ALGEBRAIC RELATIONS, TAYLOR COEFFICIENTS OF HYPERLOGARITHMS AND IMAGES BY FROBENIUS - III

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Abstract. We widen our geometric framework that enables to study the properties of multiple harmonic sums, and to understand them in terms of motives and periods. We define what we call a reindexation of the pro-unipotent fundamental groupoid of $\mathbb{P}^1 - Z$. It is a period map in a generalized sense, that enables to transport the fundamental groupoid and certain of its motivic structures into a variant. It is equipped with a "conjecture of periods".
We show results of structure on the reindexed variant of the fundamental groupoid of $\mathbb{P}^1 - Z$. We apply it, among others, to the obtention of other algebraic relations, to the theory of series of p-adic multiple zeta values, and to the interpretation in terms of motives and periods of other of our results of p-adic analysis.
Finally, we also explain, thanks to this language, how we can build ad hoc a Galois theory of iterated series adapted to prime multiple harmonic sums.

Contents

1. Introduction

1.1. **Quick overview.**

1.1.1. *General framework.* Our framework is the pro-unipotent fundamental groupoid of the curves of the type $\mathbb{P}^1$ minus a finite number of points over a number field, and the arithmetics of its periods, which are algebraic iterated integrals.
In the case of $\mathbb{P}^1 - \{0,1,\infty\}$, the periods of interest are multiple zeta values, that is to say the numbers :

$$\zeta(s_d,\ldots,s_1) = \sum_{0<n_1<\ldots<n_d} \frac{1}{n_1^{s_1}\ldots n_d^{s_d}} = (-1)^d \int_{0<t_1<\ldots<t_n<1} \wedge_{i=1}^n \frac{dt_i}{t_i - \epsilon_i} \in \mathbb{R}$$

with $d \in \mathbb{N}^*$, $s_d,\ldots,s_1 \in \mathbb{N}^*$, $s_d \geq 2$, $n = \sum_{i=1}^d s_i$, $(\epsilon_n,\ldots,\epsilon_1) = (\overbrace{0\ldots0}^{s_d-1}1,\ldots,\overbrace{0\ldots0}^{s_1-1}1)$.
In the more general case of a curve $\mathbb{P}^1 - Z$, with $Z = \{0,z_1,\ldots,z_r,1,\infty\} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$, $r \in \mathbb{N}$, analogues of multiple zeta values are the values of hyperlogarithms at tangential base points, that we will denote by $\mathrm{Li}\begin{pmatrix} z_{i_d},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{pmatrix}(z_{i_{d+1}})$.
They are defined by iterated integrals, on a special path, of the form :

$$\mathrm{Li}\begin{pmatrix} z_{i_d},\ldots,z_{i_1} \\ s_d,\ldots,s_1 \end{pmatrix}(z_{i_{d+1}}) = \int_0^z \omega_{z_{i_n}}(t_n) \int_0^{t_n} \ldots \int_0^{t_3} \omega_{z_{i_2}}(t_2) \int_0^{t_2} \omega_{z_{i_1}}(t_1)$$

1.1.2. *The algebraic interpretation of a p-adic analytic equality.* The origin of these three papers is the our work of p-adic analysis on $\pi_1^{un}(\mathbb{P}^1 - Z)$. We have proved an unexpected equality relating two different objets in a very specific way. The first object is the multiple harmonic sums associated with $\mathbb{P}^1 - Z$:

$$H_n\begin{pmatrix} z_{i_{d+1}},\ldots,z_1 \\ s_d,\ldots,s_1 \end{pmatrix} = \sum_{0<n_1<\ldots<n_r<n} \frac{\big(z_{i_2}/z_{i_1}\big)^{n_1}\ldots\big(z_{i_{d+1}}/z_{i_d}\big)^{n_d}\big(1/z_{i_{d+1}}\big)^n}{n_1^{s_1}\ldots n_d^{s_d}}$$

where $d \in \mathbb{N}^*$, $i_1,\ldots,i_{d+1} \in \{1,\ldots,r\}$, $s_d,\ldots,s_1 \in \mathbb{N}^*$, and $n \in \mathbb{N}^*$.
The second one is p-adic hyperlogarithms at tangential base points. We found that multiple harmonic sums have an expression in terms of infinite p-adic sums of p-adic hyperlogarithms at tangential base-points. The formula is the simplest in the case where n is a power of a prime number p : if $\tilde{w} = \begin{pmatrix} z_{i_{d+1}},\ldots,z_1 \\ s_d,\ldots,s_1 \end{pmatrix}$, then :

(1) $\quad (p^k)^{s_d+\ldots+s_1} H_{p^k}(\tilde{w}) =$

$$z_{i_{d+1}}^{-p^k}(-1)^d \sum_{\substack{0\leq d'\leq d \mid z_{i_{d'}}=z_{i_{d+1}} \\ l_{d'+1},\ldots,l_d\geq 0}} \prod_{i=d'}^d \binom{-s_i}{l_i}(-1)^{s_i}\,\mathrm{Li}_{p,-k}\begin{pmatrix} z_{i_{d'+1}}\ldots\ldots z_{i_{d+1}} \\ s_{d'+1}+l_{d'+1},\ldots,s_d+l_d \end{pmatrix}(z)$$

$$\times\, \mathrm{Li}_{p,-k}\begin{pmatrix} z_{i_{d'}},\ldots,z_{i_1} \\ s_{d'},\ldots,s_1 \end{pmatrix}(z)$$

where the index $-k$ in $\mathrm{Li}_{p,-k}$ refers to the power of Frobenius.

In the case of $\mathbb{P}^1 - \{0,1,\infty\}$ and $k=1$, this gives, where ζ_p denotes the p-adic analogues of multiple zeta values (here, $z_{i_1} = \ldots = z_{i_d} = 1$) :

$$(2) \quad p^{s_d+\ldots+s_1} H_p(s_d,\ldots,s_1)$$
$$= (-1)^d \sum_{d'=0}^{d} \sum_{l_{d'+1},\ldots,l_d \geq 0} \prod_{i=d'}^{d} \binom{-s_i}{l_i} \zeta_p(s_{d'+1}+l_{d'+1},\ldots,s_d+l_d)\zeta_p(s_{d'},\ldots,s_1)$$

In part I and II, we explained that, although this equality involves infinite sums, it "reflects algebraic relations" : we can built, via both sides of the equality, two variants of multiple zeta values, two periods of the same motive, a "Taylor period" and a "p-adic period".
This motive does not lie in the usual algebra of motivic hyperlogarithms, but in its completion relatively to the weight filtration. This slightly larger context has several particular aspects : one of them is that it makes sense to define period maps and period conjectures in the context, not of algebras, but of topological algebras. This leads to interpret motivically some of the information on the valuation on multiple harmonic sums.

1.1.3. *Part I.* In part I, we defined what a "prime multiple harmonic sum motive"

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}[\tilde{w}]$$

It admits a formal complex period, a formal p-adic period : they lie respectively in

$$(\mathcal{Z}/\zeta(2)\mathcal{Z})[[\Lambda]], \mathcal{Z}_p[[\Lambda]]$$

where $\mathcal{Z}$, resp. $\mathcal{Z}_p$, is the $\mathbb{Q}$-algebra of complex, respectively p-adic hyperlogarithms.
It also admits what we call a "Taylor period", which is the following :

$$(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}[\tilde{w}] = \bigg(p^k H_{p^k}[\tilde{w}]\bigg)_p \in \prod_p \overline{\mathbb{Q}_p}$$

where $\tilde{w}$ is an index $\begin{pmatrix} z_{i_{d+1}},\ldots,z_1 \\ s_d,\ldots,s_1 \end{pmatrix}$ and $k \in \mathbb{N}^*$ is the number of iterations of Frobenius.
We showed that the motive and its periods satisfy certain algebraic relations which are variants of the usual algebraic relations between multiple zeta values, that conjecturally generate all their relations. As for the Taylor period, the relations can be obtained by considering Taylor coefficients of algebraic relations between multiple polylogarithms. All those results are for us a justification for this definition of this motive and this periods.
We also defined period maps and stated the associated conjectures of periods, except for the "Taylor period" which we delayed to part II.

1.1.4. *Part II.* A generic subject of part II is the Taylor period map and its period conjecture, in the context of complete topological algebras. We show how delicate it is to formulate - our formulation at the end of part II is not as precise than the other ones - and what questions it englobes.

We study in part II the relations between the prime multiple harmonic sum motive

and its periods and, others motives and periods : the usual hyperlogarithm and multiple zeta motives, and the one attached to the finite multiple zeta values of Kaneko and Zagier. In particular, we study lift of congruences between finite multiple zeta values, and how the part I combined to our results of p-adic analysis yields a theory of series for p-adic hyperlogarithms. Finally, we explain that some precise parts of the information on the valuation of multiple harmonic sums could have a motivic avatar.
We explain how the question of the Taylor period map in the context of complete topological algebras enables to condensate several questions at the same time, on lift of congruences, their rational coefficients, and the valuation of multiple harmonic sums.

1.1.5. *Part III.* We are now going to see that the amount of information that can be formulated through motives and periods - periods in a generalized sense - is wider than what we saw in part I and II.
We are going to obtain, among other things, an essentialization of our computations of parts I and II, and a more canonical framework to understand prime multiple harmonic sum motive.

1.2. **Heuristics for part III.**

1.2.1. *A canonical framework around the prime multiple harmonic sum motive.* There is actually an empty space between the prime multiple harmonic sum motive and the pro-unipotent fundamental groupoid of $\mathbb{P}^1 - Z$. Let us recall that the relation between the two uses maps of reindexations of differential forms and of paths, that we have denoted respectively by Σ_ω and Σ_γ in part I. We are going to define a "reindexed fundamental groupoid", which keeps all the structures of the motivic fundamental group that enable to study the algebra and arihmetics of periods, and which is more directly related to the prime multiple harmonic sum motive : the role played by multiple zeta values in the usual context will be played by the prime multiple harmonic sums. It is no more a groupoid, but a finite sequence of schemes - which still contains the same arithmetic information - it is the pushed forward version of the pro-unipotent fundamental groupoid by a "reindexation" Σ :

$$\Sigma_{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}} : \pi_1^{un,dR}(X_Z) \to \Sigma_{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}}\big(\pi_1^{un,dR}(X_Z)\big)$$

1.2.2. *The completed Hopf algebra of the pro-unipotent fundamental groupoid.* An essential object in this paper will be the completion of the graded Hopf algebra of the fundamental groupoid at its canonical base point, i.e. the weight-adically complete shuffle Hopf algebra

$$\widehat{\mathcal{O}(\pi_1^{un,dR}(X_Z,\mathrm{can}))}$$

It appeared in part I, only as the target of certain re-indexation maps of differential forms Σ_ω. In the present paper, it will appear slightly more intrinsically. Intuitively, it is the algebra of functions on a "completed pro-unipotent fundamental groupoid" ; but, a priori, there is no natural notion of points of an hypothetical completed pro-unipotent fundamental groupoid. What we want to point out is that, after having applied certain reindexation maps Σ, the operation of "completing" the reindexed fundamental group $\Sigma\big(\pi_1^{un,dR}(X_Z)\big)$ can become natural.

Indeed, in our example of prime multiple harmonic sums, the completion of the reindexed fundamental group is the natural receptacle, not of prime multiple harmonic sums, but of the algebra of absolutely convergent infinite sums of prime multiple harmonic sums. Unlike for the usual context of $\pi_1^{un,dR}(X_Z)$, this algebra occurs naturally, because the valuation of multiple harmonic sums is lower bounded by their weight and because the relations between prime multiple harmonic sums involve infinite sums - or instead because the definition of the prime multiple harmonic sum motive uses a certain completion.
What we could call the absolute "completed pro-unipotent fundamental groupoid" is thus the collection of all such possible examples of completions $\Sigma\big(\widehat{\pi_1^{un,dR}(X_Z)}\big)$ that we can build naturally - involving both a comparison between the valuation and the weight filtration (we can also say the Hodge filtration instead) and a natural apparition of sums of series. Each of these examples yields a map from the universal complete algebra $\mathcal{O}(\widehat{\pi_1^{un,dR}(X_Z, \text{can})})$ to a complete topological algebra which can genuinely be viewed, in a natural way, as an algebra of functions of a certain group, whose points are generating series of periods, and which is equipped with motivic tools to study their arithmetics.

1.2.3. *The reindexation map as a period map.* In this part III, we will take, as a substitute to the motivic Galois action on the fundamental group, the Ihara action. Using it, we are going to see that the p-adic analytic equality (1) which is the origin of this work has an algebraic analogue : the equality between two different push forwards of the Ihara action ; one acts naturally on Taylor coefficients of hyperlogarithms (left hand side of the equality (1)), and the other one which acts naturally on the infinite sums of p-adic multiple zeta values (right hand side of the equality (1)). We will view the fact that the two are equal despite that they are defined differently as an additional argument to the consistency to the notion of Taylor period, as a notion by itself, and not a simple byproduct of the p-adic (or we should say adelic) period.

We are going to interpret it in terms of motives and periods : let us view the two push-forwards of the Ihara action as its two "periods", and the two pushing forward operations as "period maps", associated with the reindexation $\Sigma_{(\text{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}}}$.
Let us push further the interpretation : the result of part II that the prime multiple harmonic sum motives of $\mathbb{P}^1 - \{0,1,\infty\}$ generate the weight-adic completion of the Hopf algebra of motivic multiple zeta values, combined with the equality of the two "periods" of the Ihara action is reinterpreted as the validity of a "conjecture of periods" for the reindexation - it is accessible because the target of the map is still motivic and there are no issues of transcendence. It provides the validity of this reindexation.
A period map transports algebraic information, conjecturally to the identical. A reindexation of the fundamental group also has the same fundamental property ; we will consider the reindexed fundamental group as the "period" of the fundamental group under the reindexation.

1.2.4. *Reindexations of algebraic relations.* In the literature, there are many examples of relations between multiple zeta values, proved by elementary algebraic of analytic methods, and which have been later retreived as consequences of the double shuffle relations. The emblematic example, proved by Hoffman and Ohno by using partial fractions, is called "the cyclic sum formula" and is the following : for $d \in \mathbb{N}^*$, and $s_d, \ldots, s_1 \in (\mathbb{N}^*)^d$ such that $s_d \geq 2$, we have :

$$(3) \quad \sum_{k=1}^{d} \zeta(s_k + 1, s_{k-1}, \ldots, s_1, s_d, \ldots, s_{k+1}) = \sum_{1 \leq k \leq d} \sum_{j=0}^{s_k - 2} \zeta(s_k - j, s_{k-1}, \ldots, s_1, s_d, \ldots, s_{k+1}, j+1)$$

We will obtain in this part III a variant of the cyclic sum formula for prime multiple harmonic sums and finite multiple zeta values. It is implicitly a byproduct of the reindexation for algebraic relations that we will define.

A known method for retrieveing such equalities as byproducts of the double shuffle relations are apparented to our methods of part I. In their paper [IKZ] on double shuffle relations, Ihara[1], Kaneko and Zagier retrieve the cyclic sum formula and many other similar ones by certain specific automorphisms of the algebra of indices, which involve - although with a different point of view - a weight-adic completion.

Thus, our framework of reindexations is also an attempt to unify and essentialize our computations of part I, some computations of Ihara-Kaneko-Zagier and other computations in the same style ; we will take into account not only double shuffle relations and their variants, but also, Kashiwara-Vergne relations (and the motivic Galois action). It is an attempt to provide a systematic method of computation for proving variants of algebraic relations.

1.2.5. *A Galois theory for series.* Our reindexed fundamental group will have the following particularity : its structures such as Frobenius action or Ihara action on prime multiple harmonic sums will admit an extremely concise formula in terms of series, and their indices lying in $\mathbb{N}$.
This is not at all a usual phenomenon : the Ihara action, or its dual, the Goncharov coaction, are defined in the context of the motivic fundamental group, or at least of its Hodge realization, and have a priori nothing to do with series.

We are going to turn these observations into definitions and conjectures, expressed again by motives, periods, and period maps ; they will provide a sort of motivic Galois theory for prime multiple harmonic sums.

1.3. **Notations.** For a fully detailed account of the notations, see part I, §1 and §2. We give here the ones which will be specifically useful for this part. We will give additional notations when necessary throughout the paper.

[1]This is K.Ihara, distinct from Y.Ihara who defined the motivic Galois action on $\pi_1^{un}(\mathbb{P}^1 - \{0,1,\infty\})$ of the previous paragraph

1.3.1. *For the fundamental group.* Here, let us take again a curve $\mathbb{P}^1 - Z$ with Z a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$.
We denote the elements of Z through $Z = \{0, z_1, \dots, z_r, 1, \infty\}$, with $r \in \mathbb{N}$.

The de Rham version of the pro-unipotent fundamental groupoid will be denoted by $\pi_1^{un,dR}$. A tangential base-point $\vec{v}$ at a point x will be denoted by $\vec{v}_x$. The canonical base-point will be denoted by can.

Let e_Z be the alphabet $\{e_0, e_{z_1} \dots, e_{z_r}, e_1\}$. The shuffle Hopf algebra over $\mathbb{Q}$ relative to e_Z is denoted by $\mathcal{H}_{ш}(e_Z)$.
Let the non-commutative algebra of formal power series with variables the letters of e_Z, and coefficients in an algebra R, be denoted by

$$R\langle\langle e_Z\rangle\rangle = R\langle\langle e_0, e_{z_1} \dots, e_{z_r}, e_1\rangle\rangle$$

An element of it can be written uniquely as

$$f = f[\emptyset] + \sum_{\substack{s_d,\dots,s_0 \in \mathbb{N}^* \\ i_d,\dots,i_1 \in Z - \{0,1,\infty\}}} f[e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^{s_0-1}] e_0^{s_d-1} e_1 \dots e_0^{s_1-1} e_1 e_0^{s_0-1}$$

For each couple of base-points u, v, and for each algebra R, we have

$$\pi_1^{un,dR}(X_Z, u, v)(R) \subset R\langle\langle e_Z\rangle\rangle$$

For $i \in \{0, \dots, r+1\}$, we have linear maps

$$\partial_{e_i} \ , \ \tilde{\partial}_{e_i} : \mathcal{H}_{ш}(e_Z) \to \mathcal{H}_{ш}(e_Z)$$

defined by the removing of the letter the most at the left or at the right of a word : they are characterized uniquely by the following equality : for all $w \in \mathcal{H}_{ш}(e_Z)$,

$$w = \sum_{i=0}^{r+1} e_i \partial_{e_i}(w) = \sum_{i=0}^{r+1} \tilde{\partial}_{e_i}(w) e_i$$

The weight of a word on e_Z is its number of letters. The depth of a word w on e_Z is its number of letters distinct from e_0.

1.3.2. *Periods and their generating series.* Generating series of (complex, p-adic, motivic) hyperlogarithms, resp. multiple zeta values are elements

$$\Phi_{0z} \in \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(\mathbb{C}) \subset \mathbb{C}\langle\langle e_Z\rangle\rangle \quad , \quad (\Phi_{0z})_{p,-k} \in \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(\overline{\mathbb{Q}_p}) \subset \overline{\mathbb{Q}_p}\langle\langle e_Z\rangle\rangle$$

$$\Phi_{0z}^{\mathcal{M}} \in \pi_1^{un,mot}(X_Z, \vec{1}_0, \vec{1}_z) \in \mathcal{Z}_{\mathcal{M}}\langle\langle e_Z\rangle\rangle$$

resp., with $Z = \{0, 1, \infty\}$,

$$\Phi \in \pi_1^{un,dR}(X_Z, \vec{1}_0, -\vec{1}_1)(\mathbb{R}) \subset \mathbb{R}\langle\langle e_0, e_1\rangle\rangle, \ \Phi_{p,-k} \in \pi_1^{un,dR}(X_Z, \vec{1}_0, -\vec{1}_1)(\mathbb{Q}_p) \subset \mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle,$$

$$\Phi^{\mathcal{M}} \in \pi_1^{un,mot}(X_Z, \vec{1}_0, -\vec{1}_1) \subset \mathcal{Z}_{\mathcal{M}}\langle\langle e_0, e_1\rangle\rangle,$$

where $k \in \mathbb{N}^*$ is the number of iterations of Frobenius, and z is the extremity of the path of integration appearing in the definition of §1.1.
The fact that these are generating series of periods must be understood via the following

correspondence between words on e_Z and indices introduced before :

$$\left(\begin{smallmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{smallmatrix} \right) \leftrightarrow e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}$$

Namely, we have, for all indices :

$$\zeta(s_d, \dots, s_1) = (-1)^d \Phi[e_0^{s_d-1} e_1 \dots e_0^{s_1-1} e_1] \quad \text{etc.}$$

The factor $(-1)^{\text{depth}}$ comes from that the series expansion with respect to z of a differential form $\frac{dz}{z-z_i}$, which occurs in the iterated integral representation of multiple zeta values, comes with a negative sign. Finally, the p-adic, resp. motivic analogues of multiple zeta values are denoted by $\zeta_{p,-k}$, resp. $\zeta_{\mathcal{M}}$. The p-adic, resp. motivic analogues of hyperlogarithms are denoted by $\text{Li}_{p,-k}$, $\text{Li}^{\mathcal{M}}$.
We denote by $\zeta_{p,-\infty}$, resp. $\text{Li}_{p,-\infty}$ the numbers obtained from $\zeta_{p,-k}$, $\text{Li}_{p,-k}$ by taking limits $k \to \infty$. These are the inverse for the Ihara action of the numbers reflecting the Frobenius-invariant path in the fundamental group.
We will denote Kaneko-Zagier's finite multiple zeta values by $\zeta_{\mathcal{A}}$; their motivic versions by $\zeta_{\mathcal{A}}^{\mathcal{M}}$. They have complex and p-adic analogues, which we will denote by $\zeta_{\mathcal{A}}^{\mathcal{Z}/\zeta(2)\mathcal{Z}}$, and $\zeta_{\mathcal{A}}^{\mathcal{Z}_p}$.

1.4. **Outline.** This third part is less computational than the two others and, although we still make some computations, we give a wider role to definitions and interpretations, marked with the notions of motives and periods. The conclusion of these definitions and interpretations is a Galois theory of series at the end. Below, we mark in bold only three computations.

Until now, the motivic objects that we defined to apply them to prime multiple harmonic sums passed only through the way of infinite sums of p-adic hyperlogarithms. We restricted the use of Taylor coefficients to proving algebraic relations.
In §2 we abolish this situation. We define two variants of the Ihara action. One is adapted to the way of Taylor coefficients and arises naturally from the proof of our p-adic theorem stated above. The other uses the way of infinite sums of p-adic hyperlogarithms. We are going to prove that, although the two are defined differently, they are equal.

Theorem 1. Two "periods" of the Ihara action, $\circ_{Taylor}$ and $\circ_{\Lambda-adic}$ defined differently, and they are equal.

This is an algebraic analogue of the p-adic equality which is the origin of this work.

In §3, we recall the basics on the schemes of solutions to the usual sets of relations, such as DMR for the double shuffle relations. We define the variants arising of those schemes from part I.

We define a variant DMRS and we explain its analogy with a scheme close but not exactly equal to the one of solutions to the Kashiwara-Vergne equations.

We prove a result on those sets of relations.

Proposition 2. The shuffle relation for the Λ-adic periods of the prime multiple harmonic sum motive is equivalent to the shuffle relation for multiple zeta values.

It has an application to the algebraic theory of series of p-adic multiple zeta values : by this proposition, the shuffle relations of prime multiple harmonic sums, which can be read quite explicitly, can be seen as an indirect way to read on series the shuffle relation for p-adic multiple zeta values. This partial answer to the question of reading explicitly algebraic relations goes in a different direction from the one that we gave in II.

In §4, we discuss in general the reindexations of algebraic relations of the fundamental groupoid. It is a sort of systematic rule of computation for deriving variants of algebraic relations.
We also define a notion of "formal algebraic relations" : relations involving infinite sums of periods but that "come from geometry". We give two significative counter-examples, one complex and one p-adic, of relations involving infinite sums that are not algebraic relations. Inspired by the notion of reindexation, we prove the following result :

Proposition 3. There exists a variant of the cyclic sum formula (3) for prime multiple harmonic sums and finite multiple zeta values.

In §5, we define the fundamental group reindexed according to the prime multiple harmonic sum motive :

$$\Sigma_{\mathrm{Li}^{\mathcal{M}}_{\mathcal{O},\mathrm{prime}}}[\tilde{w}]$$

We describe, in two distinct ways, its structures : Ihara action, Frobenius, existence and unicity of invariant path by Frobenius, expansion of Frobenius with respect to the number of iterates. First, in §5.2, in the way of the fundamental groupoid, up to the reindexation ; secondly, in §5.3, relying on our p-adic analytic work, in an elementary way.
We interpret in §5.4 this elementary description in terms of periods. This gives an interpretation in terms of motives and periods of some analogies observed in our p-adic analytic work.

In §6, we explain, using again our p-adic analytic work, how the elementary description can be "indexed by $\mathbb{N}$".
We define then a framework which works as a motivic Galois theory of series to describe the properties of multiple harmonic sums. It is supported by a period conjecture which reformulate the one of the previous parts.
In §6.1 we define a variant of the fundamental groupoid "indexed by $\mathbb{N}$" ; in §6.2 we define a class of relations between prime multiple harmonic sums in terms of this framework of series.

We conclude in §7 on the motivic Galois theory of series that we have obtained.

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2. The Ihara action and the prime multiple harmonic sum motive

2.1. **Preliminaries on the de Rham fundamental groupoid.** Recall from part I that the de Rham fundamental groupoid of $\mathbb{P}^1 - Z$ admits, as base points, the rational points of $\mathbb{P}^1 - Z$, the non zero rational tangent vectors $\vec{v}_z$ at z to $\mathbb{P}^1$, and a canonical base point can. The pro-unipotent affine group scheme $\pi_1^{un,dR}(\mathbb{P}^1 - Z, \text{can})$ over $\mathbb{Q}$ is described explicitly as follows.

Definition 2.1. Let $\mathcal{H}_{ш}(e_Z)$ be the $\mathbb{Q}$-vector space $\mathbb{Q}\langle (e_{z_i})_{i=0,\dots,r+1}\rangle = \mathbb{Q}\langle e_Z\rangle$, freely generated by words over e_Z, including the empty word. It is graded by the length of words called the "weight" of words. It is a Hopf algebra, called the shuffle Hopf algebra, endowed with :
i) The shuffle product ш defined by, for all words $u_1 \dots u_m$, $u_{m+1}\dots u_{m+m'}$ over e_Z :

$$(u_1 \dots u_m) \text{ ш } (u_{m+1}\dots u_{m+m'}) = \sum_{\substack{\sigma \text{ permutation of } \{1,\dots,m+m'\} \\ \sigma(1)<\dots\sigma(m) \\ \sigma(m+1)<\dots<\sigma(m+m')}} u_{\sigma^{-1}(1)} \dots u_{\sigma^{-1}(m+m')}$$

ii) The deconcatenation coproduct $\Delta_{\text{dec}} : u_1 \dots u_m \mapsto \sum_{k=0}^{r} u_1 \dots u_k \otimes u_{k+1}\dots u_m$
iii) the counit ϵ equal to the augmentation morphism
iv) the antipode $S : u_m \dots u_1 \mapsto (-1)^m u_1 \dots u_m$.

Proposition 2.2. i) We have : $\pi_1^{un,dR}(X_Z, \text{can}) = \text{Spec}(\mathcal{H}_{ш}(e_Z))$
ii) The points of the group $\pi_1^{\text{dR}}(X, \text{can})$ are the grouplike series :

$$\pi_1^{un,dR}(X_Z, \text{can})(R) = \{f \in R\langle\langle e_Z\rangle\rangle \text{ s.t. } \Delta_{ш}(f) = f \otimes f, \epsilon(f) = 1\}$$

iii) We have

$$\text{Lie}(\mathcal{H}_{ш}^{\vee}(Z)) \otimes_{K(Z)} R = \{f \in R\langle\langle e_Z\rangle\rangle \text{ s.t. } \Delta_{ш}(f) = f \otimes 1 + 1 \otimes f\}$$

Proposition 2.3. For all points x, y, $\pi_1^{un,dR}(\mathbb{P}^1 - Z, x, y)$, has a canonical path ${}_x1_y$. These paths are compatible with the groupoid structure : we have for all x, y, z, $({}_x1_y).({}_y1_z) = ({}_x1_z)$. They induce canonical isomorphisms of schemes

$$\pi_1^{un,dR}(X_Z, x, y) \simeq \pi_1^{un,dR}(X_Z, \text{can})$$

We can work most of the time using this identification and make most of the computations in $\pi_1^{un,dR}(X_Z, \text{can})$.

2.2. **The Ihara action.**

2.2.1. *Introduction.* We will use the following operation as a substitute to the motivic Galois action of Goncharov that we used in I. It is essentially dual to it.
The computation will be through the trivialization at 0 of the fundamental torsor of paths that start at 0, obtained by left multiplication of a scheme $\pi_1^{un,dR}(X, x, \vec{1}_0)$ by the canonical path ${}_{\vec{1}_0}1_x$.

2.2.2. *The Ihara action on the restricted groupoid.* We consider the following set of couples of tangential base points of $\pi_1^{un,dR}(\mathbb{P}^1 - Z)$:

$$T = \{(\vec{1}_0, \vec{1}_0)\} \cup \{(\vec{1}_0, \vec{1}_z) \mid z \in Z - \{0, \infty\}\} \cup \{(\vec{1}_z, \vec{1}_z) \mid z \in Z - \{0, \infty\}\}$$

Definition 2.4. Let $\pi_1^{un,dR}(X_Z, t)_{t \in T}$ be the subgroupoid of $\pi_1^{un,dR}(X_Z)$ generated by the schemes $\pi_1^{un,dR}(X_Z, x, y)$ with $(x, y) \in T$. It is the "restricted fundamental groupoid".

The Ihara action is primarily an action of a certain group of automorphisms of the restricted groupoid. It factorizes the motivic Galois action of the pro-unipotent part of the motivic Galois group associated to the de Rham realization.
In the following, all the schemes of $\pi_1^{un,dR}(X_Z, t)_{t \in T}$ are identified to $\pi_1^{un,dR}(X_Z, \text{can})$ by the canonical isomorphisms. The case of roots of unity, where particular phenomena happen, is treated in [DG].

Definition 2.5. Let V' the group of sequences $v = (v_t)_{t \in T} \in \prod_{t \in T} \text{Aut}(\pi_1^{un,dR}(X_Z, t))$ such that :
1) v is compatible with the groupoid structure of $\pi_1^{un,dR}(X_Z, t)_{t \in T}$
2) For all $i \in \{0, \ldots, r+1\}$, Lie $v_{(\vec{1}_z, \vec{1}_z)}$ maps $e_{z_i} \mapsto e_{z_i}$

Definition 2.6. Here we assume that $Z = \{0, \infty\} \cup \mu_N$ with $N \in \mathbb{N}^*$. Let ξ be a primitive root of unity over $\mathbb{Q}$. Let V be the subgroup of V' which consists of the automorphisms which are, moreover, compatible with the automorphism $(z \mapsto \xi z)_*$ of the restricted groupoid.

Proposition 2.7. (Ihara, Deligne-Goncharov)
The map

$$\begin{aligned} V &\to \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_1) \\ v &\mapsto v({}_{\vec{1}_1} 1_{\vec{1}_0}) \end{aligned}$$

is an isomorphism of schemes. Via the identification $V \simeq \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_1)$ given by this isomorphism, it induces a group law on $\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_1)$, called the Ihara group law.

Notation 2.8. We will denote the Ihara action, both on the restricted groupoid and the full groupoid in the general case of $\mathbb{P}^1 - Z$, by $\circ_I$.

Proposition 2.9. (Ihara, Deligne-Goncharov)
i) In the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, the action of V, identified to $\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_1)$, on $\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_1)$ is given by

$$g \mapsto \big(f \mapsto g \circ_I f = g.f(e_0, g^{-1} e_1 g)\big)$$

and its action on $\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_0)$ is given by

$$g \mapsto \big(f \mapsto g \circ_I f = f(e_0, g^{-1} e_1 g)\big)$$

ii) In the general case of $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N)$, $N \in \mathbb{N}^*$, denote, for g a point of $\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_1)$, the image of g by the automorphism $(z \mapsto \xi^k z)_*$ of the fundamental groupoid by g_{ξ^k}. Then the analogues of the two actions above are, respectively,

$$g \mapsto \big(f \mapsto g \circ_I f = g.f(e_0, g_\xi^{-1} e_\xi g_\xi, \ldots, g_{\xi^{N-1}}^{-1} e_{\xi^{N-1}} g_{\xi^{N-1}}, g^{-1} e_1 g)\big)$$
$$g \mapsto \big(f \mapsto g \circ_I f = f(e_0, g_\xi^{-1} e_\xi g_\xi, \ldots, g_{\xi^{N-1}}^{-1} e_{\xi^{N-1}} g_{\xi^{N-1}}, g^{-1} e_1 g)\big)$$

Proof. [DG], §5, Proposition 5.11. □

Let us go back to the general case on X_Z. We only need a weaker version of this statement.

Proposition 2.10. The map

$$i : \begin{array}{l} V' \mapsto \prod_{z \in Z - \{0,\infty\}} \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z) \\ v \mapsto \big(v_{0z} = v_{(\vec{1}_0, \vec{1}_z)}({}_{\vec{1}_0}1_{\vec{1}_z})\big)_{z \in Z - \{0,\infty\}} \end{array} \tag{4}$$

is injective.
Precisely, let $z \in Z - \{0, \infty\}$. The action of V' on $\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)$ is given by

$$(v_{0z_1}, \ldots, v_{0z_r}, v_{01}) \circ_I f_z = v_{0z}.f_z(e_0, v_{0z_1}^{-1} e_{z_1} v_{0z_1}, \ldots, v_{0z_r}^{-1} e_{z_r} v_{0z_r}, v_{01}^{-1} e_1 v_{01})\big) \tag{5}$$

The action of V' on $\pi_1^{un,dR}(X_Z, \vec{1}_0)$ is given by

$$(v_{0z_1}, \ldots, v_{0z_r}, v_{01}) \circ f = f(e_0, v_{0z_1}^{-1} e_{z_1} v_{0z_1}, \ldots, v_{0z_r}^{-1} e_{z_r} v_{0z_r}, v_{01}^{-1} e_1 v_{01})\big) \tag{6}$$

Proof. Same with the classical statement. □

2.2.3. *On the whole groupoid.* Now we consider the similar action in the whole groupoid. It is sufficient for the general case, and adapted to our purposes to describe of the fundamental torsor $\pi_1^{un,dR}(X_Z, *, \vec{1}_0)$ of paths that start at $\vec{1}_0$ - it is a torsor under $\pi_1^{un,dR}(X_Z, \vec{1}_0)$. The case of $\mathbb{P}^1 - \{0, 1, \infty\}$ is treated in [Br2], §6.3, equation (6.6). We need to single out the action on the restricted groupoid, and consider the product

$$\pi_1^{un,dR}(X_Z, *, \vec{1}_0) \times \pi_1^{un,dR}(X_Z, t)_{t \in T}$$

Definition 2.11. Let x be a point of X_Z. Let V''_x be the group of sequences $v = (v_x, (v_t)_{t \in T}) \in \text{Aut}(\pi_1^{un,dR}(X_Z, \vec{1}_0, x)) \times \prod_{t \in T} \text{Aut}(\pi_1^{un,dR}(X_Z, t))$ such that :
1) v is compatible with the groupoid structure of $\pi_1^{un,dR}(X_Z, x, 0) \times \pi_1^{un,dR}(X_Z, t)_{t \in T}$
2) For all $i \in \{0, \ldots, r+1\}$, $\text{Lie}\, v_{(\vec{1}_z, \vec{1}_z)}$ maps $e_{z_i} \mapsto e_{z_i}$.

Proposition 2.12. The map

$$i_x : \begin{array}{c} V''_x \mapsto \text{Aut}(\pi_1^{un,dR}(X_Z, \vec{1}_0, x)) \times \prod_{z \in Z - \{0,\infty\}} \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z) \\ v \mapsto \big(v_{0x} = v_{(\vec{1}_0, x)}({}_{\vec{1}_0}1_x), \big(v_{0z} = v_{(\vec{1}_0, \vec{1}_z)}({}_{\vec{1}_0}1_{\vec{1}_z})\big)_{z \in Z - \{0,\infty\}}\big) \end{array} \tag{7}$$

is injective. Precisely, the action of V'' on $\pi_1^{un,dR}(X_Z, \vec{1}_0, x)$ is given by

$$\big(v_{0x}, (v_{0z_1}, \ldots, v_{0z_r}, v_{01})\big) \circ_I f_x = v_{0x}.f_z(e_0, v_{0z_1}^{-1} e_{z_1} v_{0z_1}, \ldots, v_{0z_r}^{-1} e_{z_r} v_{0z_r}, v_{01}^{-1} e_1 v_{01})\big) \tag{8}$$

and the action of V''_x on the restricted groupoid is already described as the action of V'.

Proof. Same with the previous statement. □

2.3. **Interlude on the relation between multiple harmonic sums and p-adic hyperlogarithms.** Here, we recall some of the principles of the proof of the equality (1) which is the origin of these three papers. This explains both the terminology "images by Frobenius" that we have used since the beginning, and the origin of some of the definitions below in §2.4. For the simplicity of the formulas let us write it for $\mathbb{P}^1 - \{0,1,\infty\}$, and the first power of Frobenius.

The following is a consequence of the study of p-adic overconvergent differential equation, the equation of horizontality of the Frobenius map $F_* : \pi_1^{un,dR}(X_Z) \to \pi_1^{un,dR}(X_Z^{(p)})$ with respect to the KZ connexion. Denote by Li the series expansion of multiple polylogarithms at $z = 0$, and recall that we denote by $[z^m]$ the operation of taking the coefficient of degree m in the series expansion $\sum_{m\geq 0} a_m z^m$. Let a word $w_l \in \mathcal{H}_{\text{ш}}(e_Z)$, of the form $e_0^{l-1}e_1e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1$. By minorations of valuations we can prove that the overconvergent factor tends to 1 in a certain limit and we have :

$$(9) \quad \lim_{l\to\infty} \mathrm{Li}(z^p)\big(e_0, \Phi_{p,-1}^{-1}e_1\Phi_{p,-1}\big)[w_l][z^{pn}](pn)^{\mathrm{weight}(w_l)} = \lim_{l\to\infty} \mathrm{Li}(z)(pe_0, pe_1)[w_l][z^{pn}](pn)^{\mathrm{weight}(w_l)}$$

The term of the right hand side $\mathrm{Li}(z)(pe_0, pe_1)$ actually does not depend on l, and is equal to $(pn)^{s_d+\dots+s_1}H_{pn}(s_d,\dots,s_1)$.
On the left hand side, note that , for all l,

$$(10) \quad \mathrm{Li}(z^p)\big(e_0, \Phi_{p,-1}^{-1}e_1\Phi_{p,-1}\big)[w_l][z^{pn}](pn)^{\mathrm{weight}(w_l)} = \mathrm{Li}(z)\big(pe_0, p\Phi_{p,-1}^{-1}e_1\Phi_{p,-1}\big)[w_l][z^n]n^{\mathrm{weight}(w_l)}$$

This last expression mixes coefficients of Li and coefficients of $\Phi_{p,-1}^{-1}e_1\Phi_{p,-1}$. The part of the expression related to Li does not depend on l. When taking the limit $l \to \infty$, the part related to $\Phi_{p,-1}^{-1}e_1\Phi_{p,-1}$ will give infinite sums of p-adic multiple zeta values, giving the equality (1).

The terminology "images by Frobenius" comes from that $\mathrm{Li}(z^p)\big(e_0, \Phi_{p,-1}^{-1}e_1\Phi_{p,-1}\big)$ is essentially the image of $\mathrm{Li}(z)$ by the Frobenius isomorphism, where the action

$$F_* : \pi_1^{un,dR}(\mathbb{P}^1 - \{0,1,\infty\}, \vec{1}_0, \vec{1}_0)(\mathbb{Q}_p) \to \pi_1^{un,dR}(\mathbb{P}^1 - \{0,1,\infty\}, \vec{1}_0, -\vec{1}_1)(\mathbb{Q}_p)$$

maps $e_0 \mapsto \frac{1}{p}e_0$ and $e_1 \mapsto \frac{1}{p}\Phi_{p,-1}^{-1}e_1\Phi_{p,-1}$.

2.4. Application to the prime multiple harmonic sum motive.

2.4.1. *Introduction.* Let W the set of indices $\tilde{w}$ of the form $\begin{pmatrix} z_{i_{d+1}} \dots z_{i_1} \\ s_d \dots s_1 \end{pmatrix}$ with $i \in \{1,\dots,r+1\}$ and $s_i \in \mathbb{N}^*$, including the "empty words" $\begin{pmatrix} z_{i_{d+1}} \end{pmatrix}$ (the $d = 0$ case).

Definition 2.13. Let $\mathbb{A}^W$ be the scheme $\mathbb{A}^{\mathbb{N}}$ over $\mathbb{Q}$ where $\mathbb{N}$ is viewed as being in bijection with W.

For each $N \in \mathbb{N}^*$, we have a point

$$H_N \in \mathbb{A}^W(\overline{\mathbb{Q}})$$

whose coordinates are multiple harmonic sums of upper bound N.

We denote by $1 \in \mathbb{A}^W(\overline{\mathbb{Q}})$ the point whose all components are 0, except for those of the empty word which are 1. Note that we have :

$$1 = H_1$$

2.4.2. *Pushing forward by the map* $\Phi_{0z} \mapsto \Phi_{0z}^{-1} e_z \Phi_{0z}$. For each $z \in Z - \{0, \infty\}$, we consider the map

$$\Sigma_\gamma(z) : \begin{array}{c} \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(R) \to \mathrm{Lie}^\vee \pi_1^{un,dR}(X_Z, \vec{1}_0) \otimes R \\ \Phi_{0z} \mapsto \Phi_{0z}^{-1} e_z \Phi_{0z} \end{array}$$

Let Σ_γ be the product of all $\Sigma_\gamma(z)$ for all z.

Proposition 2.14. We have a commutative diagram

$$\begin{array}{ccc} V'(R) \times \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(R) & \overset{\circ}{\longrightarrow} & \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z) \\ \downarrow \Sigma_\gamma \times \Sigma_\gamma(z) & & \downarrow \Sigma_\gamma(z) \\ \mathrm{Lie}^\vee V' \otimes R \times \mathrm{Lie}^\vee \pi_1^{un,dR}(X_Z, \vec{1}_0) \otimes R & \overset{\circ_{\Sigma_\gamma(z)}}{\longrightarrow} & \mathrm{Lie}^\vee \pi_1^{un,dR}(X_Z, \vec{1}_0) \otimes R \end{array}$$

where $\circ_{\Sigma_\gamma(z)}$ is defined by the formula :

$$(u_{0z_1}, \dots, u_{0z_{r+1}}) \circ_{\Sigma_\gamma(z)} f = f(e_0, u_{0z_1}, \dots, u_{0z_{r+1}})$$

Proof. Follows directly from the formula for the Ihara action of V' of the previous paragraph. □

Definition 2.15. We will call $\circ_{\Sigma_\gamma}$ the symmetric Ihara action.

The exponential variant is also useful. Let

$$\Sigma_\gamma^{mu}(z) : \begin{array}{c} \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(R) \to \pi_1^{un,dR}(X_Z, \vec{1}_0)(R) \\ \Phi_{0z} \mapsto \Phi_{0z}^{-1} e^{\mu e_z} \Phi_{0z} \end{array}$$

Let Σ_γ^μ be the collection of all $\Sigma_\gamma^\mu(z)$ for all z.

Proposition 2.16. We have a commutative diagram

$$\begin{array}{ccc} V'(R) \times \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(R) & \overset{\circ_I}{\longrightarrow} & \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z) \\ \downarrow \Sigma_\gamma^\mu \times \Sigma_\gamma^\mu(z) & & \downarrow \Sigma_\gamma^\mu(z) \\ \pi_1^{un,dR}(X_Z, \vec{1}_0)(R) \times \pi_1^{un,dR}(X_Z, \vec{1}_0)(R) & \overset{\circ_{\Sigma_\gamma^\mu(z)}}{\longrightarrow} & \pi_1^{un,dR}(X_Z, \vec{1}_0)(R) \end{array}$$

where $\circ_{\Sigma_\gamma^\mu}$ is defined by :

$$(u_{0z_1}, \dots, u_{0z_{r+1}}) \circ_{\Sigma_\gamma(z)} f = f(e_0, u_{0z_1}, \dots, u_{0z_{r+1}})$$

Proof. Clear. □

2.4.3. *Version adapted to the images by Frobenius.* We go one step further and push forward $\circ_{\Sigma_\gamma}$. For each $z \in Z - \{0, \infty\}$, let

$$\Sigma_{\omega,Lie}(z) : \begin{array}{c} W \to \mathcal{H}_{ш}(e_Z)[[\Lambda]] \\ \begin{pmatrix} z, z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix} \mapsto \frac{\Lambda^{1+\sum s_i}}{1-\Lambda e_0} e_{z_i} e_0^{s_d - 1} e_{z_{i_d}} \dots e_0^{s_1 - 1} e_{z_{i_1}} \end{array}$$

Its dual is a map

$$\Sigma_{\omega,Lie}^\vee(z) : \mathrm{Lie}^\vee \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)(R) \to \mathbb{A}^W(R[[\Lambda]])$$

Proposition 2.17. There exists a unique map $\circ_{\Lambda-adic}$ such that the following diagram is commuative.

$$\begin{array}{ccc} \mathrm{Lie}^{\vee} V'(R) \times \mathrm{Lie}^{\vee} \pi_1^{un,dR}(X_Z, \vec{1}_0)(R) & \overset{\circ_{\Sigma_{\gamma(z)}}}{\longrightarrow} & \mathrm{Lie}^{\vee} \pi_1^{un,dR}(X_Z, \vec{1}_0)(R) \\ \downarrow \mathrm{id} \times \Sigma^{\vee}_{\omega,Lie}(z) & & \downarrow \Sigma^{\vee}_{\omega,Lie}(z) \\ \mathrm{Lie}^{\vee} V'(R) \times \mathbb{A}^W(R[[\Lambda]]) & \overset{\circ_{\Lambda-adic}}{\longrightarrow} & \mathbb{A}^W(R[[\Lambda]]) \end{array}$$

Proof. Follows directly from the formula for the symmetric Ihara action. $\square$

Recall that the Λ-adic periods of the prime multiple harmonic sum motive are the numbers

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{Z}_p[[\Lambda]]}_{O,\mathrm{prime}}[\tilde{w}] = (\Phi_{0z}^{-1} e_z \Phi_{0z}) \left[\frac{1}{1-\Lambda e_0} e_z w \right]$$

These form a point of $\mathbb{A}^W$ which is naturally subject to the Ihara action.

We note that the case of $\tilde{w} = \emptyset$ yields $(\Phi_{0z}^{-1} e_z \Phi_{0z}) \left[\frac{1}{1-\Lambda e_0} e_z \right] = 1$, which is coherent with our convention that the component of empty words of a point of $\mathbb{A}^W$ is 1.

Remark 2.18. The $\circ_{\Lambda-adic}$ action on 1 amounts to the application of $\Sigma^{\vee}_{\omega,\mathrm{Lie}}(z)$: for all g, we have

$$g \circ_{\Lambda-adic} 1 = \Sigma^{\vee}_{\omega,\mathrm{Lie}}(z)(g)$$

2.4.4. *Version adapted to the Taylor coefficients.*

2.4.4.a. **Setup for the Ihara action on Taylor coefficients**

We denote by $\mathbb{C}_{\infty} = \mathbb{C}$, and let p be a prime number or ∞. Choose any branch of the logarithm on $\mathbb{C}_p$. Let $\epsilon \in \mathbb{R}^{+*}$ and $\mathcal{D} = \{x \in \mathbb{C}_p \mid 0 < |x| < \epsilon\} \subset X_Z(\mathbb{C}_p)$ be a punctured disk around 0. Let the set $s_{an}(\mathcal{D}, \pi_1)$ of sections of the fundamental torsor of paths $\pi_1^{un,dR}(X_Z, \vec{1}_0, *)$ over $\mathcal{D}$, that are analytic, with a logarithmic singularity at 0.

Fact 2.19. The set $s_{an}(\mathcal{D}, \pi_1)$ is in bijection with a subgroup

$$G_{\mathcal{D}} \subset \pi_1^{un,dR}(X_Z, \mathrm{can})(\overline{\mathbb{Q}}[[x]][\log(x)])$$

characterized by a convergence condition depending on $\mathcal{D}$, via the map

$$L \in G_{\mathcal{D}} \mapsto \left(x \in \mathcal{D} \overset{L}{\to} \pi_1^{un,dR}(X_Z, \mathrm{can})(\overline{\mathbb{Q}}[[x]][\log(x)]) \to \pi_1^{un,dR}(X_Z, \vec{1}_0, x)(\mathbb{C}_p) \right) \tag{11}$$

where the last arrow is the composition of the substitution of x to X, and the canonical isomorphism between $\pi_1^{un,dR}(X_Z, \mathrm{can})$ and $\pi_1^{un,dR}(X_Z, \vec{1}_0, x)$.
We will not choose here a particular convergence condition ; only work abstractly with $\mathcal{D}$.

Now let $s(\mathcal{D}, V'')$ be the set of sections on $\mathcal{D}$ (not necessarily analytic) of the bundle V'' appearing in §2.2.3. It acts on $s_{an}(\mathcal{D}, \pi_1)$ by the Ihara action, and this action factorizes through the map

$$s(\mathcal{D}, V'') \hookrightarrow s(\mathcal{D}, \pi_1) \times \prod_{t \in T} \pi_1^{un,dR}(X_Z, t)$$

where $s(\mathcal{D}, \pi_1)$ is the set of sections (not necessarily analytic) on $\mathcal{D}$ of the fundamental torsor at $\vec{1}_0$.

The proof of the relation between multiple harmonic sums and p-adic hyperlogarithms indicates that, for our purposes, we can restrict to the term of order zero with respect to the action of $s(\mathcal{D},\pi_1)$, that is, replace it by 1. A full formula would necessitate to write a Taylor expansion of the analytic sections in $s(\mathcal{D},V'')$.
The Ihara action on the term of order zero is described by the following proposition. Let

$$\text{pr} : \text{Aut}(\pi_1^{un,dR}(X_Z,\vec{1}_0,x)) \times \prod_{z\in Z-\{0,\infty\}} \pi_1^{un,dR}(X_Z,\vec{1}_0,\vec{1}_z) \to \text{Aut}(\pi_1^{un,dR}(X_Z,\vec{1}_0,x))$$

be the natural projection.

Proposition 2.20. Let x a point of X_Z. Let i_x be the map of equation (7). The subscheme

$$V'_x = \ker(\text{pr}\circ i_x) \subset V''_x$$

is canonically isomorphic (via a certain isomorphism f) to the scheme V' of §2.2.2, which does not depend on x ; its Ihara action on $\pi_1^{un,dR}(X_Z,\vec{1}_0,x)$ is equal to the Ihara action of V' on $\pi_1^{un,dR}(X_Z,\vec{1}_0)$. Precisely, there is a commutative diagram

$$\begin{array}{ccc} V'_x \times \pi_1^{un,dR}(X_Z,\vec{1}_0,x) & \overset{\circ}{\longrightarrow} & \pi_1^{un,dR}(X_Z,\vec{1}_0,x) \\ \downarrow f\times \text{can} & & \downarrow \text{can} \\ V'\times \pi_1^{un,dR}(X_Z,\vec{1}_0) & \overset{\circ}{\longrightarrow} & \pi_1^{un,dR}(X_Z,\vec{1}_0) \end{array}$$

where can is the canonical isomorphism induced by the canonical base-point.

Proof. The fact that V'_x is canonically isomorphic to V' follows from their definition as groups of sequences of automorphisms. The commutativity of the diagram follows from the formulas (6) and (8). □

Corollary 2.21. Composing the canonical isomorphism $\pi_1^{un,dR}(X_Z,\vec{1}_0,x) \simeq \pi_1^{un,dR}(X_Z,\vec{1}_0)$ for all $x\in\mathcal{D}$, with the isomorphism (11), the term of the Ihara action on $s_{an}(\mathcal{D},\pi_1)$ of order zero with respect to the action of $s(\mathcal{D},\pi_1)$ is the Ihara action over a subgroup of

$$\pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])$$

Below, we will not pick a particular $\mathcal{D}$: our object of interest is now the Ihara action on $\pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])$.

2.4.4.b. **Taking Taylor coefficients and limits**

For all this paragraph, we fix an $n\in\mathbb{N}^*$.

Definition 2.22. Let, for each $l\in\mathbb{N}^*$, a map of "coefficient of degree n in the series expansion"

$$\begin{array}{ccc} \pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)]) & \overset{(l)}{\underset{\text{deg n}}{\longrightarrow}} & \mathbb{A}^W(\overline{\mathbb{Q}}) \\ L & \longmapsto & L^{(l)}_{\text{deg n}} = \left(n^{l+\text{weight}(w)} L[x^n\log(x)^0][e_0^{l-1}w]\right)_{\tilde{w}\in W} \end{array}$$

where, for each $\tilde{w} = \begin{pmatrix} z_{i_{d+1}},\dots,z_{i_1} \\ s_d,\dots,s_1 \end{pmatrix}$, the associated w is $e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}$.

Proposition 2.23. Let the S be the subset of $\pi_1^{un,dR}(X_Z,\vec{1}_0)(R[[X]][\log(X)])$ made of the elements f such that $f^{(l)}_{\text{deg n}}$ does not depend on l. There is a unique map $\circ_{\text{deg n}}$ characterized by the commutative diagram

$$\begin{array}{ccc} V'(R)\times S & \overset{\circ}{\longrightarrow} & S \\ \downarrow \text{id} \times^{(l)}_{\text{deg n}} & & \downarrow {}_{\text{deg n}} \\ V'(R[[\Lambda]])\times \mathbb{A}^{W}(\overline{\mathbb{Q}}) & \overset{\circ^{(l)}_{\text{deg n}}}{\longrightarrow} & \mathbb{A}^{W}(\overline{\mathbb{Q}}) \end{array}$$

The following definition, which involves a limit, can be defined directly on the level of the Hopf algebras, by involving a completion. Since we have chosen the point of view of groups, let us consider the map

$$V'(R)\hookrightarrow V'(R)[[\Lambda]]$$

which maps each component $f=\sum_w f[w]w$ of $i(V')$ to $\sum_w f[w]w\Lambda^{\text{weight}}$ and view $\circ^{(l)}_{\text{deg n}}$ through this embedding. Then it makes sense to define :

Definition 2.24. Let

$$\circ_{Taylor,n}=\lim_{l\to\infty}\circ^{(l)}_{\text{deg n}}:V'(R)\times\mathbb{A}^{W}(\overline{\mathbb{Q}})\longrightarrow\mathbb{A}^{W}(\overline{\mathbb{Q}}[[\Lambda]])$$

2.4.5. *Theorem.* We can now state the theorem.

Theorem 1. We have :

$$\circ_{Taylor,n}|_{S_{\text{deg n}}}=\circ_{\Lambda-adic}$$

Proof. Follows from the definitions of the two Ihara actions, and the usual combinatorics of the composition of formal series $f(e_0,g)$ in terms of subwords and quotient words, using the key hypothesis that we have restricted to elements of $\pi_1^{un,dR}(X_Z,\vec{1}_0)(R[[x]][\log(x)])$ whose image by ${}^{l}_{\text{deg n}}$ is independent of l. □

This is an algebraic analogue of the equality between Taylor period and the p-adic period of the prime multiple harmonic sum motive. We see it as an additional reason to view the Taylor period as a period by itself and not just a byproduct of the p-adic period. We also view it as a reason why it makes sense to push-forward the fundamental group by this map.
It seems to us that we can see it in the language of motives and periods, by saying that $\circ_{Taylor,n}|_{S_{\text{deg n}}}$ and $\circ_{\Lambda-adic}$ are two different "periods" of the Ihara action by the push-forward, which are equal.

3. On the sets of solutions to the relations of part I - p-adic applications

In this part, for the simplicity of the formulas, we consider the case of $\mathbb{P}^1-\{0,1,\infty\}$.

3.1. **Preliminaries.** We have recalled the definition of the shuffle Hopf algebra in §2.1. Let us now recall the definition of the quasi-shuffle Hopf algebra.

Definition 3.1. The quasi-shuffle or series shuffle graded Hopf algebra $\mathcal{H}_*$ is the $\mathbb{Q}$-vector space $\mathbb{Q}\langle(y_s)_{s\in\mathbb{N}^*}\rangle$ of words over Y, including the empty word $y_0=1$, graded by

the length of words. It is endowed with the following structures :
i) the quasi-shuffle product $*$, defined recursively by, for w_1, w_2 words, and $s, s' \in \mathbb{N}^*$,

$$y_s w_1 * y_{s'} w_2 = y_s(w_1 * y_{s'} w_2) + y_{s'}(y_s w_1 * w_2) + y_{s+s'}(w_1 * w_2)$$

Each word appearing in the expression of $w_1 * w_2$ as sum of words is called a "series shuffle element" of (w_1, w_2).
ii) the deconcatenation coproduct Δ_{dec} relative to words in the y_s's
iii) the counit ϵ equal to the augmentation morphism
iv) and the antipode given by the two following formulae

$$z_{s_d,\ldots,s_1} = \sum_{\substack{1 \leq l \leq d \\ 1=i_1<i_2<\ldots<i_{l+1}=d}} y_{\sum_{i=i_l}^{i_{l+1}=d} s_i} \ldots y_{\sum_{i=i_1}^{i_2} s_i}$$

Then

$$S(y_{s_d} \ldots y_{s_1}) = (-1)^d z_{s_1,\ldots,s_d} = \sum_{\substack{l \geq 1 \\ y_{s_d}\ldots y_{s_1} = w_l \ldots w_1}} (-1)^l w_l * \ldots * w_1$$

Fact 3.2. The completed dual $\widehat{\mathcal{H}_*^\vee}$ of $\mathcal{H}_*$ is the non-commutative algebra of series $\mathbb{Q}\langle\langle (y_s)_{s \in \mathbb{N}} \rangle\rangle$, equipped with the (continuous) coproduct Δ_*, satisfying $\Delta_*(y_n) = \sum_{k=0}^n y_k * y_{n-k}$.

Definition 3.3. i) Let $\mathrm{inv} : \mathcal{H}_* \to \mathcal{H}_*$ the unique linear map sending $y_{s_d} \ldots y_{s_1} \mapsto (-1)^{s_1+\ldots+s_d} y_{s_d} \ldots y_{s_1}$
ii) Let $(\Sigma_\omega)_* : \mathcal{H}_* \to \widehat{\mathcal{H}_*}$ the unique continuous linear map sending : $y_{s_d} \ldots y_{s_1} \mapsto \sum_{l_1,\ldots,l_d \geq 0} \Lambda^{l_1+\ldots+l_d} \prod_{i=1}^d \binom{-s_i}{l_i} y_{s_d+l_d} \ldots y_{s_1+l_1}$.

Let also $r : R\langle\langle e_0, e_1 \rangle\rangle \to R\langle\langle (y_n)_{n \in \mathbb{N}} \rangle\rangle$ be the unique continuous linear map defined on words by $\begin{cases} e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1 \mapsto y_{s_d} \ldots y_{s_1} \\ w = z e_0 \mapsto 0 \end{cases}$.

We review the usual sets of solutions to the algebraic relations of I.

Definition 3.4. (Drinfeld) Let M_μ be the scheme of couples (μ, Φ) satisfying the associator relations.

Definition 3.5. (Racinet : [Ra], IV, 1.2, définition 1.3)
Let R a $\mathbb{Q}$-algebra. Let $\mathrm{DMR}(R)$ be the set of couples (Φ, Φ_*) of $R\langle\langle e_0, e_1 \rangle\rangle \times R\langle\langle (y_n)_{n \in \mathbb{N}} \rangle\rangle$, satisfying : $\Phi[\emptyset] = 1$, $\Phi[e_0] = \Phi[e_1] = 0$, and

$$\Delta_ш \Phi = \Phi \otimes \Phi$$

$$\Delta_* \Phi_* = \Phi_* \otimes \Phi_*$$

$$\Phi_* = e^{\sum_{n \geq 2} \frac{(-1)^{n-1}}{n} \Phi[e_0^{n-1} e_1] e_1^n} r(\Phi)$$

Drinfeld has shown in [Dr] that the preservation of the associator relations by a law of composition that is nothing but the Ihara group law. Racinet has shown in [Ra] similar results for double shuffle relations.

3.2. **Defintions and comments.** Our proofs of part I, theorem 1 and theorem 3 and the definitions of DMR and M lead naturally to the following definitions.

3.2.1. *The variants adapted to* $\Phi^{-1}e_1\Phi$.

Definition 3.6. Let DMRS be the scheme defined as follows : for R a $\mathbb{Q}$-algebra, DMRS(R) is the set of couples $(\Theta, \Theta_*^{(l)}) \in R\langle\langle e_0, e_1\rangle\rangle \times (R\langle\langle (y_n)_{n\in\mathbb{N}}\rangle\rangle)^{\mathbb{N}}$ such that :

$$\Delta_{ш}\Theta = \Theta \otimes 1 + 1 \otimes \Theta$$

$$\text{For all } l \in \mathbb{N}, \qquad \Delta_*(\Theta_*^{(l)}) = \Theta_*^{(l)} \otimes \Theta_*^{(l)}$$

$$\text{For all } l \in \mathbb{N}, \qquad \Theta_*^{(l)}[s_d, \ldots, s_1] \equiv \Theta[e_0^l e_1 e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1] \mod \Theta[e_1 e_0 e_1]$$

Part of our proof of theorem 1 in part I, §4 reformulates as :

Proposition 3.7. We have a map DMR $\to$ DMRS, given by

$$(f, f_*) \mapsto \big(f^{-1}e_1 f, \big(\operatorname{inv}(f_*(l))f_*\big)_{l\in\mathbb{N}}\big)$$

where $f_*(l)$ is the coefficient of Λ^l in $(\Sigma_\omega)_*(f)$.

Remark 3.8. This definition of DMRS can be made without any reference to the fact that Θ can be of the form $\Phi^{-1}e_1\Phi$ with Φ a grouplike series.
This is not true anymore if we deal with the full setting $\zeta^{\mathcal{M}}(2) \neq 0$, we have to make a reference to Φ.

The second fact that we want to point out is a remark rather than a definition. Following [AET], the following equation is satisfied by the monodromy automorphism μ_Φ attached to an associator Φ :

$$\text{"}\operatorname{Ad}\Phi(t_{12}, t_{23}) \circ \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}\text{"} \tag{12}$$

It is close but not exactly equivalent to the Kashiwara-Vergne relations.

Since both DMRS and this relation carry a family of algebraic relations for the prime multiple harmonic sum motive, as we saw in part I, we find that there is an analogy between the map DMR $\to$ DMRS and the map from the scheme of associators to the scheme of solutions to this equation.
It is surprising because, usually we always think of three families of universal algebraic relations : double shuffle, associator and Kashiwara-Vergne relations. Here, it seems as if there was four different schemes : DMR, M, DMRS, and the scheme of solutions to (12) that we could denote by MS.

3.2.2. *The variants adapted to prime multiple harmonic sums.*

Definition 3.9. Let DMR$_{har}$ be the scheme of solutions to the double shuffle equations of prime multiple harmonic sums of theorem 1 of part I.

Definition 3.10. Let KV$_{har}$ be the scheme of solutions to the Kashiwara-Vergne equations of prime multiple harmonic sums of theorem 3 of part I.

Both are subschemes of $\mathbb{A}^W$.

3.3. **The case of the shuffle relation and application to the theory of series of p-adic multiple zeta values.** Let us recall Deligne-Goncharov's question on the theory of series of p-adic multiple zeta values ([DG], S5.28) : "*il serait intéressant aussi* *de disposer pour ces coefficients* [p-adic multiple zeta values] *d'expressions p-adiques qui rendent clair qu'ils vérifient des identités du type* [series shuffle relations]".

The same question can be asked for all the families of relations that p-adic multiple zeta values satisfy.
We gave a partial answer to it in part II, based on the expression of the Frobenius, and in particular of prime multiple harmonic sums, with respect to the number of iterations of Frobenius, with coefficients expressed in terms of the Frobenius-invariant paths, i.e. Furusho's p-adic multiple zeta values.

Here we give another approach to the same problem. We construct a bridge, in the case of the shuffle relation :

$$\left\{ \begin{array}{c} \text{shuffle relation for} \\ \text{the p-adic formal period of} \\ \text{the prime multiple harmonic sum motive} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{shuffle relation for} \\ p\text{-adic multiple zeta values} \end{array} \right\}$$

Thus, it is a bridge from relations between prime multiple harmonic sums to relations between p-adic multiple zeta values *up to* the comparison between several periods of a same motive.

Proposition 3.11. Let f be a point of $\pi_1^{un,dR}(\mathbb{P}^1 - \{0,1,\infty\}, \text{can})$, satisfying $f[e_1^n] = 0$ for all $n \in \mathbb{N}^*$.
i) We have the following equivalence :
f satisfies the shuffle relation , i.e. $\Delta_{ш}(f) = f \otimes f \Leftrightarrow f^{-1}e_1 f$ satisfies the shuffle relation modulo products, i.e. $\Delta_{ш}(f) = f \otimes 1 + 1 \otimes f$.
ii) We have the following equivalence :
$f^{-1}e_1 f$ satisfies the shuffle relation modulo products $\Leftrightarrow$ the map

$$w \in \ker \tilde{\partial}_{e_1} \mapsto (f^{-1}e_1 f)\left[\frac{1}{1-\Lambda e_0} e_1 w\right]$$

satisfies the shuffle relation of prime multiple harmonic sums, i.e.

$$(\text{Li}\,\mathcal{T})_{O,\text{prime}}[w \text{ ш } w'] = (\text{Li}\,\mathcal{T})_{O,\text{prime}}[(\Sigma_\omega)_* \circ \text{inv})(w')w]$$

and we have, for all words $w \in \ker \tilde{\partial}_{e_1}$,

$$(f^{-1}e_1 f)[w \text{ ш } e_0] = 0$$

Proof. i) The fact that $f^{-1}e_1 f$ is a Lie series is equivalent to say that $\Delta_{ш}(f)(f \otimes f)^{-1}$ commutes to $\Delta_{ш}(e_1)$. Thus the statement is equivalent to the following one : let $u \in R\langle\langle e_0, e_1\rangle\rangle \otimes R\langle\langle e_0, e_1\rangle\rangle$. Then u commutes to $\Delta_{ш}(e_1)$ if and only if $u \in R\langle\langle e_1\rangle\rangle \otimes R\langle\langle e_1\rangle\rangle$. Let us prove it. For u in $R\langle\langle e_1\rangle\rangle \otimes R\langle\langle e_1\rangle\rangle$, we have :

$$(\Delta_{ш}(e_1)u)[w \otimes w'] = u[\partial_{e_1}(w) \otimes w'] + u[w \otimes \partial_{e_1}(w')]$$

$$(u\Delta_{ш}(e_1))[w \otimes w'] = u[\tilde{\partial}_{e_1}(w) \otimes w'] + u[w \otimes \tilde{\partial}_{e_1}(w')]$$

Let $(w, w') \in W \times W$, where W is the set of words in e_0, e_1, with at least one among w, w' not of the form e_1^N, $N \geq 0$ - we can assume that it is w - we show that $u[w \otimes w'] = 0$.

$$u[w \otimes w'] = u[\tilde{\partial}_{e_1}(we_1) \otimes w']$$

$$= (u\Delta_{ш}(e_1))[we_1 \otimes w'] - u[we_1 \otimes \tilde{\partial}_{e_1}(w')] = (\Delta_{ш}(e_1)u)[we_1 \otimes w'] - u[we_1 \otimes \tilde{\partial}_{e_1}(w')]$$
$$= u[\partial_{e_1}(w)e_1 \otimes w'] + u[we_1 \otimes \partial_{e_1}(w')] - u[we_1 \otimes \tilde{\partial}_{e_1}(w')]$$

Because of the hypothesis on w, the index of nilpotence for ∂_{e_i} is strictly smaller for $\partial_{e_1}(w)e_1$ than for w. The result then follows by induction on $m+m'+k$ where m,n,k, are respectively the smallest integers satisfying :

$$\partial_{e_1}^{m}(w) = 0, \partial_{e_1}^{m'}(w') = 0, (\tilde{\partial}_{e_1})^k(w') = 0$$

ii) The fact that $f^{-1}e_1f$ satisfies the shuffle relation modulo product is equivalent to the fact that, for all words w,w', we have $(f^{-1}e_1f)[w \text{ ш } w'] = 0$.

We have to go back to the proof of the shuffle relations for the prime multiple harmonic sum motive : part I, §4. There, we have shown that, for all w,w' words, and $s \in \mathbb{N}^*$, the following formal infinite sum of words :

$$-\frac{1}{1-\Lambda e_0}e_1\left[(e_0^{s-1}e_1w)шw' - wш\left(\frac{e_0^{s-1}e_1}{(1-\Lambda e_0)^s}e_1w'\right)\right]$$

was a linear combination of shuffles, more precisely equal to

$$\sum_{k=0}^{s-1}(e_0^k e_1 w) \text{ ш } (-1)^{s-k}\frac{e_0^{s-1-k}}{(1-\Lambda e_0)^{s-k}}e_1w'$$

Thus, the shuffle relation for $w \mapsto (f^{-1}e_1f)\left[\frac{1}{1-\Lambda e_0}e_1w\right]$ is true if and only if, for all $s \in \mathbb{N}$, for all w,w' words, we have :

$$(f^{-1}e_1f)\left[\sum_{k=0}^{s-1}(e_0^k e_1 w) \text{ ш } (-1)^{s-k}\frac{e_0^{s-1-k}}{(1-\Lambda e_0)^{s-k}}e_1w'\right] = 0$$

It remains to show that these linear combinations of shuffles generate all the possible shuffles. Now, for each $l \in \mathbb{N}$, the coefficient of Λ^l in this linear combination is of the form :

$(e_0^{s-1}e_1w)ш(e_0^l e_1 w') + \sum_{0\leq s'<s} c_{s'}(e_0^{s'-1}e_1w)ш(e_0^{l+s-s'}e_1w') = 0$ with $c_{s'} \in \mathbb{Q}$. This shows that for all $z,z' \in \ker\tilde{\partial}_{e_1}$, $zшz'$ is a linear combination of the shuffles of the statement, by induction on the index of nilpotence of z for ∂_{e_0}. □

Let us enlarge the definition of the prime multiple harmonic sum motive to non-convergent words, i.e. words w such that $w \notin \ker\tilde{\partial}_{e_1}$

Definition 3.12. For w any word on e_Z of the form e_1w', let :

$$(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{\mathcal{O},\mathrm{prime}}[\tilde{w}] = \left((\Phi^{\mathcal{M}})^{-1}e_1\Phi^{\mathcal{M}}\left[\frac{1}{1-\Lambda e_0}e_1w'\right]\right)$$

Then, the shuffle relation remains true, and the Kashiwara-Vergne relations also remain true : both proofs are independent of the words involved being or not in $\ker\tilde{\partial}_{e_1}$. On the other hand, by nature, the series shuffle relation is a statement concerning words in $\ker\tilde{\partial}_{e_1}$. If we want the equality between the Taylor period and the p-adic period to remain true, we have to consider, instead of prime multiple harmonic sums, polynomials on a formal variable representing log, with coefficients being prime multiple harmonic sums.

The application of this definition to the purposes of the present paragraph is that it enables to state the previous proposition without the artificial additional condition "$(f^{-1}e_1f)[w ш e_0]=0$"

Proposition 3.13. The terms of the equivalence of ii) of the previous proposition are also equivalent to "the map $w \mapsto (f^{-1}e_1f)\left[\frac{1}{1-\Lambda e_0}e_1 w\right]$ satisfies the extended integral shuffle relation of the prime multiple harmonic sum motive".

Proof. Indeed, the conjunction of : for all words $w,w' \in \ker\tilde{\partial}_{e_0}$, $(f^{-1}e_1f)[w ш w']=0$, and for all words $w \in \ker\tilde{\partial}_{e_0}$, $(f^{-1}e_1f)[w ш e_0]=0$, is equivalent to : for all words w,w' (not necessarily in $\ker\tilde{\partial}_{e_0}$), $(f^{-1}e_1f)[w ш w']=0$. □

We apply this proposition to $f=\Phi_{p,-k}\in\pi_1^{un,dR}(\mathbb{P}^1-\{0,1,\infty\},\vec{1}_0,-\vec{1}_1)(\mathbb{Q}_p)$, the generating series of p-adic multiple zeta values : we see that the shuffle relations of multiple harmonic sums, which can be visualized completely explicitly, can be seen as an indirect way to vizualise the shuffle relation of p-adic multiple zeta values.

This approach has the advantage, relatively to the one of part II, that it leads, starting with the shuffle relation of prime multiple harmonic sums, directly to the genuine shuffle relation for p-adic multiple zeta values, instead of a variant as in part II. On the other hand, we are obligated, as in the other approach, to pass through a conjectural step.
If we would not have had the notion of prime multiple harmonic sum motive, we would have said that this conjectural step is essentially the use of the weight homogeneity of algebraic relations.
Here, we can say that this conjectural step is the isomorphism between the Taylor period and the formal p-adic period of the prime multiple harmonic sum motive. This formulation seems much better, because because we see that it is much more specific.

4. Reindexations of algebraic relations on $\pi_1^{un}(\mathbb{P}^1-Z)$

4.1. **Reindexation of algebraic relations.**

4.1.1. *Definition.*

Notation 4.1. i) In this part, to shorten the formulas, we will denote the canonical scheme $\pi_1^{un,dR}(X_Z,\text{can})$ by $\Pi^{un}(X_Z)$.
ii) Let $\mathcal{O}_*(\Pi^{un}(X_Z))$ be the vector subspace of the shuffle Hopf algebra $\mathcal{O}(\Pi^{un}(X_Z))$, generated by words whose first letter at the right is not a e_0, i.e. the subspace $\ker\tilde{\partial}_{e_0}$. It is also another name for the vector space underlying the quasi-shuffle Hopf algebra.

We have the natural inclusion $\mathcal{O}_*(\Pi^{un}(X_Z))\hookrightarrow\mathcal{O}(\Pi^{un}(X_Z))$. On the other hand, we have a surjective linear map

$$\mathcal{O}(\Pi^{un}(X_Z))\twoheadrightarrow\mathcal{O}_*(\Pi^{un}(X_Z))$$

defined by

$$e_0^{s_d-1}e_{z_{i_d}}\dots e_0^{s_1-1}e_{z_{i_1}}e_0^{l}\mapsto\sum_{\substack{l_1,\dots,l_d\geq 0\\ l_1+\dots+l_d=l}}\prod_{i=1}^{d}\binom{-s_i}{l_i}e_0^{s_d+l_d-1}e_{z_{i_d}}\dots e_0^{s_1+l_1-1}e_{z_{i_1}}$$

Given a linear map $f : \mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\widehat{\Pi^{un}(X_Z)})$, we will denote by f_* the composition of f by those two maps :

$$f_* : \mathcal{O}_*(\Pi^{un}(X_Z)) \hookrightarrow \mathcal{O}(\Pi^{un}(X_Z)) \xrightarrow{f} \mathcal{O}(\widehat{\Pi^{un}(X_Z)}) \hookrightarrow \mathcal{O}_*(\widehat{\Pi^{un}(X_Z)})$$

Separately, given f, a linear map $\mathcal{O}(\Pi^{un}(X_Z))) \to \mathcal{O}(\widehat{\Pi^{un}(X_Z)})$, or a bilinear map defined on $\mathcal{O}(\Pi^{un}(X_Z)) \times \mathcal{O}(\Pi^{un}(X_Z))$, the corresponding weight-adic completion will be denoted by $\hat{f}$, and similarly $\hat{f}_*$ will be the completion of $\hat{f}_*$.

Goncharov's coaction $\Delta^{\mathcal{M}}$ (see part I, §2 for the formula) can be considered as a map

$$\mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\Pi^{un}(X_Z)) \otimes \mathcal{O}(\Pi^{un}(X_Z))$$

It can be factorized in a natural way as

$$\Delta = (\mathrm{id} \otimes ш) \circ \Delta_T$$

with $\Delta_T : \mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\Pi^{un}(X_Z)) \otimes T(\mathcal{O}(\Pi^{un}(X_Z)))$, where T denotes the tensor algebra, and $ш : T(\mathcal{O}(\Pi^{un}(X_Z))) \to \mathcal{O}(\Pi^{un}(X_Z))$ is the shuffle product of all tensor components.

Let us consider a linear map

$$\Sigma_\omega : \mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\widehat{\Pi^{un}(X_Z)})$$

Definition 4.2. i) The map Σ_ω is a reindexation relative to the double shuffle equations if there exists a couple of bilinear maps

$$\begin{cases} B_ш : \mathcal{O}(\Pi^{un}(X_Z)) \times \mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\Pi^{un}(X_Z)) \times \mathcal{O}(\Pi^{un}(X_Z)) \\ B_* : \mathcal{O}_*(\Pi^{un}(X_Z)) \times \mathcal{O}_*(\Pi^{un}(X_Z)) \to \mathcal{O}_*(\Pi^{un}(X_Z)) \times \mathcal{O}_*(\Pi^{un}(X_Z)) \end{cases}$$

such that :

$$\hat{ш} \circ (\Sigma_\omega \otimes \Sigma_\omega) = \Sigma_\omega \circ ш \circ B_ш \tag{13}$$

$$* \circ (\Sigma_\omega \otimes \Sigma_\omega) = \Sigma_\omega \circ * \circ \hat{B}_* \tag{14}$$

ii) The map Σ_ω is a reindexation relative to algebraic automorphisms if, for each linear map $\tau : \mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\Pi^{un}(X_Z))$ induced by an algebraic automorphism of X_Z, there exists a linear map

$$L_\tau : \mathcal{O}(\Pi^{un}(X_Z)) \to \mathcal{O}(\Pi^{un}(X_Z))$$

satisfying :

$$\hat{\tau} \circ \Sigma_\omega = \Sigma_\omega \circ \tau \circ L_\tau \tag{15}$$

iii) The map Σ_ω is a reindexation with respect to the motivic Galois coaction if there exists a linear map

$$L_\Delta : T(\mathcal{O}(\Pi^{un}(X_Z))) \to T(\mathcal{O}(\Pi^{un}(X_Z)))$$

such that we have

$$\Delta \circ f = ш \circ \tilde{f} \circ L_\Delta \circ \Delta_T$$

with $\tilde{f}; T(\mathcal{O}(\Pi^{un}(X_Z))) \to T(\mathcal{O}(\Pi^{un}(X_Z)))$ is a sum of tensor products of f and $\mathrm{id}_{\mathcal{O}(\Pi^{un}(X_Z))}$.

4.1.2. *Examples.* The part I provided us two examples that we recall below.

The simplest way to modify an iterated integral is to modify the path of integration : it is a small part of the fundamental groupoid, that is the image of the topological fundamental groupoid in the Betti fundamental groupoid.
The pro-unipotent fundamental group is a quite rigid object from the point of view the of choice of those paths, and this is especially true in the case of curves $\mathbb{P}^1 - Z$. Here, basically, the only natural domains of (iterated) integration are the paths from a tangential base point $\pm\vec{1}$ at $z \in Z$ to another, and the loops around such a base point.

Example 4.3. The example of this kind that we have from the proofs of part I, and also their rewriting in 3, is the one of the map

$$\Phi \mapsto \Phi^{-1} e^{2i\pi e_1} \Phi$$

or the Lie algebra version :

$$\Phi \mapsto \Phi^{-1} e^{2i\pi e_1} \Phi$$

The first version consists in passing from an iterated integral on the straight path γ from 0 to 1 to an iterated integral on $\gamma^{-1} \circ c \circ \gamma$, where c is a loop around 1 ; i.e. the monodromy of ∇_{KZ} transported at $0, 0$.

Example 4.4. The second example that we studied in part I is given by

$$\Sigma_{\omega,\mathrm{Lie}} : w \mapsto \frac{1}{1 - \Lambda e_0} e_{z_i} w$$

For those two examples, we have treated the aspect "motivic Galois action" by the dual point of view of the Ihara action in the §2 of this part III.

4.2. **Formal algebraic relations.** There are several examples of "relations" between multiple zeta values that involve infinite sums. There is no general reason to believe that all of them should come from geometry.
Using the notion of reindexation, we can define a subclass of them for which we know that they are the consequences of algebraic relations.

4.2.1. *Definition.*

Definition 4.5. A formal algebraic relation between multiple zeta values is a relation obtained by applying Φ_{0z} to equations (13) or (14).

We call it "formal" as a reference to the formal schemes that are defined as the sets of their solutions (they are formal completions with respect to the weight), and "algebraic" because they are the consequence of algebraic relations.

4.2.2. *Examples.* The part I has provided a family of examples of formal algebraic relations attached to the prime multiple harmonic sum motive. The simplest to write is surely the following :

$$H_p(s_1)H_p(s_2) = H_p(s_1, s_2) + H_p(s_2, s_1) + H_p(s_1 + s_2)$$

Viewing it as a relation on p-adic multiple zeta values via the expression of H_p in terms of infinite sums of p-adic multiple zeta values, the part I proves that this is a formal algebraic relation.

4.2.3. *Counter-examples.* We give two counter-examples to formal algebraic relations, i.e. relations between multiple zeta values that involve infinite sums but that are not - up to the usual conjectures - formal algebraic relations.
The first counter-example concerns complex multiple zeta values ; it is the one that we find the most significative to our knowledge. The second counter example is the only one that we know in the p-adic setting, given that our theory of series for p-adic multiple zeta values is more recent.

4.2.3.a. **A complex counter-example**

The following equality is classical : for all $s \in \mathbb{N}^*$,

$$1 = \sum_{l \in \mathbb{N}} \frac{(s+l-1)\dots s(s-1)}{(l+1)!} \big(\zeta(s+l)-1\big)$$

This cannot be a formal algebraic relation, simply because there are infinitely many terms of weight 0 (recall that rational numbers are multiple zeta values of weight 0).

The analogue in depth ≥ 2 is due, to our knowledge, to Goncharov and independently to Ecalle : for all $s_1,\dots,s_d \in \mathbb{N}^*$ with $s_d \geq 2$:

$$\zeta(s_d,\dots,s_3,s_1+s_2-1) = \sum_{l \in \mathbb{N}} \frac{(s_1+l-1)\dots s_1(s_1-1)}{(l+1)!} \zeta(s_d,\dots,s_2,s_1+l)$$

Here, assume that it is a formal algebraic relation. By the conjecture of weight homogeneity of algebraic relations, the relation should still be true after identifying the terms of each value of the weight. This would imply the vanishing of $\zeta(s_d,\dots,s_3,s_1+s_2-1)$, which is the term of weight $(\sum_{i=1}^d s_i)-1$, and of each $\frac{(s_1+l-1)\dots s_1(s_1-1)}{(l+1)!}\zeta(s_d,\dots,s_2,s_1+l)$, which is the term of weight $(\sum_{i=1}^d s_i)+k$, whereas they are all real numbers >0.

4.2.3.b. **A p-adic counter-example**

In the p-adic setting, let us recall our theorem, for simplicity in the case of $\mathbb{P}^1 - \{0,1,\infty\}$: for all indices $s_1,\dots,s_d$, primes p, $k \in \mathbb{N}^*$, we have

$$(p^k)^{s_d+\dots+s_1} H_{p^k}(s_d,\dots,s_1) =$$

$$\sum_{l \in \mathbb{N}} \sum_{\substack{l_1,\dots,l_d \geq 0 \\ l_1+\dots+l_d = l}} \Bigg(\sum_{i=0}^{d} \prod_{j=i}^{d} \binom{-s_i}{l_i} \Bigg) \zeta_{p,-k}(s_{i+1}+l_{i+1},\dots,s_d+l_d)\zeta_{p,-k}(s_i,\dots,s_1)$$

But, this time, view it in each $\mathbb{Z}_p$ instead of $\prod_p \mathbb{Z}_p$.

If this was a formal algebraic relation, this would imply the vanishing of each part of given weight.
The part of weight 0 is the rational number $(p^k)^{s_d+\dots+s_1} H_{p^k}(s_d,\dots,s_1)$; it is a real

number > 0 as soon as the sum is non-empty, i.e. as soon as $p^k > d$.
In all remaining cases, it would imply the vanishing of each

$$\sum_{l_1+\ldots+l_d=l} \left(\sum_{i=0}^{d} \prod_{j=i}^{d} \binom{-s_i}{l_i} \right) \zeta_{p,-k}(s_{i+1}+l_{i+1},\ldots,s_d+l_d)\zeta_{p,-k}(s_i,\ldots,s_1)$$

We think that even in those cases, this vanishing of all those p-adic multiple zeta values at the same time contradicts the usual conjectures, for example the one on the dimension of $\mathbb{Q}$-vector spaces on p-adic multiple zeta values.

4.3. **Further examples.** First, let us mention that the paper [IKZ] can be read in the context of reindexations.

4.3.1. *Essentializing the example of part I.* A part of the properties of the examples of reindexations of part I come from what follows.

For all words w in $\mathcal{H}_{ш}(e_Z)$, we have :

$$\frac{1}{1+e_0} ш\, w(e_{z_0}, e_{z_1}, \ldots, e_{z_{r+1}}) = w\big(\frac{1}{1+e_0}e_{z_0}, \frac{1}{1+e_0}e_{z_1}, \ldots, \frac{1}{1+e_0}e_{z_{r+1}}\big)\frac{1}{1+e_0}$$
$$= (\Sigma_\omega)_*(w)\frac{1}{1+e_0}$$

This implies the following formula $(\Sigma_\omega)_*$, as in [IKZ] :

(16) $$(\Sigma_\omega)_* : w \mapsto \big(\frac{1}{1+e_0} ш\, w\big)(1+e_0)$$

We can ask ourselves what happens if we replace $\frac{1}{1+e_0}$ by a more general word x.

Proposition 4.6. Let the following equation on $x \in O(\widehat{\pi_1^{un,dR}(X_Z}, \text{can}))$:

(17) For all words w $\in \mathcal{H}_{ш}(e_Z)$, $x ш\, w(e_0, e_{z_1}, \ldots, e_1) = w(xe_0, xe_{z_1}, \ldots, xe_1)x$

The solutions to this equation are the words x such that, if x_k is their weight k part, we have, for all $k \in \mathbb{N}$, $x_k = x_1^k$, i.e. these are the words of the following form

$$(1 - \sum_{z\in Z-\{\infty\}} \lambda_z e_z)^{-1}$$

with, for $z \in Z - \{\infty\}$, $\lambda_z \in \mathbb{Q}$.

Proof. We take $w = e_{z_i}$ in (17). We obtain

(18) $$x ш\, e_{z_1} = x\ e_{z_i}\ x$$

Let $x \in O(\widehat{\pi_1^{un,dR}(X_Z}, \text{can}))$ satisfying (18). For $k \in \mathbb{N}$, let x_k be the weight k part of x satisfying (17) ; we have $x = \sum_{k=0}^{\infty} x_k$. The term x_0 is of the form $\lambda_0.\emptyset$ where $\lambda_0 \in \mathbb{Q}$ and $\emptyset$ is the empty word. Considering the part of weight one of (18) gives $x_0 e_{z_i} = x_0^2 e_{z_i}$, thus, $x_0 = 0$ or $x_0 = 1$. We distinguish those two cases.
i) Assume $x_0 = 0$. Recall Radford's theorem that the shuffle algebra $\mathcal{H}_{ш}(e_Z)$ is a free polynomial algebra over Lyndon words. In particular we have the implication : for all $y \in \mathcal{H}_{ш}(e_Z)$, $y ш\, e_{z_i} = 0 \Rightarrow y = 0$. By induction on k, considering the weight $k+1$ part of (18) and using this implication, we obtain that $x_0 = \ldots = x_{k-1} = 0$.

ii) If $x_0 = 1$, considering the weight $k+1$ part of (18), we obtain by induction on k, that x_k is both of the form $x_1^{k-1}u$ and $u'x_1^{k-1}$, hence $x_k = x_1^k$. This implies $x = (1-x_1)^{-1}$. Conversely, a word satisfying $x_k = x_1^k$ for all $k \in \mathbb{N}$ clearly satisfies the equation. □

Remark 4.7. i) This means that much of the combinatorics of part I remain true when replacing $(1-\Lambda e_0)^{-1}$ by

$$(1 - \sum_{z \in Z - \{\infty\}} \Lambda_{z_i} e_{z_i})^{-1}$$

where Λ_{z_i} are formal variables.
ii) On the other hand the multiplication by $(1 - \sum_{z \in Z-\{\infty\}} \Lambda_{z_i} e_{z_i})^{-1}$ admits a natural factorization by the multiplication by $(1-\Lambda_0 e_0)^{-1}$, and of course also by each $(1-\lambda_z e_z)^{-1}$ given that all the letters of e_Z play a symmetric role : we have

$$\begin{aligned}(1-\Lambda_0 e_0)(1-\sum_{z \in Z-\{\infty\}} \Lambda_z e_z)^{-1} &= (1-\sum_{z \in Z-\{\infty\}} \Lambda_z e_z + \sum_{z \in Z-\{0,\infty\}} \Lambda_z e_z)(1-\sum_{z \in Z-\{\infty\}} \Lambda_z e_z)^{-1} \\ &= 1 + \sum_{z \in Z-\{0,\infty\}} \Lambda_z e_z (1 - \sum_{z \in Z-\{\infty\}} \Lambda_z e_z)^{-1}\end{aligned}$$

whence :

$$(1-\sum_{z \in Z-\{\infty\}} \Lambda_z e_z)^{-1} = (1-\Lambda_0 e_0)^{-1} + (1-\Lambda_0 e_0)^{-1} \sum_{z \in Z-\{0,\infty\}} \Lambda_z e_z (1-\sum_{z \in Z-\{\infty\}} \Lambda_z e_z)^{-1}$$

Such a factorization has been used to write the one dimensional part of the Kashiwara-Vergne equations.

4.3.2. *On the cyclic sum formula .* The following computation it implicitly a byproduct of a reindexations. To simplify the formulas we take the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, but the general case is similar.
The cyclic sum formula is an emblematic example of a formula that can proved by an ad hoc elementary method and can be retrieved as a consequence of the double shuffle equations. We show that a counterpart exists on the level of prime multiple harmonic sums and finite hyperlogarithms.

Notation 4.8. i) We will modify the notation of $(\Sigma_\omega)_*$, and denote it by $(\Sigma_\omega)_{*,\Lambda}$. We can substitute to Λ a complex or p-adic number λ, in which case the application will be denoted by $(\Sigma_\omega)_{*,\lambda}$.
ii) The coefficient of Λ^l, $l \in \mathbb{N}$, in a power series $S \in R[[\Lambda]]$, for R a ring, will be denoted by $S\big[\Lambda^l\big]$.

Proposition 4.9. (Cyclic sum formula for prime multiple harmonic sums) We have, for all $d \in \mathbb{N}^*$, $(s_d, \ldots, s_1) \in (\mathbb{N}^*)^d$:

$$\begin{aligned}(19) \quad (-1)^{s_d+\ldots+s_1} (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}} (\Sigma_\omega)_* (1, s_1, \ldots, s_{d-1}) \big[\Lambda^{s_d-1}\big]) \\ = -(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}} (s_d, \ldots, s_1)\end{aligned}$$

$$-\sum_{n=1}^{d} \sum_{t_n=0}^{s_n-1} (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}} \bigg((\Sigma_\omega)_{*,1} \Big(\Sigma_{*,\Lambda}(s_{d-1}, \ldots, s_{n+1})[\Lambda^{t_n}] \Big), s_n - t_n + 1, s_{n-1}, \ldots, s_1 \bigg)$$

Proof. Let us express the inverse of the generating series of hyperlogarithms in two different ways ; first using the antipode of the shuffle Hopf algebra ; secondly by the usual operation of inversion of a formal series. We have, on the one hand :

$$\mathrm{Li}^{-1}[e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1] = (-1)^{s_d+\dots+s_1}\,\mathrm{Li}[e_1e_0^{s_1-1}\dots e_1e_0^{s_d-1}]$$

and, on the other hand,

$$\text{(20)}\quad \mathrm{Li}^{-1}[e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1] = -\,\mathrm{Li}[e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1]$$
$$-\sum_{n=1}^{d-1}\sum_{t_n=0}^{s_n-1}\mathrm{Li}[e_0^{t_k}e_1e_0^{s_{k+1}-1}\dots e_0^{s_{d-1}-1}e_1e_0^{s_d-1}]\,\mathrm{Li}[e_0^{s_k-t_k-1}e_1\dots e_0^{s_1-1}e_1]$$

We translate this equality on the sums of Taylor coefficients of order $0<n<p^k$. To deal with the Taylor coefficients of the product of two hyperlogarithms, we use the formula of part I, §3.2.2. ☐

Corollary 4.10. (Alternative symmetry formula for prime multiple harmonic sums)

$$\text{(21)}\quad (-1)^{s_1+\dots+s_d}(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}(s_1,\dots,s_d) - (\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}(s_d,\dots,s_1)$$
$$= \sum_{n=1}^{d}\sum_{t_n=1}^{s_n}(\mathrm{Li}\,\mathcal{T})_{O,\mathrm{prime}}\Big((\Sigma_\omega)_{*,1}(s_d,\dots,s_{n+1},s_n-t_n),t_n+1,s_{n-1},\dots,s_1\Big)$$

Proof. This is obtained by translating on the Taylor coefficients of order p^k the equality :

$$\text{(22)}\quad (-1)^{\sum_{i=1}^d s_i+1}\,\mathrm{Li}[e_1e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1] = \mathrm{Li}^{-1}[e_1e_0^{s_1-1}e_1\dots e_0^{s_d-1}e_1]$$
$$= -\,\mathrm{Li}[e_1e_0^{s_d-1}e_1\dots e_0^{s_1-1}e_1]$$
$$-\sum_{n=1}^{d}\sum_{t_n=0}^{s_n-1}(-1)^{\sum_{i=n}^d s_i-t}\,\mathrm{Li}[e_0^{s_n-1-t}e_1e_0^{s_{n+1}-1}e_1\dots e_0^{s_d-1}e_1]\,\mathrm{Li}[e_0^te_1e_0^{s_{n-1}-1}e_1\dots e_0^{s_1-1}e_1]$$

☐

Proposition 4.11. (Cyclic sum formula in the finite case)
For all $s_d,\dots,s_1\in\mathbb{N}^*$, p prime :

$$\text{(23)}\quad (-1)^{s_1+\dots+s_d}\sum_{i=1}^{d}s_iH_p(s_1,\dots,s_i+1,\dots,s_d)$$
$$\equiv \sum_{n=1}^{d}\sum_{t_k=0}^{s_k-1}H_p\big(s_d,\dots,s_{n+1},s_n-t_n),t_n+1,s_{n-1},\dots,s_1\big)\mod p$$

Proof. Compare the terms of weight $s_d+\dots+s_1$ and $s_d+\dots+s_1+1$ in the symmetry formula 2') of theorem 1 and proposition 4.10 ; divide by $p^{s_d+\dots+s_1+1}$ and take the reduction modulo p. ☐

Remark 4.12. Dividing by $p^{s_1+\dots+s_d}$ and taking the reduction modulo p, we obtain a proof of $H_p(s_d,\dots,s_1)\equiv(-1)^{s_d+\dots+s_1}H_p(s_1,\dots,s_d)\mod p$ which does not involve the change of variable $(n_d,\dots,n_1)\mapsto(p-n_d,\dots,p-n_1)$.

5. The reindexed fundamental group $\Sigma_{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{PRIME}}}\big(\pi_1^{un}(\mathbb{P}^1 - Z)\big)$

5.1. **Definition.** In §2, we wrote the Ihara action on $\pi_1^{un,dR}(X_Z)$ by considering the product of the fundamental torsor of paths starting at $\vec{1}_0$ by the restricted groupoid at the usual tangential base-points $t \in T$:

$$\pi_1^{un,dR}(X_Z, \ast, \vec{1}_0) \times \pi_1^{un,dR}(X_Z, t)_{t\in T}$$

Then, we considered separately the two factors of the products.
i) From the fundamental torsor of paths at $\vec{1}_0$, we kept the germs of analytic sections at 0, equipped with its Ihara action. We explained in §2.4.4.a that, given the p-adic analytic proof of the relation between prime multiple harmonic sums and p-adic hyperlogarithms, we could restrict ourselves to a "zeroth order term" of the Ihara action with respect to one of its factors, on these germs of sections. This part of the Ihara action was equivalent to the Ihara action on :

$$\pi_1^{un,dR}(X_Z, \vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])$$

Finally, a certain byproduct of it was equipped with what we called a "Taylor version" of the Ihara action.
ii) On the other hand, from the restricted groupoid, which is associated with the base points

$$T = \{(\vec{1}_0, \vec{1}_0)\} \ \cup \ \{(\vec{1}_0, \vec{1}_z) \mid z \in Z - \{0,\infty\}\} \ \cup \ \{(\vec{1}_z, \vec{1}_z) \mid z \in Z - \{0,\infty\}\}$$

we considered specifically the subset $\{(\vec{1}_0, \vec{1}_z) \mid z \in Z - \{0,\infty\}\}$; and we applied, to the corresponding schemes, two successive reindexation maps

$$\Sigma^{\vee}_{\omega,\mathrm{Lie}} \circ \Sigma_{\gamma} : \pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z)) \to \mathbb{A}^W$$

Let us rename : $\Sigma_{\Lambda-\mathrm{adic}} = \Sigma^{\vee}_{\omega,\mathrm{Lie}} \circ \Sigma_{\gamma}$.

Definition 5.1. Let us denote by

$$\Sigma_{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\mathrm{prime}}}\pi_1^{un,dR}(X_Z)$$

and call the reindexed pro-unipotent fundamental groupoid of X_Z along the prime multiple harmonic sum motive,the product

$$\pi_1^{un,dR}(X_Z, \vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)]) \quad \times \quad \Sigma_{\Lambda-\mathrm{adic}}(\pi_1^{un,dR}(X_Z, \vec{1}_0, \vec{1}_z))_{z\in Z-\{0,\infty\}} \tag{24}$$

We keep the minimal information that we need to have a substitute to the fundamental groupoid directly adapted to prime multiple harmonic sums. It is only one possible convention among others, since we have not defined in a precise way a general notion of reindexation. We could have chosen an object which is closer to the genuine fundamental groupoid. Actually, the general spirit that we follow is to show how we can transform the fundamental groupoid until a form where it is not recognizable at all, yet still carries all its structures necessary to describe a family of periods which is essentially equivalent to the basic one, multiple zeta values.

5.2. **Geometric description.** Now we describe the reindexed fundamental group.

5.2.1. *Ihara action.* The Ihara action on the two factors of (24) is described as follows.
- The Ihara action on $\pi_1^{un,dR}(X_Z, \vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])$ is given directly by the usual Ihara

action on $\pi_1^{un,dR}(X_Z,\vec{1}_0)$. It has, as a particular byproduct, the action $\circ_{Taylor}$ of §2.
- The Ihara action on $\Sigma_{\Lambda-\text{adic}}(\pi_1^{un,dR}(X_Z,\vec{1}_0,\vec{1}_z))_{z\in Z-\{0,\infty\}}$ is the action $\circ_{\Lambda-\text{adic}}$ introduced in §2.

5.2.2. *Canonical element.* The reindexed fundamental group $\Sigma_{\Lambda-\text{adic}}(\pi_1^{un,dR}(X_Z,\vec{1}_0,\vec{1}_z))$ contains as a canonical element

$$\Sigma_{\Lambda-\text{adic}}(\Phi_{0z})$$

Consider the p-adic version of this element depends on a choice of a power of Frobenius $k\in\mathbb{Z}\cup\{\pm\infty\}-\{0\}$, and can be denoted by $(\Sigma_{\Lambda-\text{adic}}\Phi_{0z})_{p,k}$. Our theorem expressing prime multiple harmonic sums in terms of p-adic iterated integrals is that, when $k\in -\mathbb{N}^*$, we have, for all indices :

$$(\Sigma_{\Lambda-\text{adic}}\Phi_{0z})_{p,k}\Big[\begin{array}{c} z_{i_d},\dots,z_1 \\ s_d,\dots,s_1\end{array}\Big] = p^k H_{p^k}\Big[\begin{array}{c} z,z_{i_d},\dots,z_1 \\ s_d,\dots,s_1\end{array}\Big]$$

The actions $\circ_{Taylor}$ and $\circ_{\Lambda-adic}$, of p-adic iterated integrals $(\Phi_{0z})_{p,-1}$, on this p-adic version with $-k$, send it to the analogue element with $-(k+1)$. In particular, this preserves all the algebraic relations of part I, II and III between prime multiple harmonic sums, which are true for all k at the same time.

5.2.3. *Canonical connexion.* The factor $\pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])$ keeps a track of ∇_{KZ} :

Definition 5.2. Let

$$\nabla_{\text{KZ}}^{Taylor}:\pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])\to\pi_1^{un,dR}(X_Z,\vec{1}_0)\Big(\overline{\mathbb{Q}}[[x]][\log(x)]\frac{dx}{x}\Big)$$

$$L\mapsto\Big(\frac{dx}{x}e_0-\sum_{z\in Z-\{0,\infty\}}\sum_{n\geq 0}x^n z^{-(n+1)}dxe_z\Big)L$$

5.2.4. *Frobenius.* Assume now that X_Z is defined on an absolutely non ramified discrete valuation ring, let $X_Z^{(p)}$ be its extension of scalars by Frobenius, and let $F:X_Z\overset{z\mapsto z^p}{\longrightarrow}X_Z^{(p)}$ be the relative Frobenius map. The explicit p-adic computations of iterated integrals are based on the fact that there is an isomorphism

$$F_*:\pi_1^{un,dR}(X_Z)\overset{\sim}{\longrightarrow}F^*\pi_1^{un,dR}(X_Z^{(p)})$$

which is horizontal for the KZ connexion, where the pull-back by Frobenius $F^*\pi_1^{un,dR}(X_Z)$ is constructed analytically ([D], §11). In general, the analytic F^* does not coincide with the one that can be defined directly algebraically.

The factor $\pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])$ keeps a track of this Frobenius structures as follows.

Let us define the analogue of the map F here can be defined $x\mapsto x^p$.
This leads immediately to a notion of F^*.
An analogue of F_* is obtained by taking the image of F_* by $\Sigma_{\Lambda,adic}$.
Following the notation of [D], §13, let ϕ be the inverse of F, and let the associated ϕ^* and ϕ_*

The expression of prime multiple harmonic sums in terms of p-adic hyperlogarihms is the origin of these three papers, and is written as :

$$(pn)^{\text{weight}} H_{pn} = n^{\text{weight}} \big(\Phi_{0z}^{-1} e_z \Phi_{0z}\big) \circ_H n^{\text{weight}} H_n \tag{25}$$

where $\circ_H$ is a certain "harmonic Ihara action". It was obtained as a certain limit case, in which the overconvergent factor becomes trivial, of the equation of horizontality of F_*. In this setting, we can rewrite it as follows : for all $n \in \mathbb{N}^*$,

$$\phi^*(n^{\text{weight}} H_n) = \phi_*(n^{\text{weight}} H_n)$$

It is the analogue of the equation of horizontality of Frobenius in this setting.

The image of the invariant by Frobenius in $\pi_1^{un,dR}(X_Z, \vec{1}_0)$ by $\Sigma_{\Lambda-\text{adic}}$ is the element :

$$\big(\lim_{k\rightarrow\infty} (p^k)^{\text{weight}} H_{p^k}[\tilde{w} = \Big[\begin{array}{c} 1, z_{i_d}, \ldots, z_1 \\ s_d, \ldots, s_1 \end{array} \Big]]\big)_{\tilde{w}}$$

5.2.5. *The completion.* Consider the algebras of functions on the fundamental groupoid $\pi_1^{un,dR}(X_Z)$. It has a natural weight-adic completion. Apply to it the reindexation map : we obtain the completed version of the algebras of functions on $\Sigma_{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{O,\text{prime}}} \pi_1^{un,dR}(X_Z)$. It is the natural receptacle for the convergent infinite sums of prime multiple harmonic sums $(p^k)^{\text{weight}(\tilde{w})} H_{p^k}(\tilde{w})$.

5.2.6. *Application to non-prime multiple harmonic sums.*

5.2.6.a. **Introduction**

Let us consider now the multiple harmonic sums, $H_n[\tilde{w}]$, where n is any element of $\mathbb{N}^*$, not necessarily a power of a prime number. We follow here a suggestion of Pierre Cartier. For all p, a prime and $k \in \mathbb{N}^*$ such that p^k divides n, there is a p-adic formula given by (25) expressing H_n in terms of $H_{np^{-k}}$ and p-adic hyperlogarithms associated to the $-k$-th power of Frobenius. In other words, if we consider the prime decomposition of n, which we will write as $n = p_1^{k_1} \ldots p_r^{k_r}$, there is an expression of all multiple harmonic sums $H_n[\tilde{w}]$, inductive on the prime decomposition of n, in terms of p_i-adic iterated integrals, $i \in \{1, \ldots, r\}$. It can be written in a way that, at first sight, mixes p_i-adic numbers with different i :

$$(n^{\text{weight}} H_n[\tilde{w}])_w = \tau(p_1^{k_1} \ldots p_{r-1}^{k_{r-1}})(\Phi_{0z})_{p_r,k_r} \circ_h \ldots \circ_h (\Phi_{0z})_{p_1,k_1} \circ_h 1 \tag{26}$$

The well-definedness of the formula comes from that each $\tau(p_1^{k_1} \ldots p_{i-1}^{k_{i-1}})(\Phi_{0z})_{p_i,k_i} \circ_h \ldots \circ_h (\Phi_{0z})_{p_1,k_1} \circ_h 1$, $i \in \{1, \ldots, r-1\}$, has coefficients in $\overline{\mathbb{Q}}$.

In part II, §6.1, we have explained how this formula implies constraints on the p_i-adic valuations of H_n. We discuss here the meaning of this formula from the points of view of algebraic relations and the motivic Galois action.

5.2.6.b. **Interpretation in terms of algebraic relations**

We have seen that the Ihara action $\circ_{\Lambda-\text{adic}}$ of p-adic hyperlogarithms on prime multiple harmonic sums amounts to a Frobenius action, that increases the power of Frobenius, and perserves the algebraic relations of prime multiple harmonic sums. It is legitimate to ask whether we have a similar phenomenon for general multiple harmonic sums.

Actually, this question has no sense, except for the series shuffle relations, for which the series shuffle relation is preserved since it is true for all $n \in \mathbb{N}^*$: for all words w, w',

$$H_n(w * w') = H_n(w)H_n(w')$$

All the other relations between non-prime multiple harmonic sums involve different values of n at the same time. The integral shuffle relation, for example, can be written as

$$H_n(w \sqcup\!\sqcup w') = \sum_{k=1}^{n-1} H_k(w)H_{n-k}(w')$$

and the multiplication of n by a number p^k does not send the interval $\{1, \ldots, n\}$ to the interval $\{1, \ldots, p^k n\}$.

We believe that the fact that only the series shuffle relation is preserved is not an absurd accident. We will define, in the next paragraph, a framework of a Galois theory of series for multiple harmonic sums, in which the series shuffle relation will play the role that is played by the integral shuffle relation for iterated integrals. The integral shuffle relation is true for all iterated integrals and the points of the fundamental groupoid are precisely the non-commutative formal power series that satisfy the integral shuffle relation. In the setting of Galois theory of series, the multiplication of the upper bound by a power of a prime number will be then an analogue of a morphism of the fundamental groupoid.

5.2.6.c. **View of H_n as 'iterated periods' of $\circ_{\Lambda-\mathrm{adic}}$**

Here, the role of the motivic Galois action is played by the action $\circ_{\Lambda-\mathrm{adic}}$. The particular phenomenon is that, starting with the trivial element 1 pf $\mathbb{A}^W$, applying the action $\circ_{\Lambda-\mathrm{adic}}$ of p-adic hyperlogarithms successively yields multiple harmonic sums by the formula (26), which are algebraic whence the possibility to iterate the action. We will say that multiple harmonic sums H_n as "iterated periods" of the couple $(\circ_{\Lambda-\mathrm{adic}}, 1)$.

5.3. **Elementary description and interpretation in terms of periods.** Here, we recall some other parts of our p-adic analytic work 'the Frobenius horizontal isomorphism of the pro-unipotent fundamental group of curves $\mathbb{P}^1 - Z$'. In the parts II and IV of this work [J2], [J4], we showed analogies between the results of two types computations : one "geometric", in the framework of the pro-unipotent fundamental group ; the other one made purely with multiple harmonic sums and elementary operations. In those papers, this was just an analogy. We will suggest here a formulation of this analogy in terms of motives and periods, using again the term "period" in a generalized sense as in §2 : a period of a motivic Galois action rather than a motive as the multiple zeta motive.

5.3.1. *The mutiplication by a prime number of the upper bound of multiple harmonic sums.* In the part II of our p-adic analytic work [J2], we made a computation on multiple harmonic sums which is parallel to the one giving the formula (25) : there is a

way to express each $(p^k n)^{\text{weight}} H_{p^k n}[\tilde{w}]$ in terms of $n^{\text{weight}} H_n[\tilde{w}]$ and absolutely convergent p-adic series whose terms are $\mathbb{Q}$-linear combinations (in $\prod_p \overline{\mathbb{Q}_p}$) of elements $(p^k)^{\text{weight}} H_{p^k}[\tilde{w}]$.
To interpret it, we might say that this operation is a "period" of the Ihara action $\circ_{\Lambda-\text{adic}}$.

5.3.2. *Elementary description of the expansion with respect to k.* In part IV of our p-adic analytic work [J4], we have given a computation on the fundamental group expressing how iterated integrals depend on the power of Frobenius. This applies in particular to prime multiple harmonic sums, via their expression as infinite sums of p-adic iterated integrals at tangential base-points. This showed an expansion of prime multiple harmonic sums in terms of their upper bound p^k. For prime multiple harmonic sums, we also made a parallel computation involving elementary operations giving a similar result. We might say that the expansion obtained by elementary means is a period of the expansion of Frobenius in terms of its power.

5.3.3. *Algebraic relations in terms of series.* All the proofs of algebraic relations, which are proofs in the context of the fundamental groupoid, can be expressed in terms of elementary operations on finite sums.
- The series shuffle relations is by nature a property of the series expansion.
- Since Euler, it is known that the integral shuffle relation can be proved in terms of the series expression.
- Hoffman's method to prove the one dimensional part of Kashiwara-Vergne equations for prime multiple harmonic sums, which he calls the duality theorem and which has been lifted p-adically by Rosen, uses Newton series.

This suggests that, when pushing forward the fundamental group by the reindexation, we may also push forward the rules of computation. We will essentialize this idea in the next paragraph.

6. Interpretation of the elementary description of $\Sigma_{(\mathrm{Li}\,\mathcal{T})^{\mathcal{M}}_{\mathcal{O},\text{PRIME}}}\big(\pi_1^{un}(\mathbb{P}^1 - Z)\big)$: the role of indices in $\amalg_{d\geq 1}(\mathbb{N})^d$

We are now going to define a language that enables to write in a simple way the formulas arising in the previous paragraph, which describe elementarily the reindexed fundamental group. The notions of this language, which are indexed entirely by elements of $\mathbb{N}$, work as a pseudo-motivic setting, including substitutes to the motivic Galois coaction on prime multiple harmonic sums, and "conjectures of periods" that reformulate the conjecture of the previous paragraph. Thus, everything happens as if there was a conceptual framework behind this language.

6.1. **Interpretation in terms of indices on $\mathbb{N}$.**

6.1.1. *The "pro-unipotent paths on $\mathbb{N}$".* In the analogy and the correspondence with the usual setting that we derive here, $\mathbb{N}$ will appear as a sort of "fundamental open affine subset" of a natural "space" that will play the role of $\mathbb{P}^1 - Z$.

Definition 6.1. Let $n, m \in \mathbb{N}$ with $n < m$.
i) A *path* from n to m is an element of $]n, m[$.

ii) A *pro-unipotent path* from n to m with $n < m$ is an increasing sequence of elements of $]n, m[$, which we will denote by $n < n_1 = \ldots = n_{i_1} < n_{i_1+1} = \ldots = n_{i_2} < \ldots < n_{i_{r-1}+1} = \ldots = n_{i_r} < m$.

Let us denote the set of pro-unipotent paths from n to m by $P_{n,m}$.
Let the depth of a pro-unipotent path be the number of $<$ minus one. Let $P^{(d)}_{n,m}$ be the part of depth d of $P_{n,m}$: we have $P_{n,m} = \amalg_d P^{(d)}_{n,m}$.

Note that $P^{(d)}_{n,m}$ is non empty if and only if $|m - n| > d$.

Definition 6.2. i) Let the associative composition $P_{n,m} \times P_{m,l} \to P_{n,l}$ be the one given, if $n < m < l \in \mathbb{N}$, by concatenation of sequences :

$$(n < n_1 = \ldots < \ldots n_d < m).(m < m_1 = \ldots < \ldots m_{d'} < l)$$
$$= (n < n_1 = \ldots < \ldots < n_d < m < m_1 = \ldots < \ldots m_{d'} < l)$$

ii) Let the associative and commutative "pre- series shuffle product" $P_{n,m} \times P_{n,m} \to$(free $\mathbb{Z}$-module over $P_{n,m}$) be given by the logical operation "or" : for example

$$(n < n_1 < m) \times (n < m_1 < m)$$
$$= (n < n_1 < m_1 < m) + (n < m_1 < n_1 < m) + (n < n_1 = m_1 < m)$$

and where a pro-unipotent path whose depth exceeds $|m - n|$ is send to 0.

When we will refer to the $\mathbb{Z}[P_{n,m}]$ below, it will mean the free $\mathbb{Z}$-module over $P_{n,m}$, equipped with the pre-series shuffle product, which is a bilineary map $\mathbb{Z}[P_{n,m}] \times \mathbb{Z}[P_{n,m}] \to \mathbb{Z}[P_{n,m}]$. This is a parallel to the shuffle product which applies to iterated integrals.
The composition of paths extends uniquely to a bilineary map $\mathbb{Z}[P_{n,m}] \times \mathbb{Z}[P_{m,l}] \to \mathbb{Z}[P_{n,l}]$.

Definition 6.3. Let us say that two pro-unipotent paths in $P_{n,m}$ (having the same number of letters) are equivalent, which we will denote by the symbol $\sim$, if we obtain the same sequence of symbols $=$ and $<$ when removing the letters n_i.

Proposition 6.4. i) This is clearly an equivalence relation.
ii) The set of equivalence classes identifies to the set of words on the alphabet $\{<, =\}$.

We denote the set of equivalence classes of unipotent paths from n to m by $P_{n,m}/\sim$.

Proposition 6.5. The composition of paths and the pre - series shuffle product pass to the quotient and induce maps

$$P_{n,m}/\sim \times P_{m,l}/\sim \;\longrightarrow\; P_{n,l}/\sim$$
$$\mathbb{Z}[P_{n,m}/\sim] \times \mathbb{Z}[P_{n,m}/\sim] \;\longrightarrow\; \mathbb{Z}[P_{n,m}/\sim]$$

They also induce maps between equivalence classes on $P_{n,m}$ obtained by summing on all the paths of an equivalence class.

Remark 6.6. Our way to define functions over an homotopy class of paths (that will play the role of homotopy-invariant iterated integrals) will be to sum over all the paths of a homotopy class.

6.1.2. *Descending the de Rham setting.* By keeping only the series expansion of differential forms $\sum_{n\geq -1} a_n z^n dz$ at 0 and 1, and considering maps $n \mapsto a_n$ we obtain a map from $O(\pi_1^{dR}(\mathbb{P}^1 - \{0,1,\infty\})$ to the following.

First, to $\mathbb{A}^1/\overline{\mathbb{Q}}$, associate the set $\mathbb{N}$ and the $\mathbb{Q}$-vector space of functions $\mathcal{F}_{>0}(\mathbb{N},\overline{\mathbb{Q}})$ having a strictly positive radius of convergence both in the complex and the p-adic setting for all primes p.
To $\mathbb{G}_m/\overline{\mathbb{Q}}$, associate the set $\mathbb{N}\cup\{-1\}$ and the extended $\mathbb{Q}$-vector space $\mathcal{F}_{>0}(\mathbb{N}\cup\{-1\},\mathbb{Q})$ of functions satisfying the same condition of convergence (the convergence condition does not affect the value at -1).

Now, to $(\mathbb{A}^1 - \{0 = z_0, z_1, \ldots, z_r, z_{r+1} = 1\})/\mathbb{Q}$, associate $r+2$ copies of $\mathbb{N}\cup\{-1\}$, labeled by $0, z_1, \ldots, z_r, z_{r+1}$ and transition maps

$$\mathcal{F}_{>0}(\mathbb{N}\cup\{-1\},\mathbb{Q})(z_i) \to \mathcal{F}_{>0}(\mathbb{N}\cup\{-1\},\mathbb{Q})(z_j)$$

such that the transition map from z_i to 0 sends

$$\mathbb{1}_{n=-1} \text{ at } z_i \text{ to } n \mapsto -z_i^{-n} \times \mathbb{1}_{n\geq 0} \text{ at } 0$$

The definition of the transition maps, and their view as arising from transition maps on the level of the $\mathbb{N}\cup\{-1\}$ themselves, can be stated, for example, by embedding $\mathbb{N}\cup\{-1\}$ into a vector space of infinite formal sums of elements of $\mathbb{N}$, to suppress the additional issues of convergence. Since, for the present paper, we will discuss only computations, that are made inside the "affine subset" $\mathbb{N}\cup\{-1\}$ at 0, we do not need to enter into such a formalism and leave it to a future paper.

6.1.3. *The correspondence.* We have the natural "cohomological equivalence" $\sim$ of elements of $\mathcal{F}_{>0}(\mathbb{N}\cup\{-1\},\mathbb{Q})$ defined by two functions having the same value at -1.
In the case of $\mathbb{P}^1 - \{0,1,\infty\}$, iterated integration induces a correspondence between the data above modulo this cohomological equivalence and the "pro-unipotent paths of $\mathbb{N}$" of the previous paragraph, that we do not need to write explicitly, and that recasts the Taylor expansion of hyperlogarithms. It essentially consists in the identification from the words on $(=,<)$ to words on (e_0, e_1). In the general case we would have had to add to the letter $<$ an additional label z_i.

What we will use is that this correspondence factorizes through the following other correspondence : we consider now functions $f_i : \mathbb{N} \to \mathbb{Q}$ which are no more coefficients of series expansions of a $\frac{dz}{z-z_i}$, but a piece of the result of the iterated integration.

Given a path $\gamma = (n_j)_j \in P_{n,m}$ with $n < m$, regroup the indices n_i which are equal to each other, and order them according to the order of $\mathbb{N}$. This gives $depth(\gamma)$ equivalence classes which we denote by $1(\gamma) < \ldots < d(\gamma)$. Given an index j, let $i = step(j,\gamma)$ be the element of $\{1,\ldots,d\}$ such that n_j occurs in $i(\gamma)$. It depends only on the homotopy class of γ.

Definition 6.7. Let, for each value of the depth d, the "summation" coupling :

$$(27) \qquad \big([\gamma'] \in P^{(d)}_{n,m}, (f_1,\dots,f_d) \in \mathcal{F}(\mathbb{N},\mathbb{Q})^d\big) \overset{S}{\longmapsto} \sum_{\substack{\gamma=(n_i)\in \\ \text{homotopy class } [\gamma']}} \prod_i f_{\text{step}(i,[\gamma'])}(n_i)$$

In the case of $f_1 = \ldots = f_d : l \mapsto \frac{1}{l}$, the result of the coupling map is the multiple harmonic sum function and its shifted versions :

$$H_{n<m}(s_d,\dots,s_1) = \sum_{n<n_1<\ldots n_d<m} \frac{1}{n_1^{s_1}\ldots n_d^{s_d}}$$

6.1.4. *Pro-unipotent paths on $\mathbb{N}$ "relative to a subspace" and associated "motivic Galois coaction".* In this paragraph, let M be a subset of $\mathbb{N}$.

Our setting here is all the sets of pro-unipotent paths $P_{n,m}$ with extremities in M, i.e. such that $(n,m) \in M^2$.

We first define a notion of restriction of a path to M.

Definition 6.8. Let $\gamma = (n_j) \in P_{n,m}$, with $(n,m) \in M^2$. The restriction of γ to M, denoted by

$$\gamma|_M$$

is the subsequence of γ made of the n_j's such that $n_j \in M$.

Now we define a notion of homotopy of pro-unipotent paths relative to M.

Definition 6.9. We say that two pro-unipotent paths $\gamma_1, \gamma_2 \in P_{n,m}$, with $(n,m) \in M^2$ are homotopic relative to M if and only if
i) they are homotopic
ii) their steps that are in M are the same

Proposition 6.10. This defines clearly an equivalence relation.

We will denote this equivalence relation by $\sim_M$.

Definition 6.11. Let, for $n,m \in M$, the function

$$\Delta_M : \mathbb{Z}[P_{n,m}] \mapsto \mathbb{Z}[P_{n,m}] \otimes \big(\otimes_{n<l<l'<m} \mathbb{Z}[P_{l<l'}]\big)$$
$$\gamma \mapsto \gamma|_M \otimes \big(\otimes_{i=1^{r(p)-1}} \underbrace{n_{i(p)} < \ldots < n_{i+1(p)}}_{\text{subpath of } \gamma \text{ from } n_{i(p)} \text{ to } n_{i+1(p)}}\big)$$

where

$$\gamma|_M = (0 < n_{1(p)} = \ldots = n_{1(p)} < \ldots < n_{r(p)} = \ldots = n_{r(p)} < n)$$

In the case of prime multiple harmonic sums, Δ_M is compatible with the coupling.

6.1.5. *Prime multiple harmonic sums.* Let $k \in \mathbb{N}^*$ (the power of Frobenius).

Consider the set of parts $p^k\mathbb{N}$, for all primes p.

The coupling (27) applied to functions $f_1, \ldots, f_d$ all equal to $l \mapsto \frac{p^k}{l}$ and classes of $P^{(d)}_{0,p^k}$ yields the prime multiple harmonic sum function :

$$(s_d, \ldots, s_1) \mapsto (p^k)^{s_d + \ldots + s_1} H_{p^k}(s_d, \ldots, s_1)$$

Definition 6.12. Let $\mathcal{H}ar$ be the $\mathbb{Z}$-algebra generated by the constant function $n \mapsto 1$ and the maps $n \mapsto \frac{p^k}{n}$.

Remark 6.13. The algebra $\mathcal{H}ar$ contains certain binomial coefficient functions

$$n \mapsto n \binom{p^k}{n} = \frac{p^k (p^k - 1) \ldots (p^k - n + 1)}{1 \ldots (n-1)} = (1 - \frac{p^k}{1}) \ldots (1 - \frac{p^k}{n-1})$$

Remark 6.14. If we consider the weight-adic completion of $\mathcal{H}ar$, the obtained algebra also contains the series expansion of p-adic multiple zeta values. For example, the series expansion of p-adic multiple zeta values of depth one was : for all $s \in \mathbb{N}^*$ such that $s \geq 2$:

$$\zeta_{p,-k}(s) = \frac{1}{s-1} (p^k)^s \sum_{N \geq -1} (p^k)^N (-1)^N \binom{N+s}{s-1} B_{N+1} \sum_{0<n<p^k} \frac{1}{n^{s+N}}$$

And it can be rewritten as :

$$\zeta_{p,-k}(s) = \sum_{0<n<p^k} \frac{1}{s-1} \sum_{N \geq -1} (-1)^N \binom{N+s}{s-1} B_{N+1} (\frac{p^k}{n})^{s+N}$$

We will consider the map Δ_M with $M = p^k \mathbb{N}$.

Definition 6.15. By summing on an equivalence class, we obtain a map which passes to the quotient :

$$\Delta_{p^k \mathbb{N}, \sim} : \mathbb{Z}[P_{n,m} / \sim] \mapsto \mathbb{Z}[P_{n,m} / \sim] \otimes \big(\otimes_{n<l<l'<m} \mathbb{Z}[P_{l<l'} / \sim] \big)$$

6.1.6. *Application.* The elementary computation of [J2] evoked in §5.3.1, that expresses the multiplication by p^k of the upper bound of a multiple harmonic sum, can be expressed efficiently in terms of $\Delta_{p^k \mathbb{N}, \sim}$. This adds up to the fact that Δ_p applied to prime multiple harmonic sums amounts to change the power of Frobenius and preserves all known algebraic relations.

This means that, for numbers obtained by coupling elements of $\mathcal{H}ar$ and pro-unipotent paths on $\mathbb{N}$, applying Δ_p yields numbers obtained by coupling elements of the weight-adic completion of $\mathcal{H}ar$ and pro-unipotent paths on $\mathbb{N}$, and we have a notion of "algebraic relation preserved by $\Delta_{p\mathbb{N}}$".

Question 6.16. ("conjecture of periods" in this case) Are all the relations preserved by $\Delta_{p\mathbb{N}}$?

6.2. Interpretation in terms of a class of operations and of relations between multiple harmonic sums.

6.2.1. *Usual operations on periods.* Let us recall general statements on periods, from the paper of Kontsevich and Zagier [KZ] on this subject.
The most elementary definition of periods is the following ([KZ], §1.1) : "A period is

a complex number whose real and imaginary parts are values of absolutely convergent iterated integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients."

Then, the most elementary way to formulate the conjecure that all algebraic relations between periods should come from their integral representation is the following ([KZ], §1.2, conjecture 1) : if a period has two different integral expressions, we could pass from one to another by using the following rules :
- additivity of the integral with respect to the integrand and the domain of integration.
- invertible change of variable
- the "Newton-Leibniz formula", i.e. the expression of the difference of a value of f at two points by the integral of the derivative of f.

6.2.2. *An algebra of functions and a class of operations on multiple harmonic sums.* Given the previous paragraphs, we are led to ask for an analogue of this conjecture, of course not in the general case of all series, but in the very particular case of prime multiple harmonic sums.

We define first an algebra of functions on $\mathbb{N}$. A priori, there are expressions of prime multiple harmonic sums other than the usual one and we cannot consider only the algebra $\mathcal{H}ar$ of the previous paragraph.

Definition 6.17. Let the field generated by the following functions on $\mathbb{N}$: polynomial functions, functions $k \mapsto (-1)^k$, and $k \mapsto k!$.

Definition 6.18. Consider the following operations :
i) Additivity of the summation with respect to the domain of summation - in the case of iterated sums, this includes the series shuffle product.
ii) Additivity of the summation with respect to the summand.
iii) Change of variable.
iv) Limits in $\prod_p \mathbb{Q}_p$; this includes absolutely convergent sums and the expansion $\frac{1}{1-x} = \sum_{n\geq 0} x^n$ for $x = (x_p)_p \in \prod_p \mathbb{Q}_p$ such that for all p, $|x_p|_p < 1$.

Question 6.19. If we have an equality between prime multiple harmonic sums in $\prod_p \mathbb{Q}_p$, can we pass from one to another using those operations between sums defined via the field of functions above ?

In other terms, does the class of algebraic relations obtained this way generates all relations between prime multiple harmonic sums ? If not, does it become true with certain restrictions, such as restrictions appearing in the conjecture associated to the Taylor period map in II ?

7. Conclusion : example of a motivic Galois theory of series

Usually, the introductions to multiple zeta values start with their expression as series

$$\sum_{0<n_1<\ldots<n_d} \frac{1}{n_1^{s_1}\ldots n_d^{s_d}}$$

shortly followed by their expression as iterated integrals, of the form

$$(-1)^d \int_{0<t_1<\ldots<t_n<1} \wedge_{i=1}^{n} \frac{dt_i}{t_i-\epsilon_i}$$

There is thus at first an apparent symmetry between the two expressions ; moreover, the usual introductions continue then by stating that each of these expressions yields a shuffle product and a shuffle relation, giving rise to the double shuffle relations. At some point, a radical dissymetry appears, because the iterated integral expression is the starting point to the study of multiple zeta values as periods, whereas the series shuffle relation is mostly left aside.

At such a point, we asked ourselves, as did other people, the following naive question : is there an analogous Galois theory for iterated sums ?

The usual answer to this question is that there is of course no Galois theory for series.

Here, as a conclusion to these three papers, let us explain what we call a motivic Galois theory of series in the case of prime multiple harmonic sums.

In the diagram below, the highest level will be refered to at the zeroth step, etc. and the lowest one to the third step.

There is a passage, in three steps, from the usual fundamental groupoid (precisely, the torsor of paths starting at 0 *multiplied by the groupoid restricted to usual tangential base points) to a collection of elementary algebras of functions on* $\mathbb{N}$ *:*

$$\pi_1^{un,dR}(X_Z,0,\ast)\times\prod_{t\in T}\pi_1^{un,dR}(X_Z,t)$$

$$\downarrow$$

$$\pi_1^{un,dR}(X_Z,\vec{1}_0)(\overline{\mathbb{Q}}[[x]][\log(x)])\times\prod_{z\in Z-\{0,\infty\}}\Sigma_{\Lambda-\text{adic}}\pi_1^{un,dR}(X_Z,\vec{1}_0,\vec{1}_z)$$

$$\downarrow$$

$$\prod_{z\in Z-\{\infty\}}\mathbb{A}^W(\overline{\mathbb{Q}}[[x]][\log(x)])$$

$$\downarrow$$

Elementary algebras of functions on $\mathbb{N}$

with,

a) for the two first steps :

i) the arrow (the first one is the reindexation $\Sigma_{(\mathrm{Li}\,\mathcal{T})^{\sqcup\!\sqcup}_{O,prime}}$ *of §5, the second one is described in §5.4) is subject to a commutative diagram involving the motivic Galois action and a pushed forward version of it*

ii) we make a "conjecture of periods" for the arrow, establishing the consistence of the *setting, i.e. the view of the pushed forward action as a motivic Galois action*

Before stating the property satisfied by the third step, let us mention the status of those two "conjectures of periods" : the one of the first arrow, which lies purely in the motivic framework, is proved by parts II and III ; our hope in the conjecture of periods of the second arrow comes from the main results of parts I and II. The equality between the substitutes of the motivic Galois action at the first and second step comes from our p-adic work.

At the end of the two first arrows, we have object attached to prime multiple harmonic sums, keeping a track of the motivic Galois action and the conjectures of periods, and having the particularity that it is purely expressed in terms of Taylor coefficients at 0 of hyperlogarithms. We add the following observation.

b) We can define, at the third step, a substitute to a part of the motivic fundamental groupoid of X_Z that is indexed purely in terms of $\mathbb{N}$; it is compatible with the usual fundamental groupoid.
Precisely, it is a substitute to : the variety X_Z, its Betti and de Rham fundamental groupoid, its motivic Galois action on multiple zeta values, and the class of motivic relations between multiple zeta values

(formula for the analogue of the Galois action on the "pro-unipotent paths on $\mathbb{N}$" :

$$"\Delta_p : \gamma \mapsto \gamma|_{p\mathbb{N}} \otimes \big(\otimes_{i=1}^{r(p)-1} \underbrace{n_{i(p)} < \ldots < n_{i+1(p)})}_{\text{subpath of } \gamma \text{ from } n_{i(p)} \text{ to } n_{i+1(p)}} ")$$

It satisfies :
i) the third arrow (described in §6.1, and which goes actually from the third step to the second step) is compatible with Δ_p on the third step and the avatar of the Galois action on the second step.
ii) there is thus a "conjecture of periods" for the arrow, it reformulates the conjecture of periods at the second step.

Moreover, it becomes natural in this context to define a class of relations between multiple harmonic sums that is an analogue of the class of relations between periods arising from elementary manipulations of integrals, and to formulate the conjecture of periods in this terms.

As a conclusion, we can say that, in the example of prime multiple harmonic sums, we can build ad hoc of a motivic Galois theory of series.

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