Phase structure of the Born-Infeld-anti-de Sitter black holes probed by non-local observables

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ABSTRACT: With the non-local observables such as two point correlation function and holographic entanglement entropy, we probe the phase structure of the Born-Infeld-anti-de Sitter black holes. We find for the case $bQ > 0.5$, the phase structure is similar to that of the Reissner-Nordström-AdS black hole, namely the black hole undergoes a Hawking-Page phase transition, a first order phase transition, and a second order phase transition. While for the case $bQ < 0.5$, we find there is a new branch for the infinitesimally small black hole so that a pseudo phase transition emerges besides the original first order phase transition. For the first order phase transition and the pseudo phase transition, the equal area law is checked, and for the second order phase transition, the critical exponent of the analogous heat capacity is obtained in the neighborhood of the critical points. All the results show that the phase structure of the non-local observables is the same as that of the thermal entropy regardless of the size of the boundary region in the field theory.

Contents

1. Introduction

Investigation on phase transition of an AdS space time has attracted attention of many theoretical physicists recently. The main motivation maybe steps from the existence of the AdS/CFT correspondence [1, 2, 3], which relates a black hole in the AdS space time to a thermal system without gravity. For in this case, some interesting but intractable phenomena in strongly coupled system become to be tractable easily in the bulk. To probe these fascinating phenomena in field theory, one should employ some non-local observables such as two point correlation function, Wilson loop, and holographic entanglement entropy, which are dual to the geodesic length, minimal area surface, and minimal surface area in the bulk individually. It has been shown that these observables can probe the non-equilibrium thermalization behavior [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], superconducting phase transition [15, 16, 17, 18, 19, 20, 21, 22], and cosmological singularity [23, 24].

In this paper, we intend to use the non-local observables to probe the phase structure of the Born-Infeld-anti-de Sitter black holes. Usually, phase structure of a black hole is understood from the viewpoint of thermodynamics. For an uncharged AdS black hole, it has been found that there is a phase transition between the thermal gas in AdS space and Schwarzschild AdS black holes [25], which was interpreted as the confinement/deconfinement phase transition in the dual gauge field theory later [26]. As the charge is endowed with, the AdS black hole will undergo a Van der Waals-like phase transition before it reaches the stable state in the entropy-temperature plane [27]. Specifically speaking, there exists a critical charge, and for the case that the charge of the black hole is smaller than the critical

charge, the black hole undertakes a first order phase transition. As the charge increases to the critical charge, the phase transition is second order, while the charge exceeds the critical charge, there is not phase transition and the black hole is always stable. The Van der Waals-like phase transition can be observed in many circumstances. In [28], it was found that a 5-dimensional neutral Gauss-Bonnet black hole demonstrates the Van der Waals-like phase transition in the $T - \alpha$ plane, where T is the Hawking temperature and α is the Gauss-Bonnet coupling parameter. In [29], the Van der Waals-like phase transition was also observed in the $Q - \Phi$ plane, where Q is electric charge and Φ is the chemical potential. Treating the negative cosmological constant as the pressure P and its conjugate as the thermodynamical volume V, the Van der Waals-like phase transition has also been observed in the $P - V$ plane recently[30, 31, 32, 33, 34, 35, 36, 37, 38].

Very recently, with the entanglement entropy as a probe, [40] investigated the phase structure of the Reissner-Nordström-AdS black hole and found that there was also a Van der Waals-like phase transition in the entanglement entropy-temperature plane [1]. They also obtained the critical exponent of the heat capacity for the second order phase transition in the neighborhood of the critical points. To further confirm the similarity of the phase structure between the thermal entropy and entanglement entropy, [41] checked the equal area law later and found that it held for the first order phase transition in the entanglement entropy-temperature plane. Now [40] has been generalized to the extended space time [42], massive gravity [43] , as well as Weyl gravity [44], and all the results showed that the entanglement entropy exhibited the same phase structure as that of the thermal entropy.

In this paper, besides the entanglement entropy, we will employ the two point correlation function to probe the phase structure of the black holes. We choose the Born-Infeld-anti-de Sitter black holes as the gravity model, which is a solution of the Einstein-Born-Infeld action. There have been many works to study phase structure of the Born-Infeld-anti-de Sitter black holes [45, 46, 47, 48]. The results showed that both the Born-Infeld parameter b and charge Q affect the phase structure, and to assure the existence of a non-extremal black hole, one should impose the condition $bQ > 0.5$. In this paper, we find for the case $bQ > 0.5$, the phase structure of the black hole is similar to that of the Reissner-Nordström-AdS black hole in the entropy-temperature plane. While for the case $bQ < 0.5$, there is a novel phase structure, which has not been observed previously. Specially speaking, a new branch emerges compared with that of the Reissner-Nordström-AdS black hole so that there are two unstable regions and two phase transition temperature correspondingly. All these phase structure are probed by the nonlocal observables such as two point correlation function as well as holographic entanglement entropy, and phase structure of the nonlocal observables are found to be the same as that of the thermal entropy.

Our paper is outlined as follows. In the next section, we will review the thermodynamic properties of the Born-Infeld-anti-de Sitter black hole firstly, and then study its phase structure in $T - S$ plane in a fixed charge ensemble. In Section 3, we will employ the two point

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[1]In fact, the Van der Waals phase transition has also been observed for a zero temperature system in the framework of holography [39].

correlation function and holographic entanglement entropy to probe the phase structure of the Born-Infeld-anti-de Sitter black hole. In each subsection, the equal area law is checked and the critical exponent of the heat capacity is obtained. The last section is devoted to discussions and conclusions.

2. Thermodynamic phase transition of the Born-Infeld-anti-de Sitter black hole

2.1 Review of the Born-Infeld-anti-de Sitter black hole

The 4-dimensional Born-Infeld AdS black hole is a solution of the following action [49]

$$S = \int d^4x \sqrt{-g} \left[\frac{R - 3\Lambda}{16\pi G} + \frac{b^2}{4\pi G} \left(1 - \sqrt{1 + \frac{2F}{b^2}}\right) \right], \tag{2.1}$$

in which $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, R is scalar curvature, G is the gravitational constant, Λ is the cosmological constant that relates to the AdS radius as $\Lambda = -3/l^2$, and b is the Born-Infeld parameter which relates to the string tension α' as $b = 1/(2\pi\alpha')$. Explicitly, the solutions of the Born-Infeld AdS black hole can be written as [50, 51]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.2}$$

where

$$f(r) = \frac{4Q^2 \, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{Q^2}{b^2 r^4}\right)}{3r^2} + \frac{2b^2 r^2}{3}\left(1 - \sqrt{\frac{Q^2}{b^2 r^4} + 1}\right) + \frac{r^2}{l^2} - \frac{2M}{r} + 1, \tag{2.3}$$

in which $_2F_1$ is the hypergeometric function. From Eq.(2.3), we know that in the limit $b \to \infty$, $Q \neq 0$, the solution reduces to the the Reissner-Nordström-AdS black hole, and in the limit $Q \to 0$, it reduces to the Schwarzschild AdS black hole.

The ADM mass of the black hole, defined by $f(r_+) = 0$, is given by

$$M = \frac{4l^2 Q^2 \, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{Q^2}{b^2 r_+^4}\right) - 2b^2 l^2 r_+^4 \sqrt{\frac{Q^2}{b^2 r_+^4} + 1} + 2b^2 l^2 r_+^4 + 3l^2 r_+^2 + 3r_+^4}{6l^2 r_+}, \tag{2.4}$$

in which r_+ is the event horizon of the black hole. The Hawking temperature defined by $T = \frac{f'(r)}{4\pi}\big|_{r_+}$ can be written as

$$T = \frac{l^2\left(1 - 2b^2 r_+^2\left(\sqrt{\frac{Q^2}{b^2 r_+^4} + 1} - 1\right)\right) + 3r_+^2}{4\pi l^2 r_+}, \tag{2.5}$$

where we have used Eq.(2.4). In addition, according to the Bekenstein-Hawking entropy area relation, we can get the black hole entropy
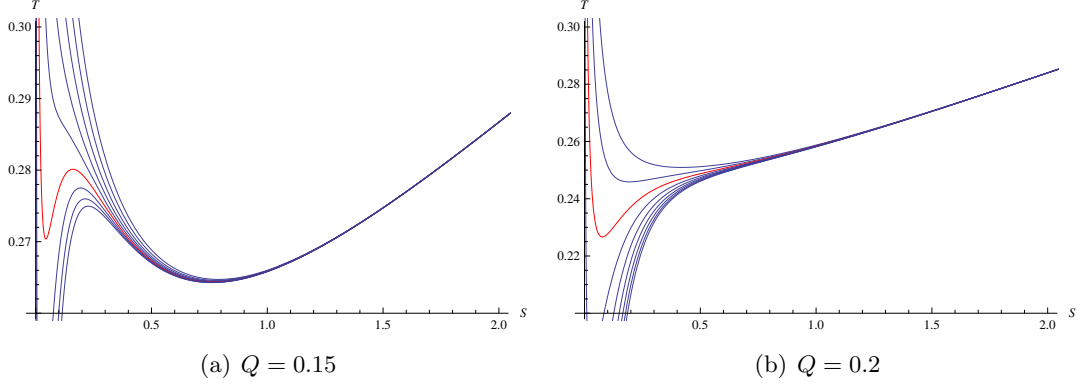
$$S = \pi r_+^2. \tag{2.6}$$

(a) $Q = 0.15$ (b) $Q = 0.2$

Figure 1: Relations between entropy and temperature for different b at a fixed Q. The curves from top to down correspond to the case b varies from 1.5 to 3.5 with step 0.25.

Inserting Eq.(2.6) into Eq.(2.5), we can get the entropy temperature relation, namely

$$T = \frac{-2b^2 l^2 S \sqrt{\frac{\pi^2 Q^2}{b^2 S^2} + 1} + 2b^2 l^2 S + \pi l^2 + 3S}{4\pi^{3/2} l^2 \sqrt{S}}. \tag{2.7}$$

It is obvious that besides the charge Q, the Born-Infeld parameter b also affects the phase structure of this space time in the $T - S$ plane.

2.2 Phase transition of thermal entropy

Based on Eq.(2.7), we will discuss the phase structure of the Born-Infeld-anti-de Sitter black hole. We are interested in how b or Q affects it. For a fixed charge Q, the effect of b on the phase structure is shown in Figure 1. It is obvious that for the small b, the phase structure resembles as that of the Schwarzschild AdS black hole, and for large b, it resembles as that of the Reissner-Nordström-AdS black hole as expected. From (b) of Figure 1, we can see that as b increases, the minimum temperature of the space time decreases, and the entropy, also the event horizon, becomes smaller. That is, the Born-Infeld parameter b promotes the formation of an AdS space time. More interestingly, for the smaller charge, we find there is a novel phase transition, which is labeled by the red solid line in (a) of Figure 1. From this curve, we know that there are two unstable regions. Namely the black hole will undergo the following transition: unstable-stable-unstable-stable. Compared with that of the Reissner-Nordström-AdS black hole, a new branch emerges at the onset of the phase transition. The influence of the charge on the phase structure for a fixed Born-Infeld parameter is plotted in Figure 2. Apparently, the phase structure is similar to that of the Schwarzschild AdS black hole for the small charges, and Reissner-Nordström-AdS black hole for large charges. From (a) of Figure 2, we observe that the larger the charge is, the lower the minimum temperature will be. In other words, the charge will promote the formation of an AdS space time, which has the same effect as that of the Born-Infeld parameter on the phase structure of the black hole. For the case $b = 4$ in (b) of Figure 2, we see that for some intermediate values of charge Q, there is
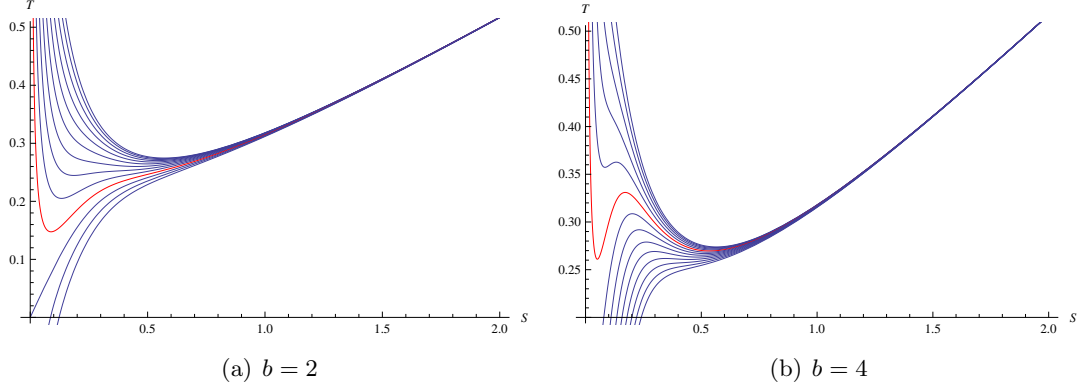
(a) $b = 2$ (b) $b = 4$

Figure 2: Relations between entropy and temperature for different Q at a fixed b. The curves from top to down in (a) correspond to that Q varies from 0.05 to 0.3 with step 0.02, and in (b) they correspond to the case Q varies from 0.065 to 0.195 with step 0.01.
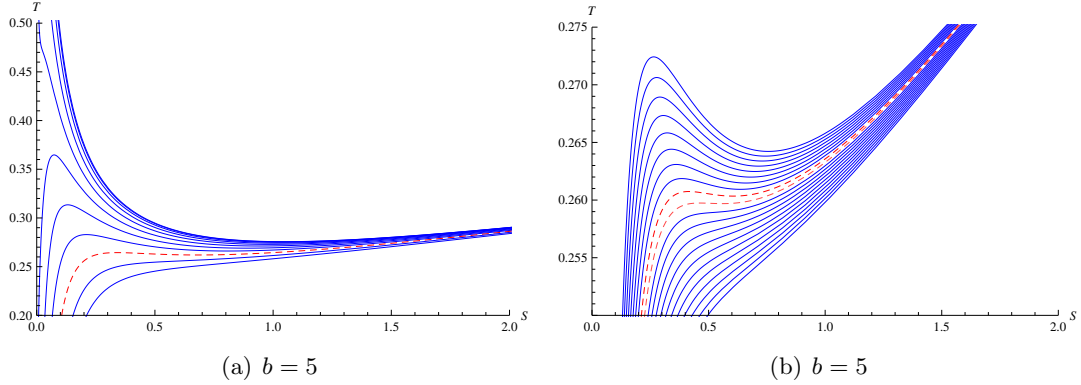

(a) $b = 5$ (b) $b = 5$

Figure 3: Relations between entropy and temperature for different Q at a fixed b. The curves from top to down in (a) correspond to that Q varies from 0.02 to 0.2 with step 0.02, and in (b) they correspond to that Q varies from 0.015 to 0.19 with step 0.002.

also a similar phase structure as that in (b) in Figure 1, which is labeled by the red solid line in (b) of Figure 2. Note that however, for the large b, the novel phase structure disappears, which is shown in Figure 3. For in this case, the Born-Infeld parameter is large enough, so that the phase structure resembles completely as that of the Reissner-Nordström-AdS black hole.

Next, we will study detailedly the phase structure of the Born-Infeld AdS black hole for the case $b = 4$ and $b = 5$ respectively. To finish it, we should first find the critical charge for a fixed Born-Infeld parameter by the condition

$$(\frac{\partial T}{\partial S})_Q = (\frac{\partial^2 T}{\partial S^2})_Q = 0. \tag{2.8}$$

However, from Eq.(2.5), we find it is hard to get the analytical result directly. Taking the case $b = 5$ as an example, we will show how to get it numerically next. We first plot a series
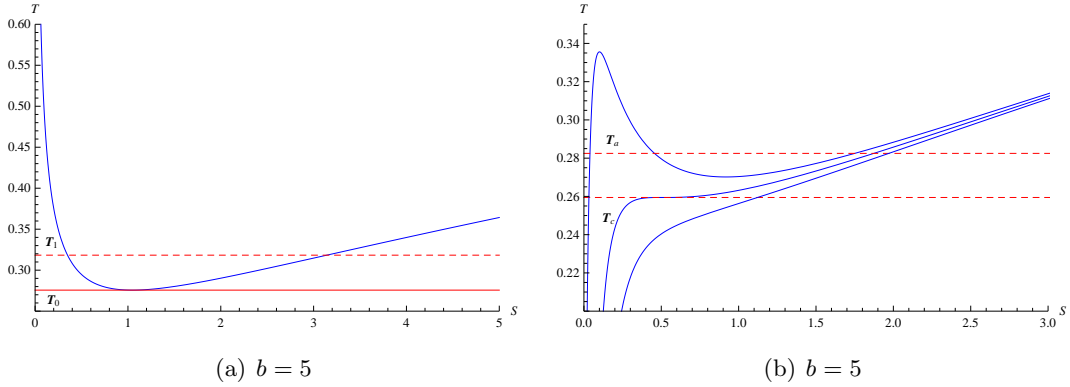
(a) $b = 5$ (b) $b = 5$

Figure 4: Relations between entropy and temperature for different Q at $b = 5$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red dashed lines from top to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.

of curves for different charges in the $T - S$ plane, which is shown in (a) of Figure 3, and read off the charge which satisfies likely the condition $(\frac{\partial T}{\partial S})_Q = 0$. With that rough value, we plot a bunch of curves in the $T - S$ plane once with smaller step so that we can get the most likely value of charge. From (b) of Figure 3, we find the critical charge should be $0.168 < Q_c < 0.17$, which are labeled by the red dashed lines in (b) of Figure 3. Lastly, we adjust the value of Q_c by hand to find the only solution S_c which satisfies Eq.(2.8), which produces $Q_c = 0.168678344129$, $S_c = 0.510691$. Substituting these critical values into Eq.(2.5), we further get the critical temperature $T_c = 0.259444$.

Having obtained the critical charge, we can plot the isocharge curves for the case $b = 5$ in the $T - S$ plane, which is shown in Figure 4. We observe that the black hole undergoes a Hawking-Page phase transition and a Van der Waals-like phase transition. Specifically speaking, for the case $Q = 0$, there is a minimum temperature $T_0 = \frac{\sqrt{3}}{2\pi}$ [52], which is indicated by the red dashed line in (a) of Figure 4. When the temperature is higher than T_0, there are two additional black hole branches. The small branch is unstable while the large branch is stable. The Hawking-Page phase transition occurs at the temperature given by $T_1 = \frac{1}{\pi}$ [52], which is indicated by the red dotted line. The Hawking-Page phase transition also can be observed in the $F - T$ plane, in which F is the Helmholtz free energy defined by[2]

$$F = \frac{8l^2 Q^2 \, _2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{Q^2}{b^2 r_+^4}\right) + l^2 r_+^2 \left(2b^2 r_+^2 \left(\sqrt{\frac{Q^2}{b^2 r_+^4} + 1} - 1\right) + 3\right) - 3r_+^4}{12l^2 r_+}. \tag{2.9}$$

From (a) of Figure 5, we know that there is a minimum temperature T_0, and above this temperature, there are two branches. The lower branch is stable always. The Hawking-Page phase transition occurs at T_1 for in this case the free energy vanishes.

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[2]Note that here we have implicitly chosen the pure AdS as the reference spacetime because the free energy vanishes for pure AdS by this formula.
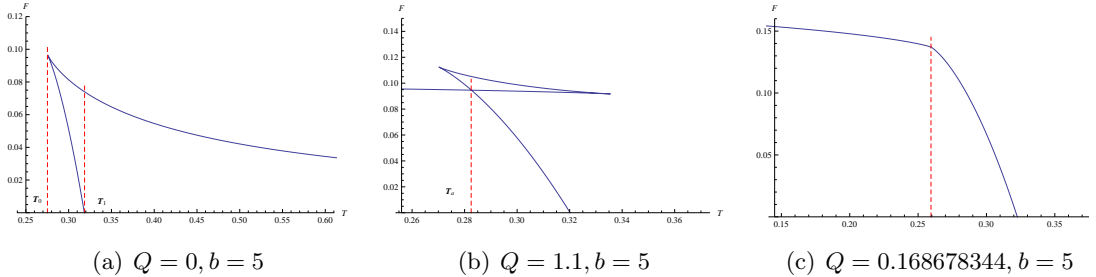
Figure 5: Relations between the free energy and temperature for different charges at $b = 5$.

For the case $Q \neq 0$, the phase structure is similar to that of the Van der Waals phase transition, which is shown in (b) of Figure 4. The solid blue lines from top to down correspond to the isocharges for the case $Q = 0.11, 0.168678344129, 0.21$. We can see that black holes endowed with different charges have different phase structures. For the small charge, there is an unstable black hole interpolating between the stable small hole and stable large hole. The small stable hole will jump to the large stable hole at the critical temperature T_a. As the charge increases to the critical charge, the small hole and the large hole merge into one and squeeze out the unstable phase so that an inflection point emerges. The divergence of the heat capacity in this case implies that the phase transition is second order. As the charge exceeds the critical charge, we simply have one stable black hole at each temperature. The Van der Waals-like phase transition can also be observed from the $F - T$ relation. From (b) of Figure 5, we observe that there is a swallowtail structure, which corresponds to the unstable phase in the top curve in (b) Figure 4. The critical temperature $T_a = 0.2825$ is apparently the value of the horizontal coordinate of the junction between the small black hole and the large black hole. As the temperature is lower than the critical temperature T_a, the free energy of the small black hole is lowest, so the small hole is stable. As the temperature is higher than T_a, the free energy of the large black hole is lowest, so that the large hole dominates thereafter. The non-smoothness of the junction indicates that the phase transition is first order. From (c) of Figure 5, we know that there is an inflection point, which corresponds to the inflection point in the middle curve in (b) of Figure 4. The horizontal coordinate of the inflection point corresponds to the second order phase transition temperature T_c.

Similarly, we also can study the phase structure for the case $b = 4$ in the $T - S$ plane. To finish it, we should first find the critical charge. Adopting the same strategy as that of the case $b = 5$, we find $Q_c = 0.16987452395$, $S_c = 0.502683752119928$, $T_c = 0.259166$. For the case $Q = 0$, the phase structure is the same as that in (a) of Figure 4. We will not repeat it here. For the case $Q \neq 0$, we find besides the Van der Waals-like phase transition, there is a novel phase transition for $Q = 0.115$ as observed in Figure 1 and Figure 2, which is plotted in (a) of Figure 6. That is, at the beginning of the phase transition, an infinitesimally small black hole branch emerges. It is unstable and its life is very short so that it becomes to a small stable black hole quickly. In addition, we find there are two unstable region, and correspondingly in the $F - T$ plane, we observe two swallowtail structures, which is shown in (a) of Figure 7. The
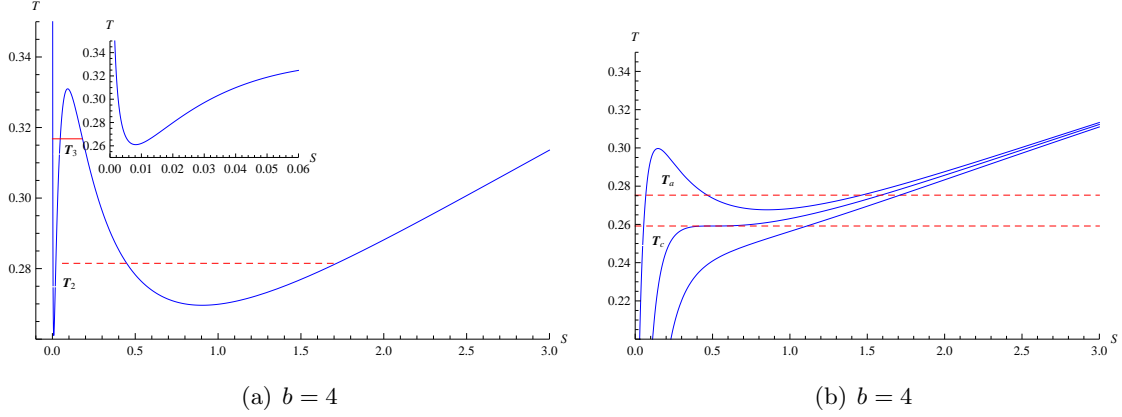
Figure 6: Relations between entropy and temperature for different Q at $b = 4$. The red solid line and dashed line in (a) correspond to pseudo phase transition temperature and first order phase transition temperature. The red dashed lines from top to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.
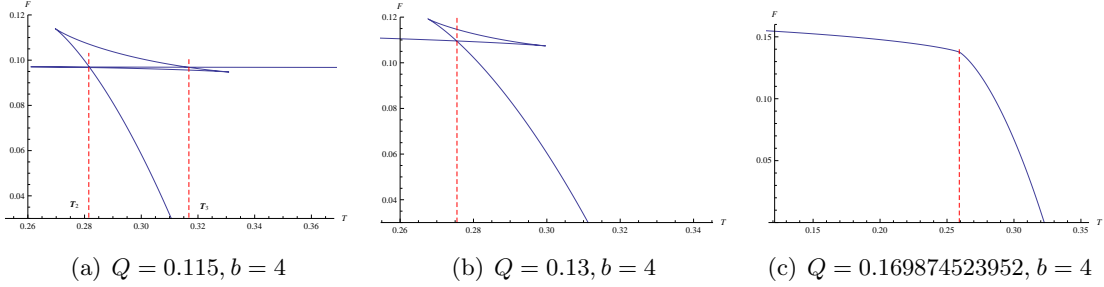

Figure 7: Relations between the free energy and temperature for different charges at $b = 4$.

horizontal coordinate of the junction of the swallowtail corresponds to the phase transition temperature, thus besides the first order phase transition temperature between the small black hole and large black hole, labeled by T_2, there is a new phase transition temperature, labeled by T_3. It should be stressed that the new phase transition can not take place actually, for above the critical temperature T_2, the free energy of the large black hole is lowest so that the space time is dominated by the large black hole, which can be seen in (a) of Figure 7. Thus, we call thus phase transition pseudo phase transition. We observe that there is also a Van der Waals-like phase transition, which is plotted in (b) of Figure 6. The solid blue lines from top to down correspond to the isocharges for the case $Q = 0.13, 0.16987452395, 0.21$. The first order phase transition temperature T_a can be read off from the horizontal coordinate of the junction of the swallowtail structure in (b) of Figure 7, and the second order phase transition temperature T_c can be read off from the horizontal coordinate of the inflection point in (c) of Figure 7.

As done in [41], we will also check numerically whether Maxwell's equal area law holds

	S_{min}	S_{max}	A_L	A_R
$b = 5, Q = 0.11, T_a = 0.2825$	0.039159	1.7506571	0.4837	0.4835
$b = 4, Q = 0.13, T_a = 0.2757$	0.0685377	1.46304	0.3840	0.3840
$b = 4, Q = 0.115, T_2 = 0.2817$	0.0210019	1.72464	0.480	0.480
$b = 4, Q = 0.115, T_3 = 0.3166$	0.00212107	0.182767	0.05717	0.05719

Table 1: Check of the equal area law in the $T - S$ plane.

for the first order phase transition and pseudo phase transition, which states

$$A_L \equiv \int_{S_{min}}^{S_{max}} T(S, Q)dS = T_x(S_{max} - S_{min}) \equiv A_R, \qquad (2.10)$$

in which $T(S, Q)$ is defined in Eq.(2.7), T_x is the phase transition temperature, and S_{min}, S_{max} are the smallest and largest values of unstable region which satisfies the equation $T(S, Q) = T_x$. Usually for this equation, there are three roots S_1, S_2, S_3, so $S_{min} = S_1$, $S_{max} = S_3$, $T_x = T_a$. But for the case $b = 4, Q = 0.115$, there are four roots. One can see from (a) of Figure 6 that $S_{min} = S_1$, $S_{max} = S_3$ for the pseudo phase transition with phase transition temperature $T_x = T_3$, and $S_{min} = S_2$, $S_{max} = S_4$ for the first order phase transition with phase transition temperature $T_x = T_2$. The calculated results for the first order phase transition and pseudo phase transition are listed in Table 1. From this table, we can see that A_L equals A_R for different b in our numeric accuracy. The equal area law therefore holds.

For the second order phase transition, we know that the heat capacity is divergent near the critical point and the critical exponent is $-2/3$. As stated in [40], near the critical point, there is always a linear relation

$$\log | T - T_c |= 3 \log | S - S_c | +constant, \qquad (2.11)$$

with 3 the slope. It is not difficult to show that for the case $b = 4, 5$ in our gravity model, the temperature and entropy also satisfy this linear relation near the critical point. Next, we will take Eq.(2.11) as a reference to check whether there is a similar relation for the second order phase transition in the two point correlation function-temperature plane as well as entanglement entropy-temperature plane.

3. Phase transition in the framework of holography

Having obtained the phase structure of thermal entropy of the Born-Infeld AdS black hole in the $T - S$ plane, we will study the phase structure of two point correlation function and entanglement entropy in the fled theory to see whether they have the similar phase structure and critical behavior.

3.1 Phase structure probed by two point correlation function

According to the AdS/CFT correspondence, in the large Δ limit, the equal time two point correlation function can be written as [53]

$$\langle \mathcal{O}(t_0, x_i) \mathcal{O}(t_0, x_j) \rangle \approx e^{-\Delta L}, \tag{3.1}$$

where Δ is the conformal dimension of scalar operator $\mathcal{O}$ in the dual field theory, L is the length of the bulk geodesic between the points (t_0, x_i) and (t_0, x_j) on the AdS boundary. In our gravity model, we can simply choose $(\phi = \frac{\pi}{2}, \theta = 0)$ and $(\phi = \frac{\pi}{2}, \theta = \theta_0)$ as the two boundary points. Then with θ to parameterize the trajectory, the proper length is given by

$$L = \int_0^{\theta_0} \mathcal{L}(r(\theta), \theta) d\theta, \quad \mathcal{L} = \sqrt{\frac{\dot{r}^2}{f(r)} + r^2}, \tag{3.2}$$

in which $\dot{r} = dr/d\theta$. Imagining θ as time, and treating $\mathcal{L}$ as the Lagrangian, one can get the equation of motion for $r(\theta)$ by making use of the Euler-Lagrange equation. Then with the following boundary conditions

$$\dot{r}(0) = 0, r(0) = r_0, \tag{3.3}$$

we can get the numeric result of $r(\theta)$. To explore whether the size of the boundary region affects the phase structure, we will choose $\theta_0 = 0.2, 0.3$ as two examples. Note that for a fixed θ_0, the geodesic length is divergent, so it should be regularized by subtracting off the geodesic length in pure AdS with the same boundary region, denoted by $L_0{}^3$. To achieve this, we are required to set a UV cutoff for each case, which is chosen to be $r(0.199)$ and $r(0.299)$, respectively for our two examples. The regularized geodesic length is labeled as $\delta L \equiv L - L_0$. In addition, during the numerics, we will set the AdS radius l to be 1.

We plot the relation between T and δL for the case $b = 5$ in Figure 8 and Figure 9 . The solid blue line in (a) corresponds to the isocharge for the case $Q = 0$, and the dashed blue lines from top to down in (b) correspond to that $Q = 0.11, 0.168678344129, 0.21$. From these figures, one can see that δL demonstrates a similar phase structure as that of the thermal entropy in Figure 4. Namely the two point correlation function can probe both the Hawking-Page phase transition and Van der Waals-like phase transition. Precisely speaking, we find that the minimum temperature T_0 as well as Hawking-Page phase transition temperature T_1 in (a), the first order phase transition temperature T_a, and second order phase transition temperature T_c in (b) are exactly the same as those in $T - S$ plane. This conclusion will not be affected as θ_0 varies in a reasonable region, which can be seen from Figure 8 and Figure 9.

With the two point correlation function, we also can probe the phase structure of the case $b = 4$, which is shown in Figure 10 and Figure 11. The solid blue line in (a) corresponds to the isocharge for the case $Q = 0.115$, and the solid blue lines from top to down in (b) correspond

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[3]L_0 can be obtained easily for there is an analytical result for $r(\theta)$ in the pre AdS, namely $r_{AdS}(\theta) = l[(\frac{\cos\theta}{\cos\theta_0})^2 - 1]^{-1/2}$ [56, 57].
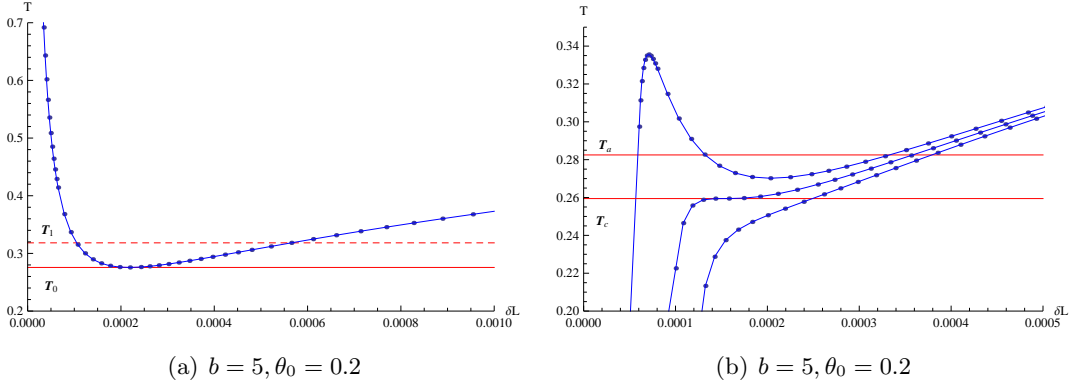
(a) $b = 5, \theta_0 = 0.2$

(b) $b = 5, \theta_0 = 0.2$

Figure 8: Isocharges in the $\delta L - T$ plane for the case $b = 5, \theta_0 = 0.2$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red solid lines from top to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.

(a) $b = 5, \theta_0 = 0.3$

(b) $b = 5, \theta_0 = 0.3$

Figure 9: Isocharges in the $\delta L - T$ plane for the case $b = 5, \theta_0 = 0.3$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red solid lines from top to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.
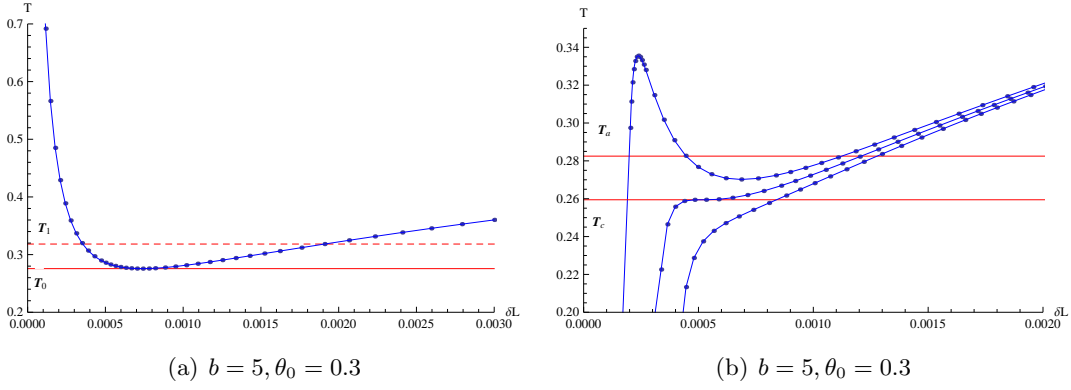
to that $Q = 0.13, 0.16987452395, 0.21$. As the case as the thermal entropy in the $T - S$ plane, we find there ia also a novel phase structure besides the Van der Waals-like phase transition in the $\delta L - T$ plane. There are also two unstable region, correspondingly two phase transition temperature, labeled by T_2, T_3. For the phase transition temperature mentioned above, it is easy to check T_0 by locating the position of local minimum. But in order to locate T_2, T_3, and T_a precisely, we are required to examine the equal area law for the first order phase transition as well as pseudo phase transition. And to locate T_c, we should obtain the critical exponent $-2/3$ for the second order phase transition, which are documented as follows.

In the $\delta L - T$ plane, we define the equal area law as

$$A_L \equiv \int_{\delta L_{min}}^{\delta L_{max}} T(\delta L) d\delta L = T_x(\delta L_{max} - \delta L_{min}) \equiv A_R, \tag{3.4}$$

(a) $b = 4, \theta_0 = 0.2$
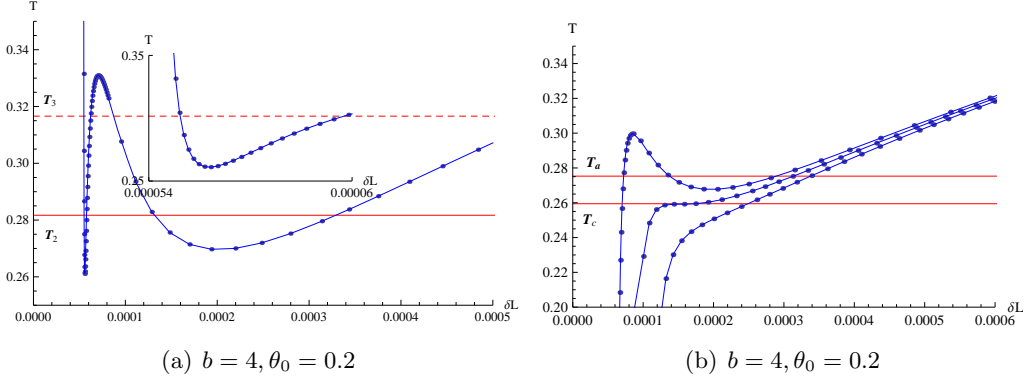

(b) $b = 4, \theta_0 = 0.2$

Figure 10: Isocharges in the $\delta L - T$ plane for the case $b = 4, \theta_0 = 0.2$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red solid lines from above to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.
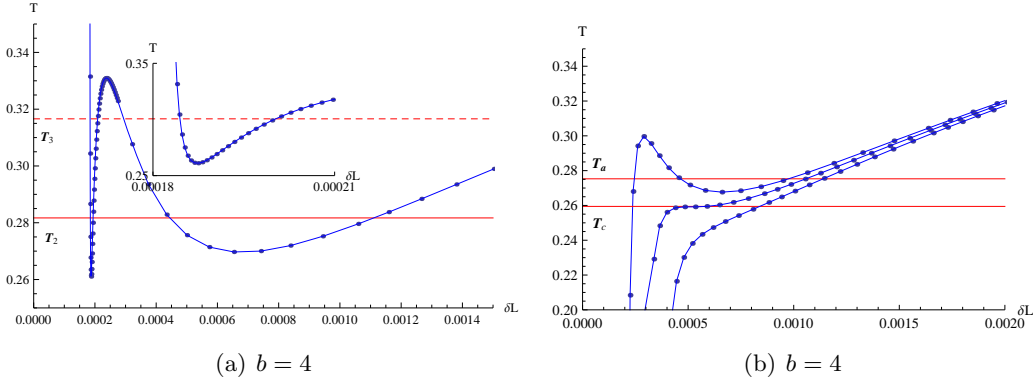

(a) $b = 4$

(b) $b = 4$

Figure 11: Isocharges in the $\delta L - T$ plane for the case $b = 4, \theta_0 = 0.3$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red solid lines from top to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.

in which $T(\delta L)$ is an Interpolating Function obtained from the numeric result, T_x is the phase transition temperature, and δL_{min}, δL_{max} are the smallest and largest values of the unstable region which satisfy the equation $T(\delta L) = T_x$. Similar to the equal area law in the $S - T$ plane, we know that for $b = 5, Q = 0.11$ and $b = 4, Q = 0.13$, $T_x = T_a$, while for $b = 4, Q = 0.115$, $T_x = T_2$ for the first order phase transition and $T_x = T_3$ for the pseudo phase transition. For different b and θ_0, the calculated results are listed in Table 2. It is obvious that A_L equals A_R for different b and θ_0 in a reasonable accuracy. That is, in the $T - \delta L$ plane, the equal area law holds, and it is independent of the Born-Infeld parameter as well as the size of the boundary region.

In order to get the critical exponent for the second order phase transition in the $T - \delta L$

	δL_{min}	δL_{max}	A_L	A_R
$b=5, Q=0.11, T_a=0.2825, \theta_0=0.2$	0.00005901	0.0003329	0.0000777	0.0000774
$b=5, Q=0.11, T_a=0.2825, \theta_0=0.3$	0.0001982	0.001124	0.000263	0.000261
$b=4, Q=0.13, T_a=0.2757, \theta_0=0.2$	0.00007227	0.0002890	0.0000598	0.0000596
$b=4, Q=0.13, T_a=0.2757, \theta_0=0.3$	0.0002453	0.0009755	0.000201	0.000201
$b=4, Q=0.115, T_2=0.2817, \theta_0=0.2$	0.00005509	0.0003290	0.0000774	0.0000772
$b=4, Q=0.115, T_2=0.2817, \theta_0=0.3$	0.0001939	0.001110	0.000259	0.000258
$b=4, Q=0.115, T_3=0.3166, \theta_0=0.2$	0.00005485	0.00008727	0.0000103	0.0000103
$b=4, Q=0.115, T_3=0.3166, \theta_0=0.3$	0.0001842	0.00003476	0.0000349	0.0000348

Table 2: Check of the equal area law in the $T - \delta L$ plane.

plane, we should first define an analogous heat capacity

$$C = T\frac{\partial \delta L}{\partial T}. \tag{3.5}$$

Provided a similar relation as showed in Eq.(2.11) is satisfied, one can get the critical exponent immediately. So next, we are interested in the logarithm of the quantities $T - T_c$, $\delta L - \delta L_c$, in which T_c is the critical temperature mentioned previously, and L_c is obtained numerically by the equation $T(\delta L) = T_c$. We plot the relation between $\log |T - T_c|$ and $\log |\delta L - \delta L_c|$ for different θ_0 and b in Figure 12, where these straight lines can be fitted as

$$\log |T - T_c| = \begin{cases} 25.0567 + 3.0113 \log |\delta L - \delta L_c|, & \text{for } b=5, \theta_0=0.2, \\ 21.207 + 3.0402 \log |\delta L - \delta L_c|, & \text{for } b=5, \theta_0=0.3, \\ 23.8282 + 3.01531 \log |\delta L - \delta L_c|, & \text{for } b=4, \theta_0=0.2, \\ 20.3704 + 3.03817 \log |\delta L - \delta L_c|, & \text{for } b=4, \theta_0=0.3. \end{cases} \tag{3.6}$$

It is obvious that the slope is about 3, which indicates that the critical exponent is $-2/3$ nearly for the analogous heat capacity, and the phase transition is also second order with the phase transition temperature T_c.

3.2 Phase structure probed by holographic entanglement entropy

According to the formula in [54, 55], holographic entanglement entropy can be given by the area A_Σ of a minimal surface Σ anchored on the boundary entangling surface $\partial\Sigma$, namely

$$S = \frac{A_\Sigma(t)}{4G}, \tag{3.7}$$

where G is the Newton's constant. We will take the region Σ to be a spherical cap on the boundary delimited by $\theta \leq \theta_0$[4]. Then based on the definition of area and (2.2), (3.7) can be

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[4]To avoid contamination of the entanglement entropy by the thermal entropy, we will choose a small entangling region as pointed out in [40, 58].
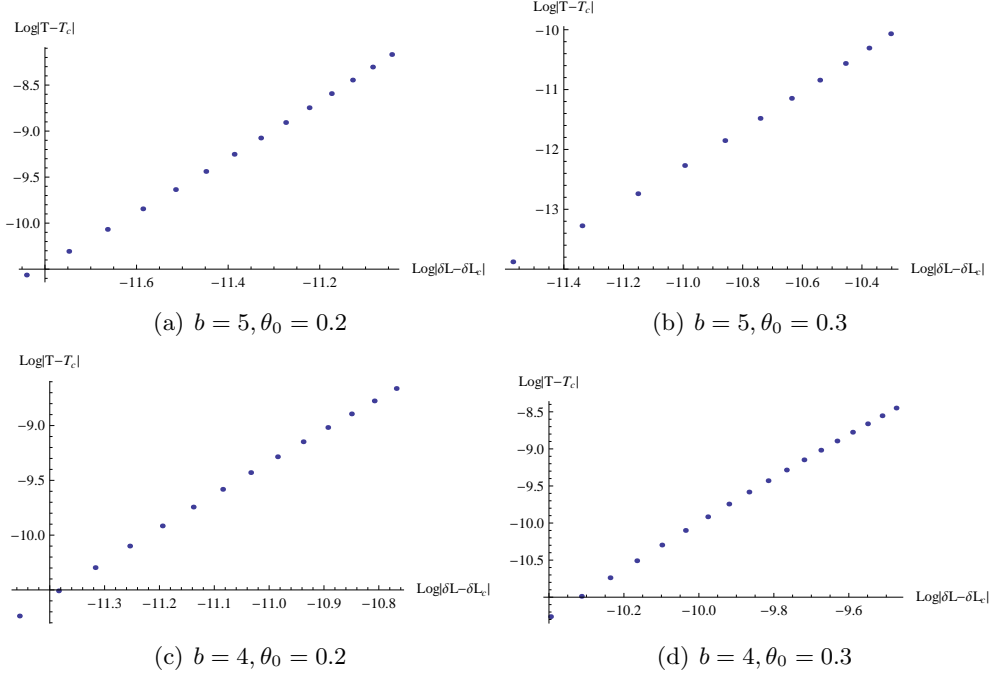
(a) $b = 5, \theta_0 = 0.2$

(b) $b = 5, \theta_0 = 0.3$

(c) $b = 4, \theta_0 = 0.2$

(d) $b = 4, \theta_0 = 0.3$

Figure 12: Relations between $\log | T - T_c |$ and $\log | \delta L - \delta L_c |$ for different b and θ_0.

rewritten as

$$S = \frac{\pi}{2} \int_0^{\theta_0} r \sin \theta \sqrt{\frac{(r')^2}{f(r)} + r^2}, \qquad (3.8)$$

in which $r' = dr/d\theta$. Making use of the Euler-Lagrange equation, one can get the equation of motion of $r(\theta)$. With the boundary conditions in Eq.(3.3), we can obtain the numeric result of $r(\theta)$ immediately. Note that the entanglement entropy is divergent at the boundary, so it should be regularized by subtracting off the entanglement entropy in pure AdS with the same entangling surface and boundary values. We label the regularized entanglement entropy as δS. We choose the size of the boundary region as $\theta_0 = 0.2$ and set the UV cutoff in the dual field theory to be $r(0.199)$.

The isocharges for the cases $b = 5$ and $b = 4$ on the entanglement entropy-temperature plane are plotted in Figure 13 and Figure 14. As the same as that of the two point correlation function, for $b = 5$, we are interested in that the black holes have charges $Q = 0, 0.11, 0.168678344129, 0.21$, and for $b = 4$, we choose $Q = 0.115, 0.13, 0.16987452395, 0.21$. From Figure 13, we know that the entanglement entropy exhibits a Hawking-Page phase transition, and a Van der Waals-like phase transition as the charge increases form 0 to 0.21. The entanglement entropy also exhibits the novel phase structure as that of the thermal entropy as well as two point correlation function, which is shown in (a) of Figure 14.

We will also employ Maxwell's equal area law to locate the first order phase transition temperature, namely T_a in (b) of Figure 13 and Figure 14, T_2 in (a) of Figure 14, and pseudo

(a) $b = 5, \theta_0 = 0.2$
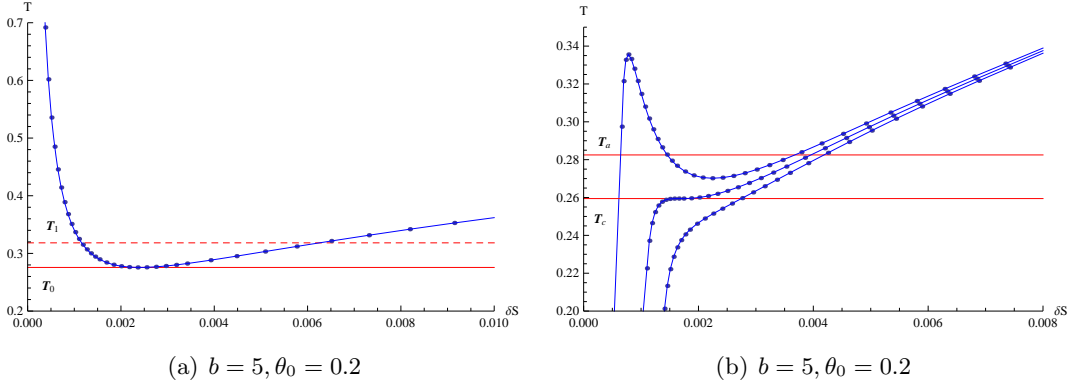
(b) $b = 5, \theta_0 = 0.2$

Figure 13: Isocharges in the $\delta S - T$ plane for the case $b = 5, \theta_0 = 0.2$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red solid lines from top to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.

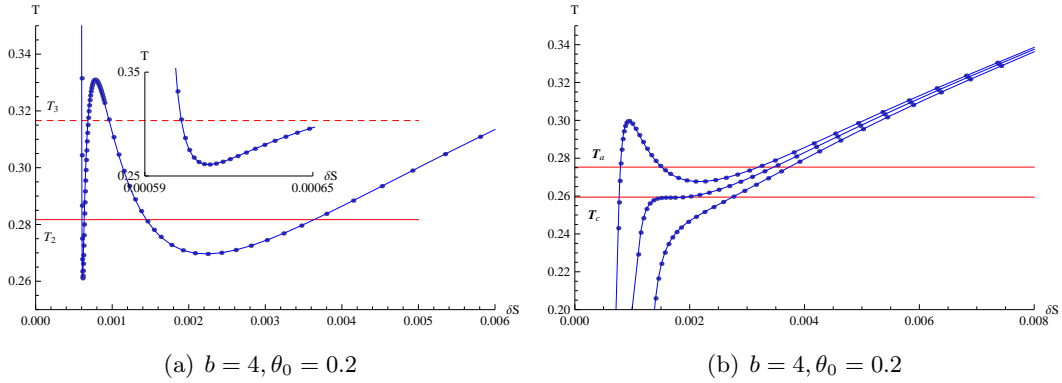(a) $b = 4, \theta_0 = 0.2$

(b) $b = 4, \theta_0 = 0.2$

Figure 14: Isocharges in the $\delta S - T$ plane for the case $b = 4, \theta_0 = 0.2$. The red solid line and dashed line in (a) correspond to the minimum temperature and Hawking-Page phase transition temperature. The red solid lines from above to down in (b) correspond to the first order phase transition temperature and second order phase transition temperature.

phase transition temperature T_3. In the $\delta S - T$ plane, we define the equal area law as

$$A_L \equiv \int_{\delta S_{min}}^{\delta S_{max}} T(\delta S) d\delta S = T_x(\delta S_{max} - \delta S_{min}) \equiv A_R, \tag{3.9}$$

in which $T(\delta S)$ is an Interpolating Function obtained from the numeric result, and δS_{min}, δS_{max} are the smallest and largest roots of the unstable regions which satisfy the equation $T(\delta S) = T_x$. Surely the values of the phase transition temperature T_x depends on the choice of Q and b. For different Q and b, the results of δS_{min}, δS_{max} and A_L, A_R are listed in Table (3). It is obvious that A_L equals nearly A_R for a fixed Q and b. That is, the equal area law is also valid for the first order phase transition and pseudo phase transition in the entanglement entropy-temperature plane. This result is the same as that of the thermal entropy as well as

	δS_{min}	δS_{max}	A_L	A_R
$b=5, Q=0.11, T_a=0.2825, \theta_0=0.2$	0.00064474	0.0036781	0.000860	0.000857
$b=4, Q=0.13, T_a=0.2757, \theta_0=0.2$	0.00076078	0.0032308	0.000681	0.000681
$b=4, Q=0.115, T_2=0.2817, \theta_0=0.2$	0.00060499	0.0036363	0.000857	0.000853
$b=4, Q=0.115, T_3=0.3166, \theta_0=0.2$	0.00060237	0.00095971	0.000113	0.000112

Table 3: Check of the equal area law in the $T - \delta S$ plane.

(a) $b=5, \theta_0=0.2$ (b) $b=4, \theta_0=0.2$

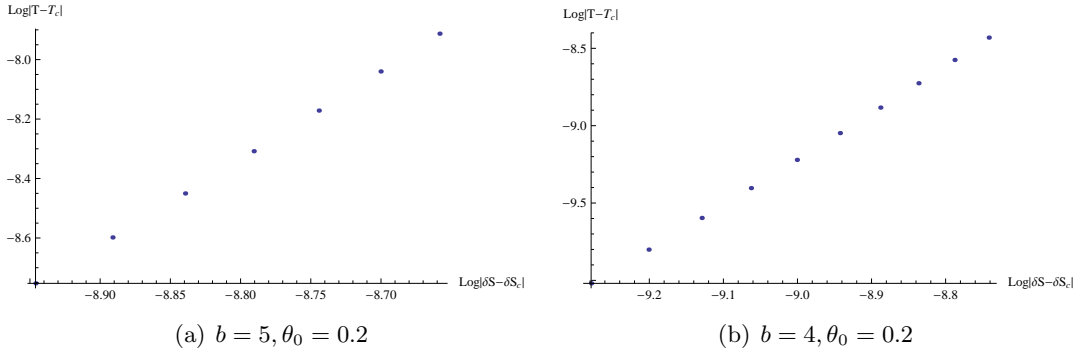**Figure 15:** Relations between $\log |T - T_c|$ and $\log |\delta S - \delta S_c|$ for different b and θ_0.

two point correlation function.

To confirm that T_c is the second order phase transition temperature in the $\delta S - T$ plane, we should explore whether it satisfies a similar relation as Eq.(2.11). For different b and Q, the relation between $\log |T - T_c|$ and $\log |\delta S - \delta S_c|$ are plotted in Figure 14, in which S_c is the critical entropy obtained numerically by the equation $T(\delta S) = T_c$. The analytical result of these curves can be fitted as

$$\log |T - T_c| = \begin{cases} 17.3605 + 2.91954 \log |\delta S - \delta S_c|, & \text{for } b=5, \theta_0 = 0.2, \\ 17.2131 + 2.93553 \log |\delta S - \delta S_c|, & \text{for } b=4, \theta_0 = 0.2. \end{cases} \tag{3.10}$$

We find for a fixed b and Q, the slope is always about 3, which is consistent with that of the thermal entropy. That is, the entanglement entropy also exhibits a second order phase transition, with phase transition temperature T_c, as that of the thermal entropy and two point correlation function.

4. Concluding remarks

Investigation on the phase structure of a back hole is of great importance for it reveals whether and how a black hole emerges. Usually, it is realized by studying the relation between some thermodynamic quantities in a fixed ensemble. In this paper, we found that the phase structure of a back hole also can be probed by the two point correlation function and holographic entanglement entropy, which provides a new strategy to understand the phase structure of the black holes from the viewpoint of holography.

Specially speaking, we first investigated the phase structure of the Born-Infeld-anti-de Sitter black holes in the $T - S$ plane in the fixed charge ensemble. We found that the phase structure of the black hole depends on not only the value of Q or b, but also the combination of bQ. For the case $b = 5$, the black hole resembles as the Reissner-Nordström-AdS black hole, and it undergoes the Van der Waals-like phase transition as its charge satisfies the non-extremal condition $bQ > 0.5$. For the case $b = 4$, besides the Van der Waals-like phase transition, we also observed a novel phase structure for the case $bQ < 0.5$. With the two point correlation function and holographic entanglement entropy, we further probed the phase structure of the Born-Infeld-anti-de Sitter black holes and found that both the probes exhibited the same phase structure as that of the thermal entropy for cases $bQ > 0.5$ and $bQ < 0.5$ regardless of the size of the boundary region. This conclusion was reinforced by checking the equal area law for the first order phase transition as well as pseudo phase transition, and calculating the critical exponent of the analogous heat capacity for the second order phase transition.

In previous investigation on the phase structure of the Born-Infeld-anti-de Sitter black holes, authors concentrated mainly on that of $bQ > 0.5$ [45, 46, 47, 48]. In this paper we found that the phase structure for the case $bQ < 0.5$ is also interesting. As $bQ \ll 0.5$, the black hole resembles as the Schwarzschild AdS black hole as expected, and there is a minimum temperature. We found the larger the charge or the Born-Infeld parameter is, the smaller the minimum temperature will be. That is, both the charge and the Born-Infeld parameter promote the formation of an AdS black hole. As bQ approaches to 0.5, a new extremal small black hole branch emerges compared with that of the Reissner-Nordström-AdS black hole, so that there are two unstable regions and correspondingly two phase transition temperature. The high temperature phase transition was found to be pseudo for in this case the space time was dominated by the large black hole, which was observed from the $F - T$ relation.

Acknowledgements

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