THE BELOSHAPKA'S MAXIMUM CONJECTURE IS CORRECT

MASOUD SABZEVARI

ABSTRACT. Applying the Élie Cartan's classical method, we show that the biholomorphic equivalence problem to a totally nondegenerate Beloshapka's universal model of CR dimension one and codimension k, whence of real dimension $2+k$, is reducible to some absolute parallelisms, namely to $\{e\}$-structures on some prolonged manifolds of real dimensions either $3+k$ or $4+k$. The proof relies on some weight analysis of the structure equations associated to the mentioned problem of equivalence. Thanks to achieved results, we prove the Beloshapka's maximum conjecture about the rigidity of his CR models of certain lengths equal or greater than three. Here, we mainly deal by CR models of fixed CR dimension one though the results seem quietly generalizable by means of some analogous proofs.

1. INTRODUCTION

In 2004, Valerii Beloshapka established in [3] his universal model surfaces associated to totally nondegenerate CR manifolds of arbitrary CR dimensions and codimensions and designed an effective method to construct them. It was in fact along the celebrated approach initiated first by Henri Poincaré [25] in 1907 to study real submanifolds in the complex space $\mathbb{C}^2$ by means of the associated model surface, namely the *Heisenberg sphere*. Several years later in 1974, Chern and Moser in their seminal work [9] notably developed this approach by associating appropriate models to arbitrary totally nondegenerate real *hypersurfaces* in complex spaces. In this framework, many questions about automorphism groups, classification, invariants and others, concerned the (holomorphic) transformations of real submanifolds in a certain complex space can be reduced to similar problems about the associated models.

But — to the best of the author's knowledge — the Beloshapka's work can be considered as the most general establishment in this setting whereas he provided appropriate models to totally nondegenerate CR manifold of *arbitrary* dimensions. These models are all homogeneous, of finite type and enjoy several other *nice* properties ([3, Theorem 14]) that exhibit their significance. Two totally nondegenerate germs are holomorphically equivalent whenever their associated models are as well. Moreover, they are most symmetric nondegenerate surfaces in the sense that the dimension of the group of automorphisms associated to a totally nondegenerate germ does not exceed that of its model.

For a CR model M of CR dimension n and codimension k in coordinates $(z_1,\ldots,z_n,w_1,\ldots,w_k)$, a holomorphic vector field:

$$\mathsf{X} := \sum_{j=1}^{n} Z^j(z,w)\frac{\partial}{\partial z_j} + \sum_{l=1}^{k} W^l(z,w)\frac{\partial}{\partial w_l}$$

is called an *infinitesimal CR automorphism* whenever its real part is tangent to M, that is $(\mathsf{X}+\overline{\mathsf{X}})|_M \equiv 0$. The collection of all infinitesimal CR automorphisms associated to M form a Lie algebra, denoted by $\mathfrak{aut}_{CR}(M)$, that parameterizes the family of maps taking a corresponding totally nondegenerate germ to another. This Lie algebra, which is in fact the CR symmetry Lie algebra of M in the terminology of Sophus Lie's symmetry theory [18], is finite dimensional, of polynomial type and graded — in the sense of Tanaka — of a form like (*cf.* [3, 28]):

$$\mathfrak{aut}_{CR}(M) := \underbrace{\mathfrak{g}_{-\rho}\oplus\cdots\mathfrak{g}_{-1}}_{\mathfrak{g}_-}\oplus\mathfrak{g}_0\oplus\underbrace{\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_\varrho}_{\mathfrak{g}_+}, \qquad \varrho,\rho\in\mathbb{N} \tag{1}$$

Date: 2016-1-7.

2010 *Mathematics Subject Classification.* 32V40, 58A15, 57S25.

with:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

In this case, the integer ρ which is called the *length* of M, is in fact the maximum weight of the complex variables appearing among its defining equations. If one prefers to view the real analytic CR generic model M in a purely intrinsic way, one may consider the local Lie group $\mathsf{Aut}_{CR}(M)$, associated to $\mathfrak{aut}_{CR}(M)$, comprising automorphisms of the CR structure, namely of local $\mathscr{C}^\infty$ diffeomorphisms $h : M \to M$ satisfying:

$$h_*(T^cM) = T^cM, \quad \text{and} \quad h_* \circ J = J \circ h_*,$$

where T^cM is the complex tangent bundle of M and J is the associated structure map. In other words, h belongs to $\mathsf{Aut}_{CR}(M)$ if and only if it is a (local) biholomorphism of M ([19]). Accordingly, one may write the associated Lie group $\mathsf{Aut}_{CR}(M)$ as:

$$\mathsf{Aut}_{CR}(M) := \mathsf{G}_- \oplus \mathsf{G}_0 \oplus \mathsf{G}_+. \tag{2}$$

Beloshapka in [3] showed that for a CR model M of CR dimension n and codimension k — and whence of real dimension $2n + k$ — then the Lie group G_- associated to the above $(2n + k)$-dimensional subalgebra $\mathfrak{g}_-$ of $\mathfrak{aut}_{CR}(M)$ acts on M freely and can naturally be identified with M, itself. Also, G_0 associated to the subalgebra $\mathfrak{g}_0$ comprises all *linear* automorphisms of M in the isotropy subgroup $\mathsf{Aut}_0(M)$ of $\mathsf{Aut}_{CR}(M)$ at $0 \in M$ while G_+, associated to $\mathfrak{g}_+$, comprises as well all *nonlinear* ones.

Determining such Lie algebras of infinitesimal CR automorphisms is a question which lies pivotally at the heart of the problem of classifying local analytic CR manifolds up to biholomorphisms (*see e.g.* [6] and the references therein). In fact, the groundbreaking works of Sophus Lie and his followers (Friedrich Engel, Georg Scheffers, Gerhard Kowalewski, Ugo Amaldi and others) showed that the most fundamental question in concern here is to draw up lists of possible such Lie algebras which would classify all possible manifolds according to their CR symmetries. Moreover, having in hand these algebras may also help one to treat the problem of constructing (canonical) Cartan geometries on certain classes of CR manifolds ([22, 30]) or to construct the so-called moduli spaces of model real submanifolds ([26]).

In the computational point of view and though computing the nonpositive part $\mathfrak{g}_- \oplus \mathfrak{g}_0$ of the above algebra $\mathfrak{aut}_{CR}(M)$ is convenient — in particular by means of the algorithm designed in [28] — but unfortunately computing $\mathfrak{g}_+$ admits tremendous and much complicated computations which rely on constructing and solving some arising systems of partial differential equations ([16, 21, 27, 29]). Nevertheless, after several years of experience of computing these algebras in various dimensions, Beloshapka in [1] conjectured that[1];

Conjecture 1.1. [Beloshapka's Maximum Conjecture] *Each CR model M of the length $\rho \geq 3$ has rigidity; that is: in its associated Lie algebra $\mathfrak{aut}_{CR}(M)$ as* (1)*, the subalgebra $\mathfrak{g}_+$ is trivial or equivalently $\varrho = 0$.*

Holding this conjecture true may bring about having several other facts about CR models or their associated totally nondegenerate CR manifolds (*see e.g.* [2]). At this time, there are just a few considerable results that verify this conjecture in some specific cases. For instance, Gammel and Kossovskiy [13] confirmed it in the length $\rho = 3$. Also, Mamai in [16] proved this conjecture for the models of the fixed CR dimension $n = 1$ and codimensions $k \leq 13$. Some more relevant (partial) results in this setting are also as follows:

- If $\rho = 2$, then $\varrho \leqslant 2$ ([5, p. 32]).
- If $\rho = 4$, then $\varrho \leqslant 1$ ([4, Corollary 7]).
- If $\rho = 5$, then $\varrho \leqslant k$, where k is the CR codimension of M ([31, Proposition 2.2]).

In almost all of these works the results are achieved by means of computing *directly* the associated Lie algebras of infinitesimal CR automorphisms. But the much difficulty of this method, lying in the incredible *differential-algebraic* complexity involved (*cf.* [27]), may plainly convince oneself requiring some other ways to attack this conjecture.

[1] Although Beloshapka introduced his conjecture in 2012 but he and his students had been aware of it since several years before. For example see [2, 13, 16]

On the other hand, recently in [21] and in particular in §5 of this paper (or in §12 of its expanded version), we attempted to study the biholomorphic equivalence problem to the 5-dimensional length 3 cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ of codimension 3, represented in coordinates (z, w_1, w_2, w_3) as the graph of:

$$\begin{cases} w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\ w_2 - \overline{w}_2 = 2i\, z\overline{z}(z + \overline{z}), \\ w_3 - \overline{w}_3 = 2\, z\overline{z}(z - \overline{z}). \end{cases}$$

As we observed (*see* [21, Theorem 5.1]), the associated 7-dimensional Lie algebra:

$$\mathfrak{aut}_{CR}(M^5_{\mathsf{c}}) := \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

computed in §3 of this paper, is isomorphic to the Lie algebra defined by the final *constant type* structure equations of the mentioned equivalence problem to M^5_{c}. This observation was our original motivation to look upon the Cartan's classical approach of solving biholomorphic equivalence problems as an appropriate way to consider the Beloshapka's maximum conjecture. Examining this idea on some other CR models like those studied in [22, 24, 26, 29] also convinced us more about the effectiveness of this approach to suit our purpose. In fact, the systematic approach developed among the recent years by Joël Merker, Samuel Pocchiola and the present author provides a unified way toward treating the wide variety of biholomorphic equivalence problems between CR manifolds.

Cartan's classical method for solving equivalence problems includes three major parts: *absorption*, *normalization*, *prolongation*, and usually, all steps require advanced computations, the size of which increases considerably as soon as the dimension of CR manifolds increases, even by one unit. In particular, among the absorbtion-normalization steps, one encounters some arising polynomial systems in which its solution determine the value of some group parameters associated to the problem. But, solving such appearing systems may cause some unavoidable and serious algebraic complexity.

As it is quietly predictable, one of our main obstacles to proceed along the aim of proving the Beloshapka's maximum conjecture by applying the Cartan's classical approach, is actually solving the already mentioned arising polynomial systems in this general manner, namely the outcome of normalizing the group parameters. In order to bypass and manipulate this critical complexity, our main weapon is in fact some helpful results achieved by a careful *weight analysis* on the preliminary equipments of constructing the associated structure equations to the problem of equivalence, under study. Such analysis notably enables us to provide an appropriate weighted homogeneous subsystem of the already mentioned polynomial system. This much more convenient subsystem is in fact *deceptively hidden* inside the original one and opens our way to find the desired general outcome of the normalization process.

This paper is organized as follows. In the next preliminary section 2, we present a brief description of constructing defining equations of the Beloshapka's CR models in CR dimension 1.

Then in section 3, we attempt to find certain expressions of structure equations associated to the biholomorphic equivalence problem between an arbitrary CR model M_k, of codimension k and any other totally nondegenerate CR manifold of the same codimension. For this aim, we provide first some preliminary equipments such as an initial frame on the model, its dual coframe and the associated Darboux-Cartan structure. Moreover, we find the ambiguity matrix g of the mentioned equivalence problem as an invertible $(2+k) \times (2+k)$ lower triangular matrix of the form (*cf.* (14)):

$$\begin{pmatrix} a_1^p \overline{a}_1^q & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \dots & a_1 \overline{a}_1^2 & 0 & 0 & \dots & 0 \\ \mathsf{a}_\bullet & \vdots & \dots & 0 & a_1^2 \overline{a}_1 & 0 & \dots & 0 \\ \mathsf{a}_\bullet & \vdots & \dots & -\overline{a}_3 & a_3 & a_1 \overline{a}_1 & 0 & 0 \\ \mathsf{a}_\bullet & \vdots & \dots & \overline{a}_4 & a_5 & -\overline{a}_2 & \overline{a}_1 & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \dots & \overline{a}_5 & a_4 & a_2 & 0 & a_1 \end{pmatrix},$$

where some powers of a_1 and $\overline{a}_1$ are visible only at its diagonal.

The main focus of section 4 is on a weight analysis on the structure equations, constructed in the preceding section. In particular, after appropriate weight association to the appearing group parameters and also after inspecting carefully the inverse of the ambiguity matrix $\mathbf{g}$, we discover that all the torsion coefficients through the structure equations are of the same weight zero (*see* Proposition 4.5).

Next in section 5, we consider the outcome of the absorption and normalization steps on the constructed structure equations. It is in this section that we extract a subtle weighted homogeneous subsystem of the polynomial system, arising among the absorbtion and normalization steps. As the result of solving this subsystem by means of some computational techniques from *weighted algebraic geometry* ([11]) and in particular by *dehomogenizing* the associated weighted homogeneous variety (*see* subsection 5.2), we find out that;

Proposition 1.2. *(see Proposition 5.5) All the appearing group parameters $a_2, a_3, \ldots$ vanish identically after sufficient steps of absorption and normalization.*

This gives also an appropriate form of the final constant type structure equations (*see* Proposition 5.6). Concerning the only not-yet-determined parameter a_1, we also discover that it is either normalizable to a real (or imaginary) group parameter or it is never normalizable (*see* Corollary 5.7). In the former case, the structure group G of the above ambiguity matrices will be reduced to G^{red} of real dimension 1 while in the later case G^{red} is of real dimension 2. Next, we start the last part, namely prolongation, of the Cartan's method. Accordingly, our equivalence problem to our arbitrary CR model M_k converts by that to the prolonged space $M_k \times G^{\mathsf{red}}$ of real dimension either $3+k$ or $4+k$. We conclude that;

Theorem 1.1. *(see Theorem 5.1) The biholomorphic equivalence problem of a totally nondegenerate CR model M_k of codimension k and real dimension $2+k$ is reducible to some absolute parallelisms, namely to some certain $\{e\}$-structures on prolonged manifolds of real dimensions either $3+k$ or $4+k$.*

In section 6, then we start to utilize the achieved results to prove the Beloshapka's maximum conjecture. According to the principles of the Cartan's theory ([23]), once we receive the final constant type structure equations of the equivalence problem to each CR model M_k, then one can plainly attain the structure of its symmetry Lie algebra $\mathfrak{aut}_{CR}(M_k)$. Computing and inspecting this algebra, then we realize that it is graded, without any positive part (*see* Proposition 6.1) as was the assertion of the Beloshapka's maximum conjecture.

Finally in appendix A, we illustrate the results by considering the length 4 and 8-dimensional CR model M_6 of CR codimension $k=6$.

It may be worth to notice at the end of this section that though the main purpose of this paper is to prove the Beloshapka's maximum conjecture in CR dimension 1, but in fact we achieve a more general and stronger fact about his CR models, namely Theorem 1.1. Even more, we show that;

Corollary 1.3. *(see Proposition 6.1) The associated Lie algebra $\mathfrak{aut}_{CR}(M_k)$ of a k-codimensional weight ρ model M_k is graded of the form:*

$$\mathfrak{aut}_{CR}(M_k) := \underbrace{\mathfrak{g}_{-\rho} \oplus \ldots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0$$

where $\mathfrak{g}_-$ is $(2+k)$-dimensional and where $\mathfrak{g}_0$ is Abelian of dimension either 1 or 2. Thus, we have:

$$\dim\big(\mathfrak{aut}_{CR}(M_k)\big) = 3+k \quad or \quad 4+k.$$

As a homogeneous space, each Beloshapka's CR model M_k can be considered as a quotient space (*see* the paragraph after equation (2)):

$$M_k \equiv \frac{\mathsf{Aut}_{CR}(M_k)}{\mathsf{Aut}_0(M_k)} \cong \mathsf{G}_-$$

of the CR automorphism group $\mathsf{Aut}_{CR}(M_k)$, corresponding to $\mathfrak{aut}_{CR}(M_k)$ by its isotropy subalgebra $\mathsf{Aut}_0(M_k)$ at the origin, corresponding to $\mathfrak{g}_0 \oplus \mathfrak{g}_+$. The above corollary states, in a more precise manner, that such isotropy group is just G_0, corresponding to the Abelian algebra $\mathfrak{g}_0$ and comprises only *linear*

CR automorphisms $h : M_k \to M_k$, preserving the origin. Even more precisely, in this case that $\dim \mathfrak{g}_0$ is either 1 or 2, then G_0 can be identified with the matrix Lie group $\mathsf{GL}(1,\mathbb{R}) = (\mathbb{R}^*, \times)$ in the former case and $\mathsf{GL}(1,\mathbb{C}) = (\mathbb{C}^*, \times)$ in the latter case (*see e.g.* [16, Theorem p. 102]).

2. BELOSHAPKA'S MODELS

In this preliminary section, let us explain the method of constructing defining equations of the Beloshapka's CR models in CR dimension 1. For more general and detailed explanation, we refer the reader to [3, 15]. In each fixed CR codimension k, a certain Beloshapka's model M_k is in fact a real submanifold in the complex ambient space $\mathbb{C}^{1+k}$ equipped with the coordinates $(z, w_1, \ldots, w_k)$ and represented as the graph of some k real-valued polynomial functions. Throughout constructing these defining polynomials and to each appearing complex variables x, it will be assigned a so-called *weight* number $[\mathsf{x}]$. Recall that for a monomial $\mathsf{x}_1^{\alpha_1} \cdots \mathsf{x}_n^{\alpha_n}$, the associated weight is defined as $\sum_{i=1}^n \alpha_i [\mathsf{x}_i]$. Moreover, a polynomial is called *weighted homogeneous* of the weight w whenever all of its monomials are of this weight. In this paper and for each complex variable x, we assign the same weight $[\mathsf{x}]$ to its conjugation $\overline{\mathsf{x}}$ and its real and imaginary parts.

Definition 2.1. (*see* [14]) An arbitrary $\mathscr{C}^2$ complex function $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ in terms of the canonical coordinates $(\mathsf{z}_1, \ldots, \mathsf{z}_n)$ is called *pluriharmonic* on its domain Ω whenever we have:

$$\frac{\partial^2 f}{\partial \mathsf{z}_i \, \partial \overline{\mathsf{z}}_j} \equiv 0$$

for each $i, j = 1, \ldots, n$.

In the case that f is real-valued, then locally, pluriharmonicity of f is equivalent to state that it is the real part of a holomorphic function ([14, Propoition 2.2.3]).

By convention, here we assign to the complex variable z the weight $[z] = 1$. The weights of the next complex variables $w_1, w_2, \ldots$, which are absolutely bigger than 1, will be determined as follows, step by step. At the first onset that only the weight of the single variable z is known, let $\mathscr{N}_2$ be a basis for the set of all non-pluriharmonic real-valued polynomials of the homogeneous weight 2, in terms of the complex variables z and $\overline{z}$. A careful inspection shows that $\mathscr{N}_2$ comprises merely the single term:

$$\mathscr{N}_2 := \{z\overline{z}\}.$$

Then, since the cardinality of this set is $k_2 = 1$, then we assign immediately the weight 2 to the only first complex variable w_1, *i.e.* $[w_1] = 2$.

At the moment, two of the complex variables z and w_1 have received their weight numbers. Then, consider the next set $\mathscr{N}_3$ as a basis for the collection of all real-valued polynomials of the weight 3, in terms of the variables $z, \overline{z}$ and $\operatorname{Re} w_1$, which are non-pluriharmonic on the submanifold represented as the graph of the weight two homogeneous polynomial:

$$\operatorname{Im} w_1 = z\overline{z}$$

in $\mathbb{C}^2$. Again, a careful inspection shows that:

$$\mathscr{N}_3 := \{\operatorname{Re} z^2\overline{z} = \frac{z^2\overline{z} + z\overline{z}^2}{2}, \quad \operatorname{Im} z^2\overline{z} = \frac{z^2\overline{z} - z\overline{z}^2}{2i}\}.$$

This time, since the cardinality of $\mathscr{N}_3$ is $k_3 = 2$, then immediately we assign the weight 3 — namely the weight of the monomials in $\mathscr{N}_3$ — to the next two complex variables w_2 and w_3.

Inductively, assume that $\mathscr{N}_{j_0}$ is the last constructed collection of non-pluriharmonic polynomials. This means that so far all the complex variables $z, w_1, w_2, w_3, \ldots, w_{\mathsf{r}}$ have received their weight numbers where $\mathsf{r} := \sum_{i=2}^{j_0} k_i$ and where k_i is the cardinality of the set $\mathscr{N}_i$. To construct the next collection $\mathscr{N}_{j_0+1}$ and for the sake of clarity, let us show the k_i elements of each $\mathscr{N}_i$ as:

$$\mathscr{N}_i := \{\mathsf{t}_1^i, \mathsf{t}_2^i, \ldots, \mathsf{t}_{k_i}^i\}. \tag{3}$$

Also for each $\ell = 2, \ldots, j_0$, let $\mathbf{w}^\ell = (w_l, \ldots, w_{l+k_\ell-1})^t$ be the k_ℓ-tuple of all complex variables $w_1, \ldots, w_{\mathsf{r}}$ of the same weight ℓ and consider:

$$A_\ell = \begin{pmatrix} a^\ell_{11} & \ldots & a^\ell_{1,k_\ell} \\ \vdots & \vdots & \vdots \\ a^\ell_{k_\ell 1} & \ldots & a^\ell_{k_\ell k_\ell} \end{pmatrix}$$

as some real $k_\ell \times k_\ell$ matrix of the maximum $\mathrm{Rank}(A_\ell) = k_\ell$. Then, the sought set $\mathscr{N}_{j_0+1}$ is a basis for the collection of all real-valued polynomials of the weight $j_0 + 1$, in terms of the already weight determined variables $z, \overline{z}, \mathrm{Re}\, w_1, \mathrm{Re}\, w_2, \ldots, \mathrm{Re}\, w_{\mathsf{r}}$, which are non-pluriharmonic on the submanifold represented as the graph of some r weighted homogeneous polynomial functions:

$$\mathrm{Im}\, \mathbf{w}^\ell = A_\ell \cdot \begin{pmatrix} \mathsf{t}^\ell_1 \\ \vdots \\ \mathsf{t}^\ell_{k_\ell} \end{pmatrix}, \qquad {\scriptstyle (\ell=2,\ldots,j_0)},$$

in $\mathbb{C}^{\mathsf{r}+1}$. Here, $\mathrm{Im}\, \mathbf{w}^\ell$ is the k_ℓ-tuple of imaginary parts of $\mathbf{w}^\ell$. If the cardinality of $\mathscr{N}_{j_0+1}$ is k_{j_0+1}, then one assigns the weight $j_0 + 1$ to all the next complex variables:

$$w_{\mathsf{r}+1}, \ldots, w_{\mathsf{r}+k_{j_0+1}}.$$

2.1. **Constructing the defining equations.** After associating appropriate weights to the complex variables $z, w_\bullet$, we are ready to explain the procedure of constructing defining polynomials of a Beloshapka's model $M_k \subset \mathbb{C}^{k+1}$ of CR codimension k. One notices that in this case, we need only the associated weights to the complex coordinates $z, w_1, \ldots, w_k$ of M_k. Then, we have to construct the above sets $\mathscr{N}_i$ until we arrive at the stage $i = \rho$ where the integer ρ satisfies the two inequalities:

$$k_2 + \ldots + k_{\rho-1} \lneqq k \leqslant k_2 + \ldots + k_{\rho-1} + k_\rho. \tag{4}$$

In this case, the chain of associated weights to the complex variables $z, w_1, \ldots, w_k$ is ascending and the last variable w_k is of the maximum weight ρ.

Definition 2.2. The above unique integer ρ is called the *length* of the CR model M_k, in question.

Now, for each $\ell = 2, \ldots, \rho - 1$, consider the k_ℓ-tuple $\mathbf{w}^\ell = (w_l, \ldots, w_{l+k_\ell-1})$ and the $k_\ell \times k_\ell$ matrix A_ℓ as above. For $\ell = \rho$ and since in this case the number of the present weight ρ variables among $w_1, \ldots, w_k$ is $m = k - \sum_{i=2}^{\rho-1} k_i \leqslant k_\rho$, then consider the m-tuple $\mathbf{w}^\rho$ as $\mathbf{w}^\rho = (w_{k-m+1}, \ldots, w_k)$. Also let:

$$A_\rho = \begin{pmatrix} a^\rho_{11} & \ldots & a^\rho_{1,k_\rho} \\ \vdots & \vdots & \vdots \\ a^\rho_{m1} & \ldots & a^\rho_{mk_\rho} \end{pmatrix}$$

be a certain real $m \times k_\rho$ matrix of the maximum $\mathrm{Rank}(A_\rho) = m$. Then, the defining equations of our CR model M_k can be represented in the following matrix form (*see* (3)):

$$\mathrm{Im}\, \mathbf{w}^\ell = A_\ell \cdot \begin{pmatrix} \mathsf{t}^\ell_1 \\ \vdots \\ \mathsf{t}^\ell_{k_\ell} \end{pmatrix}, \qquad {\scriptstyle (\ell=2,\ldots,\rho)}. \tag{5}$$

As we observe, in a fixed codimension k one may find infinite number of CR models M_k determined by different values of the above matrix entries a^ℓ_{ij}. Nevertheless, possibly many of them are equivalent, up to some biholomorphic change of coordinates. For example in codimension $k = 3$, CR models $M_3 \subset \mathbb{C}^4$ can be represented as the graph of some three defining polynomials:

$$\begin{aligned}\frac{w_1-\overline{w}_1}{2i}&=a\,z\overline{z},\\ \frac{w_2-\overline{w}_2}{2i}&=a_{11}\,(z^2\overline{z}+z\overline{z}^2)+ia_{12}\,(z^2\overline{z}-z\overline{z}^2),\qquad (a,a_{ij}\in\mathbb{R}),\\ \frac{w_3-\overline{w}_3}{2i}&=a_{21}\,(z^2\overline{z}+z\overline{z}^2)+ia_{22}\,(z^2\overline{z}-z\overline{z}^2).\end{aligned}$$

But by some simple biholomorphic changes of coordinates like those presented at the page 50 of [21], one shows that they are biholomorphically equivalent to the so-called *5-cubic* model:

$$M^5_{\mathsf{c}}:\qquad \left[\begin{aligned}\tfrac{w_1-\overline{w}_1}{2i}&=z\overline{z},\\ \tfrac{w_2-\overline{w}_2}{2i}&=z^2\overline{z}+z\overline{z}^2,\\ \tfrac{w_3-\overline{w}_3}{2i}&=i\,(z^2\overline{z}-z\overline{z}^2).\end{aligned}\right.$$

Anyway, in this paper we do not stress on such biholomorphic normalizations since it does not matter whether the under consideration defining equations are normalized or not in our case of proving the Beloshapka's maximum conjecture.

Summing up the above procedure, then each arbitrary CR model $M_k\subset\mathbb{C}^{1+k}$ of codimension k and of the length ρ can be represented as the graph of some k certain real-valued defining functions:

$$\tag{6} M_k:\quad\begin{cases}w_1-\overline{w}_1=2i\,\Phi_1(z,\overline{z}),\\ \qquad\vdots\\ w_j-\overline{w}_j=2i\,\Phi_j(z,\overline{z},w,\overline{w}),\\ \qquad\vdots\\ w_k-\overline{w}_k=2i\,\Phi_k(z,\overline{z},w,\overline{w}),\end{cases}$$

where each Φ_j is a weighted homogeneous polynomial of the weight $[w_j]$, in terms of the complex variables $z,\overline{z}$ and real variables $\mathrm{Re}\,w_i=\frac{w_i+\overline{w}_i}{2}$ with $[w_i]\lneqq[w_j]$. As one observes, the defining equations of M_k are in fact those of a certain $(k-1)$-codimensional model M_{k-1}, added just by the last equation $w_k-\overline{w}_k=2i\,\Phi_k(z,\overline{z},w,\overline{w})$.

Definition 2.3. Let M be an arbitrary CR manifold of CR dimension n and codimension k and assume that $T^{1,0}M$ and $T^{0,1}M=\overline{T^{1,0}M}$ are the holomorphic and antiholomorphic subbundles of the complexified bundle $\mathbb{C}\otimes TM$. Then, M is called *totally nondegenerate* whenever $\mathbb{C}\otimes TM$ can be generated by means of the minimum possible number of iterated Lie brackets between the generators of $T^{1,0}M+T^{0,1}M$, growing up through the chain:

$$D_1\subsetneq D_2\subsetneq\ldots\subsetneq D_\rho=\mathbb{C}\otimes TM$$

with $D_1:=T^{1,0}M+T^{0,1}M$ and $D_j=D_{j-1}+[D_1,D_{j-1}]$.

All the Beloshapka's CR models M_k are totally nondegenerate ([3]) and the number ρ in the above chain is actually equal to the length of M_k, or equivalently to the (maximum) weight of the last complex variable w_k among its defining equations. Even more, as is proved in [3] (*see* also [21]) and after some holomorphic polynomial changes of coordinates, every totally nondegenerate CR model of CR dimension 1 and codimension k, can be represented as the graph of some k defining equations of the form:

$$M:=\quad\begin{cases}w_1-\overline{w}_1=2i\,\Phi_1(z,\overline{z})+\mathrm{O}(2),\\ \qquad\vdots\\ w_k-\overline{w}_k=2i\,\Phi_k(z,\overline{z},w,\overline{w})+\mathrm{O}([w_k]),\end{cases}$$

where the polynomials $\Phi_\bullet$ are precisely those considered in (6) and where $\mathrm{O}(t)$ is some certain sum of monomials of the weights $\gneqq t$. Therefore, it is completely reasonable to consider each CR manifold M_k as some suitable model for the above totally nondegenerate CR manifolds M.

Remark 2.4. Instead of the above Beloshapka's algebraic method for constructing defining equations of a totally nondegenerate CR model M_k, Joël Merker in [17] has introduced a more geometric way to construct them by considering the affect of totally nondegeneracy on the converging power series expansions of the desired defining equations.

3. Constructing the associated structure equations

Studying equivalences between geometric objects by means of the Cartan's classical approach entails first some preliminary equipments, the last of which is the construction of the structure equations associated to the problem. In the current case of biholomorphic equivalence between CR manifolds, we follow the systematic method developed among the recent years by Joël Merker, Samuel Pocchiola and the present author in [24, 26, 21, 29]. It includes four major steps to bring us to the stage of constructing the required structure equations:

- Finding an appropriate initial frame of the model and computing its Lie commutators.
- Passage to the dual coframe and computing the associated Darboux-Cartan structure.
- Finding the ambiguity matrix of the equivalence problem, in question.
- Constructing the desired structure equations.

3.1. **Associated initial frames for the complexified tangent bundles.** *From now on*, let us fix $M_k \subset \mathbb{C}^{k+1}$ as a $(2+k)$-dimensional and weight ρ Beloshapka's CR model in CR dimension 1 and codimensions k, represented as the graph of some k polynomial functions as (6) in coordinates $(z, w_1, \ldots, w_k)$. As we mentioned in the preceding section, each real-valued polynomial $\Phi_j(z, \overline{z}, \overline{w})$ is a weighted homogeneous polynomial of the weight equal to $[w_j]$. These polynomials are all $\mathrm{O}(2)$ and thus we can apply the analytic implicit function theorem in order to solve these equations for the k variables w_j , $j = 1, \ldots, k$. Performing this, we obtain a collection of k complex defining equations of the shape:

$$M_k : \quad \{ \; w_j = \Theta_j(z, \overline{z}, \overline{w}) \qquad {\scriptstyle (j=1,\ldots,k)}, \tag{7}$$

where each *complex-valued* polynomial function Θ_j is in terms of $z, \overline{z}, \overline{w}_j$ and some other conjugated variables $\overline{w}_\bullet$ of absolutely lower weights than $[w_j]$. One notices that similar to the case of real-valued functions $\Phi_\bullet$, also each complex-valued Θ_j is a weighted homogeneous polynomial of the weight $[w_j]$. To prove this assertion, we make a plain induction on the associated weights to the complex variables $w_\bullet$. More precisely, in the first weight 2, we have the simple defining equation:

$$w_1 = 2i\, z\overline{z} + \overline{w}_1 =: \Theta_1(z, \overline{z}, \overline{w}_1)$$

and hence Θ_1 is of the expected weight $[w_1] = 2$. Now, let w_j be a weight $n+1$ complex variable and, as the induction hypothesis, assume that for each variable w_i of the weight $[w_i] = 2, \ldots, n$, the associated complex-valued function Θ_i is of the weight $[w_i]$. Then the j-th corresponding defining equation:

$$w_j - \overline{w}_j = 2i\, \Phi_j(z, \overline{z}, w_\bullet, \overline{w}_\bullet),$$

of (6) — where Φ_j admits just complex variables $w_\bullet$ and $\overline{w}_\bullet$ of the weights absolutely less than $[w_j]$ — converts into the form:

$$w_j = 2i\, \Phi_j(z, \overline{z}, \Theta_\bullet(z, \overline{z}, \overline{w}), \overline{w}_\bullet) + \overline{w}_j =: \Theta_j(z, \overline{z}, \overline{w}).$$

Here, the polynomial Φ_j is of the weight homogeneity $[w_j]$ and we have replaced each complex variable w_i, in its expression, by some equal complex function Θ_i of the same weight. Such substitutions do not change the weight homogeneity of Φ_j and hence Θ_j will be remained as a weighted homogeneous complex polynomial of the same weight as Φ_j.

Having in hand the complex defining polynomials (7) of the CR model M_k and according to [20, 21], then the associated holomorphic and antiholomorphic tangent bundles $T^{1,0}M_k$ and $T^{0,1}M_k$, can be generated respectively by the single vector fields:

$$\mathscr{L} := \frac{\partial}{\partial z} + \sum_{j=1}^{k} \frac{\partial \Theta_j}{\partial z}(z, \overline{z}, \overline{w}) \frac{\partial}{\partial w_j} \quad \text{and} \quad \overline{\mathscr{L}} := \frac{\partial}{\partial \overline{z}} + \sum_{j=1}^{k} \frac{\partial \overline{\Theta}_j}{\partial \overline{z}}(z, \overline{z}, w) \frac{\partial}{\partial \overline{w}_j}. \tag{8}$$

Weight association. Let x be one of the complex variables $z, w_1, \ldots, w_k$ or one of their conjugations of the weight $[\mathsf{x}]$. Then we assign the weight $-[\mathsf{x}]$ to the standard vector filed $\frac{\partial}{\partial \mathsf{x}}$.

Notice that for $F(\mathsf{x}, \overline{\mathsf{x}})$ as a weighted homogeneous polynomial, then each differentiation of the shape F_{x_i} or $F_{\overline{\mathsf{x}}_i}$ decreases its weight by $[x_i]$ numbers. Then, by a glance on the above expressions of $\mathscr{L}$ and $\overline{\mathscr{L}}$ one finds them as two weighted homogeneous fields of the same weight -1.

According to the totally nondegeneracy of the Beloshapka's models, it is possible to construct a frame for the complexified bundle $\mathbb{C} \otimes TM_k$ of the weight ρ CR model M_k by means of the (minimum number) $2 + k$ iterated Lie brackets between $\mathscr{L}$ and $\overline{\mathscr{L}}$ of the lengths from 1 through ρ (*cf.* [3, 28]).

3.1.1. *Notations.* Henceforth, let us denote by $\mathscr{L}_{1,1}$ and $\mathscr{L}_{1,2}$, the above vector field $\mathscr{L}$ and its conjugation $\overline{\mathscr{L}}$, respectively. Then, the desired initial frame on M_k can be constructed by the iterated Lie brackets of these two vector fields, up to the length ρ. Let us denote by $\mathscr{L}_{\ell,i}$, the i-th appearing vector filed obtained by an iterated Lie bracket of the length ℓ between these two fields. For example, the next and third appearing vector filed can be computed as the (length two) iterated Lie bracket:

$$\mathscr{L}_{2,3} = [\mathscr{L}_{1,1}, \mathscr{L}_{1,2}].$$

In the case that i is not important and by abuse of notation, we denote it just by $\mathscr{L}_\ell$ an arbitrary iterated Lie brackets of the length ℓ which actually is a vector field obtained (inductively) as:

$$\mathscr{L}_\ell := [\mathscr{L}_{1,i_1}, \underbrace{[\mathscr{L}_{1,i_2}, [\ldots, [\mathscr{L}_{1,i_{\ell-1}}, \mathscr{L}_{1,i_\ell}]]]]}_{\mathscr{L}_{\ell-1}} \qquad (i_j = 1, 2, \quad \ell = 1, \ldots, \rho). \tag{9}$$

Notice that from the length $\ell = 2$ to the end, one will not observe any coefficient of $\frac{\partial}{\partial z}$ or $\frac{\partial}{\partial \overline{z}}$ in the expression of $\mathscr{L}_\ell$. In fact, this is a plain consequence of the fact that in the above expressions (8) of $\mathscr{L}_{1,1}$ and $\mathscr{L}_{1,2}$ these coefficients are constant, namely 1.

Accordingly then in each point $p \in M_k$, near the origin, the Lie algebra $\mathfrak{h} := \mathbb{C} \otimes TM_k$ is graded (in the sense of Tanaka) of the form:

$$\mathfrak{h} := \mathfrak{h}_{-\rho} \oplus \mathfrak{h}_{-\rho+1} \oplus \cdots \oplus \mathfrak{h}_{-1}$$

where $\mathfrak{h}_{-\ell}$ is the collection of all vector fields $\mathscr{L}_\ell$ of the length ℓ. For example, we have $\mathfrak{h}_{-1} = \langle \mathscr{L}_{1,1}, \mathscr{L}_{1,2} \rangle$, $\mathfrak{h}_{-2} = \langle \mathscr{L}_{2,3} \rangle$ and so on.

Lemma 3.1. *Each length ℓ iterated vector field $\mathscr{L}_\ell$ is homogeneous of the weight $-\ell$.*

Proof. We proceed by a plain induction on the length ℓ of vector fields. For its base step, we saw that the two vector fields $\mathscr{L}_{1,1}$ and $\mathscr{L}_{1,2}$ of the length $\ell = 1$ are of the homogeneous weight -1. Also by computing the Lie bracket $\mathscr{L}_{2,3} = [\mathscr{L}_{1,1}, \mathscr{L}_{1,2}]$, one easily verifies that the only length 2 field $\mathscr{L}_{2,3}$ is of the weight -2. For the next lengths and as our induction hypothesis, assume that all length ℓ vector fields:

$$\mathscr{L}_\ell := \sum_i \varphi_i \frac{\partial}{\partial w_i}$$

are weighted homogeneous of the weight $-\ell$. Hence the nonzero polynomial coefficients $\varphi_i(z, \overline{z}, w, \overline{w})$ are homogeneous of the weight $[w_i] - \ell$. Now, consider the new appearing vector field $\mathscr{L}_{\ell+1} = [\mathscr{L}_1, \mathscr{L}_\ell]$ of the weight $\ell + 1$. Then applying the Leibniz rule on the present expressions of the weighted homogeneous vector fields $\mathscr{L}_1$ in (8) and $\mathscr{L}_\ell$, this Lie bracket manifests itself as a weight $-(\ell + 1)$ homogeneous vector field, as was expected. □

3.2. **The associated initial coframes and their Darboux-Cartan structures.** For $\ell = 1, \dots, \rho$ and $i = 1, \dots, 2+k$, let us denote by $\sigma_{\ell,i}$ the dual 1-form associated to the initial vector field $\mathscr{L}_{\ell,i}$. Since the collection of the weighted homogeneous vector fields $\{\mathscr{L}_{1,1}, \dots, \mathscr{L}_{\rho,2+k}\}$ forms a frame for the complexified bundle $\mathbb{C} \otimes TM_k$, then the dual set $\{\sigma_{1,1}, \dots, \sigma_{\rho,2+k}\}$ is a coframe for it.

Lemma 3.2. *Given a frame $\big\{\mathscr{V}_1, \dots, \mathscr{V}_n\big\}$ on an open subset of $\mathbb{R}^n$ enjoying the Lie structure:*

$$\big[\mathscr{V}_{i_1}, \mathscr{V}_{i_2}\big] = \sum_{k=1}^{n} c_{i_1,i_2}^k \mathscr{V}_k \qquad (1 \leqslant i_1 < i_2 \leqslant n),$$

where the c_{i_1,i_2}^k are certain functions on $\mathbb{R}^n$, the dual coframe $\{\omega^1, \dots, \omega^n\}$ satisfying by definition:

$$\omega^k\big(\mathscr{V}_i\big) = \delta_i^k$$

enjoys a quite similar Darboux-Cartan structure, up to an overall minus sign:

$$d\omega^k = -\sum_{1 \leqslant i_1 < i_2 \leqslant n} c_{i_1,i_2}^k \, \omega^{i_1} \wedge \omega^{i_2} \qquad (k = 1 \cdots n).$$

Proceeding along the same lines as the proof of Lemma 3.1, one verifies that the Lie bracket $[\mathscr{L}_\ell, \mathscr{L}_{\ell'}]$ of the two weight $-\ell$ and $-\ell'$ initial fields is again homogeneous of the weight $-(\ell + \ell')$. Then, as a local section of $\mathbb{C} \otimes TM_k$, it should be generated as some combination of the length $-(\ell + \ell')$ initial vector fields in the frame $\{\mathscr{L}_{1,1}, \dots, \mathscr{L}_{\rho,2+k}\}$. Thanks to the above Lemma, then it follows the Darboux-Cartan structure of our initial coframe;

Proposition 3.3. *The exterior differentiation of each 1-form σ_ℓ dual to the weight $-\ell$ initial vector field $\mathscr{L}_\ell$ is of the form:*

$$d\sigma_\ell := \sum_{\beta+\gamma=\ell} \mathsf{c}_{\beta,\gamma} \, \sigma_\beta \wedge \sigma_\gamma,$$

for some constant complex integers $\mathsf{c}_{\beta,\gamma}$. This equivalently means that in the expression of each corresponding Lie bracket $[\mathscr{L}_\beta, \mathscr{L}_\gamma]$, with $\beta + \gamma = \ell$, one sees the coefficient $-\mathsf{c}_{\beta,\gamma}$ of $\mathscr{L}_\ell$.

Weight association. Naturally, we associate the weight $-\ell$ to a certain 1-form $\sigma_{\ell,i}$ and its differentiation $d\sigma_{\ell,i}$ associated to the weight $-\ell$ homogeneous vector field $\mathscr{L}_{\ell,i}$.

Another simple but quietly useful result is as follows;

Lemma 3.4. *For each weight $-\ell$ initial 1-form $\sigma_{\ell,i}$ with $\ell \neq 1$, there is a weight $-(\ell - 1)$ initial 1-form $\sigma_{\ell-1,j}$ where either $\sigma_{\ell-1,j} \wedge \sigma_{1,1}$ or $\sigma_{\ell-1,j} \wedge \sigma_{1,2}$ is visible uniquely in the Darboux-Cartan structure of $d\sigma_{\ell,i}$.*

Proof. This is a consequence of the fact that in the procedure of constructing our initial frame, each weight $-\ell$ vector field $\mathscr{L}_{\ell,i}$ is constructed as the Lie bracket between $\mathscr{L}_{1,1}$ or $\mathscr{L}_{1,2}$ and a *unique* weight $-(\ell - 1)$ vector filed $\mathscr{L}_{\ell-1,j}$. Then, Lemma 3.2 implies the desired results. □

3.3. **Associated ambiguity matrix.** After providing the above appropriate initial frame and coframe on the complexified Tangent bundle $\mathbb{C} \otimes TM_k$, now this is the time of seeking the associated ambiguity matrix of the problem which actually encodes biholomorphic equivalences to M_k. The procedure of constructing this matrix is demonstrated in our recent works [29, 24, 21, 26] in the specific cases of $k = 1, 2, 3, 4$. Let us explain it here in the general case of the CR models M_k. Assume that:

$$\begin{aligned} h\colon \quad M_k &\longrightarrow M_k' \\ (z, w) &\longmapsto \big(z'(z, w),\, w'(z, w)\big) \end{aligned}$$

is a (biholomorphic) equivalence map between our $(2+k)$-dimensional CR model M_k and another arbitrary $(2+k)$-dimensional totally nondegenerate and CR generic submanifold $M'_k \subset \mathbb{C}^{1+k}$ of codimension k and in canonical coordinates $\big(z', w_1', \ldots, w_k'\big)$. Naturally, we assume that M_k' is also equipped with a frame of $2+k$ *lifted* vector fields:

$$\Big\{\mathbf{L}_{1,1},\, \mathbf{L}_{1,2},\, \mathbf{L}_{2,3},\, \mathbf{L}_{3,4},\, \mathbf{L}_{3,5}, \ldots, \mathbf{L}_{\rho,2+k}\Big\}$$

where, as before, $\mathbf{L}_{1,1}$ and $\mathbf{L}_{1,2} = \overline{\mathbf{L}_{1,1}}$ are local generators of $T^{1,0}M_k'$ and $T^{0,1}M_k'$, respectively and where each other vector field $\mathbf{L}_{\ell,i}$ can be computed as an iterated Lie bracket between $\mathbf{L}_{1,1}$ and $\mathbf{L}_{1,2}$ of the length ℓ, exactly as (9) for constructing the initial vector filed $\mathscr{L}_{\ell,i}$. The differentiation of the biholomorphism h:

$$h_*: \quad TM_k \longrightarrow TM_k'$$

induces a push-forward complexified map, still denoted by the same symbol with the customary abuse of notation ([7]):

$$\begin{aligned} h_*: \quad \mathbb{C}\otimes TM_k &\longrightarrow \mathbb{C}\otimes TM_k', \\ \mathsf{z}\otimes \mathscr{X} &\longmapsto \mathsf{z}\otimes h_*(\mathscr{X}). \end{aligned}$$

Our current purpose is to seek the associated matrix to this linear map between the complexified vector spaces $\mathbb{C}\otimes TM_k$ and $\mathbb{C}\otimes TM_k'$.

According to principles in CR geometry ([7, 19]), the differentiation h_* transfers every generator of $T^{1,0}M_k$ to a vector field in the same bundle $T^{1,0}M_k'$. Hence for the single generator $\mathscr{L}_{1,1}$ of $T^{1,0}M_k$, there exists some *nonzero* function $a_1 := a_1(z', w')$ with:

$$h_*(\mathscr{L}_{1,1}) = a_1\,\mathbf{L}_{1,1}. \tag{10}$$

Moreover, h_* preserves the conjugation, whence for $\mathscr{L}_{1,2} := \overline{\mathscr{L}_{1,1}}$, we have:

$$h_*(\mathscr{L}_{1,2}) = \overline{a}_1\,\mathbf{L}_{1,2}.$$

The third vector field in the basis of $\mathbb{C}\otimes TM_k$ is the *imaginary field* $\mathscr{L}_{2,1} := [\mathscr{L}_{1,1}, \mathscr{L}_{1,2}]$ which is our only length two vector field. Here we have:

$$h_*(\mathscr{L}_{2,3}) = h_*\big([\mathscr{L}_{1,1}, \mathscr{L}_{1,2}]\big) = \big[h_*(\mathscr{L}_{1,1}), h_*(\mathscr{L}_{1,2})\big] = \big[a_1\,\mathbf{L}_{1,1}, \overline{a}_1\,\mathbf{L}_{1,2}\big],$$

and if we expand it, then we obtain:

$$\begin{aligned} h_*(\mathscr{L}_{2,3}) &= a_1\overline{a}_1\big[\mathbf{L}_{1,1}\,\mathbf{L}_{1,2}\big] \underbrace{-\overline{a}_1\,\mathbf{L}_{1,2}(a_1)}_{=:\,a_2}\mathbf{L}_{1,1} + a_1\,\mathbf{L}_{1,1}\big(\overline{a}_1\big)\mathbf{L}_{1,2} \\ &=: a_1\overline{a}_1\mathbf{L}_{2,3} + a_2\,\mathbf{L}_{1,1} - \overline{a}_2\,\mathbf{L}_{1,2}, \end{aligned} \tag{11}$$

for a certain function $a_2 := a_2(z', w')$.

Next in the length three, we have two iterated Lie brackets:

$$\mathscr{L}_{3,4} := [\mathscr{L}_{1,1}, \mathscr{L}_{2,3}] \quad \text{and} \quad \mathscr{L}_{3,5} := [\mathscr{L}_{1,2}, \mathscr{L}_{2,3}],$$

where $\mathscr{L}_{3,5} = \overline{\mathscr{L}_{3,4}}$. In a similar fashion of computations, one finds:

$$\begin{aligned} h_*(\mathscr{L}_{3,4}) := a_1^2\overline{a}_1\,\mathbf{L}_{3,4} &+ \underbrace{\big(a_1\,\mathbf{L}_{1,1}\big(a_1\overline{a}_1\big) - a_1\overline{a}_2\big)}_{=:a_3}\mathbf{L}_{2,3} + \\ &+ \underbrace{\big(-a_1\overline{a}_1\,\mathbf{L}_{2,3}(a_1) + a_1\,\mathbf{L}_{1,1}(a_2) - a_2\,\mathbf{L}_{1,1}(a_1) + \overline{a}_2\,\mathbf{L}_{1,2}(a_1)\big)}_{=:a_4}\mathbf{L}_{1,1} - \\ &\underbrace{-a_1\mathbf{L}_{1,1}\big(\overline{a}_2\big)}_{=:a_5}\mathbf{L}_{1,2}, \end{aligned} \tag{12}$$

for some three certain complex functions $a_j := a_j(z', w'),\ j = 3, 4, 5$. By conjugation, we also have:

$$h_*(\mathscr{L}_{3,5}) = a_1\overline{a}_1^2\,\mathbf{L}_{3,5} - \overline{a}_3\,\mathbf{L}_{2,3} + \overline{a}_5\,\mathbf{L}_{1,1} + \overline{a}_4\,\mathbf{L}_{1,2}.$$

Proceeding along the same lines of computations, then one finds the value of the linear differentiation h_* on all the basis fields $\mathscr{L}_\ell$ of $\mathbb{C}\otimes TM_k$. By a careful inspection of this procedure one observes that;

Lemma 3.5. *For a fixed length ℓ initial vector field $\mathscr{L}_{\ell,i}$, the differentiation map h_* transfers it to a combination like:*

$$h_*(\mathscr{L}_{\ell,i}) := a_1^p\overline{a}_1^q\,\mathbf{L}_{\ell,i} + \sum_{l<\ell} \mathsf{a}_{r_j}\,\mathbf{L}_{l,r}, \qquad \text{with} \quad p+q=\ell$$

where a_{r_j}s are some (possibly zero) complex functions in terms of the target coordinates (z', w'). In other words, $h_*(\mathscr{L}_{\ell,i_0})$ is a combination of some coefficient $a_1^p\overline{a}_1^q$ of the corresponding lifted vector field $\mathbf{L}_{\ell,i_0}$ and some other ones of absolutely smaller lengths $l<\ell$.

Accordingly, our sought invertible matrix associated to the linear map h_* is a $(2+k)\times(2+k)$ upper triangular matrix of the form:

$$\begin{pmatrix} \mathscr{L}_{\rho,i} \\ \mathscr{L}_{\rho-1,j} \\ \vdots \\ \mathscr{L}_{3,5} \\ \mathscr{L}_{3,4} \\ \mathscr{L}_{2,3} \\ \mathscr{L}_{1,2} \\ \mathscr{L}_{1,1} \end{pmatrix} = \begin{pmatrix} a_1^p\overline{a}_1^q & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet \\ 0 & a_1^{p'}\overline{a}_1^{q'} & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet \\ 0 & 0 & \ddots & \mathsf{a}_\bullet & \ldots & \ldots & \ldots & \mathsf{a}_\bullet \\ 0 & \ldots & 0 & a_1\overline{a}_1^2 & 0 & -\overline{a}_3 & \overline{a}_4 & \overline{a}_5 \\ 0 & \ldots & 0 & 0 & a_1^2\overline{a}_1 & a_3 & a_5 & a_4 \\ 0 & 0 & \ldots & \ldots & 0 & a_1\overline{a}_1 & -\overline{a}_2 & a_2 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & \overline{a}_1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & a_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{L}_{\rho,i} \\ \mathbf{L}_{\rho-1,j} \\ \vdots \\ \mathbf{L}_{3,5} \\ \mathbf{L}_{3,4} \\ \mathbf{L}_{2,3} \\ \mathbf{L}_{1,2} \\ \mathbf{L}_{1,1} \end{pmatrix}. \tag{13}$$

If on the main diagonal of the matrix and in front of $\mathscr{L}_{\ell,r}$ we have $a^{p_r}\overline{a}^{q_r}$, then we have $p_r+q_r=\ell$ where p_r and q_r are respectively the numbers of the appearing $\mathscr{L}_{1,1}$ and $\mathscr{L}_{1,2}$ in the iterated Lie bracket of constructing $\mathscr{L}_{\ell,r}$ (*cf.* (9)).

As a result of *explicitness* in the already procedure of constructing the above desired matrix, we have also the following key observation;

Lemma 3.6. *In the case that both the group parameters a_2 and a_3, appeared in* (11) *and* (12)*, vanish then all the next parameters $a_4, a_5, \ldots$ vanish identically. Moreover as a nonzero complex function, if the first group parameter a_1 is constant then all the next group parameters $a_2, a_3, a_4, \ldots$ vanish identically, too.*

Proof. To prove the assertion, first we claim that in the above procedure of computing the value of h_* on the initial vector fields, all the appeared group parameters a_j with $j>1$ are some combinations of the iterated $\{\mathbf{L}_{1,1}, \mathbf{L}_{1,2}\}$-differentiations of the first parameter a_1 and its conjugation $\overline{a}_1$. One can check the correctness of this claim by a careful glance on the explicit expressions of a_2, a_3, a_4, a_5 in (11) and (12), reminding that by our assumption on the lifted vector fields, every $\mathbf{L}_\ell$ with $\ell>1$ is in fact an iterated Lie bracket between two fundamental fields $\mathbf{L}_{1,1}$ and $\mathbf{L}_{1,2}$ (*cf.* (9)). We prove our claim by an induction on the length of the initial fields and by perusing the image of $\mathscr{L}_{\ell+1} = [\mathscr{L}_{1,1}, \mathscr{L}_{\ell,i}]$ under the linear map h_* (similar argument holds in the case that $\mathscr{L}_{\ell+1} = [\mathscr{L}_{1,2}, \mathscr{L}_\ell]$). As our induction hypothesis, assume that all the group parameters $a_\bullet$ appearing among computing the value of h_* on each initial field $\mathscr{L}_l$ of the length less or equal to ℓ is an iterated $\{\mathbf{L}_{1,1}, \mathbf{L}_{1,2}\}$-differentiation of a_1. Then for the next length $\ell+1$ vector field $\mathscr{L}_{\ell+1}$ and according to the above Lemma 3.5 we have:

$$h_*(\mathscr{L}_{\ell+1}) = \big[h_*(\mathscr{L}_{1,1}), h_*(\mathscr{L}_{\ell,i})\big] = \big[a_1\mathbf{L}_{1,1}, a_1^p\overline{a}_1^q\,\mathbf{L}_{\ell,i} + \sum_{l<\ell} \mathsf{a}_{r_j}\,\mathbf{L}_{l,r}\big],$$

where, by hypothesis induction, the appearing coefficients a_{r_j} are some combinations of the iterated $\{\mathbf{L}_{1,1}, \mathbf{L}_{1,2}\}$-differentiations of a_1. Then, computing this Lie bracket by means of the Leibniz rule, one finds the new appearing coefficients, namely new group parameters, again as some combinations of the iterated $\{\mathbf{L}_{1,1}, \mathbf{L}_{1,2}\}$-differentiations of a_1. This completes the proof of our claim.

Thus, if a_1 is constant then, clearly all the next parameters a_j vanish, identically. Moreover, according to (11) and (12) we have:

$$a_2 = -\overline{a}_1\,\mathbf{L}_{1,2}(a_1),$$
$$a_3 = a_1\,\mathbf{L}_{1,1}\big(a_1\overline{a}_1\big) - a_1\overline{a}_2,$$

and by the assumption $a_1 \neq 0$, then vanishing of a_2 and a_3 implies that — reminding $\mathbf{L}_{1,2} = \overline{\mathbf{L}_{1,1}}$:

$$\mathbf{L}_{1,1}(\overline{a}_1) \equiv 0, \quad \mathbf{L}_{1,1}(a_1) \equiv 0, \quad \mathbf{L}_{1,2}(a_1) \equiv 0, \quad \mathbf{L}_{1,2}(\overline{a}_1) \equiv 0.$$

Thus according to our claim, if a_2 and a_3 vanish then all the next group parameters a_j vanish, identically. $\square$

Weight association. Let a_j be a group parameter which has been appeared among computing the value of h_* on a length ℓ initial vector field $\mathscr{L}_\ell$. Then, from now on, we associate the weight ℓ to this group parameter and its conjugation $\overline{a}_j$. For example, according to (10), (11) and (12) we have:

$$[a_1] = 1, \quad [a_2] = 2, \quad [a_3] = [a_4] = [a_5] = 3, \quad \cdots .$$

According to this association, the nonzero group parameters at each row of the above matrix (13) have equal weight since they are actually coefficient functions appearing among computing the value of h_* on an initial vector field of the similar length.

For each lifted vector field $\mathbf{L}_{\ell,i}$, let us consider $\Gamma_{\ell,i}$ as its associated dual 1-form. Similar to initial 1-forms $\sigma_{\ell,i}$, here we also associate the weight $-\ell$ to each *lifted* 1*-form* $\Gamma_{\ell,i}$. The ambiguity matrix of our equivalence problem, in question, is in fact the invertible matrix associated to the dual linear map of h_*. Then, in terms of the dual basis of 1-forms it becomes, after a plain matrix transposition, as:

$$\tag{14} \begin{pmatrix} \Gamma_{\rho,i} \\ \vdots \\ \Gamma_{\rho-1,j} \\ \vdots \\ \Gamma_{3,5} \\ \Gamma_{3,4} \\ \Gamma_{2,3} \\ \Gamma_{1,2} \\ \Gamma_{1,1} \end{pmatrix} = \underbrace{\begin{pmatrix} a_1^p\overline{a}_1^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathsf{a}_\bullet & a_1^{p'}\overline{a}_1^{q'} & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \ddots & 0 & 0 & \ldots & \ldots & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \ldots & a_1\overline{a}_1^2 & 0 & 0 & \ldots & 0 \\ \mathsf{a}_\bullet & \vdots & \ldots & 0 & a_1^2\overline{a}_1 & 0 & \ldots & 0 \\ \mathsf{a}_\bullet & \vdots & \ldots & -\overline{a}_3 & a_3 & a_1\overline{a}_1 & 0 & 0 \\ \mathsf{a}_\bullet & \vdots & \ldots & \overline{a}_4 & a_5 & -\overline{a}_2 & \overline{a}_1 & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \ldots & \overline{a}_5 & a_4 & a_2 & 0 & a_1 \end{pmatrix}}_{\mathbf{g}} \cdot \begin{pmatrix} \sigma_{\rho,i} \\ \vdots \\ \sigma_{\rho-1,j} \\ \vdots \\ \sigma_{3,5} \\ \sigma_{3,4} \\ \sigma_{2,3} \\ \sigma_{1,2} \\ \sigma_{1,1} \end{pmatrix}.$$

Remark 3.7. In order to know better this matrix $\mathbf{g}$, it is important to notice that thanks to Lemma 3.5 and for each arbitrary i-th *column* of this matrix, the first nonzero entry, which is at the diagonal of the matrix, is of a form like $a_1^r\overline{a}_1^s$. Also if the i-th *row* of the left (or right) hand side vertical matrix in (14) is of the weight $-\ell$, then all the entries below this $a_1^r\overline{a}_1^s$ in $\mathbf{g}$ and in front of a weight $-\ell$ 1-form Γ_ℓ are zero. This fact is shown for example by the zero vector $\mathbf{0}$ in the first column of $\mathbf{g}$ or by the entry 0 below $a_1\overline{a}_1^2$ at the middle of this matrix.

The collection of all invertible matrices of the form $\mathbf{g}$ constitutes a finite dimensional (matrix) Lie group G, called by the *structure Lie group* of the equivalence problem to the CR model M_k.

Lemma 3.8. *All the group parameters appearing at the i-th column of $\mathbf{g}$ are of the same weight equal to the length of the* 1*-forms at the i-th row of the left (or right) hand side vertical matrices of* (14)*.*

Proof. It is a straightforward consequence of the two paragraphs mentioned before the equality (14). $\square$

Recall that (*see* (6) and the paragraph after it) the defining equations of our k-codimensional CR model $M_k \subset \mathbb{C}^{1+k}$ are precisely those of a CR model M_{k-1} of codimension $k-1$ added just by the last equation $w_k - \overline{w}_k = 2i\Phi_k(z, \overline{z}, w, \overline{w})$. Then one finds out that;

Proposition 3.9. *the $(k-1)\times(k-1)$ ambiguity matrix $\mathbf{g}_{k-1}$ associated to the CR model M_{k-1} is a submatrix of the ambiguity matrix $\mathbf{g}$ associated to M_k. More precisely, we have (cf.* (14)*):*

$$\mathbf{g} = \begin{pmatrix} a_1^p \overline{a}_1^q & 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\ \mathbf{0} & \\ \mathsf{a}_\bullet & \\ \vdots & \\ \mathsf{a}_\bullet & \mathbf{g}_{k-1} \\ \mathsf{a}_\bullet & \\ \mathsf{a}_\bullet & \\ \mathsf{a}_\bullet & \end{pmatrix}. \tag{15}$$

Proof. Let M_{k-1} be of the length $\rho' \leq \rho$. If we proceed *ab initio* along the procedure of constructing an initial frame for the CR model M_{k-1} (*cf.* subsection 3.1), then we realize that though here the expressions of initial vector fields of the appearing frame $\{\mathscr{L}_{1,1}^{\mathsf{old}}, \mathscr{L}_{1,2}^{\mathsf{old}}, \ldots, \mathscr{L}_{\rho',1+k}^{\mathsf{old}}\}$ are slightly different (at least they do not contain any coefficient of $\frac{\partial}{\partial w_k}$) but according to the totally nondegeneracy of M_{k-1} one can construct them by means of the iterated Lie brackets of $\mathscr{L}_{1,1}^{\mathsf{old}}$ and $\mathscr{L}_{1,2}^{\mathsf{old}}$ of $T^{1,0}M_{k-1}$ and $T^{0,1}M_{k-1}$; exactly as those for the initial vector fields on M_k (*cf.* (9)) — here we assign the symbol "old" to objects corresponding to M_{k-1}. More precisely, if we have $\mathscr{L}_{\ell,j}^{\mathsf{old}} = [\mathscr{L}_1^{\mathsf{old}}, \mathscr{L}_{\ell-1,i}^{\mathsf{old}}]$ then correspondingly we should have $\mathscr{L}_{\ell,j} = [\mathscr{L}_1, \mathscr{L}_{\ell-1,i}]$. Now, for a general biholomorphism $h^{\mathsf{old}} : M_{k-1} \to M'_{k-1}$ and proceeding along the same lines of computing the value of $h_*^{\mathsf{old}} : \mathbb{C}\otimes TM_{k-1} \to \mathbb{C}\otimes TM'_{k-1}$ as subsection 3.3 then, one finds that if we have (*cf.* Lemma 3.5):

$$h_*(\mathscr{L}_{\ell,i}) := a_1^p \overline{a}_1^q \, \mathbf{L}_{\ell,i} + \sum_{l<\ell} \mathsf{a}_{r_j} \, \mathbf{L}_{l,r}, \qquad {\scriptstyle (i=1,\ldots,1+k)}$$

then correspondingly we also should have:

$$h_*^{\mathsf{old}}(\mathscr{L}_{\ell,i}^{\mathsf{old}}) := a_1^p \overline{a}_1^q \, \mathbf{L}_{\ell,i}^{\mathsf{old}} + \sum_{l<\ell} \mathsf{a}_{r_j} \, \mathbf{L}_{l,r}^{\mathsf{old}}, \qquad {\scriptstyle (i=1,\ldots,1+k)},$$

though in the former case the appearing group parameter-functions are in terms of the complex variables $z, w_1, \ldots, w_{k-1}, w_k$ while in the latter case they do not admit the last one w_k. The only distinction here is that the frame of M_k has one more initial vector field, namely $\mathscr{L}_{\rho,2+k}$ for which its value under h_* should be computed, separately. This value $h_*(\mathscr{L}_{\rho,2+k})$ manifests itself as the first column of $\mathbf{g}$. □

Remark 3.10. By an inspection of the above proof, one finds that among the construction of the ambiguity matrix associated to M_{k-1}, the assigned weights to all the appearing initial vector fields, 1-forms and group parameters will be exactly as their corresponding ones in the case of M_k.

3.4. **Associated structure equations.** After providing the preliminary equipments of the (biholomorphic) equivalence problem to our CR model M_k, including its associated initial frame and coframe, Darboux-Cartan structure and the desired ambiguity matrix, now it is the time of launching the Cartan's method of solving this problem. For this aim, first we have to compute the so-called associated structure equations.

The procedure of constructing desired structure equations begins by the exterior differentiation from the both sides of the equation (14). Assuming $\Gamma := (\Gamma_{\rho,2+k}, \ldots, \Gamma_{1,1})^t$ and $\Sigma := (\sigma_{\rho,2+k}, \ldots, \sigma_{1,1})^t$ as our lifted and initial coframes, then by differentiating the equality $\Gamma = \mathbf{g}\cdot\Sigma$ we have:

$$d\Gamma = d\mathbf{g}\wedge\Sigma + \mathbf{g}\cdot d\Sigma. \tag{16}$$

For the first part $d\mathbf{g}\wedge\Sigma$ of this equation, one can replace it by:

$$\underbrace{d\mathbf{g}\cdot\mathbf{g}^{-1}}_{\omega_{\mathtt{MC}}} \wedge \underbrace{\mathbf{g}\cdot\Sigma}_{\Gamma},$$

where $\omega_{\mathtt{MC}}$ is the well-known Maurer-Cartan form of the matrix Lie group G. Since $\mathbf{g}$ is a lower triangular matrix with the powers of the form $a_1^r \overline{a}_1^s$ on its main diagonal (*cf.* (14)), then the associated Maurer-Cartan matrix is again a lower triangular matrix of the form:

$$\omega_{\mathtt{MC}} := \begin{pmatrix} p\alpha + q\overline{\alpha} & 0 & 0 & 0 & 0 \\ \delta_\bullet & p'\alpha + q'\overline{\alpha} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots \\ \delta_\bullet & \delta_\bullet & \delta_\bullet & \overline{\alpha} & 0 \\ \delta_\bullet & \delta_\bullet & \delta_\bullet & \delta_\bullet & \alpha \end{pmatrix}, \tag{17}$$

with:

$$\alpha := \frac{d\,a_1}{a_1}$$

and with $\delta_\bullet$s as some (possibly zero) certain combinations of the standard forms $da_j, j = 1, 2, \ldots$ with the coefficient functions in terms of $a_1, a_2, \ldots$. Thus, the equation (16) will take the expanded form:

$$\begin{aligned} \begin{pmatrix} d\Gamma_{\rho,i} \\ d\Gamma_{\rho-1,j} \\ \vdots \\ d\Gamma_{3,5} \\ d\Gamma_{3,4} \\ d\Gamma_{2,3} \\ d\Gamma_{1,2} \\ d\Gamma_{1,1} \end{pmatrix} &= \underbrace{\begin{pmatrix} p\alpha + q\overline{\alpha} & 0 & 0 & 0 & 0 \\ \delta_\bullet & p'\alpha + q'\overline{\alpha} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots \\ \delta_\bullet & \delta_\bullet & \delta_\bullet & \overline{\alpha} & 0 \\ \delta_\bullet & \delta_\bullet & \delta_\bullet & \delta_\bullet & \alpha \end{pmatrix}}_{\omega_{\mathtt{MC}}^{\mathsf{new}}} \wedge \begin{pmatrix} \Gamma_{\rho,i} \\ \Gamma_{\rho-1,j} \\ \vdots \\ \Gamma_{3,5} \\ \Gamma_{3,4} \\ \Gamma_{2,3} \\ \Gamma_{1,2} \\ \Gamma_{1,1} \end{pmatrix} \\ &+ \underbrace{\begin{pmatrix} a_1\overline{a}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathsf{a}_\bullet & a_1^{p'}\overline{a}_1^{q'} & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \ddots & 0 & 0 & \ldots & \ldots & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & a_1\overline{a}_1^2 & 0 & 0 & \ldots & 0 \\ \mathsf{a}_\bullet & \vdots & \mathsf{a}_\bullet & 0 & a_1^2\overline{a}_1 & 0 & \ldots & 0 \\ \mathsf{a}_\bullet & \vdots & \mathsf{a}_\bullet & -\overline{a}_3 & a_3 & a_1\overline{a}_1 & 0 & 0 \\ \mathsf{a}_\bullet & \vdots & \mathsf{a}_\bullet & \overline{a}_4 & a_5 & -\overline{a}_2 & \overline{a}_1 & 0 \\ \mathsf{a}_\bullet & \mathsf{a}_\bullet & \mathsf{a}_\bullet & \overline{a}_5 & a_4 & a_2 & 0 & a_1 \end{pmatrix}}_{\mathbf{g}} \cdot \begin{pmatrix} d\sigma_{\rho,i} \\ d\sigma_{\rho-1,j} \\ \vdots \\ d\sigma_{3,5} \\ d\sigma_{3,4} \\ d\sigma_{2,3} \\ d\sigma_{1,2} \\ d\sigma_{1,1} \end{pmatrix}. \end{aligned} \tag{18}$$

These equations are called the *structure equations* of the biholomorphic equivalence problem to M_k. The following lemma is encouraging enough to have some rigorous weight analysis on the structure equations in the next section. Recall that for each term $a_j d\sigma_{\ell,i}$, coming from the last matrix multiplication of (18), the associated weight is naturally defined as $[a_j] + [d\sigma_{\ell,i}]$.

Lemma 3.11. *All entries of the vertical matrix* $\mathbf{g} \cdot d\Sigma$ *at the right hand side of the above structure equations* (18) *are homogeneous of the equal weight zero.*

Proof. It is a straightforward consequence of Lemma 3.8, reminding that if σ_ℓ is a length ℓ initial 1-form then the assigned weight to it and its differentiation $d\sigma_\ell$ is $-\ell$. □

As one observes, the first matrix term of the structure equations (18) is only in terms of the wedge products between Maurer-Cartan and lifted 1-forms while, still, the second term $\mathbf{g} \cdot d\Sigma$ is expressed in terms of the initial 2-forms $d\sigma_\bullet$. But, using the Darboux-Cartan structure computed in Proposition 3.3, one can replace each 2-form $d\sigma_\bullet$ by some combination of the wedge products between initial 1-forms $\sigma_\bullet$. Afterward, by means of the equality:

$$\Sigma = \mathbf{g}^{-1} \cdot \Gamma,$$

it is also possible to replace each initial 1-form $\sigma_\bullet$ by some combination of the lifted 1-forms $\Gamma_\bullet$. Doing so, then all differentiations at the right hand side vertical matrix $\mathbf{g} \cdot d\Sigma$ of (18) will be expressed simultaneously in terms of the wedge products of the lifted 1-forms $\Gamma_\bullet$. Consequently, our structure equations will be independent of the initial 1-forms and can be rewritten in the form:

$$
\begin{aligned}
d\,\Gamma_{\ell,i} &:= (p_i\,\alpha + q_i\,\overline{\alpha}) \wedge \Gamma_{\ell,i} + \sum_{r,j,\; l\gneqq\ell} \delta_r \wedge \Gamma_{l,j} \\
&\quad + \sum_{l,j,m,n} T^i_{jn}(a_\bullet)\,\Gamma_{l,j} \wedge \Gamma_{m,n}, \qquad\qquad {\scriptstyle (\ell=1,\ldots,\rho,\;\; i=1,\ldots,2+k)},
\end{aligned}
\tag{19}
$$

where T^i_{jn}s are some certain functions in terms of the group parameters $a_\bullet$ which are called by the *torsion coefficients* of the problem.

Remark 3.12. Since our ambiguity matrix $\mathbf{g}$ is invertible and lower triangular with the powers $a_1^p\overline{a}_1^q$ at its diagonal, then a simple induction on the number of its column and rows shows that its inversion $\mathbf{g}^{-1}$ is again lower triangular and its non-diagonal entries are some fraction polynomial functions where their denominators are only some powers of the form $a_1^r\overline{a}_1^s$. Also, if the i-th diagonal entry of $\mathbf{g}$ is, say, $a_1^p\overline{a}_1^q$ then this entry in $\mathbf{g}^{-1}$ is $\frac{1}{a_1^p\overline{a}_1^q}$. Finally, thanks to Lemma 3.5 and again since $\mathbf{g}$ is a lower triangular matrix, then in the expression of each length ℓ lifted 1-from $\Gamma_{\ell,i}$ in the equation (14), the only appearing initial 1-form of the lengths $\leqslant \ell$ is $\sigma_{\ell,i}$. Consequently, we encounter the same fact in expressing each initial 1-form $\sigma_{\ell,i}$ in terms of the lifted ones by means of the equality $\Sigma = \mathbf{g}^{-1}\cdot\Gamma$. This means that if the i-th row σ_{ℓ,i_0} of the vertical matrix Σ is of the length ℓ then the i-th row of $\mathbf{g}^{-1}$ is of the form:

$$
\big(c_\bullet,\ldots,c_\bullet,\underbrace{0,\ldots,0}_{t_1\text{times}},\underbrace{\frac{1}{a_1^r\overline{a}_1^s}}_{i-th\;\;\text{place}},\underbrace{0,\ldots,0}_{t_2\;\;\text{times}}\big)
$$

where $r+s=\ell$ and t_1+t_2 is at least equal to the number of all initial 1-forms $\sigma_\bullet$ of the lengths $\leqslant \ell$, except σ_{ℓ,i_0}. One notices that here of course we have $t_2 = 2+k-i$.

4. Weight analysis on the structure equations

In the previous section, we assigned naturally some weights to the complex variables, initial and lifted vector fields and 1-forms, their differentiations and also to group parameters. in this section, our main aim is to show that all the appearing torsion coefficients in the constructed structure equations (19) are homogeneous of the same weight zero. These torsion coefficients are some polynomial fractions in terms of the group parameters with only some powers of a_1 and $\overline{a}_1$ in their denominators (*cf.* Remark 3.12). We also will inspect more the inverse matrix $\mathbf{g}^{-1}$.

Definition 4.1. Let:

$$
f(a_1,a_2,\cdots) = \frac{a_1^{r_1}\overline{a}_1^{s_1}a_2^{s_2}\overline{a}_2^{s_2}\cdots a_n^{r_n}\overline{a}_n^{s_n}}{a_1^r\overline{a}_1^s}
$$

be an arbitrary monomial fraction in terms of the group parameters. Then the weight of f is defined as:

$$
[f] = r_1[a_1] + s_1[\overline{a}_1] + r_2[a_2] + s_2[\overline{a}_2] + \cdots + r_n[a_n] + s_n[\overline{a}_n] - r[a_1] - s[\overline{a}_1].
$$

A weighted homogeneous polynomial fraction is a sum of some monomial fractions as above of the same weigh.

As stated in Lemma 3.8, all the nonzero entries in a fixed column of our ambiguity matrix $\mathbf{g}$ are of the same weight. Our next goal is to show that in the inverse matrix $\mathbf{g}^{-1}$, the rows enjoy a similar fact.

Lemma 4.2. *Fix an integer $i_0 = 1,\ldots,2+k$ and let $-\ell$ be the weight of the 1-form σ_ℓ at the i_0-th row of the vertical matrix:*

$$
\Sigma = \big(\sigma_{\rho,2+k},\ldots,\sigma_{1,2},\sigma_{1,1}\big)^t,
$$

appearing in (14). *Then:*

(i) *all the nonzero entries of the i_0-th row of $\mathbf{g}^{-1}$ are of the same homogeneous weight $-\ell$, too.*

(ii) *if the $(i_0 j)$-th entry $e_{i_0 j}$, at the i_0-th row of $\mathbf{g}$ is of the weight $\ell+1$, then this entry in $\mathbf{g}^{-1}$ is of the form:*

$$-\frac{e_{i_0 j}}{a_1^m \overline{a}_1^n}$$

for some constant integers m and n.

Proof. We prove the assertion by an induction on the CR codimension k of the CR model, under study. The base of this induction is provided in [21, p. 104] for $k=3$. Thanks to Proposition 3.9, one shows that if $\mathbf{g}_{k-1}^{-1}$ is the inverse of the ambiguity matrix $\mathbf{g}_{k-1}$ associated to the equivalence problem to the CR model M_{k-1} of codimension $k-1$, then we have:

$$\mathbf{g}^{-1} = \begin{pmatrix} \frac{1}{a_1^p \overline{a}_1^q} & 0 \; 0 \; \cdots \; 0 \; 0 \; 0 \; 0 \\ \mathbf{0} & \\ \mathsf{b}_j & \\ \vdots & \begin{pmatrix} & & \\ & \mathbf{g}_{k-1}^{-1} & \\ & & \end{pmatrix} \\ \mathsf{b}_4 & \\ \mathsf{b}_3 & \\ \mathsf{b}_2 & \\ \mathsf{b}_1 & \end{pmatrix}, \tag{20}$$

with $p+q=\rho$ and for some certain functions $\mathsf{b}_\bullet$. Thus according to Remark 3.10 and to prove the first part of the assertion, it suffices to show it just for the new entry b_t at the i_0-th row of $\mathbf{g}^{-1}$. According to Lemma 3.8, all the nonzero group parameters at the first column of $\mathbf{g}$ are of the same maximum weight ρ. By our induction hypothesis and except b_t, we know that all the nonzero entries at the i_0-th row of $\mathbf{g}^{-1}$ are of the same weight $-\ell$. Our aim is to show that b_t is also of this weight. Multiplying the i_0-th row of $\mathbf{g}^{-1}$ by the first column of $\mathbf{g}$ gives:

$$\mathsf{b}_t \cdot (a_1^p \overline{a}_1^q) + \Psi = 0$$

where Ψ is some function of the weight $\rho - \ell$. Taking into account that $p+q=\rho$, then the polynomial fraction $\mathsf{b}_t = -\frac{\Psi}{a_1^p \overline{a}_1^q}$ is of the homogeneous weight $\rho - \ell - \rho = -\ell$, as we expect.

For the second part of the assertion and according to our induction, it suffices to prove it only for some of the entries $e_{i_0 1}$ at the first column of $\mathbf{g}$. These entries are all of the same maximum weight ρ and hence we have to look for weight $-(\rho-1)$ rows of the inverse matrix $\mathbf{g}^{-1}$. These rows start immediately after the zero vector $\mathbf{0}$ at the first column and are in front of the weight $-(\rho-1)$ initial 1-forms $\sigma_{\rho-1,j}$ in the equation (14). Let us assume that b_r is at the i_0-th such rows of $\mathbf{g}^{-1}$. Then, this row is of the form (*cf.* Remark 3.12):

$$\big(\mathsf{b}_r, c_1, \ldots, c_t, 0, \ldots, 0, \underbrace{\frac{1}{a_1^r \overline{a}_1^s}}_{i_0\text{-th place}}, 0, \ldots, 0\big),$$

where $t+1$ is the number of the weight ρ lifted 1-forms σ_ρ. Moreover, let us assume that at the i_0-th row of the first column of $\mathbf{g}$ is $e_{i_0 1}$. Then, multiplying again the above row of $\mathbf{g}^{-1}$ to the first column of $\mathbf{g}$ implies that:

$$\mathsf{b}_r \cdot (a_1^p \overline{a}_1^q) + \frac{e_{i1}}{a_1^r \overline{a}_1^s} = 0$$

and hence:

$$\mathbf{b}_r = -\frac{e_{i1}}{a_1^{p+r} \overline{a}_1^{q+s}},$$

as was desired. □

Roughly speaking, the first part (i) of this lemma states that for each fixed row of the three matrices appearing in the equation $\Sigma = \mathbf{g}^{-1} \cdot \Gamma$, all the nonzero entries are of the same weight. Furthermore, taking into account the shape of the lower triangular matrix $\mathbf{g}^{-1}$ and by the first part of the above lemma, one observes that;

Lemma 4.3. *For each weight $-\ell$ initial 1-form $\sigma_{\ell,i}$, its expression in terms of the lifted 1-forms is as follows:*

$$\sigma_{\ell,i} := \sum_{l \gneqq \ell} \mathsf{A}^i_j(a_\bullet)\,\Gamma_{l,j} + \frac{1}{a_1^{p_i}\overline{a}_1^{q_i}}\,\Gamma_{\ell,i},$$

with $p_i + q_i = \ell$ and for some weighted homogeneous polynomial fractions A^i_j of the weight $-\ell$ where their denominators are some powers of only a_1 and $\overline{a}_1$.

Also, after expressing each initial 1-form σ in terms of the lifted ones Γ by means of the inverse matrix $\mathbf{g}^{-1}$, then the second part of Lemma 4.2 implies that;

Lemma 4.4. *If in the structure equation $d\Gamma_{\ell-1,m}$ of* (18) *we have the term $a_j d\sigma_{\ell,n}$ for some (possibly zero) group parameter a_j, then the coefficient of $\Gamma_{\ell,n}$ in the expression of $\sigma_{\ell-1,m}$ is of the form $-\frac{a_j}{a_1^r\overline{a}_1^s}$ for some constant integers r and s.*

Proof. First one notices that according to Lemma 3.11, the group parameter a_j should be of the weight ℓ. The term $a_j d\sigma_{\ell,n}$ in (18) can come only from the second part $\mathbf{g} \cdot d\Sigma$ of this matrix equation and hence the appearance of this term in the structure equation $d\Gamma_{\ell-1,m}$ means that the coefficient of $\sigma_{\ell,n}$ in the expression of $\Gamma_{\ell-1,m}$ — coming from the equality $\Gamma = \mathbf{g} \cdot \Sigma$ — is a_j:

$$\underbrace{\begin{pmatrix} \vdots \\ \vdots \\ \Gamma_{\ell-1,m} \\ \vdots \end{pmatrix}}_{\Gamma} = \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \ldots & a_j & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{\mathbf{g}} \cdot \underbrace{\begin{pmatrix} \vdots \\ \sigma_{\ell,n} \\ \vdots \\ \vdots \end{pmatrix}}_{\Sigma}.$$

As one sees in the above matrix equality, the weight ℓ group parameter a_j is settled in a certain row of $\mathbf{g}$ in front of a weight $-(\ell-1)$ row of Γ (and Σ). Then according to the second part (ii) of Lemma 4.2, we have some $-\frac{a_j}{a_1^r\overline{a}_1^s}$ in the same entry of the inverse matrix $\mathbf{g}^{-1}$. But this entry in the inverse matrix determines, through the equality $\Sigma = \mathbf{g}^{-1} \cdot \Gamma$, the coefficient of the lifted 1-form $\Gamma_{\ell,n}$ in the expression of $\sigma_{\ell-1,m}$. □

This suggests that if we are seeking the coefficient of $\Gamma_{\ell,n}$ in the expression of some $\sigma_{\ell-1,m}$, then it is opposite to the fraction of the coefficient of $d\sigma_{\ell,,n}$ in the structure equation $d\Gamma_{\ell-1,m}$ by some powers of a_1 and $\overline{a}_1$.

Now, we are ready to prove the main result of this section;

Proposition 4.5. *All torsion coefficients $T^i_{jn}(a_\bullet)$ appearing among the structure equations* (19) *are weighted homogeneous polynomial fractions of the equal weight zero where their denominators are some powers of only a_1 and $\overline{a}_1$.*

Proof. According to (18), each structure equation can be expressed as:

$$d\Gamma_{\ell,i} = (p_i\alpha + q_i\overline{\alpha}) \wedge \Gamma_{\ell,i} + \sum_{l \gneqq \ell} \delta_{i_j} \wedge \Gamma_{l,j} + \sum_{l \gneqq \ell} a_{i_j} d\sigma_{l,j} + a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\ell,i}$$

with $p_i + q_i = \ell$. Our torsion coefficients come from the last parts:

$$\sum_{l \gneqq \ell} a_{i_j} d\sigma_{l,j} + a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\ell,i} \tag{21}$$

of this equation after replacing each differentiation $d\sigma_\bullet$ according to the Darboux-Cartan structure computed in Proposition 3.3 and next substituting each initial 1-form $\sigma_\bullet$ with some combinations of lifted 1-forms $\Gamma_\bullet$ by means of the equality $\Sigma = \mathbf{g}^{-1} \cdot \Gamma$. Thanks to Lemma 3.11, the weight of the coefficient a_{i_j} in the term $a_{i_j} d\sigma_{l,j}$ of (21) is l . On the other hand, according to Proposition 3.3, we have:

$$d\sigma_{l,j} := \sum_{\beta,\gamma} \mathsf{c}_{\beta,\gamma}\,\sigma_\beta \wedge \sigma_\gamma \quad \text{with} \quad \beta + \gamma = l.$$

After replacing the expressions of σ_β and σ_γ as Lemma 4.3, such Darboux-Cartan structure takes the form:

$$d\sigma_{l,j} = \sum_{l_1,l_2 \geqslant l} \mathsf{T}^j_{m,n}(a_\bullet)\, \Gamma_{l_1,m} \wedge \Gamma_{l_2,n}$$

where the polynomial fractions $\mathsf{T}^j_{m,n}$ appear as multiplications of some weight $-\beta$ and $-\gamma$ polynomial fractions. Thus, all the coefficients $\mathsf{T}^j_{m,n}$ are of the same weight $-\beta-\gamma=-l$. Now, our torsion coefficients $T^j_{m,n}$ come from the terms $a_{i_j}\, d\sigma_{l,j}$ in the above structure equations (21) by multiplying the functions $\mathsf{T}^j_{m,n}$ by the weight l group parameter a_{i_j} and this results some weighted homogeneous polynomial fractions of the equal weight zero. □

5. Picking up an appropriate weighted homogeneous subsystem

After providing the structure equations of the biholomorphic equivalence problem to M_k, now we are ready to apply the Cartan's method which includes three major parts *absorbtion, normalization* and *prolongation*. The main result behind the first two parts is the following;

Proposition 5.1. *(see* [21, Proposition 4.7]*) In the structure equations:*

$$d\theta^i = \sum_{k=1}^{n} \left(\sum_{s=1}^{r} \mathsf{c}^i_{ks}\, \alpha^s + \sum_{j=1}^{k-1} T^i_{jk}\, \theta^j \right) \wedge \theta^k \qquad (i=1\cdots n),$$

one can replace each Maurer-Cartan form α^s and each torsion coefficient T^i_{jk} with:

(22)
$$\begin{aligned} \alpha^s &\longmapsto \alpha^s + \sum_{j=1}^{n} z^s_j\, \theta^j && (s=1\cdots r), \\ T^i_{jk} &\longmapsto T^i_{jk} + \sum_{s=1}^{r} \left(\mathsf{c}^i_{js}\, z^s_k - \mathsf{c}^i_{ks}\, z^s_j \right) && (i=1\cdots n;\ 1\leqslant j<k\leqslant n), \end{aligned}$$

for some arbitrary functions $z^\bullet_\bullet$ on the base manifold M.

This proposition permits one to substitute each Maurer-Cartan 1-form α and δ_j in the structure equations by any combination of the form:

(23)
$$\begin{aligned} \alpha &\mapsto \alpha + t_{2+k}\, \Gamma_{\rho,2+k} + \ldots + t_2\, \Gamma_{1,2} + t_1\, \Gamma_{1,1}, \\ \delta_j &\mapsto \delta_j + s^j_{2+k}\, \Gamma_{\rho,2+k} + \ldots + s^j_2\, \Gamma_{1,2} + s^j_1\, \Gamma_{1,1}, \end{aligned}$$

for some arbitrary coefficient functions $t_\bullet$ and $s^\bullet_\bullet$. In the absorption-normalization step, we can apply such substitutions (absorption) and try to convert new coefficients of the wedge products $\Gamma_{\ell_1,i_1} \wedge \Gamma_{\ell_2,i_2}$ to some constant integers — possibly zero — by appropriate determinations of the arbitrary functions $t_\bullet, s^\bullet_\bullet$. But, to convert all the already mentioned coefficients to some constant integers, it may be inadequate only appropriate determination of these arbitrary functions but also it is necessitates to determine — or normalize in this literature — some of the group parameters, appropriately. In fact, after appropriate determination of the arbitrary coefficient functions $t_\bullet$ and $s^\bullet_\bullet$, it is possible to remain still some non-constant coefficient functions of the mentioned wedge products. Such coefficients, which we call them by *essential torsion coefficients*, are some combinations of the extant torsion coefficients and are in terms of the group parameters $a_\bullet$. In the normalization step, we shall also converts these essential torsion coefficients to some constant integers — possibly zero — by appropriate determination of some group parameters.

Thus to proceed along the absorption and normalization steps, one has to solve an arising polynomial system with $t_\bullet$, $s^\bullet_\bullet$ and some of the group parameters as its unknowns. But the virtual importance of the solution of this system is not determining the coefficient functions $t_\bullet$ and $s^\bullet_\bullet$ but it is actually the found values of involving group parameters $a_\bullet$. But unfortunately, solving such arising polynomial system — specifically in this general manner — may cause some unavoidable and serious algebraic complexity, the size of which increases considerably as soon as the codimension k increases, even by one unit. To bypass and manipulate such complexity, in this section we introduce a method of picking up an appropriate and convenient subsystem for which it affords all results that we are seeking from the solution of the original system. At first, we need the following useful lemma;

Lemma 5.2. *Assume that $\sigma_{\ell-1,i}\wedge\sigma_{1,t}$, for $t=1$ or 2, is the unique appearing wedge product in the Darboux-Cartan structure of $d\sigma_{\ell,j}$, as stated in Lemma 3.4. Then, among all the expressions of differentiations $d\sigma_{l,r}$, with $l\geqslant\ell$, in terms of the wedge products of the lifted 1-forms, a nonzero coefficient of $\Gamma_{\ell-1,i}\wedge\Gamma_{1,t}$ appears uniquely in $d\sigma_{\ell,j}$. Such coefficient is a fraction of the form $\frac{1}{a_1^p\overline{a}_1^q}$ for some constant integers p and q.*

Proof. Since $\mathbf{g}^{-1}$ is a lower triangular matrix, then in the expression of each $\sigma_{\ell',r}$ by means of the equality $\Sigma=\mathbf{g}^{-1}\cdot\Gamma$, the only appearing lifted 1-form $\Gamma_{l,m}$ with $l\leqslant\ell'$ is some $\frac{1}{a_1^p\overline{a}_1^q}\Gamma_{\ell',r}$ (*cf.* Remark 3.12). In particular, the only initial 1-form having some coefficient of $\Gamma_{1,1}$ in its expression is $\sigma_{1,1}$ and this coefficient is $\frac{1}{a_1}$; also, the only initial 1-form admitting some coefficient of $\Gamma_{1,2}$ is $\sigma_{1,2}$ and this coefficient is just $\frac{1}{\overline{a}_1}$. If in the expression of a differentiation $d\sigma_{l_0,r}$, with $l_0\geqslant\ell$, it is found a nonzero coefficient of $\Gamma_{\ell-1,i}\wedge\Gamma_{1,t}$ then in its Darboux-Cartan structure, $d\sigma_{l_0,r}$ includes some nonzero coefficient of the wedge product $\sigma_{l',i}\wedge\sigma_{1,t}$ with $l'\leqslant\ell-1$. But if $\sigma_{l',i}\wedge\sigma_{1,t}$ appears in the Darboux-Cartan structure of $d\sigma_{l_0,r}$ then Lemma 3.3 implies that we should have $l'+1=l_0\geqslant\ell$ and whence $l'\geqslant\ell-1$. Consequently, we have $l'=\ell-1$ and thus $\sigma_{l',i}=\sigma_{\ell-1,i}$. But according to our assumption, $\sigma_{\ell-1,i}\wedge\sigma_{1,t}$ appears uniquely in the Darboux-Cartan structure of $d\sigma_{\ell,j}$ and hence we should have $d\sigma_{l_0,r}=d\sigma_{\ell,j}$, as desired. In addition, the coefficient $\Gamma_{\ell-1,i}\wedge\Gamma_{1,t}$ in $d\sigma_{\ell,j}$ comes from the wedge product $\sigma_{\ell-1,i}\wedge\sigma_{1,t}$ in its Darboux-Cartan structure and according to what mentioned at the beginning of the proof, it will be nothing but some fraction like $\frac{1}{a_1^p\overline{a}_1^q}$. □

5.1. **Picking up an appropriate subsystem.** Consider a weight $-(\rho-1)$ structure equation:

$$\begin{aligned} d\Gamma_{\rho-1,i_0} &= \big(p_{i_0}\alpha+q_{i_0}\overline{\alpha}\big)\wedge\Gamma_{\rho-1,i_0}+\sum_j\delta_{i_j}\wedge\Gamma_{\rho,j} \\ &\quad +\sum_r a_{i_r}\,d\sigma_{\rho,r}+a_1^{p_{i_0}}\overline{a}_1^{q_{i_0}}\,d\sigma_{\rho-1,i_0} \end{aligned} \tag{24}$$

as one of those in (18). Lemma 3.11 implies that all the above group parameters a_{i_r} are of the weight ρ and also $p_{i_0}+q_{i_0}=\rho-1$ (*see* the paragraph after equation (13)). Then, as a consequence of the above Lemma 5.2 and for each fixed term $a_{i_r}\,d\sigma_{\rho,r}$ of the above structure equation, one finds a unique wedge product:

$$\frac{a_{i_r}}{a_1^{p_\bullet}\overline{a}_1^{q_\bullet}}\,\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}, \tag{25}$$

coming from a uniquely appearing $\sigma_{\rho-1,j_r}\wedge\sigma_{1,t_r}$ in the Darboux-Cartan structure of $d\sigma_{\rho,r}$ for which no any other weight $-\rho$ differentiation $d\sigma_{\rho,r'}$ in the above structure equation gives any nonzero coefficient of such $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$.

Now, let us inspect whether it is possible for the other terms in (24) to produce any nonzero coefficient of the same product $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$. We begin with the last term $a_1^{p_{i_0}}\overline{a}_1^{q_{i_0}}\,d\sigma_{\rho-1,i_0}$. According to the constructed Darboux-Cartan structure in Proposition 3.3, if we assume that:

$$d\sigma_{\rho-1,i_0}=\sum_{l_1+l_2=\rho-1}\mathsf{c}_{l_1,l_2}\sigma_{l_1,m}\wedge\sigma_{l_2,m}$$

then to pick a wedge product of the form $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$ we have to look for the terms of the form[2] $\mathsf{c}_m\sigma_{\rho-2,m}\wedge\sigma_{1,t_r}$ in the above expression and next pick the coefficient of $\Gamma_{\rho-1,j_r}$ from the expression of $\sigma_{\rho-2,m}$. According to Lemma 4.4 and if we assume that the coefficient of $d\sigma_{\rho-1,j_r}$ in the structure equation of $d\Gamma_{\rho-2,m}$ is a weight $\rho-1$ group parameter a_{j_m}, then this desired coefficient will be of the form $-\frac{a_{j_m}}{a_1^{r_\bullet}\overline{a}_1^{s_\bullet}}$

[2]Remind that one can find the lifted 1-forms $\Gamma_{1,1}$ and $\Gamma_{1,2}$ only in the expressions of $\sigma_{1,1}$ and $\sigma_{1,2}$, respectively.

for some constant integers $r_\bullet$ and $s_\bullet$[3]. Hence the the last term $a_1^{p_{i_0}}\overline{a}_1^{q_{i_0}}\, d\sigma_{\rho-1,i_0}$ may give some term like:

$$-\Big(\sum_m \mathsf{c_m}\frac{a_{j_m}}{a_1^{r_\bullet}\overline{a}_1^{s_\bullet}}\Big)\,\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}. \tag{26}$$

for some weight $\rho-1$ group parameters a_{j_m}.

The second part $\sum_j\,\delta_{i_j}\wedge\Gamma_{\rho,j}$ of the structure equation (24) does not give any nonzero coefficient of the wedge product $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$ but, nevertheless, after the substitutions (23) in the first part $(p_{i_0}\alpha+q_{i_0}\overline{\alpha})\wedge\Gamma_{\rho-1,i_0}$, we receive two terms:

$$-\,(p_{i_0}t_1+q_{i_0}\overline{t}_2)\,\Gamma_{\rho-1,i_0}\wedge\Gamma_{1,1}\qquad\text{and}\qquad -\,(p_{i_0}t_2+q_{i_0}\overline{t}_1)\,\Gamma_{\rho-1,i_0}\wedge\Gamma_{1,2}, \tag{27}$$

where in the case that $i_0=j_r$, then they will be of the same form as the wedge product $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$, in question.

Then, all possible coefficients of the wedge product $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$ in the above weight $-(\rho-1)$ structure equation (24) are those presented in (25, 26, 27). Equating this coefficient to zero — as is the method of absorption-normalization — then one finds some fraction polynomial equation of the form:

$$\frac{a_{i_r}}{a_1^{p_\bullet}\overline{a}_1^{q_\bullet}}-\sum_m \mathsf{c_m}\frac{a_{j_m}}{a_1^{r_\bullet}\overline{a}_1^{s_\bullet}}=\mathsf{a}_{i_r}t_1+\mathsf{b}_{i_r}t_2+\mathsf{a'}_{i_r}\overline{t}_1+\mathsf{b'}_{i_r}\overline{t}_2,$$

for some (possibly zero) constants $\mathsf{a}_{i_r},\mathsf{b}_{i_r},\mathsf{a'}_{i_r},\mathsf{b'}_{i_r}$. Here, the left hand side of this equation is in fact the possible torsion coefficient of $\Gamma_{\rho-1,j_r}\wedge\Gamma_{1,t_r}$ in the structure equation (24) which comes from (25, 26). Hence according to Proposition 4.5, it is of the weight zero. Minding that here a_{i_r} is a weight ρ parameter while a_{j_m}s are of the weight $\rho-1$, then multiplying both side of this equation by the denominator $a_1^{p_\bullet}\overline{a}_1^{q_\bullet}$ gives the following equivalent *weighted homogeneous* polynomial equation — here we assign naturally the weight zero to the parameters t_1, t_2 and their conjugations:

$$a_{i_r}-\sum_m \mathsf{c}_m\, a_1^{r'_\bullet}\overline{a}_1^{s'_\bullet}a_{j_m}=a_1^{p_\bullet}\overline{a}_1^{q_\bullet}\big(\mathsf{a}_{i_0}t_1+\mathsf{b}_{i_0}t_2+\mathsf{a'}_{i_0}\overline{t}_1+\mathsf{b'}_{i_0}\overline{t}_2\big). \tag{28}$$

Proceeding along the same lines of computations, it ensues that;

Proposition 5.3. *Among the procedure of absorbtion and associated to each weight ρ group parameter a_{i_r} appearing in an arbitrary weight $-(\rho-1)$ structure equation* (24), *one finds a weighted homogeneous parametric complex polynomial equation as* (28), *expressing a_{i_r} in terms of $a_1,\overline{a}_1$, some weight $\rho-1$ group parameters a_{j_m} and two parameters t_1 and t_2.*

This constructive result is generalizable to arbitrary weights $-\ell=-(\rho-1),\ldots,-1$. To make it more precise, for each weight $-\ell$ structure equation:

$$\begin{aligned} d\Gamma_{\ell,m}&=(p_m\alpha+q_m\overline{\alpha})\wedge\Gamma_{\ell,m}+\sum_{l\gneq\ell}\delta_{i_t}\wedge\Gamma_{l,j}\\ &\quad+\sum_{l\geq\ell+2,\,n}a_{j_n}d\sigma_{l,n}+\sum_r a_{j_r}d\sigma_{\ell+1,r}+a_1^{p_m}\overline{a}_1^{q_m}\,d\sigma_{\ell,m},\end{aligned} \tag{29}$$

in (18) and *just for each* term of the form $a_{j_{r_0}}d\sigma_{\ell+1,r_0}$ in the penultimate part $\sum_r\,a_{j_r}d\sigma_{\ell+1,r}$, we shall seek the coefficient of the wedge product $\Gamma_{\ell,i_j}\wedge\Gamma_{1,t_r}$ where $\sigma_{\ell,i_j}\wedge\sigma_{1,t_r}$ is visible uniquely in the Darboux-Cartan structure of $d\sigma_{\ell+1,r_0}$. (*cf.* Lemma 3.4). Let us look for it, part by part.

According to Lemma 5.2, from the part $\sum_{l\geq\ell+2,n}\,a_{j_n}d\sigma_{l,n}+\sum_r\,a_{j_r}d\sigma_{\ell+1,r}$ one finds merely one nonzero coefficient of the form:

$$\frac{a_{j_{r_0}}}{a_1^{p_\bullet}\overline{a}_1^{q_\bullet}}\,\Gamma_{\ell,i_j}\wedge\Gamma_{1,t_r}$$

which comes just from the term $a_{j_{r_0}}\,d\sigma_{\ell+1,r_0}$. Here notice that $a_{j_{r_0}}$ is of the weight $\ell+1$.

[3]Notice also that here a_{j_m} can be zero and it does not effect our next results.

Moreover, by a similar argument like that presented before (26), one realizes that the last term $a_1^{p_m}\overline{a}_1^{q_m}\,d\sigma_{\ell,m}$ may produce some (possibly zero) coefficient of the form:

$$-\Big(\sum_m \mathsf{c_m}\frac{a_{i_m}}{a_1^{r_\bullet}\overline{a}_1^{s_\bullet}}\Big)\,\Gamma_{\ell,i_j}\wedge\Gamma_{1,t_r} \tag{30}$$

for some weight ℓ group parameters a_{i_m}.

The second term $\sum_{l\gneqq\ell}\delta_{i_t}\wedge\Gamma_{l,j}$ will not introduce any nonzero coefficient of $\Gamma_{\ell,i_j}\wedge\Gamma_{1,t_r}$ while after the substitutions (23) in the first part $(p_m\alpha+q_m\overline{\alpha})\wedge\Gamma_{\ell,m}$, one may find some terms like:

$$-\,(p_mt_1+q_m\overline{t}_2)\,\Gamma_{\ell,m}\wedge\Gamma_{1,1}\quad\text{and}\quad-\,(p_mt_2+q_m\overline{t}_1)\,\Gamma_{\ell,m}\wedge\Gamma_{1,2}, \tag{31}$$

where in the case that $m=i_j$, then they will be of the same form as the wedge product $\Gamma_{\ell,i_j}\wedge\Gamma_{1,t_r}$, in question.

Summing up the computations and by equating to zero the coefficient of $\Gamma_{\ell,i_j}\wedge\Gamma_{1,t_r}$ in the structure equation of $d\Gamma_{\ell,m}$, after absorption, then we find a weight zero homogeneous equation of the form:

$$\frac{a_{j_{r_0}}}{a_1^{p_\bullet}\overline{a}_1^{q_\bullet}}-\sum_m\mathsf{c_m}\frac{a_{i_m}}{a_1^{r_\bullet}\overline{a}_1^{s_\bullet}}=\mathsf{a}_{j_{r_0}}t_1+\mathsf{b}_{j_{r_0}}t_2+\mathsf{a'}_{j_{r_0}}\overline{t}_1+\mathsf{b'}_{j_{r_0}}\overline{t}_2,$$

where after some sufficient multiplication by powers of a_1 and $\overline{a}_1$, it converts to a weighted homogeneous parametric polynomial equation:

$$\boxed{a_{j_{r_0}}-\sum_m\mathsf{c}_m\,a_1^{r'_\bullet}\overline{a}_1^{s'_\bullet}a_{i_m}=a_1^{p_\bullet}\overline{a}_1^{q_\bullet}\big(\mathsf{a}_{j_{r_0}}t_1+\mathsf{b}_{j_{r_0}}t_2+\mathsf{a'}_{j_{r_0}}\overline{t}_1+\mathsf{b'}_{j_{r_0}}\overline{t}_2\big)}. \tag{32}$$

Here notice that $a_{j_{r_0}}$ is of the weight $\ell+1$ while the weight of a_{i_m}s is ℓ.

Proposition 5.4. *(Extension of Proposition 5.3) Let* $\ell=1,\ldots,\rho-1$. *Then, among the procedure of absorbtion and associated to each weight* $\ell+1$ *group parameter* $a_{j_{r_0}}$ *appearing in an arbitrary weight* $-\ell$ *structure equation* (29)*, one finds a weighted homogeneous parametric complex polynomial equation as* (32)*, expressing* $a_{j_{r_0}}$ *in terms of* $a_1,\overline{a}_1$*, some weight* ℓ *group parameters* a_{i_m} *and two parameters* t_1 *and* t_2*.*

Let us denote by S the weighted homogeneous parametric polynomial system[4] of equations mentioned in the above proposition and extracted throughout the weight $-1,\ldots,-(\rho-1)$ structure equations $d\Gamma_{\ell,i}$ in (18), after absorption. Among this system S, two equations coming from the last two structure equations:

$$d\Gamma_{2,3}=(\alpha+\overline{\alpha})\wedge\Gamma_{2,3}+\sum_{l\gneqq2}\delta_{i_j}\wedge\Gamma_{l,j}+\sum_{l\gneqq3}a_{i_j}\,d\sigma_{l,j}+a_3\,d\sigma_{3,4}+\overline{a}_3\,d\sigma_{3,5}+a_1\overline{a}_1\,d\sigma_{2,3},$$
$$d\Gamma_{1,1}=\alpha\wedge\Gamma_{1,1}+\sum_{l\gneqq1}\delta_{i_j}\wedge\Gamma_{l,j}+\sum_{l\gneqq2}a_{i_j}\,d\sigma_{l,j}+a_2\,d\sigma_{2,3}+a_1\,d\sigma_{1,1}$$

are of particular importance. According to our suggested method, in the weight -2 structure equation $d\Gamma_{2,3}$ we should focus on the term $a_3\,d\sigma_{3,4}$ since $d\sigma_{3,4}$, together with $d\sigma_{3,5}$, are the only weight $-(2+1)=-3$ differentiations visible in it. Since $\mathscr{L}_{3,4}=[\mathscr{L}_{1,1},\mathscr{L}_{2,3}]$, then the uniquely appearing wedge product in the Darboux-Cartan structure of $d\sigma_{3,4}$ is $\sigma_{2,3}\wedge\sigma_{1,1}$ (*cf.* Lemma 3.4 and its proof). Thus, we shall look for the (torsion) coefficient of $\Gamma_{2,3}\wedge\Gamma_{1,1}$ in this structure equation $d\Gamma_{2,3}$. Then, correspondingly we will find a weighted homogeneous equation, expressing a_3 in terms of a_1,a_2,t_1,t_2 and their conjugations. Also in the weight -1 structure equation $d\Gamma_{1,1}$ we should focus on the single term $a_2\,d\sigma_{2,3}$. The uniquely appearing wedge product in the Darboux-Cartan structure of $d\sigma_{2,3}$ is $\sigma_{1,1}\wedge\sigma_{1,2}$, then let us find the coefficient of $\Gamma_{1,1}\wedge\Gamma_{1,2}$ in this structure equation. This will give us a weighted homogeneous equation that expresses a_2

[4]Notice that this system does not involve all the group parameters $a_\bullet$. Also, it involves only the parameters t_1 and t_2 in the substitutions (23).

in terms of a_1, t_1, t_2 and their conjugations. Performing necessary computations, we respectively find the following two weight zero homogeneous equations, after applying the substitutions (23):

$$\begin{aligned} \frac{a_3}{a_1^2\overline{a}_1} + i\frac{\overline{a}_2}{a_1\overline{a}_1} &= t_1 + \overline{t}_2, \\ i\frac{a_2}{a_1\overline{a}_1} &= t_2, \end{aligned} \tag{33}$$

which give, surprisingly, the parameters t_1 and t_2 as some weight zero expressions:

$$\boxed{t_1 = \frac{a_3}{a_1^2\overline{a}_1} + 2i\frac{\overline{a}_2}{a_1\overline{a}_1}, \qquad t_2 = i\frac{a_2}{a_1\overline{a}_1}}. \tag{34}$$

Putting these expression in S and multiplying again the appearing fractional equations by some sufficient powers of a_1 and $\overline{a}_1$, then one finds it as a weighted homogeneous polynomial system *with no any parameter*. Except a_2 and a_3 that we already spent their associated equations (33) to find the expressions of the parameters t_1 and t_2, for each other involving group parameters $a_\bullet$ there exists one equation in S that expresses it in terms of some lower weight group parameters. Our next goal is to find two more polynomial equations including a_2 and a_3 as their unknowns, too.

5.1.1. *Structure equations of the weight $-\rho$.* One might be somehow surprised that since the beginning of this subsection 5.1, we have not talked about the weight $-\rho$ structure equations, yet. In fact, our main trick was to retain these structure equations for our current aim of providing two more weighted homogeneous equations containing a_2 and a_3 as their unknowns[5]. One notices that the method suggested above, can not be applied on a weight $-\rho$ structure equation:

$$d\Gamma_{\rho,i} = (p_i\alpha + q_i\overline{\alpha}) \wedge \Gamma_{\rho,i} + a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\rho,i} \tag{35}$$

since, obviously, it does not contain any term $a_\bullet\, d\sigma_\bullet$ with $d\sigma_\bullet$ of the weight $-(\rho+1)$. However, here we can think about picking up coefficients of the wedge products like $\Gamma_{\rho,i} \wedge \Gamma_{1,t}$ from $d\Gamma_{\rho,i}$ for $t = 1, 2$. Making it more precise and associated to each structure equation $d\Gamma_{\rho,i}$, one finds the Darboux-Cartan structure of $d\sigma_{\rho,i}$, visible in it, as a combination of some certain wedge products $\sigma_{\rho-1,j} \wedge \sigma_{1,t}$ (*cf.* Lemma 3.3). According to Lemma 4.4, if the coefficient of $d\sigma_{\rho,i}$ in the weight $-(\rho-1)$ structure equation $d\Gamma_{\rho-1,j}$ — considered at the beginning of this subsection — is a (possibly zero) weight ρ group parameter a_{j_r}, then the coefficient of $\Gamma_{\rho,i}$ in $\sigma_{\rho-1,j}$ is some fraction of the form $-\frac{a_{j_r}}{a_1^\bullet\overline{a}_1^\bullet}$. Moreover, since $\sigma_{\rho-1,j}$ has no any nonzero coefficient of $\Gamma_{1,t}$, then the coefficient of the sought wedge product $\Gamma_{\rho,i} \wedge \Gamma_{1,t}$ in $\sigma_{\rho-1,j} \wedge \sigma_{1,t}$ is the multiplication between the coefficient $-\frac{a_{j_r}}{a_1^\bullet\overline{a}_1^\bullet}$ of $\Gamma_{\rho,i}$ in $\sigma_{\rho-1,j}$ and the coefficient of $\Gamma_{1,t}$ in $\sigma_{1,t}$, which is $\frac{1}{a_1}$ where $t = 1$ and $\frac{1}{\overline{a}_1}$ where $t = 2$. This implies that:

(i) After absorption and equating to zero the coefficients of $\Gamma_{\rho,i} \wedge \Gamma_{1,1}$ and $\Gamma_{\rho,i} \wedge \Gamma_{1,2}$ in this structure equation $d\Gamma_{\rho,i}$, then one finds:

$$\begin{aligned} \sum_{j_r} \frac{a_{j_r}}{a_1^\bullet\overline{a}_1^\bullet} + p_i\, t_1 + q_i\, \overline{t}_2 &= 0, \\ \sum_{j'_r} \frac{a_{j'_r}}{a_1^\bullet\overline{a}_1^\bullet} + q_i\, \overline{t}_1 + p_i\, t_2 &= 0, \end{aligned} \tag{36}$$

where according to (34) they are actually two equations in terms of a_2, a_3 and some other weight ρ group parameters ones a_{j_r}.

(ii) In the system S, one finds some polynomial equations which express a_{j_r}s and a'_{j_r}s in terms of some lower weight group parameters.

[5]Actually in CR dimension 1, the reason of satisfying the Beloshapka's maximum conjecture in the lengths $\rho \geqslant 3$ refers to this part of our constructions. In fact, to provide two more weighted homogeneous equations for a_2 and a_3, we need some more structure equations than those of $d\Gamma_{2,3}, d\Gamma_{1,1}$ and $d\Gamma_{1,2} = \overline{d\Gamma_{1,1}}$. This means that we should at least have the next structure equation $d\Gamma_{3,4}$ which appears in the case of CR models which are of length $\rho \geqslant 3$.

Granted the fact that throughout the polynomial system S and for each group parameter $a_\bullet$ (except a_1, a_2, a_3) involving in it, there is some equation which expresses $a_\bullet$ in terms of some lower weight group parameters, then one can consider eventually the above two equations (36) as some equations in terms of only a_3, a_2 and a_1. Now, to finalize constructing the desired subsystem, we add these two already found equations to S.

5.2. **Solving the picked up subsystem.** After providing the weighted homogeneous polynomial system S, now let us attempt to find the weighted projective variety $\mathbf{V}(\mathscr{I})$ of the polynomial ideal $\mathscr{I} = \langle \mathsf{S} \rangle$ — namely the solution of the system S — in the weighted projective space $\mathbb{P}(1, 2, 3, 3, 3, \ldots)$ (*see e.g.* [11] for more details). Since the only weight 1 group parameter a_1 is assumed to be nonzero, then this variety does not contain any point at the infinity surface $a_1 = 0$. Assume that $\mathscr{I}^{\mathsf{aff}} \subset \mathbb{C}[a_2, a_3, \ldots, a_r]$ is the affine ideal obtained by dehomogenizing $\mathscr{I}$ by setting $a_1 = 1$. If g is a weighted homogeneous polynomial in $\mathscr{I}$, then the following relation holds between it and its dehomogenization g^{deh} (*see* [11, Theorem 5.16]):

$$g(a_1, a_2, a_3, \ldots, a_r) = a_1^{\mathsf{w-deg}} \cdot g^{\mathsf{deh}}\big(\frac{a_2}{a_1^{[a_2]}}, \frac{a_3}{a_1^{[a_3]}}, \ldots, \frac{a_r}{a_1^{[a_r]}}\big) \tag{37}$$

where $\mathsf{w} - \mathsf{deg}$ is the weight degree of the affine polynomial g^{deh}. One plainly verifies about this new affine ideal that:

- Associated to each group parameter a_j visible in it, there exists some polynomial in $\mathscr{I}^{\mathsf{aff}}$ where its expression is in terms of a_j and some other group parameters (variables) of absolutely lower weights. Moreover, these polynomials are all linear (consider the equations of S after setting $a_1 = 1$ in (32), (34), (36)).

This means that after selecting some appropriate order $\prec$ on the extant group parameters $a_\bullet$ enjoying the property that $a_i \prec a_j$ whenever $[a_i] < [a_j]$, then the affine ideal $\mathscr{I}^{\mathsf{aff}}$ is in fact in *Noether normal position* and according to the Finiteness Theorem ([10, Theorem 6 and Corollary 7, pp. 230-1]), the affine variety $\mathbf{V}(\mathscr{I}^{\mathsf{aff}})$ is zero dimensional containing just the origin[6] $(0, 0, \ldots, 0)$. Then according to the above equality (37), one concludes that the weighted projective variety $\mathbf{V}(\mathscr{I})$, or equivalently the solution set of the weighted homogeneous system S comprises some points of the concrete form:

$$\mathbf{V}(\mathscr{I}) = \{(a_1, 0, 0, \ldots, 0), \quad a_1 \neq 0\}.$$

In other words, in the solution set of our weighted homogeneous system S, all the group parameters visible in it — but not necessarily all the group parameters appearing in our ambiguity matrix — take the value zero, identically. In particular, the two fundamental group parameters a_2 and a_3 shall be zero. But, thanks to Lemma 3.6, vanishing of these two group parameters is sufficient to assert that *all* the group parameters $a_j, j \neq 1$, appearing in the ambiguity matrix g will be normalized to zero;

Proposition 5.5. *After sufficient steps of applying absorption and normalization on the structure equations of the equivalence problem to a totally nondegenerate Beloshapka's CR model M_k of CR dimension 1 and codimension k, all the appearing group parameters a_j with $j = 2, 3, 4, \ldots$ vanish, identically.*

After vanishing these group parameters, then our ambiguity matrix group G (*see* (14)) reduces to the simple diagonal matrix Lie group G^{red} comprising matrices of the form:

$$\mathbf{g}^{\mathsf{red}} := \begin{pmatrix} a_1^p \overline{a}_1^q & 0 & 0 & \cdots & 0 \\ 0 & a_1^{p'} \overline{a}_1^{q'} & 0 & \cdots & 0 \\ 0 & \cdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \overline{a}_1 & 0 \\ 0 & 0 & \cdots & 0 & a_1 \end{pmatrix}. \tag{38}$$

Also in the Maurer-Cartan matrix (17), all the Maurer-Cartan forms δ vanish identically and it reduces to a diagonal matrix with the combinations of the 1-forms α and $\overline{\alpha}$ at its diagonal. Finally, after applying

[6] One notices that according to Proposition 4.5, all the nonconstant torsion coefficients vanish by putting $a_2 = a_3 = \ldots \equiv 0$. Then the origin is a solution of the system S

vanishing of the group parameters $a_2, a_3, \ldots$, then all torsion coefficients $T^i_{j,m}$ vanish identically except those which were constant from the beginning of constructing the structure equations in (18);

Proposition 5.6. *After vanishing the group parameters $a_2, a_3, a_4, \ldots$, our structure equations convert into the simple constant type:*

$$d\,\Gamma_{\ell,i} := (p_i\,\alpha + q_i\,\overline{\alpha}) \wedge \Gamma_{\ell,i} + \sum_{\substack{l+m=\ell \\ j,n}} \mathsf{c}^i_{j,n}\,\Gamma_{l,j} \wedge \Gamma_{m,n} \quad {\scriptstyle (\ell=1,\ldots,\rho,\ \ i=1,\ldots,2+k)} \tag{39}$$

for some (possibly zero) constant complex integers $\mathsf{c}^i_{j,n}$.

Proof. According to (18), our structure equations were originally of the form:

$$d\Gamma_{\ell,i} = (p_i\alpha + q_i\overline{\alpha}) \wedge \Gamma_{\ell,i} + \underbrace{\sum_{l \gneqq \ell} \delta_{i_j} \wedge \Gamma_{l,j}}_{\circ} + \underbrace{\sum_{l \gneqq \ell} a_{i_j} d\sigma_{l,j}}_{\circ} + a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\ell,i}.$$

As mentioned, after vanishing of the group parameters $a_2, a_3, \ldots$ all the Maurer-Cartan forms $\delta_\bullet$ vanish identically and this kills the first sum $\sum_{l \gneqq \ell} \delta_{i_j} \wedge \Gamma_{l,j}$. For the second term $\sum_{l \gneqq \ell} a_{i_j} d\sigma_{l,j}$ and according to Lemma 3.11, since all the differentiations $d\sigma_{l,j}$ are of the weights $\lneqq -1$ (notice that here $l \gneqq \ell$ and $\ell \geqslant 1$) then all the group parameters a_{i_j} are of the weights $\gneqq 1$ and hence none of them is a_1. This implies the vanishing of this term, too. Then, it suffices to consider the last term $a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\ell,i}$ of the above structure equation. According to the computed Darboux-Cartan structure in Proposition 3.3 we have:

$$d\sigma_{\ell,i} := \sum_{\substack{r,s \\ \beta+\gamma=\ell}} \mathsf{c}_{r,s}\,\sigma_{\beta,r} \wedge \sigma_{\gamma,s}.$$

On the other hand, our inverse matrix $\mathbf{g}^{-1}$ is now converted to the simple form:

$$(\mathbf{g}^{\mathsf{red}})^{-1} = \begin{pmatrix} \frac{1}{a_1^p\overline{a}_1^q} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_1^{p'}\overline{a}_1^{q'}} & 0 & \cdots & 0 \\ 0 & \cdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \frac{1}{\overline{a}_1} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{a_1} \end{pmatrix}$$

and hence, if we seek nonzero coefficients of the wedge products $\Gamma_{\ell_1} \wedge \Gamma_{\ell_2}$ in $\sigma_{\beta,r} \wedge \sigma_{\gamma,s}$, then we will find nothing apart from:

$$\frac{1}{a_1^{m_r}\overline{a}_1^{n_s}}\,\Gamma_{\beta,r} \wedge \Gamma_{\gamma,s},$$

for some constant integers m_r and n_s. Then, the last term $a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\ell,i}$ can be brought into a combination of the form:

$$a_1^{p_i}\overline{a}_1^{q_i}\, d\sigma_{\ell,i} := \sum_{\substack{\beta+\gamma=\ell \\ r,s}} \mathsf{c}_{r,s}\,\frac{a_1^{p_i}\overline{a}_1^{q_i}}{a_1^{m_r}\overline{a}_1^{n_s}}\,\Gamma_{\beta,r} \wedge \Gamma_{\gamma,s}.$$

On the other hand, these coefficients $\mathsf{c}_{r,s}\frac{a_1^{p_i}\overline{a}_1^{q_i}}{a_1^{m_r}\overline{a}_1^{n_s}}$ are in fact the only remained torsion coefficients T^i_{rs} of the wedge products $\Gamma_{\beta,r} \wedge \Gamma_{\gamma,s}$, in the structure equation $d\Gamma_{\ell,i}$ and hence according to Proposition 4.5, are of the weight zero. Since they involve just weight one group parameters a_1 and $\overline{a}_1$ then, after simplifications if necessary, they will be either some constants or some fractions of a form like:

$$\mathsf{c}_{r,s}\frac{a_1^i}{\overline{a}_1^i} \quad \text{or} \quad \mathsf{c}_{r,s}\frac{\overline{a}_1^i}{a_1^i}.$$

Consequently, our structure equation $d\Gamma_{\ell,i}$ is converted into the form:

$$d\Gamma_{\ell,i} = (p_i\alpha + q_i\overline{\alpha}) \wedge \Gamma_{\ell,i} + \sum_{\substack{\beta'+\gamma'=\ell \\ r',s'}} \mathsf{c}_{r',s'}\,\Gamma_{\beta',r'} \wedge \Gamma_{\gamma',s'} + \\ + \sum_{\substack{\beta+\gamma=\ell \\ r,s}} \mathsf{c}_{r,s}\frac{a_1^i}{\overline{a}_1^i}\,\Gamma_{\beta,r} \wedge \Gamma_{\gamma,s} + \sum_{\substack{\beta+\gamma=\ell \\ r,s}} \mathsf{c}_{r,s}\frac{\overline{a}_1^j}{a_1^j}\,\Gamma_{\beta,r} \wedge \Gamma_{\gamma,s}.$$

All the appearing βs and γs in this equation are absolutely less than ℓ, whence in the case that one $\mathsf{c}_{r,s}$ is nonzero then the torsion coefficient $T^i_{r,s} = \mathsf{c}_{r,s}\frac{a_1^i}{\overline{a}_1^i}$ or $T^i_{r,s} = \mathsf{c}_{r,s}\frac{\overline{a}_1^i}{a_1^i}$ of $\Gamma_{\beta,r} \wedge \Gamma_{\gamma,s}$ is essential and can be plainly normalized to some constant, say $\mathsf{c}_{r,s}$, by determining a_1 such that $\frac{a_1}{\overline{a}_1} = 1$, *i.e.* by considering the only remaining parameter a_1 as *real*. This gives the normalization of all powers of the fraction $\frac{a_1}{\overline{a}_1}$ to 1. Thus, we receive finally just some constant coefficients of these remaining wedge products. □

What mentioned at the end of the above proof also determines the normalization of the only remaining group parameter a_1. Accordingly, this parameter is never normalizable in the case that after vanishing the group parameters $a_2, a_3, \ldots$, all the torsion coefficients of the structure equations are constant. Otherwise, a_1 will be normalized to a real group parameter.

Corollary 5.7. *There are two possibility for the normalization of the only remaining group parameter a_1. It is either normalizable to a real group parameter or it is never normalizable. The reduced structure group G^{red} (cf.* (38)*) is of real dimension* 1 *or* 2 *respectively in the former or latter cases.*

For instance, one observes in [21] that in the case of M_3, the group parameter a_1 is never normalizable while for M_4, it is normalizable to a real group parameter as is shown in [26].

To continue along the Cartan's method of solving equivalence problems and after applying all the required absorption-normalization steps, now one has to start the *prolongation* procedure. The main fact behind this step is the following fundamental proposition;

Proposition 5.8. *(see* [23, Proposition 12.1]*) Let θ and θ' be two lifted coframes on two manifolds M and M' having the same structure group G, let α and α' be the modified Maurer-Cartan forms obtained by solving the absorption equations and assume that neither group-dependent essential torsion coefficients exist nor free absorption variables remain. Then there exists a diffeomorphism $\Phi : M \to M'$ mapping θ to θ' for some choice of group parameters if and only if there is a diffeomorphism $\Psi : M \times G \longrightarrow M' \times G$ mapping the prolonged coframe $\{\theta, \alpha\}$ to $\{\theta', \alpha'\}$.* □

This permits us to substitute the current equivalence problem to the $(2+k)$-dimensional CR model M_k by that of the $(3+k)$ or $(4+k)$-dimensional prolonged space $M_k \times G^{\mathsf{red}}$. To do this, we have to add the remaining Maurer-Cartan forms α and $\overline{\alpha}$ to the original lifted coframe Γ and consider the collection $(\Gamma_{\rho,2+k}, \ldots, \Gamma_{1,1}, \alpha, \overline{\alpha})$ as the new lifted coframe associated to this prolonged space. In the case that a_1 is normalizable to a real group parameter, then of course we have $\alpha = \overline{\alpha}$. Constructing the associated structure equations to this new equivalence problem is easy, just adding:

$$d\alpha = d\Big(\frac{d\,a_1}{a_1}\Big) = 0$$

to the former structure equations. Then, the final structure equations of our new equivalence problem to the prolonged space $M_k \times G^{\mathsf{red}}$ take the following $\{e\}$-*structure* constant type:

$$\left[\begin{aligned} & d\,\Gamma_{\ell,i} = (p_i\,\alpha + q_i\,\overline{\alpha}) \wedge \Gamma_{\ell,i} + \textstyle\sum_{\substack{\ell_1+\ell_2=\ell \\ j,n}} \mathsf{c}^i_{j,n}\,\Gamma_{\ell_1,j} \wedge \Gamma_{\ell_2,n} \quad {\scriptstyle (\ell=1,\ldots,\rho,\ \ i=1,\ldots,2+k)}, \\ & d\alpha = 0, \\ & d\overline{\alpha} = 0. \end{aligned}\right. \tag{40}$$

Then we have arrived at the stage of stating the main result of this paper;

Theorem 5.1. *The biholomorphic equivalence problem to a $(2+k)$-dimensional totally nondegenerate CR model M_k of CR dimension 1 and codimension k is reducible to some absolute parallelisms, namely to some certain $\{e\}$-structures on prolonged manifolds $M_k \times G^{\sf red}$ of real dimensions either $3+k$ or $4+k$.*

Weight association. We assign naturally[7] the weight zero to our new lifted 1-forms α and $\overline{\alpha}$.

6. PROOF OF THE BELOSHAPKA'S MAXIMUM CONJECTURE

After providing key results in the previous section, now we are ready to present our proof of the Beloshapka's maximum conjecture. As we saw, the equivalence problem to a certain Beloshapka's CR model M_k is converted finally to that of a prolonged space $M^{\sf pr} := M_k \times G^{\sf red}$ with the final constant type structure equations (40).

According to [23, Theorem 8.16], if the final structure equations of an equivalence problem to an r-dimensional smooth manifold M equipped with some lifted coframe $\gamma^1, \ldots, \gamma^r$ is of the constant type:

$$d\gamma^k = \sum_{1 \leqslant i < j \leqslant r} c_{ij}^k \, \gamma^i \wedge \gamma^j \qquad\qquad (k = 1 \cdots r),$$

then M is (locally) diffeomorphic to an r-dimensional Lie group G corresponding to the Lie algebra $\mathfrak{g}$ with the basis elements $\{{\sf v}_1, \ldots, {\sf v}_r\}$ enjoying the *structure constants*:

$$\big[{\sf v}_i, {\sf v}_j\big] = -\sum_{k=1}^{r} c_{ij}^k \, {\sf v}_k \qquad\qquad (1 \leqslant i < j \leqslant r).$$

Accordingly, let us try to find the Lie algebra $\mathfrak{g}$ corresponding to the constant structure equations (40). At first, we associate to each lifted 1-form $\Gamma_{\ell,i}$ the basis element ${\sf v}_{\ell,i}$ of $\mathfrak{g}$. Also, for the new appearing lifted 1-forms α and $\overline{\alpha}$, let us associate the basis elements ${\sf v}_0$ and ${\sf v}_{\overline{0}}$. Of course, if the real part of a_1 is normalizable (*cf.* Corollary 5.7) then we dispense with ${\sf v}_{\overline{0}}$ since in this case we have $\alpha = \overline{\alpha}$. Then accordingly, our desired Lie algebra $\mathfrak{g}$ will be of dimension either $3+k$ or $4+k$. In particular, because we do not see any wedge product $\alpha \wedge \overline{\alpha}$ among the structure equations (40) then we have $[{\sf v}_0, {\sf v}_{\overline{0}}] = 0$. This means that $\{{\sf v}_0, {\sf v}_{\overline{0}}\}$ generates an *Abelian* subalgebra of $\mathfrak{g}$. Let us assign naturally the weight $-\ell$ to each basis element ${\sf v}_{\ell,i}$ and also the weight zero to ${\sf v}_0$ and ${\sf v}_{\overline{0}}$.

Each structure equation $d\Gamma_{\ell,i}$ in (40) is some constant combination of the wedge products between those lifted 1-forms for which the sum of their weights is exactly $-\ell$. This implies that the Lie bracket between each two basis elements ${\sf v}_\bullet$ of $\mathfrak{g}$ of the weights $-\ell_1$ and $-\ell_2$ will be some constant combination of those basis elements of $\mathfrak{g}$ which are of the same weight $-(\ell_1 + \ell_2)$. This means that;

Proposition 6.1. *The desired Lie algebra $\mathfrak{g}$ associated to the final structure equations* (40) *is graded (in the sense of Tanaka* [21, 22]*) of the form:*

$$\mathfrak{g} := \mathfrak{g}_{-\rho} \oplus \mathfrak{g}_{-(\rho-1)} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

with:

$$\big[\mathfrak{g}_{-\ell_1}, \mathfrak{g}_{-\ell_2}\big] = \mathfrak{g}_{-(\ell_1+\ell_2)}$$

where each $\mathfrak{g}_{-\ell}$ is generated by all basis elements ${\sf v}_{\ell,i}$ of the weight $-\ell$ and $\mathfrak{g}_0$, which is Abelian, is generated by ${\sf v}_0$ and ${\sf v}_{\overline{0}}$. Consequently, $\mathfrak{g}_-$ is $(2+k)$-dimensional where $\mathfrak{g}_0$ is of dimension either 1 or 2.

On the other hand, Corollary 14.20 of [23] says that this Lie algebra $\mathfrak{g}$ is in fact the symmetry Lie algebra of the prolonged space $M^{\sf pr} = M_k \times G^{\sf red}$ with respect to its coframe $(\Gamma_{1,1}, \ldots, \Gamma_{\rho,2+k}, \alpha, \overline{\alpha})$; that is the Lie algebra associated to the Lie group ${\sf G}$ of self-equivalences $\Phi : M^{\sf pr} \to M^{\sf pr}$, satisfying $\Phi^*(\theta) = \theta$ for each $\theta = \Gamma_{1,1}, \ldots, \Gamma_{\rho,2+k}, \alpha, \overline{\alpha}$. But, according to [23, Proposition 12.1] and also its proof, ${\sf G}$ can be identified with the CR symmetry group ${\sf Aut}_{CR}(M)$ of biholomorphic maps $h : M_k \to M_k$ and hence:

$$\mathfrak{aut}_{CR}(M_k) \cong \mathfrak{g}.$$

[7] Notice that the differentiation $d\alpha$ took the value zero, exactly as constant functions. Hence we assign the weight of constant integers to this new 1-form α and its conjugation.

This helps us to complete the proof of the Beloshapka's maximum conjecture;

Theorem 6.1. *(The Beloshapka's maximum conjecture). The Lie algebra $\mathfrak{aut}_{CR}(M_k)$ associated to a Beloshapka's CR model of CR dimension 1 and codimension k and of the length $\rho \geq 3$ — or equivalently of codimension $k \geq 2$ — contains no any homogeneous component of absolutely positive homogeneity. In other words, such CR model is rigid. Moreover, this Lie algebra is of dimension either $3+k$ or $4+k$.* $\square$

Appendix A. An example in the length four

Let us conclude this paper by an illustrative example. In fact, in this appendix we aim to check our method proposed in Section 5 by means of inspecting the equivalence problem to the length four, 8-dimensional Beloshapka's CR model $M_6 \subset \mathbb{C}^7$ of codimension $k = 6$ represented as the graph of six defining polynomials:

$$\begin{aligned} & w_1 - \overline{w}_1 = 2i\, z\overline{z}, \\ & w_2 - \overline{w}_2 = 2i\big(z^2\overline{z} + z\overline{z}^2\big), \quad w_3 - \overline{w}_3 = 2\big(z^2\overline{z} - z\overline{z}^2\big), \\ & w_4 - \overline{w}_4 = 2i\big(z^3\overline{z} + z\overline{z}^3\big), \quad w_5 - \overline{w}_5 = 2\big(z^3\overline{z} - z\overline{z}^3\big), \quad w_6 - \overline{w}_6 = 2i\, z^2\overline{z}^2. \end{aligned}$$

The assigned weights to the extant complex variables are:

$$[z] = 1, \quad [w_1] = 2, \quad [w_2] = [w_3] = 3, \quad [w_4] = [w_5] = [w_6] = 4.$$

Saving the space, we do not present the intermediate calculations. According to our computations according to what explained in Section 3, our initial frame contains eight vector fields of various lengths $-1, \ldots, -4$:

$$\begin{aligned} & \mathscr{L} := \mathscr{L}_{1,1}, \qquad \overline{\mathscr{L}} := \mathscr{L}_{1,2}, \\ & \mathscr{T} := \mathscr{L}_{2,3} = i[\mathscr{L}, \overline{\mathscr{L}}], \\ & \mathscr{S} := \mathscr{L}_{3,4} = [\mathscr{L}, \mathscr{T}], \qquad \overline{\mathscr{S}} := \mathscr{L}_{3,5} = [\overline{\mathscr{L}}, \mathscr{T}], \\ & \mathscr{U} := \mathscr{L}_{4,6} = [\mathscr{L}, \mathscr{S}], \qquad \overline{\mathscr{U}} := \mathscr{L}_{4,7} = [\overline{\mathscr{L}}, \overline{\mathscr{S}}], \qquad \mathscr{V} := \mathscr{L}_{4,8} = [\mathscr{L}, \overline{\mathscr{S}}] = [\overline{\mathscr{L}}, \mathscr{S}]. \end{aligned}$$

The other Lie brackets between these eight initial vector fields are all zero. Assume that:

$$\Sigma := \big(\underbrace{\nu_0, \mu_0, \overline{\mu}_0}_{\text{weight -4}}, \underbrace{\sigma_0, \overline{\sigma}_0}_{\text{weight -3}}, \underbrace{\rho_0}_{\text{weight -2}}, \underbrace{\zeta_0, \overline{\zeta}_0}_{\text{weight -1}}\big)^t \quad \text{is the dual coframe to} \quad \big(\mathscr{V}, \mathscr{U}, \overline{\mathscr{U}}, \mathscr{S}, \overline{\mathscr{S}}, \mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}\big)^t.$$

Then the associated Darboux-Cartan structure to this coframe is:

$$\begin{aligned} & d\nu_0 = \overline{\sigma}_0 \wedge \zeta_0 + \sigma_0 \wedge \overline{\zeta}_0, \quad d\mu_0 = \sigma_0 \wedge \zeta_0, \quad d\overline{\mu}_0 = \overline{\sigma}_0 \wedge \overline{\zeta}_0 \\ & d\sigma_0 = \rho_0 \wedge \zeta_0, \qquad d\overline{\sigma}_0 = \rho_0 \wedge \overline{\zeta}_0, \\ & d\rho_0 = i\zeta_0 \wedge \overline{\zeta}_0, \\ & d\zeta_0 = 0, \qquad d\overline{\zeta}_0 = 0. \end{aligned}$$

Assuming $\Gamma := (\nu, \mu, \overline{\mu}, \sigma, \overline{\sigma}, \rho, \zeta, \overline{\zeta}_0)^t$ as the associated lifted coframe, then our computation reveals the ambiguity matrix of the biholomorphic equivalence problem to M_6 as:

$$\Gamma = \underbrace{\begin{pmatrix} a_1^2\overline{a}_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1^3\overline{a}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1\overline{a}_1^3 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & a_6 & 0 & a_1^2\overline{a}_1 & 0 & 0 & 0 & 0 \\ \overline{a}_{13} & 0 & \overline{a}_6 & 0 & a_1\overline{a}_1^2 & 0 & 0 & 0 \\ a_{11} & a_7 & \overline{a}_7 & a_3 & \overline{a}_3 & a_1\overline{a}_1 & 0 & 0 \\ a_{12} & a_8 & \overline{a}_9 & a_4 & a_5 & a_2 & a_1 & 0 \\ \overline{a}_{12} & a_9 & \overline{a}_8 & \overline{a}_5 & \overline{a}_4 & \overline{a}_2 & 0 & \overline{a}_1 \end{pmatrix}}_{\mathbf{g}} \cdot \Sigma, \tag{41}$$

with the assigned weights:

$$[a_1]=1,\quad [a_2]=2,\quad [a_3]=[a_4]=[a_5]=3,\quad [a_6]=\ldots=[a_{13}]=4.$$

By computing the (somehow big) inverse matrix $\mathbf{g}^{-1}$, one can check also the assertion of some results like Lemmas 4.2 and 4.3. Also, our Maurer-Cartan matrix is of the form:

$$\omega_{\tt MC}:=\begin{pmatrix} 2\alpha+2\overline{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 3\alpha+\overline{\alpha} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \alpha+3\overline{\alpha} & 0 & 0 & 0 & 0 & 0\\ \delta_{13} & \delta_6 & 0 & 2\alpha+\overline{\alpha} & 0 & 0 & 0 & 0\\ \overline{\delta}_{13} & 0 & \overline{\delta}_6 & 0 & \alpha+2\overline{\alpha} & 0 & 0 & 0\\ \delta_{11} & \delta_7 & \overline{\delta}_7 & \delta_3 & \overline{\delta}_3 & \alpha+\overline{\alpha} & 0 & 0\\ \delta_{12} & \delta_8 & \overline{\delta}_9 & \delta_4 & \delta_5 & \delta_2 & \alpha & 0\\ \overline{\delta}_{12} & \delta_9 & \overline{\delta}_8 & \overline{\delta}_5 & \overline{\delta}_4 & \overline{\delta}_2 & 0 & \overline{\alpha} \end{pmatrix}.$$

Then, our structure equations will be of the form — we abbreviate the combinations of the wedge products $\delta_j\wedge\bullet$ just by some " $\cdots$ " since they will not play any important role:

$$\begin{aligned} d\nu &= (2\alpha+2\overline{\alpha})\wedge\nu + a_1^2\overline{a}_1^2\, d\nu_0,\\ d\mu &= (3\alpha+\overline{\alpha})\wedge\mu + a_1^3\overline{a}_1\, d\mu_0,\\ d\sigma &= \cdots + (2\alpha+\overline{\alpha})\wedge\sigma + a_{13}\, d\nu_0 + a_6\, d\mu_0 + a_1^2\overline{a}_1\, d\sigma_0,\\ d\rho &= \cdots + (\alpha+\overline{\alpha})\wedge\rho + a_{11}\, d\nu_0 + a_7\, d\mu_0 + \overline{a}_7\, d\overline{\mu}_0 + a_3\, d\sigma_0 + \overline{a}_3\, d\overline{\sigma}_0 + a_1\overline{a}_1\, d\rho_0,\\ d\zeta &= \cdots + \alpha\wedge\zeta + a_{12}\, d\nu_0 + a_8\, d\mu_0 + \overline{a}_9\, d\overline{\mu}_0 + a_4\, d\sigma_0 + a_5\, d\overline{\sigma}_0 + a_2\, d\rho_0 + a_1\, d\zeta_0. \end{aligned} \tag{42}$$

Now, let us proceed along the lines of subsection 5.2 to pick the appropriate weighted homogeneous system S. To do it and as is the method of absorption-normalization step, first we apply the substitutions:

$$\begin{aligned} \alpha &\mapsto \alpha + t_8\,\nu + t_7\,\mu + \ldots + t_2\,\overline{\zeta} + t_1\,\zeta,\\ \delta_j &\mapsto \delta_j + s_8^j\,\nu + s_7^j\,\mu + \ldots + s_2^j\,\overline{\zeta} + s_1^j\,\zeta, \qquad j=2,\ldots,13, \end{aligned}$$

on the above structure equations. According to our proposed method of constructing S, for the minimum weight -4 structure equations $d\nu$ and $d\mu$, we have to compute the coefficients of $\nu\wedge\{\zeta,\overline{\zeta}\}$ and $\mu\wedge\{\zeta,\overline{\zeta}\}$ $d\mu$ respectively. Moreover, in the length -3 structure equation $d\sigma$, we should pick up the coefficients of $\sigma\wedge\{\zeta,\overline{\zeta}\}$ since respectively $\sigma_0\wedge\zeta_0$ and $\sigma_0\wedge\overline{\zeta}_0$ uniquely appear in the Darboux-Cartan structure of the only extant length -4 differentiations $d\mu_0$ and $d\nu_0$ visible in this structure equation. Similarly, for the length -2 and -1 structure equations $d\rho$ and $d\zeta$, we should pick the coefficients of $\rho\wedge\{\zeta,\overline{\zeta}\}$ and $\zeta\wedge\overline{\zeta}$, respectively. Equating these coefficients to zero gives respectively:

$$\begin{aligned} &\mathsf{S}_{d\nu}:=\Big\{-\frac{\overline{a}_{13}}{a_1^2\overline{a}_1^2}=2t_1+2\overline{t}_2\Big\}, \qquad \mathsf{S}_{d\mu}:=\Big\{-\frac{a_6}{a_1^3\overline{a}_1}=3t_1+\overline{t}_2,\quad 0=\overline{t}_1+3t_2\Big\},\\ &\mathsf{S}_{d\sigma}:=\Big\{\frac{a_6}{a_1^3\overline{a}_1}-\frac{a_3}{a_1^2\overline{a}_1}=2t_1+\overline{t}_2,\quad \frac{\overline{a}_{13}}{a_1^2\overline{a}_1^2}=\overline{t}_1+2t_2\Big\},\\ &\mathsf{S}_{d\rho}:=\Big\{\frac{a_3}{a_1^2\overline{a}_1}+i\frac{\overline{a}_2}{a_1\overline{a}_1}=t_1+\overline{t}_2\Big\},\\ &\mathsf{S}_{d\zeta}:=\Big\{i\frac{a_2}{a_1\overline{a}_1}=t_2\Big\}, \end{aligned}$$

where S is the union of these five systems. After putting the obtained expressions of the parameters t_1 and t_2 into these equations and multiplying them by sufficient powers of a_1 and $\overline{a}_1$, one receives the following weighted homogeneous system:

$$\mathsf{S}:=\Big\{\overline{a}_{13}+2\,\overline{a}_1a_3+2i\,a_1\overline{a}_1\overline{a}_2=0,\quad a_6+3\,a_1a_3+5i\,a_1^2\overline{a}_2=0,\quad \overline{a}_3+i\,\overline{a}_1a_2=0,$$
$$a_6-3\,a_1a_3-3i\,a_1^2\overline{a}_2=0,\quad \overline{a}_{13}-a_1\,\overline{a}_3=0\Big\}.$$

Solving this weighted homogeneous system, first let us plainly dehomogenize it by equating the weight one parameter a_1 to 1. Then, either by hand or by means of some computer softwares, one versifies that the only solution of the obtained dehomogenized system is nothing but:

$$a_2 = a_3 = a_6 = a_{13} \equiv 0$$

which immediately implies that all the group parameters $a_2, a_3, a_4, \ldots, a_{13}$ can be normalized to zero. As we check, the only remaining group parameter a_1 is not normalizable. Applying these results and after one prolongation, the first structure equations (42) converts to the simple constant form:

$$\begin{aligned}
d\nu &= (2\alpha + 2\overline{\alpha}) \wedge \nu + \overline{\sigma} \wedge \zeta + \sigma \wedge \overline{\zeta}, \\
d\mu &= (3\alpha + \overline{\alpha}) \wedge \mu + \sigma \wedge \zeta, \\
d\sigma &= (2\alpha + \overline{\alpha}) \wedge \sigma + \rho \wedge \zeta, \\
d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\
d\zeta &= \alpha \wedge \zeta \\
d\alpha &= 0.
\end{aligned}$$

Proposition A.1. *The Lie algebra $\mathfrak{g}$ associated to these structure equations is 10-dimensional with the basis elements $\{\mathsf{v}^\nu, \mathsf{v}^\mu, \mathsf{v}^{\overline{\mu}}, \mathsf{v}^\sigma, \mathsf{v}^{\overline{\sigma}}, \mathsf{v}^\rho, \mathsf{v}^\zeta, \mathsf{v}^{\overline{\zeta}}, \mathsf{v}^\alpha, \mathsf{v}^{\overline{\alpha}}\}$ and with the Lie brackets, displayed in the following table:*

	v^ν	v^μ	$\mathsf{v}^{\overline{\mu}}$	v^σ	$\mathsf{v}^{\overline{\sigma}}$	v^ρ	v^ζ	$\mathsf{v}^{\overline{\zeta}}$	v^α	$\mathsf{v}^{\overline{\alpha}}$
v^ν	0	0	0	0	0	0	0	0	$2\mathsf{v}^\nu$	$2\mathsf{v}^\nu$
v^μ	$*$	0	0	0	0	0	0	0	$3\mathsf{v}^\mu$	v^μ
$\mathsf{v}^{\overline{\mu}}$	$*$	$*$	0	0	0	0	0	0	$\mathsf{v}^{\overline{\mu}}$	$3\mathsf{v}^{\overline{\mu}}$
v^σ	$*$	$*$	$*$	0	0	0	$-\mathsf{v}^\mu$	$-\mathsf{v}^\nu$	$2\mathsf{v}^\sigma$	v^σ
$\mathsf{v}^{\overline{\sigma}}$	$*$	$*$	$*$	$*$	0	0	$-\mathsf{v}^\nu$	$-\mathsf{v}^{\overline{\mu}}$	$\mathsf{v}^{\overline{\sigma}}$	$2\mathsf{v}^{\overline{\sigma}}$
v^ρ	$*$	$*$	$*$	$*$	$*$	0	$-\mathsf{v}^\sigma$	$-\mathsf{v}^{\overline{\sigma}}$	v^ρ	v^ρ
v^ζ	$*$	$*$	$*$	$*$	$*$	$*$	0	$-i\mathsf{v}^\rho$	v^ζ	0
$\mathsf{v}^{\overline{\zeta}}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0	$\mathsf{v}^{\overline{\zeta}}$
v^α	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0
$\mathsf{v}^{\overline{\alpha}}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

This Lie algebra is graded of the form:

$$\mathfrak{g} := \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

with $\mathfrak{g}_{-4} = \langle \mathsf{v}^\nu, \mathsf{v}^\mu, \mathsf{v}^{\overline{\mu}} \rangle$, with $\mathfrak{g}_{-3} = \langle \mathsf{v}^\sigma, \mathsf{v}^{\overline{\sigma}} \rangle$, with $\mathfrak{g}_{-2} = \langle \mathsf{v}^\rho \rangle$, with $\mathfrak{g}_{-1} = \langle \mathsf{v}^\zeta, \mathsf{v}^{\overline{\zeta}} \rangle$ and with $\mathfrak{g}_0 = \langle \mathsf{v}^\alpha, \mathsf{v}^\alpha \rangle$.

Acknowledgment. The research of the author was supported in part by a grant from IPM, No. 94510061.

REFERENCES

[1] V. K. Beloshapka, *Model-surface method: an infinite-dimensional version*, Proc. Steklov Inst. Math., **279** (2012) 14–24.
[2] V. K. Beloshapka, *A generic CR-manifold as an $\{e\}$-structure*, Russian J. Math. Phys., **14**(1) (2007) 1–7.
[3] V. K. Beloshapka, *Universal models for real submanifolds*, Math. Notes, **75**(4), (2004), 475–488.
[4] V. K. Beloshapka, *Polynomial models of real manifolds*, Izvestiya: Math., **65**(4), (2001), 641–657.
[5] V. K. Beloshapka, *Invariants of CR-manifolds associated with the tangent quadric*, Math. Notes, **59**(1), (1996), 31–38.
[6] V. K. Beloshapka, I. Kossovskiy, *Classification of homogeneous CR-manifolds in dimension 4*, Math. Anal. Appl., **374**, (2011), 655–672.
[7] A. Boggess, *CR manifolds and the tangential Cauchy-Riemann complex*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991, xviii+364 pp.
[8] É. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes II*, Ann. Scuola Norm. Sup. Pisa **1** (1932), 333–354.
[9] S. S. Chern, Y. Moser, *Real hypersurfaces in complex manifold*, Acta Math., **133**, (1974), 219–271.
[10] D. A. Cox, L. Little, D. O'Shea, *Ideals, Varieties and Algorithms: an introduction to computational algebraic geometry and commutative algebra,* 2nd edition, Springer-Verlag 1997, 536 pp.

[11] W. Decker, C. Lossen, *Computing in Algebraic Geometry: a quick start using* SINGULAR, Algorithms and Computation in Mathematics, vol. 16, Springer-Verlag, 2006, 327 pp.
[12] P. Ebenfelt, *Uniformly Levi degenerate CR manifolds: the* 5*-dimensional case*, Duke Math. J., **110**(1), (2001), 37–80.
[13] R. V. Gammel, I. Kossovskiy, *The envelope of holomorphy of a model surface of the third degree and the 'rigidity' phenomenon*, Proc. Steklov Inst. Math. **2006**, no. 2 (253), 22–37.
[14] S. G. Krantz, *Function theory of several complex variables*, 2nd ed., AMS Chelsea Pub., 2001, 564 pp.
[15] I. B. Mamai, *Moduli spaces of model surfaces with one-dimensional complex tangent*, Izvestiya: Math., **77**(2), (2013), 354–377.
[16] I. B. Mamai, *Model CR-manifolds with one-dimensional complex tangent*, Russian J. Math. Phys., **16**(1), (2009), 97–102.
[17] J. Merker, *Equivalences of* 5*-dimensional CR manifolds, III: Six models and (very) elementary normalizations*, 54 pages, arxiv.org/abs/1311.7522/
[18] J. Merker, *Sophus Lie and Friedrich Engel's Theory of Transformation Groups (Vol. I, 1888). Modern Presentation and English Translation*, Springer-Verlag Berlin Heidelberg, 2015.
[19] J. Merker, S. Pocchiola, M. Sabzevari, *Equivalences of* 5*-dimensional CR manifolds, II: General classes* I, II, III_1, III_2, IV_1, IV_2, 5 figures, 95 pages, arxiv.org/abs/1311.5669/.
[20] J. Merker, E. Porten, *Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities*, Int. Math. Res. Surv., Volume **2006**, Article ID 28295, 287 pp.
[21] J. Merker, M. Sabzevari, *Cartan equivalence problem for* 5*-dimensional bracket-generating CR-manifolds in* $\mathbb{C}^4$, J. Geo. Anal., to appear, expanded form arxiv:1401.4297v1, 172 pp.
[22] J. Merker, M. Sabzevari, *Explicit expression of Cartan's connections for Levi-nondegenerate 3-manifolds in complex surfaces, and identification of the Heisenberg sphere*, Cent. Eur. J. Math., **10**(5), (2012), 1801–1835.
[23] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge, Cambridge University Press, 1995, xvi+525 pp.
[24] S. Pocchiola, Le Problème d'èquivalence Pour les Variétés de Cauchy-Riemann en Dimension 5, Ph. D. Thesis, Paris-Sud 11, 2014.
[25] H. Poincaré, *Les fonction analytiques de deux variables et la représentation conforme*, Rend. Circ. Math. Palermo, **23**, (1907), 185–220.
[26] M. Sabzevari, *Moduli spaces of model real submanifolds: two alternative approaches*, Sci. China Math., to appear.
[27] M. Sabzevari, A. Hashemi, B. M.-Alizadeh, J. Merker, *Applications of differential algebra for computing Lie algebras of infinitesimal CR-automorphisms*, Sci. China Math., **57** (2014)9, 1811–1834.
[28] M. Sabzevari, A. Hashemi, B. M.-Alizadeh, J. Merker, *Lie algebras of infinitesimal CR-automorphisms of weighted homogeneous and homogeneous CR-generic submanifolds of* $\mathbb{C}^N$, FiloMat, to appear.
[29] M. Sabzevari, J. Merker, *The Cartan equivalence problem for Levi-non-degenerate real hypersurfaces* $M^3 \subset \mathbb{C}^2$, Izvestiya: Math., **78**(6), (2014), 1158–1194.
[30] M. Sabzevari, J. Merker, S. Pocchiola, *Canonical Cartan connections on maximally minimal generic submanifolds* $M^5 \subset \mathbb{C}^4$, Elect. Res. Ann. Math. Sci. (ERA-MS), **21**, (2014), 153–166.
[31] E. N. Shananina, *Polynomial models of degree 5 and algebras of their automorphisms*, Math. Notes, **75**(5), (2004), 702–716.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SHAHREKORD, 88186-34141 SHAHREKORD, IRAN AND SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P. O. BOX: 19395-5746, TEHRAN, IRAN
E-mail address: sabzevari@math.iut.ac.ir