DEPTH REDUCTIONS FOR ASSOCIATORS

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Abstract. We reformulate part of the associator equations in a way for which the term of highest depth is significative, implying certain properties of "depth reduction". This has natural applications to multiple zeta values, and a particular application to p-adic and finite multiple zeta values.

Contents

1. Introduction

1.1. **Multiple zeta values and the depth.** The origin of this paper is the study of multiple zeta values. These are the following real numbers :

$$\zeta(s_d,\ldots,s_1)=\sum_{0<n_1<\ldots<n_d}\frac{1}{n_1^{s_1}\ldots n_d^{s_d}}=(-1)^d\int_{0<t_1<\ldots<t_n<1}\wedge_{i=1}^{n}\frac{dt_i}{t_i-\epsilon_i}\in\mathbb{R}$$

with $d\in\mathbb{N}^*$, $s_d,\ldots,s_1\in\mathbb{N}^*$, $s_d\geq 2$, $n=\sum_{i=1}^{d}s_i$, $(\epsilon_n,\ldots,\epsilon_1)=(\overbrace{0\ldots0}^{s_d-1}1,\ldots,\overbrace{0\ldots0}^{s_1-1}1)$. The integral expression above makes multiple zeta values into periods in the sense of algebraic geometry, and gives rise to the problematic of studying the polynomial relations over $\mathbb{Q}$ that they satisfy, which is our precise motivation.

Given an index $(s_d,\ldots,s_1)$ of a multiple zeta value, the sum $s_d+\ldots+s_1$ is called its weight, and the integer d is called its depth. For theoretical reasons, conjecturally, the relations among multiple zeta values are homogeneous for the weight.
The analogue of this conjecture for the depth is not true ; for example, we have $\zeta(3)=\zeta(2,1)$. The depth plays, nevertheless, a certain role in the study of algebraic relations between multiple zeta values, for reasons that we recall below.
The depth filtration is the data of the $\mathbb{Q}$-vector spaces $V_{\leq d}$ generated by multiple zeta values of depth $\leq d$. It gives rise to the depth graded multiple zeta values, i.e. the

images of the $\zeta(s_d,\dots,s_1)$'s in $V_{\leq d}/V_{\leq d-1}$. A key fact is that the depth filtration descends to a filtration on the so-called motivic multiple zeta values : this means that depth-graded multiple zeta values can be studied as periods as well : there is a notion of motivic depth-graded multiple zeta values. These have been studied in [IKZ], [B2]. Their interest comes partly from their surprising connection with iterated integrals of modular forms.
The notion of depth reduction is subjacent - or, more precisely, essentially dual - to the one of depth-graded multiple zeta values :

Definition 1.1. A $\mathbb{Q}$-linear combination of multiple zeta values of depth $d \in \mathbb{N}^*$ is said to admit a depth reduction if it can be written as a polynomial over $\mathbb{Q}$ of multiple zeta values of depth $\leq d-1$ and $2\pi i$.

1.2. **Double shuffle relations and depth reductions.** Multiple zeta values satisfy the so called double shuffle relations ; they are described in detail in [IKZ]. They are immediate consequences of the formulas of §1.1 and they give, despite that, a conjecturally complete description of the algebraic relations over $\mathbb{Q}$ between multiple zeta values.
A further interesting aspect of double shuffle relations is that they are naturally adapted to the depth filtration. This means that their depth graded version gives instantly a good conjectural description of algebraic relations between depth graded multiple zeta values.

Provided by this framework, two different examples of depth reduction have been singled out in the literature, for their special meaning or application. We state them below, and they will reappear in the next parts. The first one is well-known :

Proposition 1.2. (Tsumura, [Ts], §1, Theorem ; Ihara-Kaneko-Zagier, [IKZ], §8, corollary 8, Panzer, [P]) Let $d \in \mathbb{N}^*$ and $s_d,\dots,s_1 \in \mathbb{N}^*$ such that $s_d+\ldots+s_1-d$ is odd. Then $\zeta(s_d,\dots,s_1)$ admits a depth reduction.

A particularity of this proposition is that it actually goes back to Euler, who defined multiple zeta values in one and two variables, and proved the part of depth one and two. The depth one part is nothing else than the famous equality, valid for all $n \in \mathbb{N}^*$,

$$\zeta(2n) = \frac{|B_{2n}|}{2(2n)!}\pi^{2n}$$

The depth two part of the statement is that multiple zeta values of depth two and odd weight admit a depth reduction ; its first example is

$$\zeta(3) = \zeta(2,1)$$

A second significative example is the following result, also announced by Zagier :

Proposition 1.3. (Yasuda, [Y1], corollary 3.3) Let $d \in \mathbb{N}^*$ and $s_d,\dots,s_1 \in \mathbb{N}^*$. Then $\zeta(s_d,\dots,s_1) + (-1)^{s_d+\dots+s_1}\zeta(s_1,\dots,s_d)$ admits a depth reduction. Precisely, there is a particular way to write in lesser depth the number

$$\sum_{k=0}^{d}(-1)^{s_{k+1}+\dots+s_d}\zeta(s_{k+1},\dots,s_d)\zeta(s_k,\dots,s_1)$$

These numbers are analogues of Kaneko-Zagier's finite multiple zeta values. For us, a particularity of this proposition is that it helps to write the reduction modulo large primes of p-adic multiple zeta values in terms of finite multiple zeta values, as we will explain in §5.

1.3. **Associators and multiple zeta values.** The notion of associator has been defined by Drinfeld in [Dr]. Primarily, it arises as part of the axioms of the definition of a quasi-triangular quasi-Hopf algebra. An interest of associator equations is that they have a lot of other different incarnations, in various mathematical contexts. For our concerns, associators are certain elements of a non-commutative algebra of formal power series

$$R\langle\langle e_0, e_1\rangle\rangle$$

where R is any $\mathbb{Q}$-algebra and e_0, e_1 are formal variables. The following theorem is non trivial and important :

Theorem 1.4. (Furusho, [F], Theorem 1.1.) Associators satisfy the double shuffle equations.

To illustrate the notion of associators, let us recall briefly its meaning in the case of multiple zeta values, which is the fundamental example given in [Dr] as well as our motivation. In [Dr],§2 is defined an element

$$\Phi_{\mathrm{KZ}} \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$$

the definition is through integration of the canonical connection ∇_{KZ} on the fundamental bundle of paths of the pro-unipotent fundamental groupoid [D] of $M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$, and $M_{0,5} = (\mathbb{P}^1 - \{0, 1, \infty\})^2 -$ diagonal. The algebraic automorphisms of $M_{0,4}$ and $M_{0,5}$ induce horizontal automorphisms of this bundle with connection ; they are, on the one hand, expressed in terms of Φ_{KZ} and, on the other hand, of finite order. This gives the associator equations, recalled in §2.2, in the case of Φ_{KZ} and the constant $m = 2i\pi$. Now, an element of $R\langle\langle e_0, e_1\rangle\rangle$ can be written uniquely in the following form, with the brackets refering to coefficients

$$f = f[\emptyset] + \sum_{\substack{d\in\mathbb{N}^* \\ s_d,\ldots,s_0\in\mathbb{N}^*}} f[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1e_0^{s_0-1}]e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1e_0^{s_0-1}$$

Finally, the integral expression of multiple zeta values of §1.1 is equivalent to

$$\zeta(s_d, \ldots, s_1) = (-1)^d\Phi_{\mathrm{KZ}}[e_0^{s_d-1}e_1 \ldots e_0^{s_d-1}e_1]$$

Thus, the associator equations for $(\Phi_{\mathrm{KZ}}, 2i\pi)$ amount to algebraic relations between multiple zeta values provided by the associator equations for Φ_{KZ} ; they provide a conjecturally complete description of the algebraic relations between multiple zeta values over $\mathbb{Q}$.

1.4. **Role of associators for depth reduction - main result.** What we develop in this paper is a partial analogue for associators relations of the depth-graded double shuffle relations, with applications to the two examples of depth reductions phenomena of Proposition 1.2 and Proposition 1.3.

The difficulty is that, unlike double shuffle equations, associator equations are not adapted to the depth filtration. Precisely, for each $n \in \mathbb{N}^*$, the highest depth appearing

in an associator equation in weight n is essentially equal to n, and the depth graded highest depth term is not significative ; that is to say, it is essentially always equal to 0, and for an obvious reason. Thus, it is necessary to slightly change the formulation of associator equations.
The way we rewrite part of the associator equations involves a couple (Φ,Φ_{∞}) of elements of $R\langle\langle e_0,e_1\rangle\rangle$ and the depth filtration for both Φ and Φ_{∞}. Such a couple is a usual data, subject to Kashiwara-Vergne equations if Φ is an associator [AET]. The equations that we will write are partly related, but not equivalent to the usual formulation of a part of Kashiwara-Vergne equations. In the case of multiple zeta values, $\Phi_{\mathrm{KZ},\infty}$ is the natural analogue of Φ_{KZ} given by iterated integrals from 0 to ∞ instead of from 0 to 1. Roughly speaking, our main result is :

Main result (theorem 3.1, §3.1, and corollaries 3.3 and 3.4, §3.3.1) The one dimensional part (in the sense recalled in §2) of the associator equations for an associator Φ can be formulated in terms of (Φ,Φ_{∞}), with significative highest depth terms, reimplying the propositions 1.2 and 1.3.

1.5. **Further comments.** The propositions 1.2 and 1.3 are congruences. The exact formulas beyond the congruences can always be written. The ones obtain via double shuffle equations and via associator equations are not the same a priori.
Furthermore, whereas the double shuffle relations are more practical for many aspects, the proof via associators for the depth reduction of the numbers

$$\sum_{k=0}^{d}(-1)^{s_{k+1}+\ldots+s_d}\zeta(s_{k+1},\ldots,s_d)\zeta(s_k,\ldots,s_1)$$

is much shorter. This is basically because these numbers are a natural quantity from the point of view of associators. This was our first observation. Our computations are actually inspired by p-adic computations, originated in the work of Ünver [U1]. At the end of the paper, we discuss the interest of the depth of Φ_{∞} in the case of p-adic multiple zeta values.

1.6. **Outline.** In §2, we recall the definition of associators. In §3, we state the results. In §4 we prove them. In §5, we explain their meaning relative to multiple zeta values.

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2. Definition of associators and notations

2.1. **Two pro-unipotent affine group schemes $\Pi_1^{un}(M_{0,4})$ and $\Pi_1^{un}(M_{0,5})$.** These are the affine algebraic groups over $\mathbb{Q}$ canonically associated to the de Rham pro-unipotent fundamental groupoid of the varieties $M_{0,4}$ and $M_{0,5}$ via the main result of [D], §12. We will use mostly the case of $M_{0,4}$.

Definition 2.1. Let $\Pi_1^{un}(M_{0,4})$ be the exponential of the pro-nilpotent Lie algebra over $\mathbb{Q}$ defined by generators e_0, e_1, e_∞ subject to the relation $e_0 + e_1 + e_\infty = 0$ - i.e. freely generated by e_0, e_1 for example.

Definition 2.2. Let $\Pi_1^{un}(M_{0,5})$ be the exponential of the pro-nilpotent Lie algebra over $\mathbb{Q}$ defined by generators $e_{i,j}, 1 \leq i,j \leq 4$, subject to the relations : $e_{ii} = 0$, $e_{ji} = e_{ij}$, $[e_{jk} + e_{jl}, e_{kl}] = 0$ if j,k,l are pairwise distinct, and $[e_{ij}, e_{kl}] = 0$ if i,j,k,l are pairwise distinct.

Proposition 2.3. We have $\Pi_1^{un}(M_{0,4}) = \text{Spec}(\mathcal{H}_{ш})$, where $\mathcal{H}_{ш}$ is defined below.

Definition 2.4. The shuffle Hopf algebra $\mathcal{H}_{ш} = \mathcal{H}_{ш}(e_0, e_1)$ over $\mathbb{Q}$ associated with the alphabet $\{e_0, e_1\}$ is the following :
i) as a the $\mathbb{Q}$-vector space, it is the one freely generated by the words on $\{e_0, e_1\}$, including the empty word ;
ii) the product ш of $\mathcal{H}_{ш}$ is defined by

$$(u_1 \ldots u_r) ш (u_{r+1} \ldots u_{r+s}) = \sum_{\substack{\sigma,\ \text{permutation of } \{1,\ldots,r+s\} \\ \sigma(1)<\ldots<\sigma(r),\ \sigma(r+1)<\ldots<\sigma(r+s)}} u_{\sigma^{-1}(1)} \ldots u_{\sigma^{-1}(r+s)}$$

iii) the coproduct of $\mathcal{H}_{ш}$ is the deconcatenation of words ;
iv) the antipode is $S : u_r \ldots u_1 \mapsto (-1)^r u_1 \ldots u_r$
v) the counity ϵ is the augmentation map of the $\mathbb{Q}$-monoid of words.

To make certain formulas more readable, we will denote a word $e_0^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1 e_0^{s_0-1}$ by $\mathbf{e}_0^{s_d-1,\ldots,s_0-1}$; when $s_0 = 1$, we may denote the same word by $\mathbf{e}_0^{s_d-1,\ldots,s_1-1} e_1$ to emphasize that it starts with e_1.

The notation with brackets for coefficients of formal series given in §1.3 extends to the whole of $\mathcal{H}_{ш}$ by linearity.

Proposition 2.5. For any $\mathbb{Q}$-algebra R, we have :

$$\text{(1)} \quad \Pi^{un}(M_{0,4})(R) = \{f \in R\langle\langle e_0, e_1 \rangle\rangle \mid \text{for all words } w, w', f[w ш w'] = f[w]f[w'], \text{ and } f[\emptyset] = 1\}$$

The equation $f[wшw'] = f[w]f[w']$ is called the shuffle equation.

We will not use the similar description of the points of $\Pi^{un}(M_{0,5})$.

2.2. **Definition of associators.**

Definition 2.6. An associator (m, Φ) (with coefficients in a $\mathbb{Q}$-algebra R) is an element of $R \times \Pi^{un}(M_{0,4})(R)$, with $\Phi[e_0] = \Phi[e_1] = 1$, such that we have, respectively in $\Pi^{un}(M_{0,4})(R)$ and $\Pi^{un}(M_{0,5})(R)$:

$$\text{(2)} \qquad \Phi(e_0, e_1)\Phi(e_1, e_0) = 1$$

$$\text{(3)} \qquad e^{\frac{m}{2} e_0} \Phi(e_\infty, e_0) e^{\frac{m}{2} e_\infty} \Phi(e_1, e_\infty) e^{\frac{m}{2} e_1} \Phi(e_0, e_1) = 1$$

$$\text{(4)} \qquad \Phi(e_{12}, e_{23} + e_{24})\Phi(e_{13} + e_{23}, e_{34}) = \Phi(e_{23}, e_{34})\Phi(e_{12} + e_{13}, e_{24} + e_{34})\Phi(e_{12}, e_{23})$$

These equations are called, respectively, the 2-cycle or duality, 3-cycle or hexagon, and 5-cycle or pentagon equations. A different presentation involves two variants of the hexagon equation instead of the duality and hexagon equation.

Proposition 2.7. The hexagon and duality equations imply :
i) for $m \neq 0$ ([AET], equation (3) p.1) :

$$(5) \quad \exp(-\Phi^{-1}(e_0,e_1)me_1\Phi(e_0,e_1)))\exp(-me_0) \\ = \exp(\frac{m}{2}e_0)\exp\left(\Phi(e_0,e_\infty)^{-1}me_\infty\Phi(e_0,e_\infty)\right)\exp(-\frac{m}{2}e_0)$$

ii) for $m = 0$ ([Dr], Proposition 5.9),

$$(6) \qquad e_0 + \Phi^{-1}(e_0,e_1)e_1\Phi(e_0,e_1) + \Phi(e_0,e_\infty)^{-1}e_\infty\Phi(e_0,e_\infty) = 0$$

When $m = 0$, the equation (6) together with the equations (2), (3), (4) actually form the definition of GRT_1 in [Dr]. The equation (6) is the coefficient of degree 1 with respect to m of equation (5). We could unify the statements for $m \neq 0$ and for $m = 0$ as part of a same statement. Nevertheless, for readability, we will distinguish the two cases in the statements and proofs.
We will refer to (5), (6) as the equations of the special automorphism, as a reference to the automorphism of Lie $\Pi^{un}(M_{0,4})$ defined by (Φ,Φ_∞) in the Kashiwara-Vergne context and a usual terminology (see [AET]).
What we study is the part of the equations above which can be expressed purely in terms of $\Pi^{un}(M_{0,4})$, i.e., all equations above except for the pentagon equation.

2.3. **Notations and conventions concerning the depth filtration and the weight.** First, the constant m in the definition must be thought of as having always depth 0 and weight 1, relatively to any equality

Notation 2.8. i) Let a collection of $\mathbb{Q}$-vector subspaces, resp. $\mathbb{Q}[m]$ sub-modules of $\mathcal{H}_ш$, resp. $\mathcal{H}_ш[m]$

$$(\mathcal{H}_ш)_{\mathbf{D}\leq d}, \quad (\mathcal{H}_ш)_{\mathbf{D}\leq d}, \quad \mathcal{H}_ш[m]_{\mathbf{D}\leq d}, \quad \mathcal{H}_ш[m]_{\mathbf{D}\geq d}$$

defined as the spans of the shuffle polynomials of elements whose depth is $\leq d$, resp. $\geq d$. Their subspans of weight $n \in \mathbb{N}^*$ are denoted by adding $\mathbf{W} = n$, i.e.

$$(\mathcal{H}_ш)_{\mathbf{D}\leq d,\mathbf{W}=n}, \quad \text{etc.}$$

ii) For u a point of $\Pi^{un}(M_{0,4})$ in any algebra, and R another algebra, the image of the previous modules by u resp. $u \otimes_{\mathbb{Q}} \mathbb{Q}[m]$, tensorized with R, is denoted by

$$R[u]^+_{\mathbf{D}\leq d}, \text{ resp. } R[u]^+_{\mathbf{D}\leq d,\mathbf{W}=n}, \quad \text{etc.}$$

Definition 2.9. If u is a point of $\Pi^{un}(M_{0,4})$, m as in the definitions, and $d > r \in \mathbb{N}^*$, let

$$DR_{d\to d-r}(u)$$

be the vector subspace of $(\mathcal{H}_ш)_{\mathbf{D}\leq d}$ generated by words w such that $u[w]$ is a polynomial over $\mathbb{Q}$ of coefficients of u or depth $\leq d-r$ and of m.

3. Results

3.1. **Highest depth parts of the one dimensional associator equations.** In the rest of the paper, R is a $\mathbb{Q}$-algebra. The points $\Pi^{un}(M_{0,4})$, when viewed as maps $\mathcal{H}_{ш} \to R$, induce maps on $\mathcal{H}_{ш}[m] \to R[m]$ by tensorisation by $\mathbb{Q}[m]$, which we will denote in the same way.

Theorem 3.1. There exist linear functions $A_m, A_m^{\infty}, B_m, B_m^{\infty} : \mathcal{H}_{ш} \to \mathcal{H}_{ш}[m]$ (see §4.4 for the definitions) such that, given $(m,\psi) \in R \times \Pi^{un}(M_{0,4})(R)$, the condition that (m,ψ) satisfies the equations of duality, hexagon and special automorphism is equivalent to

$$\text{the couple } (\psi,\psi_\infty) \text{ satisfies} \quad \psi \circ A_m = \psi_\infty \circ A_m^{\infty} \quad \text{and} \quad \psi \circ B_m = \psi_\infty \circ B_m^{\infty} \tag{7}$$

Moreover, for all words w, the highest depth terms of $A_m(w)$, $A_m^{\infty}(w)$, $B_m(w)$, $B_m^{\infty}(w)$ are given as follows.

Formulas for A_m and A_m^{∞}.

$$\begin{aligned} & A_m(\mathbf{e}_0^{s_d-1,\ldots,s_0-1}) \\ & \equiv \sum_{k=0}^{s_0-1} (1-(-1)^{(\sum_{i=0}^{d} s_i)-1-k-d}) \mathbf{e}_0^{s_d-1,\ldots,s_1-1,s_0-1-k} \left(\frac{m}{2}\right)^k \quad \text{mod } \mathcal{H}_{ш}[m]_{\mathbf{D}\leq d-1} \end{aligned} \tag{8}$$

$$A_m^{\infty}(\mathbf{e}_0^{s_d-1,\ldots,s_0-1}) \equiv \sum_{k=0}^{s_d-1} \left(\frac{m}{2}\right)^k \mathbf{e}_0^{s_d-1-k,s_{d-1}-1,\ldots,s_1-1,s_0-1} \quad \text{mod } \mathcal{H}_{ш}[m]_{\mathbf{D}\leq d-1} \tag{9}$$

Formulas for B_m and B_m^{∞}. (resp. when $m \neq 0$ and $m=0$)

$$\begin{aligned} & B_m(\mathbf{e}_0^{s_d-1,\ldots,s_0-1}) \\ & \begin{cases} \equiv \sum_{k=0}^{s_0-1} \mathbb{1}_{s_{d+1}=1} m \frac{(-m)^{s_0-1-k}}{(s_0-1-k)!} \mathbf{e}_0^{s_d-1,\ldots,s_1-1-k} - m\frac{(-m)^{s_0-1}}{(s_0-1)!} \mathbf{e}_0^{s_{d+1}-1,\ldots,s_1-1} \\ \qquad \text{mod } \mathcal{H}_{ш}[m]_{\mathbf{D}\leq d-2} \\ \text{resp. } \equiv \mathbb{1}_{s_0=1} \mathbf{e}_0^{s_{d+1}-1,\ldots,s_1-1} - \mathbb{1}_{s_d=1} e_1 e_0^{s_{d-1}-1,\ldots,s_0-1} \quad \text{mod } \mathbb{Z}[u]^{+}_{\mathbf{D}\leq d-1} \end{cases} \end{aligned} \tag{10}$$

$$\begin{aligned} & B_m^{\infty}(\mathbf{e}_0^{s_d-1,\ldots,s_0-1}) \\ & \begin{cases} \equiv \sum_{\substack{0\leq k_0 \leq s_0-1 \\ 0\leq k_d\leq s_d-1}} (-1)^{k_0} \left(\frac{m}{2}\right)^{k_d+k_0} \Big(\sum_{0\leq l\leq s_d-1-k_d} m^l \mathbf{e}_0^{s_d-1-k_d-l,s_{d-1}-1,\ldots,s_0-1-k_0} \\ - \sum_{0\leq l\leq s_0-1-k_0} \big(m^l \mathbf{e}_0^{s_d-1-k_d,s_{d-1}-1,\ldots,s_0-1-k_0-l}\Big) \quad \text{mod } \mathcal{H}_{ш}[m]_{\mathbf{D}\leq d-1} \\ \text{resp. } \equiv \mathbb{1}_{s_d\geq 2} \mathbf{e}_0^{s_d-2,s_{d-1}-1,\ldots,s_0-1} + \mathbb{1}_{s_0\geq 2} \mathbf{e}_0^{s_d-1,\ldots,s_1-1,s_0-2} \end{cases} \end{aligned} \tag{11}$$

Injecting equations (8), (9), (10),(11) in (7) turns these equations into congruences relating coefficients of ψ and ψ_∞, modulo modules of coefficients of §2.3, which we do not need to write.

3.2. **Comparison of ψ and ψ_∞.** The first particular byproduct of the congruences relating ψ and ψ_∞ is :

Proposition 3.2. With the hypothesis of the theorem, we have : for all $d \in \mathbb{N}^*$,

$$\mathbb{Z}[m][\psi]^+_{\mathbf{D}\leq d} = \mathbb{Z}[m][\psi_\infty]^+_{\mathbf{D}\leq d+1}$$

3.3. **Depth reductions.** The congruences relating coefficients of ψ and ψ_∞ also produce elements in $DR_{d\to d-1}(\psi)$, $DR_{d\to d-1}(\psi_\infty)$, and also $DR_{d\to d-2}(\psi)$.

3.3.1. *Depth reductions for ψ.* Let $(m,\psi) \in R \times \Pi^{un}(M_{0,4})(R')$ satisfying the assumptions of the theorem. Let $(s_d,\ldots,s_1)$ an index.

Corollary 3.3. If $s_d+\ldots+s_1-d$ is odd, then $(s_d,\ldots,s_1) \in DR_{d\to d-1}(\psi)$.

Corollary 3.4. We have $(s_d,\ldots,s_1)+(-1)^{s_d+\ldots+s_1}(s_1+\ldots+s_d) \in DR_{d\to d-1}(\psi)$

Formula when $m=0$. We have more precisely :

$$\psi(s_d,\ldots,s_1)+(-1)^{s_d+\ldots+s_1}\psi(s_1,\ldots,s_d) \tag{12}$$

$$\equiv (-1)^{\sum_{i=1}^d s_i} \sum_{\substack{l_1,\ldots,l_{d-1}\in\mathbb{N}\\ l_1+\ldots+l_{d-1}=s_d}} \prod_{i=1}^{d} \binom{-s_i}{l_i} \psi[\mathbf{e}_0^{s_1+l_1-1,\ldots,s_{d-1}+l_{d-1}-1}e_1]$$

$$+ \sum_{\substack{l'_2,\ldots,l'_d\in\mathbb{N}\\ l'_2+\ldots+l'_d=s_1}} \prod_{i=2}^{d} \binom{-s_i}{l_i} \psi[\mathbf{e}_0^{s_d+l_d-1,\ldots,s_2+l_2-1}e_1] \mod \mathbb{Z}[\psi]_{\mathbf{D}\leq d-2}$$

Note that the right-hand side admits a more concise expression : let $z = \mathbf{e}_0^{s_d-1,\ldots,s_1-1}$, then the right hand side is also congruent to

$$\psi[ze_0]+\psi^{-1}[e_0z] \equiv \psi[ze_0]-\psi[e_0z] \equiv \psi[ze_0]+\psi^{-1}[e_0z] \mod \mathbb{Z}[\psi]_{\mathbf{D}\leq d-2} \tag{13}$$

Recall from that $\psi(s_d,\ldots,s_1)+(-1)^{s_d+\ldots+s_1}\psi(s_1,\ldots,s_d)$ is congruent modulo ... to ... .

$$(\psi^{-1}e_1\psi)[e_1w] = \sum_{k=0}^{d}(-1)^{s_{k+1}+\ldots+s_d}\psi[\mathbf{e}_0^{s_{k+1}-1,\ldots,s_d-1}e_1]\psi[\mathbf{e}_0^{s_k-1,\ldots,s_1-1}e_1]$$

and the depth reduction appears in a natural way as a depth reduction for this number. We gave the formula only for $m=0$ because the application that we have in mind concerns p-adic multiple zeta values, for which the associated m is 0 (see §5).

Combining the two corollaries 3.3 and 3.4 gives instantly :

Corollary 3.5. Assume that $(s_1+\ldots+s_d)-d$ is even. Then $(s_d,\ldots,s_1)+(-1)^{s_d+\ldots+s_1}(s_1+\ldots+s_d) \in DR_{d\to d-2}(\psi)$.

This is also known in the double shuffle case.

3.3.2. *Depth reductions for ψ_∞.* We take (m,ψ) satisfying the assumption of the theorem, $d\in\mathbb{N}^*$, and $(s_d,\ldots,s_1)\in(\mathbb{N}^*)^d$. The following corollaries are respective counterparts of corollaries 3.4 and 3.5 for ψ_∞.

Corollary 3.6. Assume that $s_d+\ldots+s_1-d$ is odd. Then we have

$$\sum_{k=0}^{s_d-1}\Big(\frac{m}{2}\Big)^k\psi_\infty[\mathbf{e}_0^{s_d-1-k,s_{d-1}-1,\ldots,s_1-1}e_1] \in DR_{d\to d-1}(\psi_\infty)$$

The fact that ψ_∞ in depth d can be written in terms of ψ in depth $\leq d-1$, implied by the theorem, is a restrictive condition - it implies for example that ψ_∞ vanishes in depth one, which is not true in general. Intuitively, this condition means, intrinsically on ψ_∞, that ψ_∞ depends on "one too much variable" as a function of indices $s_d, \ldots, s_1$. The following corollary can be seen as a formalization of this idea.

Corollary 3.7. Assume for simplicity $m = 0$. For all $d \in \mathbb{N}^*$, $s_d, \ldots, s_0 \in (\mathbb{N}^*)^d$, $r_d, r_0 \in \mathbb{N}$:

$$(14) \quad \binom{s_0 + s_d + r_0 + r_d - 1}{s_0 + s_d - 1} \psi_\infty[\mathbf{e}_0^{s_d - 1 + r_d; s_{d-1} - 1, \ldots, s_1 - 1; s_0 - 1 + r_0}]$$

$$- \sum_{u_{d-1} + \ldots + u_1 = r_d + r_0} \prod_{k=1}^{d-1} \binom{-s_k}{u_k} \psi_\infty[\mathbf{e}_0^{s_d - 1; s_{d-1} - 1 + u_{d-1}, \ldots, s_1 - 1 + u_1; s_0 - 1}] \in DR_{d \to d-1}(\psi_\infty)$$

When we take $s_d = s_0 = 1$, we obtain that all coefficients of f_∞ can be written, modulo lesser depth, as coefficients on words of the form $\psi_\infty[e_1 \ldots e_1]$.
If, moreover, we take $r_0 = 0$, renaming $r_d = s_d - 1$ we obtain

$$(15) \quad \psi_\infty[\mathbf{e}_0^{s_d - 1; s_{d-1} - 1, \ldots, s_1 - 1} e_1]$$

$$- \frac{1}{s_d} \sum_{u_{d-1} + \ldots + u_1 = r_d + r_0} \prod_{k=1}^{d-1} \binom{-s_k}{u_k} \psi_\infty[\mathbf{e}_0^{s_d - 1; s_{d-1} - 1 + u_{d-1}, \ldots, s_1 - 1 + u_1; s_0 - 1}] \in DR_{d \to d-1}(\psi_\infty)$$

3.4. **Examples in depth 1 and 2.** Let us write the first examples of the formulas. The corollary 3.4 with $m = 0$, gives in depth one and two : for all $s \in \mathbb{N}^*$, and $(s_1, s_2) \in (\mathbb{N}^*)^2$,

$$(\psi^{-1} e_1 \psi)[e_1 e_0^{s-1} e_1] = 0$$

$$(\psi^{-1} e_1 \psi)[e_1 \mathbf{e}_0^{s_2 - 1, s_1 - 1} e_1] \equiv (-1)^{s_1} \binom{s_1 + s_2}{s_1} \psi[e_0^{s_1 + s_2 - 1} e_1]$$

The depth reduction for symmetric sums for ψ_∞ gives in depth one, combined to the equation of the special automorphism, gives in depth one : for all $s \in \mathbb{N}^*$,

$$\psi[e_0^{s-1} e_1] = \frac{1}{s-1} \psi_\infty[e_1 e_0^{s-2} e_1]$$

4. Proofs

4.1. **Preliminaries.**

4.1.1. *The shuffle equation and role of the words starting with* e_1. We will use implicitly the following standard fact in the proofs. Let u be a point of $\Pi^{un}(M_{0,4})$ satisfying $u[e_0] = u[e_1] = 0$, for all $n, d \in \mathbb{N}^*$. We recall that the shuffle equation for u is the fact that we have, for all words w, w' of $\mathcal{H}_{ш}$, $u[w ш w'] = u[w]u[w']$. It implies that :

Fact 4.1. $\mathbb{Z}[u]^+_{\mathbf{D}=d, \mathbf{W}=n}$ is generated linearly by the coefficients of the form $u[\mathbf{e}_0^{s_d - 1, \ldots, s_1 - 1} e_1]$ with $s_d \geq 2$. More precisely, we have :
1) For all $s_d, \ldots, s_1, l \in \mathbb{N}^*$:

$$u[\mathbf{e}_0^{s_d - 1, \ldots, s_1 - 1, l}] = \sum_{l_1 + \ldots + l_d = l} \prod_{i=1}^{d} \binom{-s_i}{l_i} u[\mathbf{e}_0^{s_d + l_d - 1, \ldots, s_1 + l_1 - 1} e_1]$$

2) For all words w and $k \in \mathbb{N}^*$:

$$k.u[e_1^k \mathbf{e}_0 w] + u[e_1^{k-1} e_0 (e_1 ш w)] = 0$$

4.1.2. *Formulas for the dual of maps $\Pi^{un}(M_{0,4}) \to \Pi^{un}(M_{0,4})$ induced by homographies.* The homographies $z \mapsto \frac{1}{z}$ and $z \mapsto \frac{z}{z-1}$ of $\mathbb{P}^1 - \{0,1,\infty\}$ induce by functoriality of the pro-unipotent fundamental groupoid the maps $\Pi^{un}(M_{0,4}) \to \Pi^{un}(M_{0,4})$, given on the points by, respectively, $u(e_0,e_1) \mapsto u(e_\infty,e_1)$ and $u(e_0,e_1) \mapsto u(e_0,e_\infty)$. Their respective duals are given by :

Fact 4.2. For $u \in \Pi^{un}(M_{0,4})(R)$, and $w \in \mathcal{H}_ш$, we have :

$$u(e_\infty, e_1)[w] = u[w(-e_0, -e_0 + e_1)]$$

$$u(e_0, e_\infty)[w] = u[w(e_0 - e_1, -e_1)]$$

4.1.3. *The inverse of a formal series and the depth.* We will use implicitly the fact that, for $u : \mathcal{H}_ш \to R$, the inverse of u is given by $u^{-1} = \sum_{k \geq 1} (-1)^k u^k$, and that in particular, we have

Fact 4.3. For all words w, $u^{-1}[w] \equiv -u[w] \mod \mathbb{Z}[u]^+_{\mathbf{D} \leq \mathrm{depth}(w)-1}$

4.1.4. *Injectivity of the conjugation maps.*

Fact 4.4. Let $i \in \{0,1,\infty\}$. Consider the maps $R\langle\langle e_0, e_1 \rangle\rangle \to R\langle\langle e_0, e_1 \rangle\rangle$,

$$\mathrm{conj}_i : u \mapsto u^{-1} e_i u, \quad \mathrm{conj}'_i(m) : u \mapsto u^{-1} e^{m e_i} u$$

Then, for $u, v \in \Pi^{un}(M_{0,4})(R)$, we have

$$\mathrm{conj}_i(u) = \mathrm{conj}_i(v) \Leftrightarrow \mathrm{conj}'_i(m)(u) = \mathrm{conj}'_i(m)(v) \Leftrightarrow u^{-1} v = \exp(\lambda e_i) \text{ for a } \lambda \in R$$

In particular, the restrictions of conj_i and $\mathrm{conj}'_i(m)$ to the subgroup $\{u \in \Pi^{un}(M_{0,4})(R) \mid u[e_0] = u[e_1] = 0\}$ of $\Pi^{un}(M_{0,4})(R)$ are injective.

Proof. Let (we can restrict to write the proof for $i = 1$ only) $\partial, \tilde{\partial} : \mathcal{H}_ш \to \mathcal{H}_ш$ be the linear functions defined by, for all words w, $\partial(e_1 w) = w$, $\partial(e_0 w) = 0$, $\tilde{\partial}(w e_1) = w$, $\tilde{\partial}(w e_0) = 0$ and $\partial(\emptyset) = \tilde{\partial}(\emptyset) = 0$.

1) (This is known and a similar proof appears in [U1]) Let $u \in R\langle\langle e_0, e_1 \rangle\rangle$ such that u commutes to e_1.

Let w a word which is not of the form e_1^n, $n \in \mathbb{N}^*$. It is written uniquely in the form $e_1^{a(w)} e_0 z$, with $a(w) \in \mathbb{N}$ and z a word. We have $u[w] = (u e_1)[w e_1] = (e_1 u)[w e_1] = u(\partial_{e_1}(w) e_1)$. This shows that $u[w] = 0$ for all words w containing at least one letter e_0, by induction on $a(w)$. Thus $u \in R\langle\langle e_0, e_1 \rangle\rangle$ and ; if moreover u is grouplike i.e. satisfies the shuffle equation, we have $u = \exp(\lambda e_1)$ with $\lambda \in R$.

2) Let $u \in R\langle\langle e_0, e_1 \rangle\rangle$ such that u commutes to $e^{m e_1}$. Then we also have $(e^{m e_1} - 1)u = u(e^{m e_1} - 1)$. Let a word w which contains at least one letter e_0. It is written uniquely in the form $e_1^{a(w)} z e_1^{b(w)}$ with $(a(w), b(w)) \in (\mathbb{N})^2$ and z a word such that $\partial(z) = \tilde{\partial}(z) = 0$ (i.e. $z = e_0$ or z is of the form $e_0 \ldots e_0$). We obtain $\sum_{l=1}^{a(w)} \frac{m^l}{l!} u[e_1^{a(w)-l} z e_1^{b(w)}] = \sum_{l'=1}^{b(w)} \frac{m^{l'}}{l'!} u[e_1^{a(w)} z e_1^{b(w)-l'}]$. This shows, by induction on $(a(w), b(w))$ for the lexicographical order, that $u[w] = 0$ for all words w containing at least one letter e_0. □

4.2. **Transformation of the depth by certain conjugation operations.**

4.2.1. *Conjugation operations of e^{me_x}, resp. e_x, $x \in \{0,1,\infty\}$, by a series.* Let for this paragraph, $u \in \Pi^{un}(M_{0,4})(R)$ and $m \in R-\{0\}$.

Lemma 4.5. (conjugation of e^{me_0} resp. e_0)
a) For all $d \in \mathbb{N}^*$, we have :

$$\mathbb{Z}[m][u]^+_{\mathbf{D}\leq d} = \mathbb{Z}[u^{-1}e^{me_0}u]^+_{\mathbf{D}\leq d}$$
$$\mathbb{Z}[u]^+_{\mathbf{D}\leq d} = \mathbb{Z}[u^{-1}e_0u]^+_{\mathbf{D}\leq d}$$

b) More precisely, we have, for all, $d \in \mathbb{N}^*$, $s_d,\dots,s_1 \in \mathbb{N}^*$:

$$(u^{-1}e^{me_0}u)[\mathbf{e}_0^{s_d,s_{d-1}-1,\dots,s_1-1}e_1] \equiv \sum_{k=1}^{s_d}\frac{m^k}{k!}u[\mathbf{e}_0^{s_d-k,s_{d-1}-1,\dots,s_1-1}e_1] \mod \mathbb{Z}[u]^+_{\mathbf{D}\leq d-1}$$
$$(u^{-1}e_0u)[\mathbf{e}_0^{s_d,s_{d-1}-1,\dots,s_1-1}e_1] \equiv u[\mathbf{e}_0^{s_d-1,\dots,s_1-1}e_1] \mod \mathbb{Z}[u]^+_{\mathbf{D}\leq d-1}$$

Proof. We prove first the more precise formula of b) :it follows from $u[e_0]=0$ and from the property u^{-1} recalled in §4.1.3. By Fact 4.1, the highest depth terms in the congruences of b) generate all the modules of coefficients of a) : whence a) by induction on the depth. □

Lemma 4.6. (conjugation of e^{me_1} resp. e_1)
a) We have, for all $d \in \mathbb{N}^*$,

$$\mathbb{Z}[m][u]^+_{\mathbf{D}\leq d} = \mathbb{Z}[u^{-1}e^{me_1}u]^+_{\mathbf{D}\leq d+1}$$
$$\mathbb{Z}[u]^+_{\mathbf{D}\leq d} = \mathbb{Z}[u^{-1}e_1u]^+_{\mathbf{D}\leq d+1}$$

b) More precisely, for all $s_{d+1},\dots,s_0 \in \mathbb{N}^*$, resp. $s_d,\dots,s_1 \in \mathbb{N}^*$ with $s_d \geq 2$,

$$(u^{-1}e^{me_1}u)[\mathbf{e}_0^{s_{d+1}-1,\dots,s_0-1}] \equiv \mathbb{1}_{s_{d+1}=1}m.u[\mathbf{e}_0^{s_d-1,\dots,s_0-1}] - \mathbb{1}_{s_0=1}m.u[\mathbf{e}_0^{s_{d+1}-1,\dots,s_1-1}] \mod \mathbb{Z}[m][u]^+_{\mathbf{D}\leq d-1}$$
$$-m.u[\mathbf{e}_0^{s_d-1,\dots,s_1-1}e_1] \equiv (u^{-1}e^{me_1}u)[\mathbf{e}_0^{s_d-1,\dots,s_1-1}e_1^2] \mod \mathbb{Z}[m][u]^+_{\mathbf{D}\leq d-1}$$
$$(u^{-1}e_1u)[\mathbf{e}_0^{s_{d+1}-1,\dots,s_0-1}] \equiv -\mathbb{1}_{s_0=1}u[\mathbf{e}_0^{s_{d+1}-1,\dots,s_1-1}] + \mathbb{1}_{s_d=1}u[e_1e_0^{s_{d-1}-1,\dots,s_0-1}] \mod \mathbb{Z}[u]^+_{\mathbf{D}\leq d-1}$$
$$-u[\mathbf{e}_0^{s_d-1,\dots,s_1-1}e_1] \equiv (u^{-1}e_1u)[\mathbf{e}_0^{s_d-1,\dots,s_1-1}e_1^2] \mod \mathbb{Z}[u]^+_{\mathbf{D}\leq d-1}$$

Proof. Same with the previous lemma. □

Lemma 4.7. (conjugation of e^{me_∞} resp. e_∞)
a) We have, for all $d \in \mathbb{N}^*$,

$$\mathbb{Z}[m][u]^+_{\mathbf{D}\leq d-1} = \mathbb{Z}[u^{-1}e^{me_\infty}u]_{\mathbf{D}\leq d}$$
$$\mathbb{Z}[u]^+_{\mathbf{D}\leq d-1} = \mathbb{Z}[u^{-1}e_\infty u]_{\mathbf{D}\leq d}$$

b) More precisely, for all $s_d,\dots,s_1,s_0 \in \mathbb{N}^*$, we have :

$$(u^{-1}e^{me_\infty}u)[\mathbf{e}_0^{s_d-1,\dots,s_0-1}]$$
$$\equiv \sum_{k=0}^{s_d-1}m^ku[\mathbf{e}_0^{s_d-1-k,s_{d-1}-1,\dots,s_0-1}]+\sum_{k=0}^{s_0-1}m^ku[e_0^{s_d-1,\dots,s_1-1,s_0-1-k}] \mod \mathbb{Z}[m][u]^+_{\mathbf{D}\leq d-1}$$
$$(u^{-1}e_\infty u)[\mathbf{e}_0^{s_d-1,\dots,s_0-1}]$$

$$\equiv \mathbb{1}_{s_d\geq 2} u[\mathbf{e}_0^{s_d-2,s_{d-1}-1,\dots,s_0-1}] + \mathbb{1}_{s_0\geq 2} u[\mathbf{e}_0^{s_d-1,\dots,s_1-1,s_0-2}] \mod \mathbb{Z}[u]^+_{\mathbf{D}\leq d-1}$$

Proof. Same with the previous lemma. □

4.2.2. *Other conjugation operation.* We take the same hypothesis as in the previous paragraph.

Lemma 4.8. (conjugation by $e^{\frac{m}{2}e_0}$)
a) We have, for all $d \in \mathbb{N}^*$:

$$\mathbb{Z}[m][u]^+_{\mathbf{D}\leq d} = \mathbb{Z}[e^{\frac{m}{2}e_0} u e^{-\frac{m}{2}e_0}]_{\mathbf{D}\leq d}$$

b) Precisely, we have

$$u\Big[\sum_{k=0}^{s_d-1} \big(\frac{m}{2}\big)^k \mathbf{e}_0^{s_d-1-k,s_{d-1}-1,\dots,s_0-1} + \sum_{k=0}^{s_0-1} \big(-\frac{m}{2}\big)^k \mathbf{e}_0^{s_d-1,s_{d-1}-1,\dots,s_0-1-k}\Big]$$
$$\equiv (e^{\frac{m}{2}e_0} u e^{-\frac{m}{2}e_0})[\mathbf{e}_0^{s_d-1,\dots,s_0-1}] \mod \mathbb{Z}[\frac{m}{2}][u]^+_{\mathbf{D}\leq d-1}$$

$$(e^{\frac{m}{2}e_0} u e^{-\frac{m}{2}e_0})\Big[\sum_{k=0}^{s_d-1} \big(\frac{m}{2}\big)^k \mathbf{e}_0^{s_d-1-k,s_{d-1}-1,\dots,s_0-1} + \sum_{k=0}^{s_0-1} \big(-\frac{m}{2}\big)^k \mathbf{e}_0^{s_d-1,s_{d-1}-1,\dots,s_0-1-k}\Big]$$
$$\equiv u[\mathbf{e}_0^{s_d-1,\dots,s_0-1}] \mod \mathbb{Z}[\frac{m}{2}][u]^+_{\mathbf{D}\leq d-1}$$

Proof. The first congruence of b) follows from the property of u^{-1} recalled in 3.1.3. The second congruence is obtained by writing that, if $v = e^{\frac{m}{2}e_0} u e^{-\frac{m}{2}e_0}$, we have $u = e^{\frac{-m}{2}e_0} u e^{\frac{m}{2}e_0}$, and applying the first congruence with $(v, -m)$ instead of (u, m). The b) implies a) via Fact 4.1. □

4.3. **Rewriting of the set of equations.** We replace the set of associator of equations of dimension 1 by a slightly modified version.

Proposition 4.9. Let ψ a point of $\Pi^{un}(M_{0,4})$. We have an equivalence between :
i) ψ satisfies the duality, hexagon and special automorphism equations
ii) (if $m \neq 0$) ψ satisfies the equations :

$$e^{\frac{m}{2}e_\infty}\psi(e_0, e_\infty) = \psi(e_\infty, e_1)^{-1} e^{\frac{m}{2}e_1}\psi(e_0, e_1)e^{\frac{m}{2}e_0} \tag{16}$$

$$\psi^{-1}(e_0, e_1)e^{-me_1}\psi(e_0, e_1)e^{-me_0} = e^{\frac{m}{2}e_0}\psi(e_0, e_\infty)^{-1}e^{me_\infty}\psi(e_0, e_\infty)e^{-\frac{m}{2}e_0} \tag{17}$$

ii') (if $m = 0$) ψ satisfies the equations :

$$\psi(e_0, e_\infty) = \psi(e_\infty, e_1)^{-1}\psi(e_0, e_1) \tag{18}$$

$$-e_0 - \psi^{-1}(e_0, e_1)e_1\psi(e_0, e_1) = \psi(e_0, e_\infty)^{-1}e_\infty\psi(e_0, e_\infty) \tag{19}$$

Here, equations (17) and (19) are respectively the equation of the special automorphism (5) and (6).

Proof. It is the direct consequence of the following four facts :
a) Rewriting of the hexagon equation : in both cases $m \neq 0$ and $m = 0$, we have an equivalence, for ψ a point of $\Pi^{un}(M_{0,4})^{un}$: ψ satisfies the duality and the hexagon equations ⇔ ψ satisfies the duality equation and

$$e^{\frac{m}{2}e_\infty}\psi(e_0, e_\infty) = \psi(e_\infty, e_1)^{-1} e^{\frac{m}{2}e_1}\psi(e_0, e_1)e^{\frac{m}{2}e_0}$$

b) Replacing the duality by its conjugated version : because of lemma 4.4, we have the equivalence : f satisfies the duality equation $\Leftrightarrow$ $(\psi^{-1}e_1\psi)(e_1,e_0) = \psi e_0 \psi^{-1}$ $\Leftrightarrow$ for $m \neq 0$, $(\psi^{-1}e^{me_1}\psi)(e_1,e_0) = \psi e^{me_0}\psi^{-1}$.
c) Elimination of the duality equation in the $m \neq 0$ case : it is proved in [AET] (§5.2, first proof) that the hexagon and duality equation imply the special automorphism equation. The proof in [AET] is for $m = 1$, but the same proof works for any $m \neq 0$.
d) Elimination of the duality or hexagon equation in the $m = 0$ case :
Let ψ satisfying the equation of the special automorphism with $m = 0$ (6). Then :
ψ satisfies the duality equation (2) $\Leftrightarrow$ ψ satisfies the hexagon equation (3) with $m = 0$.
Indeed, let us apply to the special automorphism equation, on the one hand, the conjugation by ψ, and, on the other hand, the change of variables $(e_0,e_1) \to (e_1,e_0)$. We obtain

$$\psi(e_0,e_1)e_0\psi(e_0,e_1)^{-1} + e_1 + \psi(e_0,e_1)\psi(e_0,e_\infty)^{-1}e_\infty\psi(e_0,e_\infty)\psi(e_0,e_1)^{-1} = 0$$

$$e_1 + \psi(e_1,e_0)e_0\psi(e_1,e_0)^{-1} + \psi(e_1,e_\infty)^{-1}e_\infty\psi(e_1,e_\infty) = 0$$

This implies the equivalence by the lemma 4.4. □

Remark 4.10. The analog of d) in the proof above for $m \neq 0$ gives only, with $X = e^{\psi(e_1,e_0)^{-1}me_1\psi(e_1,e_0)}$, $Y = e^{me_1}$, that :

$$Y^{-1} = X\left[e^{\frac{m}{2}e_1}\psi(e_1,e_\infty)^{-1}e^{me_\infty}\psi(e_1,e_\infty)e^{-\frac{m}{2}e_1}\right]$$

$$X^{-1} = Y\left[\left(\psi(e_0,e_\infty)e^{-\frac{m}{2}e_0}\psi(e_0,e_1)^{-1}\right)^{-1} e^{me_\infty}\left(\psi(e_0,e_\infty)e^{-\frac{m}{2}e_0}\psi(e_0,e_1)^{-1}\right)\right]$$

This does yields the same elimination result.

4.4. **The maps A_m,A_m^∞,B_m,B_m^∞ and their highest depth terms .** The modified hexagon equation, resp. the special automorphism equation gives rise to maps A_m,A_m^∞, resp. B_m,B_m^∞.

Definition 4.11. Let $A_m, A_m^\infty, B_m, B_m^\infty : \mathcal{H}_{\text{ш}} \to \mathcal{H}_{\text{ш}}[m]$ be the unique linear maps satisfying respectively, for all points u of $\Pi^{un}(M_{0,4})$ viewed as maps on $\mathcal{H}_{\text{ш}}[m]$:

$$u(e_0,e_1) \circ A_m = u(e_\infty,e_1)^{-1}e^{\frac{m}{2}e_1}u(e_0,e_1)e^{\frac{m}{2}e_0}$$

$$u(e_0,e_\infty) \circ A_m^\infty = e^{\frac{m}{2}e_\infty}u(e_0,e_\infty)$$

$$u(e_0,e_1) \circ B_m = \begin{cases} u^{-1}(e_0,e_1)e^{-me_1}u(e_0,e_1)e^{-me_0} & \text{if } m \neq 0 \\ -u(e_0,e_1)^{-1}e_1u(e_0,e_1) - e_0 & \text{if } m = 0 \end{cases}$$

$$u(e_0,e_\infty) \circ B_m^\infty = \begin{cases} e^{\frac{m}{2}e_0}u(e_0,e_\infty)^{-1}e^{me_\infty}u(e_0,e_\infty)e^{-\frac{m}{2}e_0} & \text{if } m \neq 0 \\ u(e_0,e_\infty)^{-1}e_\infty u(e_0,e_\infty) & \text{if } m = 0 \end{cases}$$

Proof. (of the theorem 3.1) i) follows from the proposition 4.9 and the definitions of $A_m, A_m^\infty, B_m, B_m^\infty$.
ii) The results for A and B follow from lemma 4.6. The result on B_m^∞ uses lemma 4.7. The only two operations which we have not translated on the level of coefficients in the preliminary lemmas are the right multiplication of the series u by $e^{\frac{m}{2}e_0}$ or e^{me_0}, and the left multiplication of the series u by e^{me_∞}. Their translations on the level of coefficients are straightforward. □

Proof. (of the corollary 3.2) This is a consequence of lemma 4.6 and lemma 4.7. □

4.5. **Proofs of the depth reductions.** First of all, the theorem has the first following corollary.

4.5.1. *The depth reductions for ψ.* We take (m, ψ) satisfying the assumptions of the theorem.

Proof. (of the corollary 3.3 : depth reduction for f with a parity assumption) By the formula for the congruence 8, applied to $w = e_0^{s_d-1,\ldots,s_1-1}e_1$ $(s_0 = 0)$, and the equation $\psi \circ A_m = \psi^\infty \circ A_m^\infty$, we obtain that if $s_d + \ldots + s_1 - d$ is odd, $\psi[w] \in \mathbb{Z}[m][\psi^\infty]^+_{\mathbf{D} \leq d}$. But by the equation $\psi \circ B_m = \psi^\infty \circ B_m^\infty$, we also have that $\mathbb{Z}[m][\psi^\infty]^+_{\mathbf{D}\leq d} \subset \mathbb{Z}[m][\psi]^+_{\mathbf{D}\leq d-1}$. We obtain the result. □

Proof. (of the corollary 3.4 : depth reduction for f with a parity assumption)
We apply the equality $\psi \circ B_m = \psi^\infty \circ B_m^\infty$ to the coefficients $e_1\mathbf{e}_0^{s_d-1,\ldots,s_1-1}e_1$. For those special coefficients, we obtain that they are in $\mathbb{Z}[\psi^\infty]^+_{\mathbf{D}\leq d}$. Applying the expression of ψ^∞ in depth d in terms of ψ in depth $\leq d-1$ given by $\psi \circ B_m = \psi^\infty \circ B_m^\infty$ gives the result. □

4.5.2. *The depth reductions for ψ_∞.* We take again (m, ψ) satisfying the assumptions of the theorem.

Proof. (of the corollary 3.6) : depth reduction for f with a parity assumption) Combine the result of the corollary 3.3 with the formula for the highest depth term of the equation $\psi \circ A = \psi_\infty \circ A^\infty$. □

For the depth reduction for symmetric sums on f_∞, which we write only for $m = 0$, we could have kept the counterpart on f_∞ of the corollary 3.4, as we did for the parity. Nevertheless, we found interesting to modify it and take some particular linear combinations of the highest depth term, in order to obtain an interesting formula related to the pole at 1 of the zeta function. This modification is inspired by a computation of Ünver done in depth one, and in depth two and odd weights in [U1].

Proof. (of the corollary 3.7). For convenience, we do not use here the preliminary lemmas of §4.2 and we start directly at the special automorphism equation with $m = 0$. Let $\chi = \psi^{-1}e_1\psi$ and let $\tilde{\chi} = \chi - e_1$. The equation of the special automorphism for $(m = 0, \psi)$ can be rewritten as

$$e_0\psi_\infty - \psi_\infty e_0 = -(e_1\psi_\infty - \psi_\infty e_1) + \psi_\infty\tilde{\chi} \tag{20}$$

We note that, since $\psi[e_0] = 0$, $\tilde{\chi}$ vanishes in depth 0 and 1. This and the corollary 3.2 implies that we have

$$\mathbb{Z}[\psi_\infty\tilde{\chi}]^+_{\mathbf{D}=d} \subset \mathbb{Z}[\psi_\infty]_{\mathbf{D}\leq d-2} \tag{21}$$

Let us take $d \in \mathbb{N}^*$ and $t_d, \ldots, t_0 \in \mathbb{N}$. By considering the coefficient of $\mathbf{e}^{t_d+1;t_{d-1},\ldots,t_1;t_0+1}$ in (20), we obtain

$$\psi_\infty[\mathbf{e}^{t_d;t_{d-1},\ldots,t_1;t_0+1}] - \psi_\infty[\mathbf{e}^{t_d+1;t_{d-1},\ldots,t_1;t_0}] = (\psi_\infty\tilde{\chi})[\mathbf{e}^{t_d+1;t_{d-1},\ldots,t_1;t_0+1}] \tag{22}$$

On the other hand, the shuffle relation $\psi_\infty[e_0 ш (\mathbf{e}^{t_d;t_{d-1},\dots,t_1;t_0})] = 0$ gives

$$(23)\quad (t_d+1)\psi_\infty[\mathbf{e}^{t_d+1;t_{d-1},\dots,t_1;t_0}] + (t_0+1)\psi_\infty[\mathbf{e}^{t_d;t_{d-1},\dots,t_1;t_0+1}] = -\sum_{k=1}^{d-1}(t_k+1)\psi_\infty[\mathbf{e}^{t_d;t_{d-1},\dots,t_k+1,\dots,t_1;t_0}]$$

The combination of (22) and (23) is a linear system which is inversed into the following system, after a change of variable replacing t_0 by t_0-1, resp. t_d by t_d-1 :

$$(24)\quad \psi_\infty[\mathbf{e}^{t_d;\dots,t_1;t_0}] = \sum_{k=1}^{d-1}\frac{-(t_k+1)}{t_d+t_0+1}\psi_\infty[\mathbf{e}^{t_d;\dots,t_k+1,\dots;t_0-1}] + \frac{t_d+1}{t_d+t_0+1}(\psi_\infty\tilde{\chi})[\mathbf{e}^{t_d+1;t_{d-1},\dots,t_1;t_0-1}]$$

$$\psi_\infty[\mathbf{e}^{t_d;t_{d-1},\dots;t_0}] = \sum_{k=1}^{d-1}\frac{-(t_k+1)}{t_d+t_0+1}f^\infty[\mathbf{e}^{t_d-1;\dots,t_k+1,\dots;t_0}] - \frac{t_0+1}{t_d+t_0+1}(\psi_\infty\tilde{\chi})[\mathbf{e}^{t_d-1;t_{d-1},\dots,t_1;t_0+1}]$$

Let $S = (a_{d,i}, a_{0,i})_{0\le i\le r_d+r_0}$ be any sequence of elements of $\{0,\dots,r_d\}\times\{0,\dots,r_0\}$, satisfying :

$$\begin{cases} (a_{d,0}, a_{0,0}) = (r_d, r_0) \quad \text{and} \quad (a_{d,r_d+r_0}, a_{0,r_d+r_0}) = (0,0) \\ \text{For all i } \in \{0,\dots,r_d+r_0-1\}, \quad (a_{d,i+1}, a_{0,i+1}) \in \{(a_{d,i}-1, a_{0,i}), (a_{d,i}, a_{0,i}-1)\} \end{cases}$$

The proof of the lemma follows by induction on (r_d, r_0) for the lexicographical order, using the the linear system (24), and equation (21) which enables to eliminate the terms $\psi_\infty\tilde{\chi}$: we apply the first. resp second equation of (24) inductively to $(t_d,\dots,t_0) = (s_d-1+a_{d,i}, s_{d-1}-1,\dots,s_1-1, s_0-1+a_{d,i})$ if $a_{0,i+1} = a_{0,i}-1$, resp. $a_{d,i+1} = a_{d,i}-1$. □

4.6. Variants.

Remark 4.12. The special automorphism equation combined with the duality equation relative to a couple (m,ψ) implies :

$$(25)\qquad \psi(e_0,e_1)e^{-me_0}\psi(e_0,e_1)^{-1}e^{-me_1} = e^{\frac{m}{2}e_1}\psi(e_1,e_\infty)^{-1}e^{me_\infty}\psi(e_1,e_\infty)e^{-\frac{m}{2}e_1}$$

With this version, we obtain, instead of two maps B_m and B_m^∞, a single map $C_m : \mathcal{H}_ш \to \mathcal{H}_ш[m]$, such that 25 amounts to $\psi\circ C_m = 0$. It can be used to give another proof to some of the corollaries of depth reduction.

5. Applications to multiple zeta values and comments

5.1. Applications to multiple zeta values.

5.1.1. *Multiple zeta values and the KZ associator.* Recall that Φ_{KZ} is an element of $\Pi^{un}(M_{0,4})(\mathbb{R})$ defined in [Dr], §2 and

Proposition 5.1. ([Dr], §2) Φ_{KZ} is an associator with $m = 2i\pi$.

It is called the Knizhnik-Zamolodchikov associator. We have, for all $d\in\mathbb{N}^*$, $s_d,\dots,s_1\in\mathbb{N}^*$, $s_d\ge 2$,

$$(26)\qquad \Phi_{\mathrm{KZ}}[e_0^{s_d-1,\dots,s_1-1}e_1] = (-1)^d\zeta(s_d,\dots,s_1)$$

By the facts concerning the shuffle algebra recalled in §4.1, since $\Phi_{\mathrm{KZ}}[e_0] = \Phi_{\mathrm{KZ}}[e_1] = 0$, all the coefficients of Φ_{KZ} are expressed as $\mathbb{Q}$-linear combinations of multiple zeta values.

The definition of multiple zeta values can be extended in two different ways to the case where $s_d = 1$. This yields two families of real numbers, $\zeta_{ш}(s_d, \ldots, s_1)$ and $\zeta_*(s_d, \ldots, s_1)$, indexed by $\amalg_{d\in\mathbb{N}^*}(\mathbb{N}^*)^d$, which do not coincide in general on tuples $(1, s_{d-1}, \ldots, s_1)$. The first one is defined by regularization of iterated integrals on $\mathbb{P}^1 - \{0, 1, \infty\}$ and the second one by regularization of iterated integrals. For details, see for example [C].

The formula (26) extends to all indices of $\amalg_{d\in\mathbb{N}^*}(\mathbb{N}^*)^d$ as

$$\Phi_{\mathrm{KZ}}[e_0^{s_d-1,\ldots,s_1-1}e_1] = (-1)^d \zeta_{ш}(s_d, \ldots, s_1) \tag{27}$$

5.1.2. *Separation of the even and odd powers of* $2i\pi$. Applying the results of §3 to multiple zeta values gives congruences among linear combination of multiple zeta values over $\mathbb{Q}[2i\pi]$. For example, the corollary 3.3 gives that when $s_1 + \ldots + s_d - d$ is odd, $\zeta(s_d, \ldots, s_1) \in \mathbb{Q}[2i\pi][\Phi_{\mathrm{KZ}}]^+_{\mathbf{D}\leq d-1}$.
In all such statements, $\mathbb{Q}[2i\pi]$ can be replaced by $\mathbb{Q}$, by considering the real and imaginary parts of the congruence, which separates the even and odd powers of $2i\pi$.

5.1.3. *p-adic multiple zeta values and the p-adic KZ associator.* Furusho and Deligne have defined independently two different but closely related versions, that we can denote respectively by $\Phi_{p,\mathrm{KZ}}$ and Φ_p, of a p-adic analogue of Φ_{KZ}, with coefficients in $\mathbb{Q}_p$ (in, respectively, [F1], [F2], and [DG], §5.28). Ünver has shown that $\Phi_{p,\mathrm{KZ}}$ satisfies the equations of GRT_1, i.e. is an associator for $m = 0$, in [U] ; it is thus also the case for Φ_p by [F2].

Thus, the results of this paper can be applied to p-adic multiple zeta values, and certain applications are quite specific as we explain in the next paragraph.

The associator equations are also true for motivic multiple zeta values but we will not discuss it here.

5.1.4. *Finite multiple zeta values and the reduction modulo large primes of p-adic multiple zeta values.* A notion of "finite multiple zeta values" has been recently defined by Zagier.

Definition 5.2. (Zagier) Let the $\mathbb{Q}$-algebra of integers "modulo infinitely large primes"

$$\mathcal{A} = \Big(\prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \Big) / \Big(\oplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \Big)$$

Let finite multiple zeta values be the following numbers, for all $(s_d, \ldots, s_1) \in \amalg_{d\in\mathbb{N}^*}(\mathbb{N}^*)^d$:

$$\zeta_{\mathcal{A}}(s_d, \ldots, s_1) = \Big(\big(H_p(s_d, \ldots, s_1) := \sum_{0<n_1<\ldots<n_d<p} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}} \big) \mod p \Big)_p \in \mathcal{A}$$

Kaneko and Zagier gave a precise conjecture describing finite multiple zeta values as variants of multiple zeta values. To explain it, we have proved that $H_p(s_d,\dots,s_1)$ can be written in a concise way as an infinite sum of p-adic multiple zeta values [J2] : if ζ_p denotes Deligne's p-adic multiple zeta values, then, for all $d\in\mathbb{N}^*$, $s_d,\dots,s_1\in\mathbb{N}^*$, we have :

$$(28)\quad p^{s_d+\dots+s_1}H_p(s_d,\dots,s_1)$$
$$= (-1)^d \sum_{l_{k+1},\dots,l_d\geq 0}\sum_{k=0}^{d}\prod_{i=k+1}^{d}(-1)^{s_i}\binom{-s_i}{l_i}\zeta_p(s_{k+1}+l_{k+1},\dots,s_d+l_d)\zeta_p(s_k,\dots,s_1)\in\mathbb{Q}_p$$

This had been conjectured by Yasuda and Hirose. A result of Yasuda on the valuation of p-adic multiple zeta values implies that, for $d'\in\mathbb{N}^*$, $t_{d'},\dots,t_1\in\mathbb{N}^*$ and p a prime number, we have $\zeta_p(t_{d'},\dots,t_1)\in\sum_{n\geq t_{d'}+\dots+t_1}\frac{p^n}{n!}\mathbb{Z}_p$; in particular $\zeta_p(t_{d'},\dots,t_1)\in p^{t_{d'}+\dots+t_1}\mathbb{Z}_p$ if $p>t_{d'}+\dots+t_1$, and one obtains :

$$H_p(s_d,\dots,s_1)\equiv p^{-(s_d+\dots+s_1)}\sum_{k=0}^{d}(-1)^{s_{k+1}+\dots+s_d}\zeta_p(s_{k+1},\dots,s_d)\zeta_p(s_k,\dots,s_1)\mod p$$

Thus, the finite multiple zeta values in $\mathcal{A}$ can be entirely written in terms of p-adic multiple zeta values. This formula explains the formula in Kaneko and Zagier's conjecture which involves the complex analogue modulo $\zeta(2)$ of the right hand side of the congruence above.

Because of §5.1.3, the corollary 3.4 of depth reduction for symmetric sums with $(0,\Phi_p)$ implies a depth reduction for the numbers

$$\sum_{k=0}^{d}(-1)^{s_{k+1}+\dots+s_d}\zeta_p(s_{k+1},\dots,s_d)\zeta_p(s_k,\dots,s_1)$$

This is a step in our algorithm in [J4] to write the reduction modulo large primes of all p-adic multiple zeta values in terms of finite multiple zeta values. As we explain in §5.2.1 below, the low depth case of this depth reduction (§3.4) retrieves congruences among finite multiple zeta values proved by an elementary way.

5.2. **Comparison with known formulas.**

5.2.1. *The implicit depth reduction in low depth for finite multiple zeta values.*

Facts 5.3. i) For all $s\in\mathbb{N}^*$, $\zeta_{\mathcal{A}}(s)=0$.
More precisely, $H_p(s)$ is equal to $-1\mod p$ if $p-1|s$ and $0\mod p$ otherwise.
ii) For all $s_2,s_1\in\mathbb{N}^*$, $\zeta_{\mathcal{A}}(s_2,s_1)$ depends only on the weight s_2+s_1 up to a rational coefficient, and vanishes is the weight is even.
More precisely, we have, if $p>s_1+s_2$, $H_p(s_2,s_1)\equiv(-1)^{s_1}\binom{s_1+s_2}{s_1}\frac{B_{p-s_1-s_2}}{s_1+s_2}\mod p$

The i) is very classical and ii) follows from [H], theorem 6.1.

This is in accordance with the formulas of §3.4 for the depth reduction of symmetric sums for f.

5.2.2. *Yasuda's formula for the depth reduction of symmetric sums.* In [Y1], Yasuda studies the numbers $\zeta^{\mathcal{F}}$ defined as, for all $(s_d,\ldots,s_1) \in \amalg_{d\in\mathbb{N}^*}(\mathbb{N}^*)^d$, (the series regularization of)

$$\zeta^{\mathcal{F}}(s_d,\ldots,s_1) = \lim_{N\to\infty} \sum_{\substack{n_1,\ldots,n_d\in\mathbb{Z} \\ 0<|n_1|<\ldots<|n_d|<N \\ \frac{1}{n_1}>\ldots\frac{1}{n_d}}} \frac{1}{n_1^{s_1}\ldots n_d^{s_d}}$$

Following [Y1], §6.1, beginning of the proof of Theorem 6.1, we have, for all indices :

$$\zeta^{\mathcal{F}}(s_d,\ldots,s_1) = \sum_{k=0}^{d} (-1)^{s_{k+1}+\ldots+s_d} \zeta_*(s_{k+1},\ldots,s_d)\zeta_*(s_k,\ldots,s_1)$$

and, in the $\mathbb{Q}$-algebra of multiple zeta values

$$(29)\quad \sum_{k=0}^{d}(-1)^{s_{k+1}+\ldots+s_d}\zeta_*(s_{k+1},\ldots,s_d)\zeta_*(s_k,\ldots,s_1)$$
$$\equiv \sum_{k=0}^{d}(-1)^{s_{k+1}+\ldots+s_d}\zeta_{ш}(s_{k+1},\ldots,s_d)\zeta_{ш}(s_k,\ldots,s_1) \mod (\zeta(2))$$

In corollary 3.3 he gives a formula for the depth reduction of each $\zeta^{\mathcal{F}}(s_d,\ldots,s_1)$ modulo $(\mathbb{Q}[\Phi_{\mathrm{KZ}}]_{\mathbf{D}\leq d-2}+$ the $\mathbb{Q}$-vector subspace generated by products of multiple zeta values of total weight $s_d+\ldots+s_1)$. The formula depends on the parity of $s_d+\ldots+s_1-d$ and coincides with our formula (13) only when $s_d+\ldots+s_1-d$ is odd.

5.3. **The depth of ψ_∞ in the case of p-adic multiple zeta values .**

5.3.1. *The depths of ψ_∞ and p-adic iterated integrals and the computation of p-adic multiple zeta values.* The depth of ψ_∞ appears naturally in the context of p-adic multiple zeta values for the following reasons.

The Frobenius map $z\mapsto z^p$ on $\mathbb{P}^1(\mathbb{F}_p)$ has the lift $F_{0\infty} : z\mapsto z^p$ on $\mathbb{P}^1(\mathbb{Z}_p)$; it is a good lift at 0 and ∞, but not at 1 : i.e. the preimages of 0 and ∞ by $F_{0\infty}$ are 0 and ∞ with multiplicity p whereas 1 has as preimages the p-th roots of unity.
In order to relate p-adic multiple zeta values to sums of series with rational coefficients, one has to consider the overconvergent variant of the KZ connection on the pro-unipotent fundamental group of $(\mathbb{P}^1-\{0,1,\infty\})/\mathbb{Q}_p$ (as in [U1], [J1], [J2], [J3]), and solve it on a p-adic analytic space over $\mathbb{Q}_p$. Because of the previous properties of $F_{0\infty}$, we are led, following Ünver [U1], to choose the generic fiber $X_{0\infty}$ of the formal completion of $\mathbb{P}^1/\mathbb{Z}_p$ along $(\mathbb{P}^1-\{1\})/\mathbb{F}_p$. This gives an indirect computation of Φ_p, in the sense that it passes through the intermediate object $(\Phi_p)_\infty$.

There are of course other possible choices : one can take the rigid analytic fiber of the formal completion of $\mathbb{P}^1/\mathbb{Z}_p$ along the reduction modulo p of $\mathbb{P}^1-\{\infty\}$, which gives in appearance a direct computation of p-adic multiple zeta values. However, the good lift of Frobenius on this space is the conjugation of $z\mapsto z^p$ by $z\mapsto \frac{z}{z-1}$, namely $F_{01} : z\mapsto \frac{z^p}{z^p-(z-1)^p}$. Dealing with power series $\sum a_n z^n$ twisted by F_{01} instead of $F_{0\infty}$

can be decomposed into steps which consist implicitly in passing through the value at ∞. It seems that we are obligated to refer at least implicitly to the value at ∞.

Definition 5.4. Let the depth of a word over $\{\omega_0 = \frac{dz}{z}, \omega_1 = \frac{dz}{z-1}, \omega_p = \frac{z^{p-1}dz}{z^p-1}\}$ be the sum of the numbers of letters ω_1 and of letters ω_p.

Fact 5.5. The values of $(\Phi_p)_\infty$ in depth $d \in \mathbb{N}^*$ are linear combinations of p-adic iterated integrals from 0 to ∞ on $X_{0\infty}$ of words of $\{\omega_0 = \frac{dz}{z}, \omega_1 = \frac{dz}{z-1}, \omega_p = \frac{z^{p-1}dz}{z^p-1}\}$ of depth in $\{1, \ldots, d\}$.

Because of this fact 5.5, and corollary 3.2, applied to $(m, \psi) = (0, \Phi_p)$, p-adic multiple zeta values in depth d are expressed in terms of iterated integrals whose depth is in $\{1, \ldots, d+1\}$.
This relation between the depth of p-adic multiple zeta values and the depth of those elementary p-adic integrals is crucial to obtain lower bounds on multiple zeta values which are sufficiently precise to prove equality (28).

5.3.2. *Depth reduction for ψ_∞ and the computation of p-adic multiple zeta values.* In the cases of depth one, and depth two and odd weights, Ünver modifies the expression of those iterated integrals in a way such that the indices $(s_d, \ldots, s_1)$ play a more symmetric role, and this expression retrieves the simple pole of the p-adic zeta function [U1], and a sort of pseudo-pole at $s_2 = 1$ for $\zeta(s_2, s_1)$ in odd weights.

More generally, the depth reduction of symmetric sums for ψ_∞ (corollary 3.7) leads to express any coefficient of ψ_∞ in terms of coefficients of the form $\psi_\infty[e_1 \ldots e_1]$, with rational coefficients related to the poles of zeta functions.
On the other hand, if we express fully ψ in terms of ψ_∞ by equation (5) or (6), we are led to separate the cases $s_d, \ldots, s_1 \geq 2$ and the others. The coefficients of f at the other indices are recuperated via the shuffle equation.
In particular, the values of the form $\psi[e_0^{s_d-1} \ldots e_1]$ with $s_d \geq 2$ are related to values of the form $\psi_\infty[e_0^{s_d-2} \ldots e_1]$ which is expressed in terms of values $\psi_\infty[e_1 \ldots e_1]$ with rational coefficients having as denominator $\frac{1}{s_d-1}$. Both $\frac{1}{s_d-1}$ and $\psi_\infty[e_0^{s_d-2} \ldots e_1]$ do not make sense when $s_d = 1$.

When $m = 0$ this formula is in [U1], and is related the simple pole at 1 of the p-adic zeta functions $s \mapsto L_p(s, \omega^{1-s})$.

5.4. **Comments.** Let us slightly essentialize the proofs. The associator equations of Φ_{KZ} are the consequence of the horizontality of automorphisms of $M_{0,4}$ and $M_{0,5}$ with respect to the KZ equation. One general ingredient is results of depth reduction concerning the automorphisms of $\mathcal{H}_{ш}$ that they induce, and automorphisms of the form $(-1)^{\text{depth}}\,\text{id}$, $(-1)^{\text{weight}-\text{depth}}\,\text{id}$, in the sense of the fact below.

Moreover, certain maps of this type satisfy properties, not of depth reduction but of "depth augmentation".

Fact 5.6. i) The map $f_1 : w \in \mathcal{H}_{ш} \mapsto w(e_0 - e_1, -e_1) - (-1)^{\text{depth}(w)} w \in \mathcal{H}_{ш}$, dual to $u \mapsto u(e_0, e_\infty) - u(e_0, -e_1)$, admits a "depth augmentation" : we have, for all words w

$$f_1(w) \in \mathbb{Z}[\mathcal{H}_{ш}]_{\mathbf{D} \geq \text{depth}(w)+1}$$

ii) $f_2 : w \in \mathcal{H}_{ш} \mapsto w(-e_0, -e_0 + e_1) - (-1)^{\text{weight}(w)-\text{depth}(w)} w \in \mathcal{H}_{ш}$, dual to $u \mapsto u(e_\infty, e_1) - u(-e_0, e_1)$ admits a depth reduction : we have, for all words w,

$$f_2(w) \in \mathbb{Z}[\mathcal{H}_{ш}]_{\mathbf{D} \leq \text{depth}(w)-1}$$

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