiple harmonic sums, under an appropriate hypothesis on $\mathbb{P}^1 - Z$, is expressed in a compact way in terms of the fundamental groupoid, in the following form :

$$(16) \qquad \left((p^k n)^{\mathrm{weight}} H_{p^k n}[\tilde{w}]\right)_{\tilde{w}} = n^{\mathrm{weight}} \left(\Phi_{0z}^{-1} e_z \Phi_{0z}\right)_{p,-k} \circ_H \left(n^{\mathrm{weight}} H_n[\tilde{w}]\right)_{\tilde{w}}$$

where $\circ_H$ is an operation that we call the harmonic Ihara action, and the sequences of multiple harmonic sums $\left((p^k n)^{\mathrm{weight}} H_{p^k n}[\tilde{w}]\right)_{\tilde{w}}$ and $\left(n^{\mathrm{weight}} H_n[\tilde{w}]\right)_{\tilde{w}}$ are indexed by words $\tilde{w} = \left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \right)$ such that $z_{i_{d+1}} = z$.

Given the expression of $\left(\Phi_{0z}^{-1} e_z \Phi_{0z}\right)_{p,-k}$ in terms of prime multiple harmonic sums, this indicates that some of the information on the p-adic valuation of multiple harmonic sums, not necessarily prime, should have an algebraic origin. Separately, there are bounds on the p-adic valuations of $\left(\Phi_{0z}^{-1} e_z \Phi_{0z}\right)_{p,-k}$ given by log-crystalline cohomology ; they imply constraints on the p-adic valuations of multiple harmonic sums $H_{p^k n}$ and H_n.

6.2. **Complex valuations of** $(\mathrm{Li}\,\mathcal{T})_O$ **for** n **large and** $(\mathrm{Li}\,\mathcal{T})_O^{\mathcal{M}}$ **.** Recall that $(\mathrm{Li}\,\mathcal{T})_O$ denotes the sequences of all multiple harmonic sums, not necessarily primes. We have seen that relations between prime multiple harmonic sums can be obtained by taking Taylor coefficients of algebraic relations between multiple polylogarithms.
We address the question of going backwards : how could we retrieve, from an algebraic relations between prime multiple harmonic sums, an algebraic relation which would be true for all, non necessarily prime, multiple harmonic sums, and thus true for multiple polylogarithms. We are going to see that this question is related to the valuation of multiple harmonic sums.

6.2.1. *Preliminary : the asymptotics of multiple harmonic sums in* $\mathbb{C}$ *when* $n \rightarrow \infty$. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let us see multiple harmonic sums $H_n\left(\begin{array}{c} z_{i_{d+1}}, \ldots, z_{i_1} \\ s_d, \ldots, s_1 \end{array} \right) = \sum_{0<n_1<\ldots<n_r<n} \frac{\left(z_{i_2}/z_{i_1}\right)^{n_1} \ldots \left(z_{i_{d+1}}/z_{i_d}\right)^{n_d} \left(1/z_{i_{d+1}}\right)^n}{n_1^{s_1} \ldots n_d^{s_d}}$ as functions of their upper bound n, viewed as a complex number. Here is a way to obtain their asymptotic expansion when $n \rightarrow \infty$. Such an asymptotic expansions appears in [CM] (at least in the case of $\mathbb{P}^1 - \{0, 1, \infty\}$), and also in [ABS]. There are many ways to obtain this asymptotic expansion ; we write here the one that seems to us a quick way to get to a formula.

By using the series shuffle product, we can reduce ourselves to two subcases
i) the case of the indices $\left(\begin{array}{c} z_{i_2}, z_{i_1} \\ 1 \end{array} \right)$. When $z_{i_2} = z_{i_1} = 1$, this is the harmonic series $H_n = 1 + \ldots + \frac{1}{n}$.
ii) the convergent case, i.e. the one of indices $s_d \geq 2$: it is the case where $H_n(\tilde{w})$ has a limit in $\mathbb{C}$ when $n \rightarrow \infty$.

The first case is classical. To deal with the second case, we can consider the complex infinite sum $\sum_{0<n_1<\ldots<n_d<\infty} \frac{\left(z_{i_2}/z_{i_1}\right)^{n_1} \ldots \left(z_{i_{d+1}}/z_{i_d}\right)^{n_d} \left(1/z_{i_{d+1}}\right)^n}{n_1^{s_1} \ldots n_d^{s_d}} \in \overline{\mathbb{Q}}$, consider a $n \in \mathbb{N}^*$, and cut the sum at n :

Proposition 6.1. We have :

$$\sum_{0<n_1<\ldots<n_d<\infty} \frac{\left(z_{i_2}/z_{i_1}\right)^{n_1}\ldots\left(z_{i_{d+1}}/z_{i_d}\right)^{n_d}\left(1/z_{i_{d+1}}\right)^{n}}{n_1^{s_1}\ldots n_d^{s_d}} = \tag{17}$$

$$\sum_{d'=0}^{d}\Bigg(\sum_{0<n_1<\ldots<n_{d'}<n} \frac{\left(z_{i_2}/z_{i_1}\right)^{n_1}\ldots\left(z_{i_{d'+1}}/z_{i_{d'}}\right)^{n_{d'}}}{n_1^{s_1}\ldots n_{d'}^{s_{d'}}}$$

$$\times \sum_{n\leq n_{d'+1}<\ldots<n_d<\infty} \frac{\left(z_{i_{d'+1}}/z_{i_{d'+1}}\right)^{n_1}\ldots\left(z_{i_{d+1}}/z_{i_d}\right)^{n_d}\left(1/z_{i_{d+1}}\right)^{n}}{n_{d'+1}^{s_{d'+1}}\ldots n_d^{s_d}}\Bigg)$$

This implies that the map $n\mapsto H_n\left(\begin{array}{c} z_{i_{d+1}},\ldots,z_{i_1} \\ s_d,\ldots,s_1\end{array}\right)$ is a $\mathbb{C}$-linear combination of maps

$$n\mapsto \sum_{n\leq n_{d'+1}<\ldots<n_d<\infty} \frac{\left(z_{i_{d'+1}}/z_{i_{d'+1}}\right)^{n_1}\ldots\left(z_{i_{d+1}}/z_{i_d}\right)^{n_d}\left(1/z_{i_{d+1}}\right)^{n}}{n_{d'+1}^{s_{d'+1}}\ldots n_d^{s_d}}\Bigg) \tag{18}$$

where $1\leq d'\leq d$ and maps $n\mapsto H_n[\tilde{w}']$ with $\text{depth}(\tilde{w}')<d$. it reduces us, by induction on d and application of the series shuffle product, to maps given by convergent iterated sums from n to ∞, and to the case of the harmonic series. We can treat the case of the maps (18) via the Euler Mac Laurin formula as follows. Take the following notations :

Notation 6.2. 1) Bernoulli polynomials : $B_0(x)=1$; for $n\in\mathbb{N}$, $B_n'(x)=nB_{n-1}(x)$ and $\int_0^1 B_n(x)dx=1$
2) Their periodic variants : for $n\in\mathbb{N}$, $P_n(x)=B_n(\{x\})$, where $\{x\}=x-\lfloor x\rfloor$.
3) Bernoulli numbers : b_n, $n\in\mathbb{N}$.

Notation 6.3. Let the following linear operators on the $\mathbb{C}$-vector space of $\mathcal{C}^\infty$ functions on a given interval of $\mathbb{R}$:
1) M_ϕ : multiplication by a function ϕ
2) ∂^i : derivative iterated i times.
3) $T_a:\phi\mapsto\phi-\phi(a)$, $a\in\mathbb{R}$.

Proposition 6.4. (Euler Mac-Laurin formula) For $[a,b]\subset\mathbb{R}$ a segment and $f:[a,b]\to\mathbb{C}$ a $\mathcal{C}^\infty$ -function we have, for all $k\in\mathbb{N}$:

$$\sum_{n=a}^{b-1} f(n)=\sum_{i=0}^{k}\frac{b_i}{i!}(f^{(i-1)}(b)-f^{(i-1)}(a))-\int_a^b \frac{B_k(\{1-t\})}{k!}f^{(k)}(t)dt$$

Then, writing the primitive of f which vanishes in a as $f^{(-1)}$ we obtain :

$$\sum_{n=a}^{b-1} f(n)=\sum_{i=0}^{k}\frac{b_i}{i!}(T_a\partial^i f)(b)-\int_a^b \frac{B_k(\{1-t\})}{k!}f^{(k)}(t)dt$$

It is also possible to write an iterated Euler Mac-Laurin formula :

Proposition 6.5. (iterated Euler Mac-Laurin formula) For $[a,b]\subset\mathbb{R}$ a segment, $r\in\mathbb{N}^*$ and $f_1,\ldots,f_r:[a,b]\to\mathbb{C}$ $\mathcal{C}^\infty$ -functions we have, for all $k\in\mathbb{N}$:

$$\sum_{a\leq n_1<\ldots<n_r\leq b-1} f_r(n_r)\ldots f_1(n_1)$$

$$= \sum_{i_1=0}^{k} \cdots \sum_{i_r=0}^{k} \frac{b_{i_1}}{i_1!} \cdots \frac{b_{i_r}}{i_r!} (T_a \partial^{i_r} M_{f_r} \ldots T_a \partial^{i_1} M_{f_1}(1))(b)$$

$$-\sum_{s=1}^{r-1} \sum_{a \leq n_{s+1} < \ldots < n_r \leq b-1} \left(\prod_{j=s+1}^{r} f_j(n_j) \right) \int_a^{n_{s+1}} \frac{B_k(\{1-t\})}{k!} \Big(M_{f_s} T_a \partial^{i_{s-1}} \ldots M_{f_1}(1) \Big)^{(k)} (t) dt$$

$$- \int_a^b \frac{B_k(\{1-t\})}{k!} \Big(M_{f_r} T_a \partial^{i_{r-1}} \ldots M_{f_1}(1) \Big)^{(k)} (t) dt$$

We will take $a = n$, and the limit $b \to \infty$ in the convergent case. It is possible to bound the integral rest as in the application of the usual Euler Mac-Laurin formula. This gives :

Proposition 6.6. For all indices $\tilde{w}$ of multiple harmonic sums, there is an asymptotic expansion of $H_n[\tilde{w}]$ in $\mathbb{C}$ when $n \to \infty$, of the form below, where $I \in \mathbb{N}$, and $a_{i,j} \in \mathbb{C}$:

$$\sum_{\substack{0 \leq i \leq I \\ 0 \leq j}} a_{i,j} \log(n)^i n^{-j}$$

Now, we take $f_1 : t \mapsto \frac{1}{t^{s_1}}, \ldots, f_r : t \mapsto \frac{1}{t^{s_r}}$, $s_i \in \mathbb{N}^*$. We want to give an idea of the shape of the formula in this case. If $f : t \mapsto \frac{1}{t^s}$, $s \in \mathbb{N}^*$, we have, for all $i \in \mathbb{N}$, $\frac{\partial^i f}{i!} : t \mapsto \binom{-s}{i} \frac{1}{t^{s+i}}$. (This extends to the case $i = -1$ if we pose $\binom{-s}{-1} = \frac{1}{1-s}$.) For all $i_1, \ldots, i_r \geq 0$ we obtain :

$$(T_a \frac{\partial^{i_r}}{i_r!} M_{f_r} \ldots T_a \frac{\partial^{i_1}}{i_1!} M_{f_1})(b) =$$
$$\sum_{P \subset \{1,\ldots,r\}} \prod_{j=1}^{r} \binom{-\epsilon_j(P)}{i_j} (-1)^{\sharp P} b^{-(s_{\max(P)+1} + i_{\max(P)+1} + \ldots + s_r + i_r)} a^{-(s_1 + i_1 + \ldots + s_{\max P} + i_{\max P})}$$

with $\begin{cases} \epsilon_j(P) = s_j \text{ if } j \in P, \\ \epsilon_j(P) = s_{j-1} + \epsilon_{j-1} + s_j \text{ if } j \notin P \end{cases}$ Now we take $a = n$, and take the limit $b \to \infty$.

Proposition 6.7. In the case where $b \to \infty$, the main term of the iterated Euler Mac-Laurin formula is : $\sum_{i_1=0}^{k} \cdots \sum_{i_r=0}^{k} \frac{b_{i_1}}{i_1!} \cdots \frac{b_{i_r}}{i_r!} (T_a \partial^{i_r} M_{f_r} \ldots T_a \partial^{i_1} M_{f_1}(1))(b)$

$$\sum_{i_1=0}^{k} \cdots \sum_{i_r=0}^{k} b_{i_1} \ldots b_{i_r} \frac{1}{N^{s_1+i_1+\ldots+s_r+i_r}} \sum_{P \subset \{1,\ldots,r\}, r \in P} \prod_{j=1}^{r} \binom{-\epsilon_j(P)(s_1,\ldots,s_r)}{i_j} (-1)^{\sharp P}$$

$$= \sum_{i=0}^{r \times k} \frac{1}{N^{s_1+\ldots+s_r+i}} \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\ldots+i_r = i}} b_{i_1} \ldots b_{i_r} \sum_{P \subset \{1,\ldots,r\}, r \in P} \prod_{j=1}^{r} \binom{-\epsilon_j(P)(s_1,\ldots,s_r)}{i_j} (-1)^{\sharp P}$$

6.2.2. *Properties of independence of multiple polylogarithms.* Hyperlogarithms satisfy standard properties of linear or algebraic independence that can be proven via their differential equation. By taking Taylor coefficients, this implies properties of independence of multiple harmonic sums functions.

Proposition 6.8. The maps $n \in \mathbb{N}^* \mapsto H_n[\tilde{w}] \in \mathbb{C}$ are linearly independent over $\mathbb{C}$.

This follows from the well-known linear independence of the hyperlogarithms on $\mathbb{P}^1 - Z$. Similarly, it follows form a property of hyperlogarithms.

Proposition 6.9. The only algebraic relations over $\mathbb{C}$ between the functions $n \in \mathbb{C}$ are the series shuffle relations.

6.2.3. *Application to prime multiple harmonic sums.* We fix $k \in \mathbb{N}^*$, the power of Frobenius and we consider the associated sequences of prime multiple harmonic sums, $(p^k)^{\text{weight}} H_{p^k}[\tilde{w}]$. The asymptotic expansion above gives us a way to try to go back, from certain relations between prime multiple harmonic sums, to relations between all multiple harmonic sums, i.e. relations between hyperlogarithms.

Proposition 6.10. Every algebraic relation between prime multiple harmonic sums H_{p^k} is true for the asymptotic expansion of the previous paragraph.

Proof. Take the limit $p \to \infty$. □

This leads us to ask the following question.

Question 6.11. Are the algebraic relations satisfied by the asymptotic expansion of multiple harmonic sums are also satisfied by sequences of multiple harmonic sums themselves ?

This amounts to a statement on the valuation of multiple harmonic sums. Take a linear combination of words $z = \sum_{j \in J} a_j \tilde{w}_j$, where $a_j \in \mathbb{C}$ and $\tilde{w}_j$ are words.
The condition that the asymptotic expansion of $H_n(z)$ is trivial amounts to say that : $H_n(z) =_{n \to \infty} o(\frac{1}{n^s})$ for all $s \in \mathbb{N}^*$.

Proposition 6.12. The implication $\big(H_n(z) =_{n\to\infty} o(\frac{1}{n^s})$ for all $s \in \mathbb{N}^* \Rightarrow H_n(z) = 0$ for all $n \in \mathbb{N}^*\big)$ would imply that all algebraic relations between prime multiple harmonic sums are true for general multiple harmonic sums, i.e. are relations between hyperlogarithms.

6.3. **p-adic valuations of $(\mathrm{Li}\,\mathcal{T})_{O,\mathbf{prime}}$ for p large and $\zeta_{\mathcal{A}}^{\mathcal{M}}$.** We have said in §5.1 that the question of the transcendence of finite hyperlogarithms may require the one of sequences of prime multiple harmonic sums, which is more accessible, as a lemma. An intermediate object between prime multiple harmonic sums and finite hyperlogarithms is Rosen's weighted finite multiple zeta values, generalized to all curves $\mathbb{P}^1 - Z$. Thus we can imagine, for the very long term, a process of transfer of properties of linear or algebraic independence :

$$\left\{ \begin{array}{c} \text{prime multiple} \\ \text{harmonic sums} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Rosen's weighted} \\ \text{finite MZV} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Kaneko-Zagier's} \\ \text{finite MZV} \end{array} \right\}$$

Left aside the motivic aspects, let us try here to analyse the passage from prime multiple harmonic sums to Rosen's version, from the point of view of p-adic valuations. Equip $\prod_p \overline{\mathbb{Q}_p}$ with the product topology associated with the p-adic topologies. The natural quotient map

$$\prod_p \overline{\mathbb{Q}_p} \to \prod_p \overline{\mathbb{Q}_p} / \overline{\oplus_p \overline{\mathbb{Q}_p}}$$

maps $(\mathrm{Li}\,\mathcal{T})_{O,\text{prime}}[w]$ to the variant of Rosen's weighted finite multiple zeta values [Ro] recalled in §3.1.
We would like to know under which condition this map induces an injective on the corresponding algebras. We have :

Remark 6.13. This injectivity would be true if any non-zero element $S = (S_p)_p = (\sum_{n \geq 0} a_n (p^k)^n H_{p^k}[w_n])_p \in \prod_p \overline{\mathbb{Q}_p}$, with $a_n \in \mathbb{Q}$, $w_n \in \mathcal{H}_{ш}(e_Z)$ of weight n, is not in

$\overline{\oplus_p\overline{\mathbb{Q}_p}}$. In other words : either $S=0$, or $\liminf_{p\to\infty} v_pS_p < \infty$, i.e. a subsequence of the sequence $(v_pS_p)_p$. stabilizes at a value not equal to ∞. We conjecture that at least a weak form of this fact (up to conditions on the sequences a_n, for example) is true.

6.4. **Rings adapted to the valuation of prime multiple harmonic sums.** We now introduce some algebraic objects, in an axiomatic way when possible, which could be useful one day to express some precise information on multiple harmonic sums. The ring $\varprojlim \big(\prod_p \mathbb{Z}/p^n\mathbb{Z}\big)/\big(\oplus_p \mathbb{Z}/p^n\mathbb{Z}\big)$, which is part of this setting, has been introduced by Rosen in [Ro].

6.4.1. *A family of rings describing stabilization of the valuation of prime multiple harmonic sums.* The most basic information provided by Kaneko-Zagier's conjecture on finite multiple zeta values is actually the question to know which of them vanish. For an index $w=(s_d,\ldots,s_1)$, the non-vanishing of $\zeta_{\mathcal{A}}(w)$ is equivalent to

$$H_p(w)\neq 0 \mod p \text{ for infinitely many } p$$

To our knowledge, this subject is entirely conjectural : until today, no proof of the existence of a non zero value $\zeta_{\mathcal{A}}(w)$ is known. For many indices w, it is expected, more precisely, that there are both infinitely many p's such that $H_p(w)\neq 0 \mod p$ and infinitely many p's such that $H_p(w)=0 \mod p$.

Denote the set of prime numbers by $P=\{p_1,p_2,\ldots\}$ with $p_n<p_{n+1}$ for all $n\in\mathbb{N}^*$. Let $\phi:\mathbb{N}^*\to\mathbb{N}^*$ a map satisfying, for all $n\in\mathbb{N}$, $\phi(n)<\phi(n+1)$. Denote by $\prod_\phi \overline{\mathbb{Q}_p}=\prod_{n\in\phi(\mathbb{N}^*)}\overline{\mathbb{Q}_{p_n}}$. Let now their subsets describing the stabilisation of the valuation.

Definition 6.14. Let

$$V_\phi^+\overline{\mathbb{Q}_p}\subset V_\phi\overline{\mathbb{Q}_p}\subset\prod_\phi\overline{\mathbb{Q}_p}$$

be defined as follows : $V_\phi\overline{\mathbb{Q}_p}$ is the subset of $\prod_\phi\overline{\mathbb{Q}_p}$ made of sequences $(x_{p_n})_{n\in\mathbb{N}^*}$ such that $v_{p_n}(x_{p_n})$ has a limit, distinct from $-\infty$, when $n\to\infty$, and $V_\phi^+\overline{\mathbb{Q}_p}$ its subset characterized by those sequences having a limit >0.

6.4.2. *Two complete topological rings.*

6.4.2.a. **Generalities**

Let (K_n,v_n) be a sequence of fields with discrete valuation $v_n:K_n\to\mathbb{Z}\cup\{+\infty\}$; We denote by $\mathcal{O}_n$ the ring of integers of K_n and $\mathfrak{m}_n$ its maximal ideal. We consider two topologies on the product $\prod_n K_n$. For any element u of $\prod_n K_n$, for each n, the term at K_n of u will be denoted by u_n.

1) Let, for all $x\in\prod_n K_n$:

$$v_i(x)=\inf_{n\in\mathbb{N}^*} v_n(x_n)$$

We have, for $x,y\in\prod_n K_n$ such that $v_i(x),v_i(y)>-\infty$,
i) $v_i(xy)\geq v_i(x)+v_i(y)$

ii) $v_i(x+y) \geq \min(v_i(x), v_i(y))$, which is an equality if $v_i(x) \neq v_i(y)$.
It defines the uniform topology i.e. $\prod \mathfrak{m}_n$-adic topology, generated by the open subsets

$$V'_{x_0,a} = \{x \in \prod_n K_n, v_i(x - x_0) \geq a\}$$

We have inclusions

$$\mathfrak{I} = \overline{\oplus_n K_n} \subset \{x \in \prod_n K_n, v_i(x) > -\infty\}$$

$$\mathfrak{I} \cap \prod_n \mathcal{O}_n = \overline{\oplus_n \mathcal{O}_n} \subset \prod_n \mathcal{O}_n = \{x \in \prod_n K_n, v_i(x) \geq 0\}$$

These are the inclusions of ideals into topological subrings of $\prod_n K_n$.

2) Let, for all $x \in \prod_n K_n$:

$$v_\infty(x) = \liminf_{n \to \infty} v_n(x_n) \in \mathbb{Z} \cup \{\pm\infty\}$$

We have, for $x, y \in \prod_n K_n$ such that $v_\infty(x), v_\infty(y) > -\infty$,
i) $v_\infty(xy) \geq v_\infty(x) + v_\infty(y)$
ii) $v_\infty(x+y) \geq \min(v_\infty(x), v_\infty(y))$, which is an equality if $v_\infty(x) \neq v_\infty(y)$.
Let the topology reflecting the valuations of the K_n's for all n infinitely large, generated by the

$$V_{x_0,a} = \{x \in \prod_n K_n, v_\infty(x - x_0) \geq a\}$$

We have inclusions

$$\mathfrak{I} = \{x, v_\infty(x) = +\infty\} = \overline{\oplus K_n} \subset \{x \in \prod_n K_n, v_\infty(x) \geq 0\}$$

$$\mathfrak{I} = \{x, v_\infty(x) = +\infty\} = \overline{\oplus K_n} \subset \{x \in \prod_n K_n, v_\infty(x) > -\infty\}$$

These are inclusions of ideals into topological subrings.

3) We are interested in the comparison, not of the two topologies themselves but of a certain quotient of them. The topological subrings of $\prod_n K_n$ of 1) and 2) are related by

$$\{x \in \prod_n K_n, v_i(x) \geq 0\} \subset \{x \in \prod_n K_n, v_\infty(x) \geq 0\} \tag{19}$$

$$\{x \in \prod_n K_n, v_i(x) > -\infty\} = \{x \in \prod_n K_n, v_\infty(x) > -\infty\} \tag{20}$$

By passing to the quotient by the ideals appearing in 1) and 2), these inclusions give bijections

$$\{x \in \prod_n K_n, v_i(x) \geq 0\}/(\mathfrak{I} \cap \{x \in \prod_n K_n, v_i(x) \geq 0\}) \simeq \{x \in \prod_n K_n, v_\infty(x) \geq 0\}/\mathfrak{I}$$

$$\{x \in \prod_n K_n, v_i(x) > -\infty\}/\mathfrak{I} \simeq \{x \in \prod_n K_n, v_\infty(x) > -\infty\}/\mathfrak{I}$$

Each member of these equalities being equipped with the quotient topologies arising from 1) and 2), we have :

Proposition 6.15. These bijections are isomorphisms of topological rings, and the topologies on these rings are all induced the same ultrametric distance arising both from v_i and v_∞.

These quotients have an axiomatic meaning :

Proposition 6.16. These two quotients are, respectively, the universal objects for all the following problems :
i) A quotient of $\{x \in \prod_n K_n, v_\infty(x) \geq 0\}$ (resp. $\{x \in \prod_n K_n, v_\infty(x) > -\infty\}$) in which the topology of 2) is metrisable.
ii) A quotient of $\{x \in \prod_n K_n, v_\infty(x) \geq 0\}$ (resp. $\{x \in \prod_n K_n, v_\infty(x) \geq -\infty\}$) in which there exists $C \in \mathbb{Z}$ such that each x is equal to a y satisfying $v_i(y) \geq C + v_\infty(x)$ (we can take $C = 0$).
iii) A quotient of both sides of (19) and (20) by an ideal which makes (19) and (20) into isomorphisms of topological rings.

Proof. Essentially follows from the definitions and from that $\mathfrak{I}$ is the intersection of the open subsets for the topology 2) which contain 0. □

Proposition 6.17. Let us denote these two quotients by, respectively, $\mathcal{Z}_{K_{n\to\infty}}$ et $\mathcal{Q}_{K_{n\to\infty}}$. The operation $(K_n)_{n\in\mathbb{N}} \mapsto (\mathcal{Z}_{K_{n\to\infty}}, \mathcal{Q}_{K_{n\to\infty}})$ commutes with the completion of topological rings, i.e. we have canonical isomorphisms of rings $\hat{\mathcal{Z}}_{K_{n\to\infty}} = \mathcal{Z}_{\hat{K}_{n\to\infty}}$ and $\hat{\mathcal{Q}}_{K_{n\to\infty}} = \mathcal{Q}_{\hat{K}_{n\to\infty}}$.

Proof. The completion of $\prod_n K_n$ equipped with the uniform topology is $\prod_n \hat{K}_n$ equipped with the uniform topology.
Then, the completions of the subrings $\{x \in \prod_n K_n, w'(x) \geq 0\}$, $\{x \in \prod_n K_n, w'(x) > -\infty\}$ and of the ideal $\{x \in \prod_n K_n, w(x) = +\infty\}$ are their analogues in $\prod_n \hat{K}_n$ are still completion morphisms. This implies the result. □

Corollary 6.18. In particular, we have $\hat{\mathcal{Z}}_{K_{n\to\infty}} = \lim_k (\prod_n \mathcal{O}_n/\mathfrak{m}_n^k)/\overline{(\oplus_n \mathcal{O}_n/\mathfrak{m}_n^k)}$.

Proof. The uniform topology on $\prod_n \mathcal{O}_n$ is the $\prod \mathfrak{m}_n$-adic topology. Hence the quotient topology on $\mathcal{Z}_{(K_n)}$ is the $\prod \mathfrak{m}_n/(\prod \mathfrak{m}_n \cap \overline{\oplus \mathcal{O}_n}) = \prod \mathfrak{m}_n/\overline{\oplus \mathfrak{m}_n}$-adic topology. $\hat{\mathcal{Z}}_{(K_n)}$ being complete, it is thus isomorphic to the projective limit of its quotients by $(\prod \mathfrak{m}_n/\overline{\oplus \mathfrak{m}_n})^k = \prod \mathfrak{m}_n^k/\overline{\oplus \mathfrak{m}_n^k}$ which are equal to $(\prod_n \mathcal{O}_n/\mathfrak{m}_n^k)/(\overline{\oplus_n \mathcal{O}_n}/\overline{\oplus_n \mathfrak{m}_n^k})$; in fine, $\overline{\oplus_n \mathcal{O}_n}/\overline{\mathfrak{m}_n^k}$ is equal to $\overline{\oplus_n \mathcal{O}_n/\mathfrak{m}_n^k}$, where the bar denotes the closure for the quotient topology of $\prod_n \mathcal{O}_n/\mathfrak{m}_n^k$. □

Remark 6.19. i) Let $n_0 \in \mathbb{N}^*$. Let, for all $n \in \mathbb{N}$, $\mathcal{L}_n = \mathcal{K}_{n+n_0}$. Then, there are canonical isomorphisms $\mathcal{Z}_{K_{n\to\infty}} \simeq \mathcal{Z}_{\mathcal{L}_{n\to\infty}}$ and $\mathcal{Q}_{K_{n\to\infty}} \simeq \mathcal{Q}_{\mathcal{L}_{n\to\infty}}$.
ii) The ideal $\mathfrak{D}$ (resp. $\mathfrak{D}'$) of divisors of 0 in $\mathcal{Z}_{K_{n\to\infty}}$ (resp. $\mathcal{Q}_{K_{n\to\infty}}$) is the set of elements $x = (x_n)$ such that there exists a subsequence of (x_n) which is identically 0. We note that $\mathcal{Q}_{K_{n\to\infty}}/\mathfrak{D}'$ is the field of fractions of $\mathcal{Z}_{K_{n\to\infty}}/\mathfrak{D}$.

6.4.2.b. **The case of** $(\mathbb{Q}, v_p)_{p \text{ prime}}$

Notation 6.20. In the case of $(K_n, v_n) = (\mathbb{Q}, v_p)$, the following notations seem natural to us :

$$\mathbb{Z}_{p\to\infty} = \prod_p \mathbb{Z}_p / \overline{\oplus_p \mathbb{Z}_p}$$

$$\mathbb{Q}_{p\to\infty} = \{(x_p) \in \prod_p \mathbb{Q}_p, \inf_p v_p(x_p) > -\infty\} / \overline{\oplus_p \mathbb{Q}_p}$$

Moreover $\mathbb{Z}_{p\to\infty} = \varprojlim (\prod_p \mathbb{Z}/p^n\mathbb{Z})/(\oplus_p \mathbb{Z}/p^n\mathbb{Z})$.

6.4.2.c. **A generalization including the ring** $\mathcal{A}$

The construction of $\mathcal{Z}_{K_{n\to\infty}}$ can be generalized, to every sequence of rings (A_n) equipped with an ultrametric distance bounded by 1, as$\prod_n A_n/\overline{\oplus A_n}$ where $\prod_n A_n$ is equipped with the uniform topology.

This can be applied to the all the sequences $(\mathbb{Z}/p^n\mathbb{Z})_{p \text{ prime}}$ which yields the rings $(\prod_p \mathbb{Z}/p^n\mathbb{Z})/(\oplus_p \mathbb{Z}/p^n\mathbb{Z})$ and is compatible with the equality $\prod_p \mathbb{Z}_p/\overline{\oplus_p \mathbb{Z}_p} = \varprojlim (\prod_p \mathbb{Z}/p^n\mathbb{Z})/(\oplus_p \mathbb{Z}/p^n\mathbb{Z})$.

6.4.3. *A discrete valuation ring.* Let $\mathcal{U}$, respectively $\mathcal{V}$, the subring of $\prod_n K_n$ made of the elements $x = (x_n)$ such that $(v_n(x_n))_{n\in\mathbb{N}^*}$ has a limit when $n \to \infty$ in $\mathbb{N} \cup \{+\infty\}$, respectively $\mathbb{Z} \cup \{+\infty\}$.

Definition 6.21. Let $\mathcal{O}_{K_{n\to\infty}} = \mathcal{U}/\mathfrak{I} \cap \mathcal{U} \subset \mathcal{Z}_{K_{n\to\infty}}$, and $K_{K_{n\to\infty}} = \mathcal{V}/\mathfrak{I} \cap \mathcal{V} \subset \mathcal{Q}_{K_{n\to\infty}}$.

Proposition 6.22. $\mathcal{O}_{K_{n\to\infty}}$ equipped with $x \mapsto \lim_{n\to\infty} v_n(x_n)$ is a discrete valuation ring. Its field of fraction is $K_{K_{n\to\infty}}$. Its residue field is $\mathcal{F}_{K_{n\to\infty}} = \{0\} \cup (\prod_n k_n^* / \bigoplus_n k_n^*)$.

Definition 6.23. We denote respectively by $\mathcal{O}_{p\to\infty}$, $K_{p\to\infty}$ et $\mathcal{F}_{p\to\infty}$ the completed discrete valuation subring, its field of fractions and its residue field, associated to the sequence $(\mathbb{Q}, v_p)_{p \text{ prime}}$.
We will denote by $v_{p\to\infty}$, the limit of the valuation when $p \to \infty$, if it exists, of an element of $\prod_p \mathbb{Q}_p$, as well as the valuation over $\mathcal{O}_{K_{n\to\infty}}$.

We thus have injections $\mathcal{O}_{p\to\infty} \hookrightarrow \mathbb{Z}_{p\to\infty}$, and $K_{p\to\infty} \hookrightarrow \mathbb{Q}_{p\to\infty}$.

The following result is a form of universality property of $\mathcal{O}_{K_{n\to\infty}}$ among the subrings $\mathcal{Z}_{K_{n\to\infty}}$ which are of discrete valuation.

Lemma 6.24. Let A be a subring of $\mathcal{Z}_{K_{n\to\infty}}$ on which $v(x) = \liminf v_n(x_n)$ defines a valuation. Then there exists a function $\varphi : \mathbb{N} \to \mathbb{N}$, strictly increasing, such that the intersection of A with the image of $\prod_n \mathcal{O}_{\varphi(n)}$ is equal to $\mathcal{O}_{K_{\varphi(n)\to\infty}}$.

Proof. Let A be such a subring. A is countable ; we denote the elements of $A - \{0\}$ by $x^1, \ldots, x^k, \ldots$, with $x^k = (x_n^k)_{n\in\mathbb{N}}$. For each $x^k \in A$, let $E_{x^k} = \{n \in \mathbb{N}/v_n(x_n^k) = \liminf_{n\to\infty} v_n(x_n^k)\}$; it is by definition an infinite set. The hypothesis implies that $\liminf_{n\to\infty} \sum_{j=1}^n v_n(x_n^j) = \sum_{j=1}^n \liminf_{n\to\infty} v_n(x_n^j)$. We can modify a representant of the x^j's by adding an element of $\mathfrak{J}$, which permits to assume, for all $j \in \mathbb{N}^*$, $n \in \mathbb{N}$, that $x^j_n \geq \liminf x^j_n$. We see then that $E_{x_1 \dots x_k} \subset \cap_{j=1}^k E_{x_j}$. Thus, $\cap_{j=1}^k E_{x_j}$ is an infinite set. We define by induction a function φ such that for all k, $\varphi([k,+\infty[) \subset E_{a_k} : \varphi(k+1)$ is an element of $E_{x_1} \cap \dots \cap E_{x_{k+1}}$ strictly superior to $\varphi(k)$. □

7. The Taylor period map

7.1. **Summary of the evidence for the existence of a Taylor period map.** Our results of part I and II can be seen as information towards the solution of the following precise problem : show that any relation between numbers $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}]$ is equivalent to a relation which is motivic up to the usual period conjectures on multiple zeta values and finite multiple zeta values.
A solution to this problem would justify entirely the view of $(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}]$ as "Taylor periods" of $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}]$ and the definitions of a "Taylor period map".
Let us explain first why we think that this problem can be settled positively. If we consider any identity between any multiple harmonic sums (e.g. not all primes at the same time), there is no reason for this identity coming from the pro-unipotent fundamental groupoid of $\mathbb{P}^1 - Z$: the first counter-example is our theorem expressing multiple harmonic sums in terms of p-adic hyperlogartihms, viewed in each $\overline{\mathbb{Q}_p}$ and not the product $\prod_p \overline{\mathbb{Q}_p}$; there, multiple harmonic sums, being algebraic numbers, are p-adic hyperlogarithms of weight 0.
What it seems that we obtain a very rigid object when we consider sequences of multiple harmonic sums, $\big((p^k)^{\mathrm{weight}} H_{p^k}[\tilde{w}]\big)_p$ with p varying in the set of all prime numbers. If a relation between prime multiple harmonic sums is true for all p, given a $k \in \mathbb{N}^*$, we expect that it comes from a relation between hyperlogarithms ; either by taking Taylor coefficients or by taking infinite sums of values at tangential base-points and using the expression of prime multiple harmonic sums in terms of p-adic hyperlogarithms.
Now, let us explain the role of the results of part I and II regarding this problem.
In I, we have retrieved algebraic relations between prime multiple harmonic sums, proved by elementary methods by Zhao, Hoffman and Rosen, as particular cases of our theorems 1 and 3, for which we know the geometric origin.

In II, we have seen in §6 that, embedding $\overline{\mathbb{Q}}$ in $\mathbb{C}$, given a relation between sequences $\big((p^k)^{\mathrm{weight}} H_{p^k}[\tilde{w}]\big)_p$, we can take its limit $p \to \infty$ and obtain a relation concerning the complex asymptotic expansion of multiple harmonic sums when the upper bound $n \to \infty$.
We have seen also in §4 that, given a relation between prime multiple harmonic sums which is, moreover, true for all powers of Frobenius - which is true for all our known examples - it implies, by considering the asymptotics with respect to the power of Frobenius, a relation between p-adic hyperlogarithms.
We also have seen in §2 that a relation between prime multiple harmonic sums implies, by reduction modulo large primes, a relation between finite multiple zeta values, which is conjecturally motivic by Kaneko-Zagier's conjecture which has been tested experimentally.
This last statements reduces our problem to obtain a process of lift of congruences. In

§3 we have constructed an implicit general lift of all congruences of for $\mathbb{P}^1 - \{0, 1, \infty\}$: this solves our problem up to a question of convergence of series, i.e. to a problem of controlling certain rational coefficients.

7.2. **Summary of the indications on the shape of the Taylor period map.** We have discussed in §1.2 in which form we would define a Taylor period map. This would involve two topological algebras, over a ring that has to be determined. One of the algebras is motivic and the other one is defined through prime multiple harmonic sums, their valuation and p-adic numbers. Let us now review the indications that we have concerning the shape of this Taylor period map.

We know that the p-adic valuation of a prime multiple harmonic sum $(p^k)^{\text{weight}(w)} H_{p^k}[w]$ is superior or equal to the weight of w. The principle of lift of congruences says the following : if the valuation of a polynomial of prime multiple harmonic sums of a given weight is strictly superior to the weight with certain additional assumptions, for example "for all p large enough", then, the corresponding polynomial of prime multiple harmonic sums can be expressed in higher weight. This gives a partial converse inequality comparing the weight and the valuation. Note that, given the relation between prime multiple harmonic sums and p-adic iterated integrals, and given the relation between the weight filtration and the Hodge filtration on the fundamental groupoid, this question is close to the one of studying the distance between the Hodge filtration and the slopes of Frobenius on the associated log-crystalline cohomology groups.

As for the topology on the algebra of prime multiple harmonic sums, we have is that certain particular parts of the information on the valuation on multiple harmonic sums (non-necessarily prime) should have a motivic lift. These are provided by Kaneko-Zagier's conjecture, and by §6.1, §6.2, §6.3.

As for the possible denominators of the rational coefficients, we have the following information. The universal families of algebraic relations of part I between prime multiple harmonic sums have rational coefficients in $\mathbb{Z}$, except for the Kashiwara-Vergne equations, for which only the one dimensional part has coefficients in $\mathbb{Z}$; we expect the rest of the equations to have non-trivial rational denominators of at most

$$1/\,\text{weight}!$$

This kind of denominators also appears in the divided powers of the log-crystalline cohomology associated to the pro-unipotent fundamental groupoid ; and also in our lift of congruences of §3.1. On the other hand, Rosen's work on lift of congruences [Ro] suggests that there might be lifts for which the denominators are much bigger, and do not give absolutely convergent p-adic series for all p large enough, but, instead, absolutely convergent sums in the ring $\big(\prod_p \mathbb{Z}_p\big)/\big(\overline{\oplus_p \mathbb{Z}_p}\big)$ - where the closure refers to the uniform topology on the product of all $\mathbb{Z}_p$'s.

7.3. **Primary form of the conjecture.** Now we can write the most primary version of the conjecture. We chose a power $k \in \mathbb{N}^*$ of Frobenius.
As in §1.2, we consider, for all primes p at the same time, the p-adic period map from

the weight-adically completed algebra of motivic hyperlogarithms, and viewed with a factor Λ^{weight}. It is a map :

$$\widehat{\mathcal{Z}_{\mathcal{M}}} \to \prod_p \mathcal{Z}_p[[\Lambda]]$$

Its restriction to the $\mathbb{Z}$-algebra topologically generated by the prime multiple harmonic sums motives $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}}$, and post-composed with the quotient by $(\Lambda-1)$ gives a map of $\mathbb{Z}$-algebras :

$$per : (\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}} \to \prod_p \mathcal{Z}_p[[\Lambda]]/(\Lambda-1)$$

that sends $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}}[w] \to (\mathrm{Li}\,\mathcal{T})_{O,\text{prime}}[w]$ for each index w. The image of per inside $\prod_p \mathcal{Z}_p[[\Lambda]]/(\Lambda-1)$ must be seen as included in $\prod_p \overline{\mathbb{Q}_p}$; the quotient by $\Lambda-1$ gives absolutely convergent series by the lower bounds of valuations of p-adic iterated integrals.

Take an integer $s \in \mathbb{N}^*$ and consider the restriction of per to the sub algebra of $(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}}$ generated by elements of weight $\geq s$; its images is and $\prod_p \mathcal{Z}_p[[\Lambda]]/(\Lambda-1)$; then compose it with the quotient with keeps only the primes $p > s$:

$$per' : \left((\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}}\right)_{\text{weight}\geq s} \to \prod_p \mathcal{Z}_p[[\Lambda]]/(\Lambda-1)_{\text{weight}\geq s} \twoheadrightarrow \prod_{p>s} \mathcal{Z}_p[[\Lambda]]/(\Lambda-1)_{\text{weight}\geq s}$$

Question 7.1. Do there exist :
- a ring of coefficients A,
- a sub A-algebra of the source of per resp. per',
- a quotient of the target of per, of per',
- topologies on these two algebras, where the topology on the target is defined purely in terms of prime multiple harmonic sums, and the one on the source is motivic, which makes them into complete topological A-algebras,

such that the map per_{Taylor} induced by per, or per' is an isomorphism of complete topological A-algebras ?

References

[ABS] J.Ablinger, J.BlÃijmlein, C.Schneider - Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms - arXiv:1302.0378

[AET] A.Alekseev, B.Enriquez, C.Torossian - Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations, Publ. Math. Inst. Hautes ÃĽtudes Sci. No. 112 (2010), 143-189

[Bes] A.Besser - Coleman integration using the Tannakian formalism, Math. Ann. 322 (2002) 1, 19-48.

[Br1] F.Brown - Single-valued multiple polylogarithms in one variable, C.R. Acad. Sci. Paris, Ser. I 338 (2004), 527-532.

[Br2] F.Brown - Single valued periods and motivic multiple zeta values - preprint, 2013, http://www.ihes.fr/ brown/SVZ5.pdf

[Br3] F.Brown - Depth-graded motivic multiple zeta values - http://www.ihes.fr/ brown/Depth.pdf

[Ber] P.Berthelot - Cohomologie rigide et cohomologie rigide Ãă support propre, PremiÃÍre partie, PrÃľpublication IRMAR 96-03, 89 pages (1996).

[BF] A.Besser, H.Furusho - The double shuffle relations for p-adic multiple zeta values, AMS Contemporary Math, Vol 416, (2006), 9-29.

[C] P.Cartier, Fonctions polylogarithmes, nombres polyzÃłtas et groupes pro-unipotents - SÃľminaire Bourbaki, exp. nÂř 885, 2000-2001, pp.137-173.

[CM] C.Costermans, H.N.Minh - Noncommutative algebra, multiple harmonic sums and applications in discrete probability - Journal of Symbolic Computation 44 (2009) 801-817

[CL] B.Chiarellotto, B.Le Stum - F-isocristaux unipotents - Compositio Math. 116, 81-110 (1999).

[D] P.Deligne, Le groupe fondamental de la droite projective moins trois points, Galois Groups over $\mathbb{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ. 16, Springer-Verlag, New York, 1989.

[DG] P. Deligne, A.B. Goncharov, Groupes fondamentaux motiviques de Tate mixtes, Ann. Sci. Ecole Norm. Sup. 38.1 , 2005, pp. 1-56

[Dr] V.G.Drinfeld - On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)", Algebra i Analiz, 2:4 (1990), 149-181

[F1] H.Furusho - p-adic multiple zeta values I – p-adic multiple polylogarithms and the p-adic KZ equation, Inventiones Mathematicae, Volume 155, Number 2, 253-286, (2004).

[F2] H.Furusho - p-adic multiple zeta values II – tannakian interpretations, Amer.J.Math, Vol 129, No 4, (2007),1105-1144.

[F3] H.Furusho - Pentagon and hexagon equations, Annals of Mathematics, Vol. 171 (2010), No. 1, 545-556.

[F3] H.Furusho - Double shuffle relation for associators, Annals of Mathematics, Vol. 174 (2011), No. 1, 341-360.

[FJ] H.Furusho, A.Jafari - Regularization and generalized double shuffle relations for p-adic multiple zeta values, Compositio Math. Vol 143, (2007), 1089-1107.

[H] M.Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, preprint (2004) arXiv:math/0401319 [math.NT]

[IKZ] K.Ihara, M.Kaneko, D.Zagier - Derivation and double shuffle relations for multiple zeta values, Compositio Math. 142 (2006) 307-338

[J1] D.Jarossay, *The Frobenius horizontal isomorphism of the pro-unipotent fundamental group of curves* $\mathbb{P}^1 - Z$ *- I*

[J2] D.Jarossay, *The Frobenius horizontal isomorphism of the pro-unipotent fundamental group of curves* $\mathbb{P}^1 - Z$ *- II*

[J3] D.Jarossay, *The Frobenius horizontal isomorphism of the pro-unipotent fundamental group of curves* $\mathbb{P}^1 - Z$ *- III*

[J4] D.Jarossay, *The Frobenius horizontal isomorphism of the pro-unipotent fundamental group of curves* $\mathbb{P}^1 - Z$ *- IV*

[J5] D.Jarossay, Double mÃľlange des multizÃłtas finis et multizÃłtas symÃľtrisÃľs, Comptes rendus MathÃľmatique 352 (2014), pp. 767-771

[J6] D.Jarossay, Depth reductions for associators, preprint.

[Ko] M.Kontsevich, Holonomic D-modules and positive characteristic, Japan. J. Math. 4, 1-25 (2009).

[Le] M.Levine. Tate motives and the vanishing conjectures for algebraic K-theory, in: Algebraic K-theory and algebraic topology (Lake Louise, 1991), p. 167?188. NATO Adv. Sci. but. Ser. C Math. Phys. 407, Kluwer (1993).

[OY] M. Ono, S.Yamamoto - Shuffle products of finite multiple polylogarithms. arxiv:1502.06693

[Ro] J.Rosen - Asymptotic relations for weighted finite multiple zeta values, arXiv:1309.0908 [math.NT]

[SaSe] K.Sakugawa, S-I Seki - On functional equations of finite multiple polylogarithms. arXiv:1509.07653

[Sh] A.Shiho - Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, J. Math. Soc. Univ. Tokyo 9 (2002), no. 1, 1?163

[Ts] H.Tsumura - Combinatorial relations for Euler-Zagier sums - Acta Arithmetica - 01/2004, 111(1) ; 27-42

[U1] S.Unver - p-adic multi-zeta values. Journal of Number Theory, 108, 111-156, (2004).

[U2] S.Unver - Drinfel'd-Ihara relations for p-adic multi-zeta values. Journal of Number Theory, 133, 1435-1483, (2013)

[V] V.Vologodsky, Hodge structure on the fundamental group and its application to p-adic integration, Moscow Math. J. 3 (2003), no. 1, 205?247.

[Y1] S. Yasuda - Finite real multiple zeta values generate the whole space Z, arxiv:1403.1134

[Y2] S. Yasuda - Slides of a talk given in Kyushu university, 2014, 22 august : "two conjectures on p-adic MZV and truncated multiple harmonic sums"

[Z] J. Zhao, Wolstenholme type theorem for multiple harmonic sums. Int. J. Number Theory 4, 2008, pp. 73-106.