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3-Lie_∞-algebras and 3-Lie 2-algebras *

Yanqiu Zhou, Yumeng Li and Yunhe Sheng
Department of Mathematics, Jilin University,
Changchun 130012, China
email:yqzhou15@mails.jlu.edu.cn, liym1009@mails.jlu.edu.cn, shengyh@jlu.edu.cn

Abstract

In this paper, we introduce the notions of a 3-Lie_∞-algebra and a 3-Lie 2-algebra. The former is a model for a 3-Lie algebra that satisfy the fundamental identity up to all higher homotopies, and the latter is the categorification of a 3-Lie algebra. We prove that the 2-category of 2-term 3-Lie_∞-algebras is equivalent to the 2-category of 3-Lie 2-algebras. Skeletal and strict 3-Lie 2-algebras are studied in detail. A construction of a 3-Lie 2-algebra from a symplectic 3-Lie algebra is given.

1 Introduction

The notion of a Filippov algebra, or an n-Lie algebra was introduced in [14]. It is the algebraic structure corresponding to Nambu mechanics [15, 20, 25]. Recently, due to applications in the Bagger–Lambert–Gustavsson theory of multiple M2-branes [3, 10, 16, 17, 22, 27], n-Lie algebras, or more generally, n-Leibniz algebras, are widely studied [5, 6, 7, 8, 9, 13, 18]. See the review article [11] for more details.

Recently, people study higher categorical structures with motivations from string theory. One way to provide higher categorical structures is by categorifying existing mathematical concepts. One of the simplest higher structure is a 2-vector space, which is a categorified vector space. If we further put Lie algebra structures on 2-vector spaces, then we obtain the notion of Lie 2-algebras [2]. The Jacobi identity is replaced by a natural transformation, called Jacobiator, which also satisfies some coherence laws of its own. The 2-category of Lie 2-algebras is equivalent to the 2-category of L_∞-algebras. L_∞-algebras, also called strong homotopy Lie algebras, were introduced in [24]. See [12, 21] for the cohomology and deformation theory of L_∞-algebras. As a model for "Leibniz algebras that satisfy Jacobi identity up to all higher homotopies", the notion of a strongly homotopy (sh) Leibniz algebra, or a Lod_∞-algebra was given in [19] by Livernet, which is further studied by Ammar, Poncin and Uchino in [1, 26]. In [23], the authors introduced the notion of a Leibniz 2-algebra, which is the categorification of a Leibniz algebra, and prove that the category of Leibniz 2-algebras and the category of 2-term Lod_∞-algebras are equivalent.

The aim of this paper is to provide a model for 3-Lie algebras that satisfy the fundamental identity up to all higher homotopies, and give the categorification of 3-Lie algebras. It is well-known that there is a Leibniz algebra structure on the space of fundamental objects associated to a 3-Lie

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algebra (more generally, n-Lie algebras). We define 3-Lie_∞-algebras in such a way that there is a Lod_∞-algebra structure on the graded vector space of fundamental objects. Furthermore, we put 3-Lie algebra structures on 2-vector spaces and obtain 3-Lie 2-algebras. Similar to the relation between Lie 2-algebras and 2-term L_∞-algebras, we prove that the 2-category of 2-term 3-Lie_∞-algebras and the 2-category of 3-Lie 2-algebras are equivalent. Finally, we study skeletal 3-Lie 2-algebras and strict 3-Lie 2-algebras. In particular, we show that strict 3-Lie 2-algebras are equivalent to crossed modules of 3-Lie algebras. The more general notion of crossed modules of n-Leibniz had been given in [8], and the relation with the third cohomology group is established there.

The paper is organized as follows. In Section 2, we review 3-Lie algebras and their cohomology, 2-vector spaces and Lod_∞-algebras. In Section 3, we define 3-Lie_∞-algebras. We show that given a 3-Lie_∞-algebra $(\mathcal{V}, \{l_{2n+1}\}_{n=0}^{\infty})$, we can obtain a Lod_∞-algebra $(\wedge^2\mathcal{V}, \{\mathfrak{l}_n\}_{n=1}^{\infty})$ (Theorem 3.2). Then we focus on 2-term 3-Lie_∞-algebras and show that there is a 2-category of 2-term 3-Lie_∞-algebras. In Section 4, we give the notion of a 3-Lie 2-algebra, and prove that the 2-category of 2-term 3-Lie_∞-algebras is equivalent to the 2-category of 3-Lie 2-algebras (Theorem 4.5). In Section 5, first we classify skeletal 3-Lie 2-algebras using the third cohomology group. Then we show that there is a one-to-one correspondence between strict 3-Lie 2-algebras and crossed modules of 3-Lie algebras. Finally, we construct 3-Lie 2-algebras from symplectic 3-Lie algebras via the underlying 3-pre-Lie algebras given in [4].

2 Preliminaries

2.1 3-Lie algebras and their cohomology

Definition 2.1. *A 3-Lie algebra $\mathfrak{g}$ is a vector space together with a trilinear fully skew-symmetric map $[\cdot,\cdot,\cdot] : \wedge^3\mathfrak{g} \longrightarrow \mathfrak{g}$, the 3-bracket, such that the following fundamental identity is satisfied:*

$$[x_1, x_2, [x_3, x_4, x_5]] = [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]], \ \forall x_i \in \mathfrak{g}. \quad (1)$$

Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra. For all $x, y \in \mathfrak{g}$, define $\mathrm{ad}_{(x,y)} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by $\mathrm{ad}_{(x,y)}z = [x, y, z]$. Then Eq. (1) is equivalent to that $\mathrm{ad}_{(x,y)}$ is a derivation. Elements in $\wedge^2\mathfrak{g}$ are called **fundamental objects** of the 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot])$. There is a bilinear operation $[\cdot,\cdot]_{\mathrm{F}}$ on $\wedge^2\mathfrak{g}$, which is given by

$$[\mathfrak{X}, \mathfrak{Y}]_{\mathrm{F}} = [x_1, x_2, y_1] \wedge y_2 + y_1 \wedge [x_1, x_2, y_2], \quad \forall \mathfrak{X} = x_1 \wedge x_2,\ \mathfrak{Y} = y_1 \wedge y_2. \quad (2)$$

It is well-known that $(\wedge^2\mathfrak{g}, [\cdot,\cdot]_{\mathrm{F}})$ is a Leibniz algebra, which plays important role in the theory of 3-Lie algebras.

Definition 2.2. *Let V be a vector space. A representation of a 3-Lie algebra $\mathfrak{g}$ on V is a bilinear map $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, such that*

$$\begin{aligned}
[\rho(\mathfrak{X}), \rho(\mathfrak{Y})] &= \rho([\mathfrak{X}, \mathfrak{Y}]_{\mathrm{F}}),\ \forall \mathfrak{X}, \mathfrak{Y} \in \wedge^2\mathfrak{g},\\
\rho(x, [y_1, y_2, y_3]) &= \rho(y_2, y_3)\rho(x, y_1) - \rho(y_1, y_3)\rho(x, y_2) + \rho(y_1, y_2)\rho(x, y_3),\ \forall x, y_i \in \mathfrak{g}.
\end{aligned}$$

We denote a representation by $(V; \rho)$. A p-cochain on $\mathfrak{g}$ with the coefficients in a representation $(V; \rho)$ is a linear map $\alpha^p : \wedge^2\mathfrak{g} \otimes \overset{(p-1)}{\cdots} \otimes \wedge^2\mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$. Denote the space of p-cochains by $C^p(\mathfrak{g}; V)$.

The coboundary operator $\delta : C^p(\mathfrak{g}; V) \longrightarrow C^{p+1}(\mathfrak{g}; V)$ is given by

$$
\begin{aligned}
&(\delta\alpha^p)(\mathfrak{X}_1, \dots, \mathfrak{X}_p, z) \\
= &\sum_{1\le j<k} (-1)^j \alpha^p(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_{k-1}, [\mathfrak{X}_j, \mathfrak{X}_k]_{\mathrm{F}}, \mathfrak{X}_{k+1}, \dots, \mathfrak{X}_p, z) \\
&+\sum_{j=1}^{p} (-1)^j \alpha^p(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_p, [\mathfrak{X}_j, z]) \\
&+\sum_{j=1}^{p} (-1)^{j+1} \rho(\mathfrak{X}_j)\alpha^p(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_p, z) \\
&+(-1)^{p+1}\Big(\rho(y_p, z)\alpha^p(\mathfrak{X}_1, \dots, \mathfrak{X}_{p-1}, x_p) + \rho(z, x_p)\alpha^p(\mathfrak{X}_1, \dots, \mathfrak{X}_{p-1}, y_p)\Big),
\end{aligned}
$$

for all $\mathfrak{X}_i = (x_i, y_i) \in \wedge^2\mathfrak{g}$ and $z \in \mathfrak{g}$.

2.2 2-vector spaces and Lod_∞-algebras

Vector spaces can be categorified to 2-vector spaces. A good introduction for this subject is [2]. Let Vect be the category of vector spaces.

Definition 2.3. [2] *A 2-vector space is a category in the category* Vect.

Thus a 2-vector space C is a category with a vector space of objects C_0 and a vector space of morphisms C_1, such that all the structure maps are linear. Let $s, t : C_1 \longrightarrow C_0$ be the source and target maps respectively. Let $\cdot_{\mathrm{v}}$ be the composition of morphisms. Let $1 : C_0 \longrightarrow C_1$ be the unit map, i.e. for all $x \in C_0$, $1_x \in C_1$ is the identity morphism from x to x.

It is well known that the 2-category of 2-vector spaces is equivalent to the 2-category of 2-term complexes of vector spaces. Roughly speaking, given a 2-vector space C, we have a 2-term complex of vector spaces

$$\mathrm{Ker}(s) \xrightarrow{t} C_0. \tag{3}$$

Conversely, any 2-term complex of vector spaces $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$ gives rise to a 2-vector space of which the set of objects is V_0, the set of morphisms is $V_0 \oplus V_1$, the source map s is given by $s(v + m) = v$, and the target map t is given by $t(v + m) = v + \mathrm{d}m$, where $v \in V_0,\ m \in V_1$. We denote the 2-vector space associated to the 2-term complex of vector spaces $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$ by $\mathbb{V}$:

$$\mathbb{V} = \begin{array}{c} \mathbb{V}_1 := V_0 \oplus V_1 \\ \mathrm{s}\Big\downarrow\Big\downarrow\mathrm{t} \\ \mathbb{V}_0 := V_0. \end{array} \tag{4}$$

The notion of Lod_∞-algebras, also called strongly homotopy (sh) Leibniz algebras, was introduced in [19] and well studied in [1, 26].

Definition 2.4. *A Lod_∞-algebra is a graded vector space $L = L_0 \oplus L_1 \oplus \cdots$ equipped with a system $\{l_k|\ 1 \le k < \infty\}$ of linear maps $l_k : \otimes^k L \longrightarrow L$ with degree $\deg(l_k) = k - 2$, where the exterior powers are interpreted in the graded sense and the following relation is satisfied:*

$$
\begin{aligned}
&\sum_{i+j=n+1} \sum_{j\le k\le n} \sum_{\sigma} (-1)^{(k+1-j)(j-1)} (-1)^{j(|x_{\sigma(1)}|+\cdots+|x_{\sigma(k-j)}|)} \sum_{\sigma} \mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma) \\
&l_i(x_{\sigma(1)}, \dots, x_{\sigma(k-j)}, l_j(x_{\sigma(k+1-j)}, \dots, x_{\sigma(k-1)}, x_k), x_{k+1}, \dots, x_n) = 0,
\end{aligned}
$$

where the summation is taken over all $(k-j, j-1)$-unshuffles and "$\mathrm{Ksgn}(\sigma)$*" is the Koszul sign for a permutation $\sigma \in S_k$, i.e.*

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k = \mathrm{Ksgn}(\sigma) x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)}.$$

We will denote a Lod_∞-algebra by $(\mathcal{V}, \{l_k\}_{k=1}^\infty)$.

3 3-Lie_∞-algebras

In this section, we give the notion of a 3-Lie_∞-algebra and show that associated to a 3-Lie_∞-algebra, there is naturally a Lod_∞-algebra structure on the graded space of fundamental objects. Furthermore, we focus on 2-term 3-Lie_∞-algebras and construct the corresponding 2-category.

For all $n \in \mathbb{N}$ and $1 \le j \le k \le n$, we say that $\sigma \in S_k$ is a $(k-j, j)$-unshuffle if

$$\sigma(1) \le \sigma(2) \le \cdots \le \sigma(k-j) \quad \textit{and} \quad \sigma(k-j+1) \le \sigma(k-j+2) \le \cdots \le \sigma(k).$$

Definition 3.1. *A 3-Lie_∞-algebra is a graded vector space $\mathcal{V} = V_0 \oplus V_1 \oplus \cdots$, equipped with a system $\{l_{2n+1} : n = 0, 1, 2, \cdots\}$ of linear maps: $l_{2n+1} : (\wedge^2\mathcal{V}) \otimes \overset{n}{\cdots} \otimes (\wedge^2\mathcal{V}) \wedge \mathcal{V} \longrightarrow \mathcal{V}$, with degree $n-1$, such that for all $X_a = (x_a, y_a) \in \wedge^2\mathcal{V}$ and $x \in \mathcal{V}$, the following relation is satisfied:*

$$\begin{aligned}
&\sum_{i+j=n-1} \sum_{j \le k \le n-1} \sum_{\sigma \in (k-j-1, j)\text{-}unshuffle} (-1)^{(k-j)(j)} (-1)^{(j+1)(|X_{\sigma(1)}| + \cdots + |X_{\sigma(k-j-1)}|)} \mathrm{sgn}(\sigma) \mathrm{Ksgn}(\sigma) \\
&l_{2i+1}(X_{\sigma(1)}, \ldots, X_{\sigma(k-j-1)}, \widetilde{l}_{2j+1}(X_{\sigma(k-j)}, \ldots, X_{\sigma(k-1)}, X_k), X_{k+1}, \ldots, X_{n-1}, x) \\
&+ \sum_{i+j=n-1} \sum_{\sigma \in (i,j)\text{-}unshuffle} (-1)^{(i+1)(j)} (-1)^{(j+1)(|X_{\sigma(1)}| + \cdots + |X_{\sigma(i)}|)} \mathrm{sgn}(\sigma) \mathrm{Ksgn}(\sigma) \\
&l_{2i+1}(X_{\sigma(1)}, \ldots, X_{\sigma(i)}, l_{2j+1}(X_{\sigma(i+1)}, \ldots, X_{\sigma(n-1)}, x)) = 0, \qquad (5)
\end{aligned}$$

where $\widetilde{l}_{2n+1} : \otimes^{n+1}(\wedge^2\mathcal{V}) \longrightarrow \wedge^2\mathcal{V}$ is induced by l_{2n+1} via

$$\begin{aligned}
\widetilde{l}_{2n+1}(X_1, X_2, \cdots, X_{n+1}) \;\triangleq\; & (-1)^{|x_{n+1}|(|X_1| + \ldots + |X_n| + n - 1)} x_{n+1} \wedge l_{2n+1}(X_1, \cdots, X_n, y_{n+1}) \\
& + l_{2n+1}(X_1, \cdots, X_n, x_{n+1}) \wedge y_{n+1}. \qquad (6)
\end{aligned}$$

We will denote a 3-Lie_∞-algebra by $(\mathcal{V}, \{l_{2n+1}\}_{n=0}^\infty)$. By (5), we have $l_1 \circ l_1 = 0$, which makes $(\mathcal{V}, l_1)$ being a complex of vector spaces. Thus, we will write $\mathrm{d} = l_1$ sometimes.

Similar as the fact that there is a Leibniz algebra structure on the space of fundamental objects associated to a 3-Lie algebra, we have a Lod_∞-algebra structure on the graded space of fundamental objects associated to a 3-Lie_∞-algebra.

Theorem 3.2. *Let $(\mathcal{V}, \{l_{2n+1}\}_{n=0}^\infty)$ be a 3-Lie_∞-algebra. Then, $(\wedge^2\mathcal{V}, \{\mathfrak{l}_n\}_{n=1}^\infty)$ is a Lod_∞-algebra, where $\mathfrak{l}_n = \widetilde{l}_{2n-1}$ is given by (6).*

Proof. Denote the left hand side of (5) by $\Xi(X_1,\cdots,X_{n-1},x)$, we have

$$
\begin{aligned}
&\sum_{i+j=n+1}\sum_{1\le j\le k\le n}\sum_{\sigma}(-1)^{(k+1-j)(j-1)}(-1)^{j(|X_{\sigma(1)}|+\cdots+|X_{\sigma(k-j)}|)}\sum_{\sigma}\mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma)\\
&\mathfrak{l}_i(X_{\sigma(1)},\ldots,X_{\sigma(k-j)},\mathfrak{l}_j(X_{\sigma(k+1-j)},\ldots,X_{\sigma(k-1)},X_k),X_{k+1},\ldots,X_n)\\
=\;&\sum_{i+j=n+1}\sum_{j\le k\le n-1}\sum_{\sigma}(-1)^{(k+1-j)(j-1)}(-1)^{j(|X_{\sigma(1)}|+\cdots+|X_{\sigma(k-j)}|)}\sum_{\sigma}\mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma)\\
&\Big(l_{2i-1}\big(X_{\sigma(1)},\ldots,X_{\sigma(k-j)},\widetilde{l}_{2j-1}(X_{\sigma(k+1-j)},\ldots,X_{\sigma(k-1)},X_k),X_{k+1},\ldots,X_{i+j-2},x_n\big)\wedge y_n\\
&+(-1)^{(|X_{\sigma(1)}|+\cdots+|X_{\sigma(n-1)}|+i-2+j-2)|x_n|}\\
&x_n\wedge l_{2i-1}\big(X_{\sigma(1)},\ldots,X_{\sigma(k-j)},\widetilde{l}_{2j-1}(X_{\sigma(k+1-j)},\ldots,X_{\sigma(k-1)},X_k),X_{k+1},\ldots,X_{n-1},y_n\big)\Big)\\
&+\sum_{i+j=n+1}\sum_{\sigma}(-1)^{i(j-1)}(-1)^{j(|X_{\sigma(1)}|+\cdots+|X_{\sigma(i-1)}|)}\sum_{\sigma}\mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma)\\
&\widetilde{l}_{2i-1}\Big(X_{\sigma(1)},\ldots,X_{\sigma(i-1)},l_{2j-1}(X_{\sigma(i)},\ldots,X_{\sigma(n-1)},x_n)\wedge y_n\\
&+(-1)^{|x_n|(|X_{\sigma(i)}|+\cdots+|X_{\sigma(n-1)}|+j-2)}x_n\wedge l_{2j-1}(X_{\sigma(i)},\ldots,X_{\sigma(n-1)},y_n)\Big)\\
=\;&\sum_{i+j=n-1}\sum_{0\le j\le k\le n-1}\sum_{\sigma}(-1)^{(k-j)j}(-1)^{(j+1)(|X_{\sigma(1)}|+\cdots+|X_{\sigma(k-j-1)}|)}\sum_{\sigma}\mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma)\\
&l_{2i+1}(X_{\sigma(1)},\ldots,X_{\sigma(k-j-1)},\widetilde{l}_{2j+1}(X_{\sigma(k-j)},\ldots,X_{\sigma(k-1)},X_k),X_{k+1},\ldots,X_{n-1},x_n)\wedge y_n\\
&+(-1)^{(|X_{\sigma(1)}|+\cdots+|X_{\sigma(i+j)}|+i+j-2)|x_{i+j+1}|}\\
&x_n\wedge l_{2i+1}(X_{\sigma(1)},\ldots,X_{\sigma(k-j-1)},\widetilde{l}_{2j+1}(X_{\sigma(k-j)},\ldots,X_{\sigma(k-1)},X_k),X_{k+1},\ldots,X_{n-1},y_n)\\
&+\sum_{i+j=n-1}\sum_{\sigma}(-1)^{(i+1)j}(-1)^{(j+1)(|X_{\sigma(1)}|+\cdots+|X_{\sigma(i)}|)}\sum_{\sigma}\mathrm{sgn}(\sigma)\mathrm{Ksgn}(\sigma)\\
&\{l_{2i+1}\Big(X_{\sigma(1)},\ldots,X_{\sigma(i)},l_{2j+1}(X_{\sigma(i+1)},\ldots,X_{\sigma(n-1)},x_n)\Big)\wedge y_n\\
&+(-1)^{|x_n|(|X_{\sigma(i+1)}|+\cdots+|X_{\sigma(n-1)}|+j-1)+|x_n|(|X_{\sigma(1)}|+\cdots|X_{\sigma(i)}|+i-1)}\\
&x_n\wedge l_{2i+1}\Big(X_{\sigma(1)},\ldots,X_{\sigma(i)},l_{2j+1}(X_{\sigma(i+1)},\ldots,X_{\sigma(n-1)},y_n)\Big)\}\\
=\;&\Xi(X_1,\cdots,X_{n-1},x_n)\wedge y_n\\
&+(-1)^{(|X_{\sigma(1)}|+\cdots+|X_{\sigma(n-1)}|+i+j-2)|x_n|}x_n\wedge\Xi(X_1,\cdots,X_{n-1},y_n)\\
=\;&0.
\end{aligned}
$$

Therefore, $(\wedge^2\mathcal{V},\mathfrak{l}_n)$ is a Lod_∞-algebra. ■

In particular, if we concentrate on the 2-term case, we can give explicit formulas for 2-term 3-Lie_∞-algebras as follows:

Lemma 3.3. *A 2-term 3-Lie_∞-algebra $\mathcal{V}=(V_1,V_0,\mathrm{d},l_3,l_5)$, consists of the following data:*

- *a complex of vector spaces $V_1\xrightarrow{\mathrm{d}}V_0$,*
- *completely skew-symmetric trilinear maps $l_3: V_i\times V_j\times V_k\longrightarrow V_{i+j+k}$, where $0\le i+j+k\le 1$,*
- *a multilinear map $l_5:(V_0\wedge V_0)\otimes(V_0\wedge V_0\wedge V_0)\longrightarrow V_1$,*

such that for any $x,y,x_i\in V_0$ and $f,g,h\in V_1$, the following equalities are satisfied:

(a) $\mathrm{d}l_3(x,y,f)=l_3(x,y,\mathrm{d}f)$,

(b) $l_3(f,g,h)=0; \quad l_3(f,g,x)=0$,

(c) $l_3(\mathrm{d}f,g,x)=l_3(f,\mathrm{d}g,x)$,

(d) $\mathrm{d}l_5(x_1,x_2,x_3,x_4,x_5)=-l_3(x_1,x_2,l_3(x_3,x_4,x_5))+l_3(x_3,l_3(x_1,x_2,x_4),x_5)$
$+l_3(l_3(x_1,x_2,x_3),x_4,x_5)+l_3(x_3,x_4,l_3(x_1,x_2,x_5))$,

(e) $l_5(\mathrm{d}f,x_2,x_3,x_4,x_5)=-l_3(f,x_2,l_3(x_3,x_4,x_5))+l_3(x_3,l_3(f,x_2,x_4),x_5)$
$+l_3(l_3(f,x_2,x_3),x_4,x_5)+l_3(x_3,x_4,l_3(f,x_2,x_5))$,

(f) $l_5(x_1,x_2,\mathrm{d}f,x_4,x_5)=-l_3(x_1,x_2,l_3(f,x_4,x_5))+l_3(f,l_3(x_1,x_2,x_4),x_5)$
$+l_3(l_3(x_1,x_2,f),x_4,x_5)+l_3(f,x_4,l_3(x_1,x_2,x_5))$,

(g)

$$\begin{aligned}
& l_3(l_5(x_1,x_2,x_3,x_4,x_5),x_6,x_7)+l_3(x_5,l_5(x_1,x_2,x_3,x_4,x_6),x_7)+l_3(x_1,x_2,l_5(x_3,x_4,x_5,x_6,x_7)) \\
+\ & l_3(x_5,x_6,l_5(x_1,x_2,x_3,x_4,x_7))+l_5(x_1,x_2,l_3(x_3,x_4,x_5),x_6,x_7)+l_5(x_1,x_2,x_5,l_3(x_3,x_4,x_6),x_7) \\
+\ & l_5(x_1,x_2,x_5,x_6,l_3(x_3,x_4,x_7))=l_3(x_3,x_4,l_5(x_1,x_2,x_5,x_6,x_7))+l_5(l_3(x_1,x_2,x_3),x_4,x_5,x_6,x_7) \\
+\ & l_5(x_3,l_3(x_1,x_2,x_4),x_5,x_6,x_7)+l_5(x_3,x_4,l_3(x_1,x_2,x_5),x_6,x_7)+l_5(x_3,x_4,x_5,l_3(x_1,x_2,x_6),x_7) \\
+\ & l_5(x_1,x_2,x_3,x_4,l_3(x_5,x_6,x_7))+l_5(x_3,x_4,x_5,x_6,l_3(x_1,x_2,x_7)).
\end{aligned}$$

Equations (a) and (c) tells us how the differential d and the bracket l_3 interact. Equations (d), (e) and (f) tell us that the fundamental identity no longer holds on the nose, but controlled by l_5. Equation (g) gives the coherence law that l_5 should satisfy.

Corollary 3.4. *Let $(V_1,V_0,\mathrm{d},l_3,l_5)$ be a 2-term 3-Lie$_\infty$-algebras. Then, for all $f,g,h\in V_1$, we have*

$$l_3(\mathrm{d}f,\mathrm{d}g,h)=l_3(\mathrm{d}f,g,\mathrm{d}h)=l_3(f,\mathrm{d}g,\mathrm{d}h) \tag{7}$$

We continue by defining homomorphisms between 2-term 3-*Lie*$_\infty$-algebras:

Definition 3.5. *Let $\mathcal{V}=(V_1,V_0,\mathrm{d},l_3,l_5)$ and $\mathcal{V}'=(V_1',V_0',\mathrm{d}',l_3',l_5')$ be 2-term 3-Lie$_\infty$-algebras. A homomorphism $\phi:\mathcal{V}\longrightarrow\mathcal{V}'$ consists of:*

- *a chain map $\phi:\mathcal{V}\longrightarrow\mathcal{V}'$, which consists of linear maps $\phi_0:V_0\longrightarrow V_0'$ and $\phi_1:V_1\longrightarrow V_1'$ preserving the differential;*
- *a completely skew-symmetric trilinear map $\phi_2:V_0\times V_0\times V_0\longrightarrow V_1'$,*

such that for all $x_i\in V_0$ and $h\in V_1$, we have

$$\begin{aligned}
\mathrm{d}'(\phi_2(x_1,x_2,x_3)) &= \phi_0(l_3(x_1,x_2,x_3))-l_3'(\phi_0(x_1),\phi_0(x_2),\phi_0(x_3)), \qquad (8)\\
\phi_2(x_1,x_2,\mathrm{d}h) &= \phi_1(l_3(x_1,x_2,h))-l_3'(\phi_0(x_1),\phi_0(x_2),\phi_1(h)), \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
& l_5'(\phi_0(x_1),\phi_0(x_2),\phi_0(x_3),\phi_0(x_4),\phi_0(x_5))+l_3'(\phi_2(x_1,x_2,x_3),\phi_0(x_4),\phi_0(x_5)) \\
+\ & l_3'(\phi_0(x_3),\phi_2(x_1,x_2,x_4),\phi_0(x_5))+l_3'(\phi_0(x_3),\phi_0(x_4),\phi_2(x_1,x_2,x_5)) \\
+\ & \phi_2(l_3(x_1,x_2,x_3),x_4,x_5)+\phi_2(x_3,l_3(x_1,x_2,x_4),x_5)+\phi_2(x_3,x_4,l_3(x_1,x_2,x_5)) \\
=\ & l_3'(\phi_0(x_1),\phi_0(x_2),\phi_2(x_3,x_4,x_5))+\phi_2(x_1,x_2,l_3(x_3,x_4,x_5))+\phi_1(l_5(x_1,x_2,x_3,x_4,x_5)) \qquad (10)
\end{aligned}$$

Let $\varphi : \mathcal{V} \longrightarrow \mathcal{V}'$ and $\psi : \mathcal{V}' \longrightarrow \mathcal{V}''$ be 3-Lie_∞-homomorphisms, their **composition** $((\varphi \circ \psi)_0, (\varphi \circ \psi)_1, (\varphi \circ \psi)_2)$ is given by $(\varphi \circ \psi)_0 = \varphi_0 \circ \psi_0$, $(\varphi \circ \psi)_1 = \varphi_1 \circ \psi_1$, and

$$(\varphi \circ \psi)_2(x, y, z) = \psi_2(\varphi_0(x), \varphi_0(y), \varphi_0(z)) + \psi_1(\varphi_2(x, y, z)).$$

The **identity homomorphism** $1_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$ has the identity chain map as its underlying map, together with $(1_{\mathcal{V}})_2 = 0$.

Definition 3.6. *Let $\mathcal{V}$ and $\mathcal{V}'$ be 2-term 3-Lie_∞-algebras, and $\varphi, \psi : \mathcal{V} \longrightarrow \mathcal{V}'$ be 3-Lie_∞-homomorphisms. A 3-Lie_∞-2-homomorphism $\tau : \varphi \Rightarrow \psi$ is a chain homotopy such that for all $x_1, x_2, x_3 \in V_0$, the following equation holds:*

$$\begin{aligned}(\varphi_2 - \psi_2)(x_1, x_2, x_3) \quad = \quad & l_3'(\varphi_0(x_1), \varphi_0(x_2), \tau(x_3)) + l_3'(\mathrm{d}'\tau(x_1), \tau(x_2), \varphi_0(x_3)) + c.p. \\ & -\tau(l_3(x_1, x_2, x_3)) + l_3'(\mathrm{d}'\tau(x_1), \mathrm{d}'\tau(x_2), \tau(x_3)).\end{aligned} \tag{11}$$

Now we define the vertical and horizontal composition for these 2-homomorphisms. Let $\mathcal{V}$, $\mathcal{V}'$ be 2-term 3-Lie_∞-algebras, and $\varphi, \psi, \mu : \mathcal{V} \longrightarrow \mathcal{V}'$ be 3-Lie_∞-homomorphisms. Let $\tau : \varphi \Rightarrow \psi$ and $\tau' : \psi \Rightarrow \mu$ be 3-Lie_∞-2-homomorphisms. The **vertical composition** of τ and τ', denoted by $\tau'\tau$, is given by $\tau'\tau = \tau' + \tau$.

Let $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{V}''$ be 2-term 3-Lie_∞-algebras, $\varphi, \psi, : \mathcal{V} \longrightarrow \mathcal{V}'$ and $\varphi', \psi' : V' \longrightarrow \mathcal{V}''$ 3-Lie_∞-homomorphisms, and $\tau : \varphi \Rightarrow \psi$ and $\tau' : \varphi' \Rightarrow \psi'$ 3-Lie_∞-2-homomorphisms. The **horizontal composition** of τ and τ', denoted by $\tau' \circ \tau$, is given by $\tau' \circ \tau(x) = \tau'_{\varphi_0(x)} + \varphi_1'\tau(x)$.

Finally, given a 3-Lie_∞-homomorphism φ, the **identity 2-homomorphism** $1_\varphi : \varphi \Rightarrow \varphi$ is the zero chain homotopy $1_\varphi(x) = 0$.

It is straightforward to see that

Proposition 3.7. *There is a 2-category* $2\textbf{Term}3\textbf{-Lie}_\infty$ *with 2-term 3-Lie_∞-algebras as objects, 3-Lie_∞-homomorphisms as morphisms, 3-Lie_∞-2-homomorphisms as 2-morphisms.*

4 3-Lie 2-algebras

In this section, we define 3-Lie 2-algebras, which are the categorification of 3-Lie algebras, and show that the 2-category of 3-Lie 2-algebras is equivalent to the 2-category of 2-term 3-Lie_∞-algebras.

Definition 4.1. *A 3-Lie 2-algebra consists of:*

- *a 2-vector spaces L;*
- *a completely skew-symmetric trilinear functor, the **bracket**, $[\cdot, \cdot, \cdot] : L \times L \times L \longrightarrow L$;*
- *a multilinear natural isomorphism J_{x_1,x_2,x_3,x_4,x_5} for all $x_i \in L_0$,*

$$[x_1, x_2, [x_3, x_4, x_5]] \stackrel{J_{x_1,x_2,x_3,x_4,x_5}}{\longrightarrow} [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]],$$

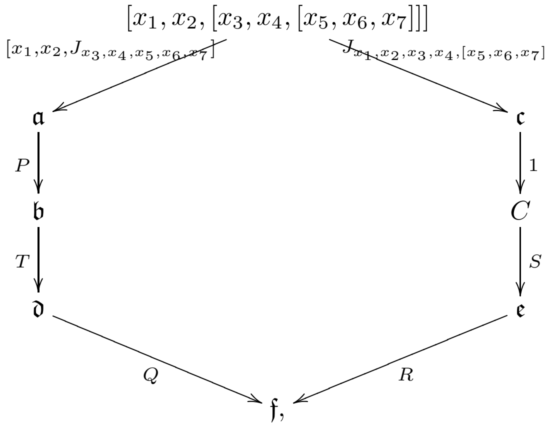
such that for all $x_1, \ldots, x_7 \in L_0$, the following fundamentor identity holds:

$$[x_1, x_2, J_{x_3,x_4,x_5,x_6,x_7}](J_{x_1,x_2,[x_3,x_4,x_5],x_6,x_7} + J_{x_1,x_2,x_5,[x_3,x_4,x_6],x_7} + J_{x_1,x_2,x_5,x_6,[x_3,x_4,x_7]})$$

$$([x_5, x_6, J_{x_1,x_2,x_3,x_4,x_7}] + 1)([x_5, J_{x_1,x_2,x_3,x_4,x_6}, x_7] + [J_{x_1,x_2,x_3,x_4,x_5}, x_6, x_7] + 1) =$$

$$J_{x_1,x_2,x_3,x_4,[x_5,x_6,x_7]}([x_3,x_4,J_{x_1,x_2,x_5,x_6,x_7}]+1)(J_{[x_1,x_2,x_3],x_4,x_5,x_6,x_7}+J_{x_3,[x_1,x_2,x_4],x_5,x_6,x_7}+ J_{x_3,x_4,[x_1,x_2,x_5],x_6,x_7}+J_{x_3,x_4,x_5,[x_1,x_2,x_6],x_7}+J_{x_3,x_4,x_5,x_6,[x_1,x_2,x_7]}), \tag{12}$$

or, in terms of a commutative diagram,



where $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f}$ *and* P,Q,R,S,T *are given by*

$$\begin{aligned}
\mathfrak{a} &= [x_1,x_2,[[x_3,x_4,x_5],x_6,x_7]]+[x_1,x_2,[x_5,[x_3,x_4,x_6],x_7]]+[x_1,x_2,[x_5,x_6,[x_3,x_4,x_7]]],\\
\mathfrak{b} &= [[x_1,x_2,[x_3,x_4,x_5]],x_6,x_7]+[[x_3,x_4,x_5],[x_1,x_2,x_6],x_7]+[[x_3,x_4,x_5],x_6,[x_1,x_2,x_7]]\\
&\quad+[[x_1,x_2,x_5],[x_3,x_4,x_6],x_7]+[x_5,[x_1,x_2,[x_3,x_4,x_6]],x_7]+[x_5,[x_3,x_4,x_6],[x_1,x_2,x_7]]\\
&\quad+[[x_1,x_2,x_5],x_6,[x_3,x_4,x_7]]+[x_5,[x_1,x_2,x_6],[x_3,x_4,x_7]]+[x_5,x_6,[x_1,x_2,[x_3,x_4,x_7]]],\\
\mathfrak{c} &= [x_3,x_4,[x_1,x_2,[x_5,x_6,x_7]]]+[[x_1,x_2,x_3],x_4,[x_5,x_6,x_7]]+[x_3,[x_1,x_2,x_4],[x_5,x_6,x_7]],\\
\mathfrak{d} &= [[x_1,x_2,[x_3,x_4,x_5]],x_6,x_7]+[[x_3,x_4,x_5],[x_1,x_2,x_6],x_7]+[[x_3,x_4,x_5],x_6,[x_1,x_2,x_7]]\\
&\quad+[[x_1,x_2,x_5],[x_3,x_4,x_6],x_7]+[x_5,[x_1,x_2,[x_3,x_4,x_6]],x_7]+[x_5,[x_3,x_4,x_6],[x_1,x_2,x_7]]\\
&\quad+[[x_1,x_2,x_5],x_6,[x_3,x_4,x_7]]+[x_5,[x_1,x_2,x_6],[x_3,x_4,x_7]]+[x_5,x_6,[[x_1,x_2,x_3],x_4,x_7]]\\
&\quad+[x_5,x_6,[x_3,[x_1,x_2,x_4],x_7]]+[x_5,x_6,[x_3,x_4,[x_1,x_2,x_7]]],\\
\mathfrak{e} &= [[x_1,x_2,x_3],x_4,[x_5,x_6,x_7]]+[x_3,[x_1,x_2,x_4],[x_5,x_6,x_7]]+[x_3,x_4,[[x_1,x_2,x_5],x_6,x_7]]\\
&\quad+[x_3,x_4,[x_5,[x_1,x_2,x_6],x_7]]+[x_3,x_4,[x_5,x_6,[x_1,x_2,x_7]]],\\
\mathfrak{f} &= [[[x_1,x_2,x_3],x_4,x_5],x_6,x_7]+[[x_3,[x_1,x_2,x_4],x_5],x_6,x_7]+[[x_3,x_4,[x_1,x_2,x_5]],x_6,x_7]\\
&\quad+[x_5,[[x_1,x_2,x_3],x_4,x_6],x_7]+[x_5,[x_3,[x_1,x_2,x_4],x_6],x_7]+[x_5,[x_3,x_4,[x_1,x_2,x_6]],x_7]\\
&\quad+[[x_3,x_4,x_5],[x_1,x_2,x_6],x_7]+[[x_3,x_4,x_5],x_6,[x_1,x_2,x_7]]+[[x_1,x_2,x_5],[x_3,x_4,x_6],x_7]\\
&\quad+[x_5,[x_3,x_4,x_6],[x_1,x_2,x_7]]+[[x_1,x_2,x_5],x_6,[x_3,x_4,x_7]]+[x_5,[x_1,x_2,x_6],[x_3,x_4,x_7]]\\
&\quad+[x_5,x_6,[[x_1,x_2,x_3],x_4,x_7]]+[x_5,x_6,[x_3,[x_1,x_2,x_4],x_7]]+[x_5,x_6,[x_3,x_4,[x_1,x_2,x_7]]],\\
P &= J_{x_1,x_2,[x_3,x_4,x_5],x_6,x_7}+J_{x_1,x_2,x_5,[x_3,x_4,x_6],x_7}+J_{x_1,x_2,x_5,x_6,[x_3,x_4,x_7]},\\
T &= [x_5,x_6,J_{x_1,x_2,x_3,x_4,x_7}]+1,\\
Q &= [x_5,J_{x_1,x_2,x_3,x_4,x_6},x_7]+[J_{x_1,x_2,x_3,x_4,x_5},x_6,x_7]+1,\\
S &= [x_3,x_4,J_{x_1,x_2,x_5,x_6,x_7}]+1,\\
R &= J_{[x_1,x_2,x_3],x_4,x_5,x_6,x_7}+J_{x_3,[x_1,x_2,x_4],x_5,x_6,x_7}+J_{x_3,x_4,[x_1,x_2,x_5],x_6,x_7}\\
&\quad+J_{x_3,x_4,x_5,[x_1,x_2,x_6],x_7}+J_{x_3,x_4,x_5,x_6,[x_1,x_2,x_7]}.
\end{aligned}$$

We continue by setting up a 2-category of 3-Lie 2-algebras.

Definition 4.2. *Given* 3*-Lie* 2*-algebras* L *and* L'*, a homomorphism* $F: L \longrightarrow L'$ *consists of:*

- *A linear functor* F *from the underlying* 2*-vector space of* L *to that of* L'*, and*

- *a completely skew-symmetric trilinear natural transformation*

$$F_2(x,y,z):[F_0(x),F_0(y),F_0(z)]'\longrightarrow F_0[x,y,z]$$

such that the following diagram commutes:

$$\begin{array}{ccc}
[F_0(x_1),F_0(x_2),[F_0(x_3),F_0(x_4),F_0(x_5)]']' & \xrightarrow{J'_{F_0(x_1),F_0(x_2),F_0(x_3),F_0(x_4),F_0(x_5)}} & \mathfrak{a} \\
\Big\downarrow{\scriptstyle [1,1,F_2]} & & \Big\downarrow{\scriptstyle [F_2,1,1]+[1,F_2,1]+[1,1,F_2]} \\
[F_0(x_1),F_0(x_2),F_0[x_3,x_4,x_5]]' & & \mathfrak{b} \\
\Big\downarrow{\scriptstyle F_2} & & \Big\downarrow{\scriptstyle F_2+F_2+F_2} \\
F_0[x_1,x_2,[x_3,x_4,x_5]] & \xrightarrow{F_1(J_{x_1,x_2,x_3,x_4,x_5})} & \mathfrak{c}\ .
\end{array}$$

where $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ *are given by*

$$\begin{aligned}
\mathfrak{a} &= [[F_0(x_1),F_0(x_2),F_0(x_3)]',F_0(x_4),F_0(x_5)]'+[F_0(x_3),[F_0(x_1),F_0(x_2),F_0(x_4)]',F_0(x_5)]' \\
&\quad +[F_0(x_3),F_0(x_4),[F_0(x_1),F_0(x_2),F_0(x_5)]']', \\
\mathfrak{b} &= [F_0[x_1,x_2,x_3],F_0(x_4),F_0(x_5)]'+[F_0(x_3),F_0[x_1,x_2,x_4],F_0(x_5)]' \\
&\quad +[F_0(x_3),F_0(x_4),F_0[x_1,x_2,x_5]]', \\
\mathfrak{c} &= F_0[[x_1,x_2,x_3],x_4,x_5]+F_0[x_3,[x_1,x_2,x_4],x_5]+F_0[x_3,x_4,[x_1,x_2,x_5]].
\end{aligned}$$

The identity homomorphism $\mathrm{Id_L}:\mathrm{L}\longrightarrow\mathrm{L}$ has the identity functor as its underlying functor, together with an identity natural transformation as $(\mathrm{Id_L})_2$. Let L,L' and L'' be 3-Lie 2-algebras, the composite of 3-Lie 2-algebra homomorphisms $F:L\longrightarrow L'$ and $G:L'\longrightarrow L''$ which we denote by $G\circ F$, is given by letting the functor $((G\circ F)_0,(G\circ F)_1)$ be the usual composition of (G_0,G_1) and (F_0,F_1), and letting $(G\circ F)_2$ be the following composite:

$$\begin{array}{ccc}
[G_0\circ F_0(x),G_0\circ F_0(y),G_0\circ F_0(z)]'' & & \\
\Big\downarrow{\scriptstyle G_2(F_0(x),F_0(y),F_0(z))} & \searrow{\scriptstyle (G\circ F)_2(x,y,z)} & \\
 & & G_0\circ F_0[x,y,z]. \\
 & \nearrow{\scriptstyle G_1(F_2(x,y,z))} & \\
G_0[F_0(x),F_0(y),F_0(z)]' & &
\end{array}$$

We also have 2-homomorphisms between homomorphisms:

Definition 4.3. *Let $F,G:L\longrightarrow L'$ be 3-Lie 2-algebra homomorphisms. A 2-homomorphism $\theta:F\Rightarrow G$ is a linear natural transformation from F to G such that the following diagram commutes:*

$$\begin{array}{ccc}
[F_0(x),F_0(y),F_0(z)]' & \xrightarrow{F_2} & F_0[x,y,z] \\
\Big\downarrow{\scriptstyle [\theta_x,\theta_y,\theta_z]'} & & \Big\downarrow{\scriptstyle \theta_{[x,y,z]}} \\
[G_0(x),G_0(y),G_0(z)]' & \xrightarrow{G_2} & G_0[x,y,z].
\end{array}$$

Since 2-homomorphisms are just natural transformations with an extra property, we vertically and horizontally compose these in the usual way, and an identity 2-homomorphism is just an identity natural transformation.

It is straightforward to see that

Proposition 4.4. *There is a* 2*-category* 3**Lie**2**Alg** *with* 3*-Lie* 2*-algebras as objects,* 3*-Lie* 2*-algebra homomorphisms as morphisms, and* 3*-Lie* 2*-algebra* 2*-homomorphisms as* 2*-morphisms.*

Now we establish the equivalence between the 2-category of 3-Lie 2-algebras and that of 2-term 3-Lie_∞-algebras. This result is based on the equivalence between 2-vector spaces and 2-term chain complexes described in Subsection 2.2.

Theorem 4.5. *The* 2*-categories* 2**Term**3**-Lie**$_\infty$ *and* 3**Lie**2**Alg** *are* 2*-equivalent.*

Proof. First we construct a 2-functor T : 2**Term**3**-Lie**$_\infty \longrightarrow$ 3**Lie**2**Alg**. Given a 2-term 3-Lie_∞-algebra $\mathcal{V}=(V_1,V_0,\mathrm{d},l_2,l_3)$, we have a 2-vector space L via (4). More precisely, $L_0=V_0, L_1=V_0\oplus V_1$, and the source and the target map are given by $s(x+f)=x$ and $t(x+f)=x+\mathrm{d}f$. Define a skew-symmetric trilinear functor $[\cdot,\cdot,\cdot]: L\times L\times L\longrightarrow L$ by

$$\begin{aligned}[x+f,y+g,z+h] &= l_3(x,y,z)+l_3(x,y,h)+l_3(x,g,z)+l_3(f,y,z)\\ &\quad +l_3(\mathrm{d}f,g,z)+l_3(\mathrm{d}f,y,h)+l_3(x,\mathrm{d}g,h)+l_3(\mathrm{d}f,\mathrm{d}g,h),\end{aligned}$$

and define the fundamentor J_{x_1,x_2,x_3,x_4,x_5} by

$$J_{x_1,x_2,x_3,x_4,x_5}=([x_1,x_2,[x_3,x_4,x_5]],l_5(x_1,x_2,x_3,x_4,x_5)).$$

The source of J_{x_1,x_2,x_3,x_4,x_5} is $[x_1,x_2,[x_3,x_4,x_5]]$ as desired. By (d) in the Lemma 3.3, its target is

$$\begin{aligned}t(J_{x_1,x_2,x_3,x_4,x_5}) &= [x_1,x_2,[x_3,x_4,x_5]]+\mathrm{d}l_5(x_1,x_2,x_3,x_4,x_5)\\ &= [[x_1,x_2,x_3],x_4,x_5]+[x_3,[x_1,x_2,x_4],x_5]+[x_3,x_4,[x_1,x_2,x_5]],\end{aligned}$$

as desired. By Conditions (e) and (f) in Lemma 3.3, we deduce that J is a natural transformation. By Condition (h) in Lemma 3.3, we can deduce that the fundamentor identity holds. This completes the construction of a 3-Lie 2-algebra $L=T(\mathcal{V})$ from a 2-term 3-Lie_∞-algebra $\mathcal{V}$.

We go on to construct a 3-Lie 2-algebra homomorphism $T(\phi):T(\mathcal{V})\longrightarrow T(\mathcal{V}')$ from a 3-Lie_∞-algebra homomorphism $\phi=(\phi_0,\phi_1,\phi_2):\mathcal{V}\longrightarrow\mathcal{V}'$ between 2-term 3-Lie_∞-algebras. Let $T(\mathcal{V})=L$ and $T(\mathcal{V}')=L'$. We define the underlying linear functor of $T(\phi)=F$ with $F_0=\phi_0$, $F_1=\phi_0\oplus\phi_1$. Define $F_2: V_0\times V_0\times V_0\longrightarrow V_0'\oplus V_1'$ by

$$F_2(x_1,x_2,x_3)=([\phi_0(x_1),\phi_0(x_2),\phi_0(x_3)]',\phi_2(x_1,x_2,x_3)).$$

Then $F_2(x_1,x_2,x_3)$ is a natural isomorphism from $[F_0(x_1),F_0(x_2),F_0(x_3)]'$ to $F_0[x_1,x_2,x_3]$, and $F=(F_0,F_1,F_2)$ is a homomorphism from L to L'. We can also prove that T preserve identities and composition of homomorphisms. So T is a functor.

Furthermore, to construct T to be a 2-functor, we only need to define T on 2-morphisms. Let $\varphi,\psi:\mathcal{V}\longrightarrow\mathcal{V}'$ be homomorphisms and $\tau:\varphi\Rightarrow\psi$ a 2-homomorphism. Then we define

$$\theta(x)=T(\tau)(x)=(\varphi_0(x),\tau(x)).$$

By (11), $T(\tau)$ is a 2-homomorphism from $T(\phi)$ to $T(\psi)$. It is obvious that T preserves the compositions and identities. Thus, T is a 2-functor from 2**Term**3**-Lie**$_\infty$ to 3**Lie**2**Alg**.

Next we construct a 2-functor S :3**Lie**2**Alg** $\longrightarrow$ 2**Term**3**-Lie**$_\infty$. Given a 3-Lie 2-algebra L, we obtain a complex of vector spaces $\mathcal{V}=S(L)$ via (3). More precisely, $V_1=\ker(s), V_0=L_0$ and $\mathrm{d}=t|_{\ker(s)}$. For all $x_1,x_2,x_3,x_4,x_5\in V_0=L_0$ and $f,g,h\in V_1\subseteq L_1$, we define l_3 and l_5 as follows:

(1) $l_3(x_1,x_2,x_3)=[1_{x_1},1_{x_2},1_{x_3}]$,

(2) $l_3(x_1,x_2,h)=[1_{x_1},1_{x_2},h]$,

(3) $l_3(x_1,f,h)=0,\ l_3(f,h,g)=0$,

(4) $l_5(x_1,x_2,x_3,x_4,x_5)=J_{x_1,x_2,x_3,x_4,x_5}-1_{s(J_{x_1,x_2,x_3,x_4,x_5})}$.

The various conditions of L being a 3-Lie 2-algebra imply that $\mathcal{V}=(V_1,V_0,\mathrm{d},l_3,l_5)$ is 2-term 3-Lie_∞-algebra. This completes the construction of a 2-term 3-Lie_∞-algebra $\mathcal{V}=S(L)$ from a 3-Lie 2-algebra L.

Let L and L' be 3-Lie 2-algebras, and $F=(F_0,F_1,F_2):L\longrightarrow L'$ a homomorphism. Let $S(L)=\mathcal{V}$ and $S(L')=\mathcal{V}'$. We go on to construct a 3-Lie_∞-homomorphism $\phi=S(F):\mathcal{V}\longrightarrow\mathcal{V}'$. Let $\phi_0=F_0$, $\phi_1=F_1|_{\ker(s)}$. Define $\phi_2:V_0\times V_0\times V_0\longrightarrow V_1'$ by

$$\phi_2(x_1,x_2,x_3)=F_2(x_1,x_2,x_3)-1_{s(F_2(x_1,x_2,x_3))}.$$

Then ϕ_2 is completely skew-symmetric, and

$$\begin{array}{rcl}\mathrm{d}'\phi_2(x_1,x_2,x_3)&=&(t'-s')F_2(x_1,x_2,x_3)\\&=&\phi_0(l_3(x_1,x_2,x_3))-l_3'(\phi_0(x_1),\phi_0(x_2),\phi_0(x_3)).\end{array}$$

The naturality of F_2 gives Equation (9) in Definition 3.5, and the fundamentor identity gives Equation (10) in Definition 3.5. Thus, $\phi=S(F)$ is a homomorphism between 2-term 3-Lie_∞-algebras.

Let $F,G:L\longrightarrow L'$ be 3-Lie 2-algebra homomorphisms and $\theta:F\Rightarrow G$ a 2-homomorphism. Let $\varphi=S(F),\psi=S(G):\mathcal{V}\longrightarrow\mathcal{V}'$ be the corresponding 3-Lie_∞-homomorphisms. We define

$$\tau(x)=S(\theta)(x)=\theta(x)-1_{s'(\theta(x))}.$$

By the commutative diagram in Definition 4.3, we can deduce that (11) holds. Thus, $\tau=S(\theta)$ is a 2-homomorphism. It is straightforward to deduce that S preserves the compositions and identities. Thus S is a 2-functor from $3\mathbf{Lie}2\mathbf{Alg}$ to $2\mathbf{Term}3\textbf{-}\mathbf{Lie}_\infty$.

In the end, it is easy to construct the natural isomorphisms $\alpha:ST\Rightarrow 1_{\mathbf{3Lie2Alg}}$ and $\beta:TS\Rightarrow 1_{\mathbf{2Term3Lie}_\infty}$. We omit details. The proof is completed. ■

5 Skeletal and strict 3-Lie 2-algebras

By Theorem 4.5, we see that 3-Lie 2-algebras and 2-term 3-Lie_∞-algebras are equivalent. Thus, we will call a 2-term 3-Lie_∞-algebra a 3-Lie 2-algebra in the sequel.

A 3-Lie 2-algebra $(V_1,V_0,\mathrm{d},l_3,l_5)$ is called **skeletal** (**strict**) if $\mathrm{d}=0$ ($l_5=0$).

In this section, first we classify skeletal 3-Lie 2-algebras via the third cohomology group. Then, we introduce the notion of a crossed module of 3-Lie algebras, and show that they are equivalent to strict 3-Lie 2-algebra. At the end, we construct a 3-Lie 2-algebra from a 3-Lie algebra with a symplectic structure.

Theorem 5.1. *There is a one-to-one correspondence between skeletal 3-Lie 2-algebras and quadruples $((\mathfrak{g},[\cdot,\cdot,\cdot]),V,\rho,\Theta)$, where $(\mathfrak{g},[\cdot,\cdot,\cdot])$ is a 3-Lie algebra, V is a vector space, ρ is a representation of $\mathfrak{g}$ on V, and Θ is a 3-cocycle on $\mathfrak{g}$ with values in V.*

Proof. Let $(V_1, V_0, \mathrm{d} = 0, l_3, l_5)$ be a skeletal 3-Lie 2-algebras. By (d) in Lemma 3.3, we see that $l_3|_{V_0}$ satisfies the fundamental identity. Thus, $(V_0, l_3|_{V_0})$ is a 3-Lie algebra. l_3 also gives rise to a map $\rho : \wedge^2 V_0 \longrightarrow V_1$ by

$$\rho(x_1, x_2)(f) = l_3(x_1, x_2, f). \tag{13}$$

By (e) and (f) in Lemma 3.3, we deduce that ρ is a representation of the 3-Lie algebra $(V_0, l_3|_{V_0})$ on V_1. Finally, by (g) in Lemma 3.3, we get that l_5 is a 3-cocycle.

Conversely, given a 3-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot])$, a representation ρ of $\mathfrak{g}$ on a vector space V, and a 3-cocycle Θ on $\mathfrak{g}$ with values in V, we define $V_0 = \mathfrak{g}$, $V_1 = V$, $\mathrm{d} = 0$, and totally skew-symmetric l_3, l_5 by

$$l_3(x, y, z) = [x, y, z], \quad l_3(x, y, f) = \rho(x, y)(f), \quad l_5 = \Theta.$$

Then, it is straightforward to see that $(V_1, V_0, \mathrm{d} = 0, l_3, l_5)$ is a skeletal 3-Lie 2-algebra. ■

Now we introduce the notion of a crossed module of 3-Lie algebras and show that they are equivalent to strict 3-Lie 2-algebras.

Definition 5.2. *A crossed module of* 3*-Lie algebras is a quadruple* $((\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}}), (\mathfrak{h}, [\cdot,\cdot,\cdot]_{\mathfrak{h}}), \mu, \alpha)$*, where* $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ *and* $(\mathfrak{h}, [\cdot,\cdot,\cdot]_{\mathfrak{h}})$ *are* 3*-Lie algebras,* $\mu : \mathfrak{g} \longrightarrow \mathfrak{h}$ *is a homomorphism of* 3*-Lie algebras, and* $\alpha : \wedge^2\mathfrak{h} \longrightarrow \mathrm{Der}(\mathfrak{g})$ *is a representation, such that for all* $x, y, z \in \mathfrak{h}, f, g, h \in \mathfrak{g}$*, the following equalities hold:*

$$\mu(\alpha(x, y)(f)) = [x, y, \mu(f)]_{\mathfrak{h}}, \tag{14}$$

$$\alpha(\mu(f), \mu(g))(h) = [f, g, h]_{\mathfrak{g}}, \tag{15}$$

$$\alpha(x, \mu(f))(g) = -\alpha(x, \mu(g))(f). \tag{16}$$

Remark 5.3. *The more general notion of crossed modules of n-Leibniz had been given in [8], and the relation with the third cohomology group is established there.*

Theorem 5.4. *There is a one-to-one correspondence between strict* 3*-Lie* 2*-algebras and crossed modules of* 3*-Lie algebras.*

Proof. Let $(V_1, V_0, \mathrm{d}, l_3, l_5)$ be a strict 3-Lie 2-algebras. Define $\mathfrak{g} = V_1, \mathfrak{h} = V_0$, and the following two bracket operations on $\mathfrak{g}$ and $\mathfrak{h}$:

$$[f, g, h]_{\mathfrak{g}} = l_3(\mathrm{d}f, \mathrm{d}g, h) = l_3(\mathrm{d}f, g, \mathrm{d}h) = l_3(f, \mathrm{d}g, \mathrm{d}h), \tag{17}$$

$$[x, y, z]_{\mathfrak{h}} = l_3(x, y, z). \tag{18}$$

It is straightforward to see that both $[\cdot,\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot,\cdot]_{\mathfrak{h}}$ are 3-Lie brackets. Let $\mu = \mathrm{d}$, by Condition (a) in Lemma 3.3, we have

$$\mu[f, g, h]_{\mathfrak{g}} = \mathrm{d}l_3(\mathrm{d}f, \mathrm{d}g, h) = l_3(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h) = [\mu(f), \mu(g), \mu(h)]_{\mathfrak{h}},$$

which implies that μ is a homomorphism of 3-Lie algebras. Define $\alpha : \wedge^2\mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\alpha(x, y)(f) = l_3(x, y, f).$$

By Condition (f) in Lemma 3.3, we deduce that $\alpha(x, y) \in \mathrm{Der}(\mathfrak{g})$. By Conditions ($e$) and ($f$) in Lemma 3.3, we can obtain that α is a representation. Furthermore, we have

$$\begin{aligned} \mu(\alpha(x, y)(f)) &= \mathrm{d}(\alpha(x, y)(f)) = \mathrm{d}l_3(x, y, f) = l_3(x, y, \mathrm{d}f) = [x, y, \mu(f)]_{\mathfrak{h}}, \\ \alpha(\mu(f), \mu(g))(h) &= l_3(\mathrm{d}f, \mathrm{d}g, h) = [f, g, h]_{\mathfrak{g}}, \\ \alpha(x, \mu(f))(g) &= l_3(x, \mu(f), g) = -l_3(x, \mu(g), f) = -\alpha(x, \mu(g))(f). \end{aligned}$$

Therefore we obtain a crossed module of 3-Lie algebras.

Conversely, a crossed module of 3-Lie algebras gives rise to a 2-term 3-Lie_∞-algebra, in which $V_1 = \mathfrak{g}$, $V_0 = \mathfrak{h}$, $\mathrm{d} = \mu$, and the totally skew-symmetric trilinear map l_3 is given by

$$\begin{aligned} l_3(x,y,z) &= [x,y,z]_{\mathfrak{h}}, \\ l_3(x,y,f) &= \alpha(x,y)(f). \end{aligned}$$

The crossed module conditions give various conditions for 2-term 3-Lie_∞-algebras. We omit details. The proof is completed. ■

Now we give an example of crossed module of 3-Lie algebras.

Example 5.5. Let $\mathfrak{g} = \mathbb{R}^3$ with a basis $\{e_1, e_2, e_3\}$, and the 3-Lie bracket is given by $[e_1, e_2, e_3]_{\mathfrak{g}} = e_1$. Let $\mathfrak{h} = \mathbb{R}^3$ with a basis $\{e^1, e^2, e^3\}$, and the 3-Lie bracket is given by $[e^1, e^2, e^3]_{\mathfrak{h}} = e^1$. Define $\mu : \mathfrak{g} \longrightarrow \mathfrak{h}$ by $\mu(e_i) = e^i$. Obviously, μ is an homomorphism. Define $\alpha : \wedge^2\mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\alpha(e^1,e^2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha(e^2,e^3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha(e^3,e^1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

More precisely, $\alpha(e^1,e^2)(e_3) = e_1$, $\alpha(e^2,e^3)(e_1) = e_1$, $\alpha(e^3,e^1)(e_2) = e_1$. It is straightforward to deduce that α is a representation and take values in $\mathrm{Der}(\mathfrak{g})$. Furthermore, (14)-(16) are also satisfied. Thus, $(\mathfrak{g}, \mathfrak{h}, \mu, \alpha)$ constructed above is a crossed module of 3-Lie algebras.

At the end of this section, we construct a strict 3-Lie 2-algebra from a 3-Lie algebra with a symplectic structure. The main ingredient in this construction is the underlying 3-pre-Lie algebra structure.

Definition 5.6. [4] *Let A be a vector space with a multilinear map $\{\cdot,\cdot,\cdot\} : A \otimes A \otimes A \to A$. $(A, \{\cdot,\cdot,\cdot\})$ is called a* **3-pre-Lie algebra** *if the following identities hold:*

$$\begin{aligned} \{x,y,z\} &= -\{y,x,z\}, && (19) \\ \{x_1,x_2,\{x_3,x_4,x_5\}\} &= \{[x_1,x_2,x_3]_C,x_4,x_5\} + \{x_3,[x_1,x_2,x_4]_C,x_5\} \\ &\quad +\{x_3,x_4,\{x_1,x_2,x_5\}\}, && (20) \\ \{[x_1,x_2,x_3]_C,x_4,x_5\} &= \{x_1,x_2,\{x_3,x_4,x_5\}\} + \{x_2,x_3,\{x_1,x_4,x_5\}\} \\ &\quad +\{x_3,x_1,\{x_2,x_4,x_5\}\}. && (21) \end{aligned}$$

where $[\cdot,\cdot,\cdot]_C$ is given by

$$[x,y,z]_C = \{x,y,z\} + \{y,z,x\} + \{z,x,y\}, \quad \forall x,y,z \in A, \tag{22}$$

Proposition 5.7. [4] *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Then, $(A, [\cdot,\cdot,\cdot]_C)$ is a 3-Lie algebra, where $[\cdot,\cdot,\cdot]_C$ is the induced 3-commutator given by Eq. (22). Furthermore, $L : \wedge^2 A \longrightarrow \mathfrak{gl}(A)$, which is defined by*

$$L_{x,y}z = \{x,y,z\}, \quad \forall\ x,y,z \in A, \tag{23}$$

is a representation of the 3-Lie algebra $(A, [\cdot,\cdot,\cdot]_C)$ on A.

Definition 5.8. *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra. A* **symplectic structure** *on $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is a nondegenerate skew-symmetric bilinear form $\omega : \wedge^2\mathfrak{g} \longrightarrow \mathbb{R}$, such that for all $x,y,z,w \in \mathfrak{g}$, the following identity hold:*

$$\omega([x,y,z],w) - \omega([x,y,w],z) + \omega([x,z,w],y) - \omega([y,z,w],x) = 0. \tag{24}$$

Proposition 5.9. [4] *Let $(\mathfrak{g},[\cdot,\cdot,\cdot])$ be a 3-Lie algebra, and ω a symplectic structure on $(\mathfrak{g},[\cdot,\cdot,\cdot])$. Then, there exists a compatible 3-pre-Lie algebra structure on $\mathfrak{g}$ given by*

$$\omega(\{x,y,z\},w)=-\omega(z,[x,y,w]),\quad \forall x,y,z,w\in\mathfrak{g}. \tag{25}$$

Theorem 5.10. *Let $(\mathfrak{g},[\cdot,\cdot,\cdot])$ be a 3-Lie algebra, ω a symplectic structure on $(\mathfrak{g},[\cdot,\cdot,\cdot])$, and $(\mathfrak{g},\{\cdot,\cdot,\cdot\})$ the induced 3-pre-Lie algebra given in Proposition 5.9. On the complex of vector spaces $\mathfrak{g}^*\stackrel{{\omega^\sharp}^{-1}}{\longrightarrow}\mathfrak{g}$, define l_3, which is totally skew-symmetric, by*

$$\left\{\begin{array}{lcll} l_3(x,y,z) &=& [x,y,z], & \forall\ x,y,z\in\mathfrak{g}\\ l_3(x,y,\xi) &=& L^*_{x,y}\xi, & \forall\ x,y\in\mathfrak{g},\ \xi\in\mathfrak{g}^*\\ l_3(x,\xi,\eta) &=& 0, & \forall\ x\in\mathfrak{g},\ \xi,\eta\in\mathfrak{g}^*\\ l_3(\xi,\eta,\gamma) &=& 0, & \forall\ \xi,\eta,\gamma\in\mathfrak{g}^*,\end{array}\right. \tag{26}$$

where L^* is the dual representation of L and $\omega^\sharp:\mathfrak{g}\longrightarrow\mathfrak{g}^*$ is given by $\omega^\sharp(x)(y)=\omega(x,y)$. Then, $(\mathfrak{g}^*,\mathfrak{g},\mathrm{d}={\omega^\sharp}^{-1},l_3)$ is a strict 3-Lie 2-algebra.

Proof. Obviously, we only need to show that Conditions (a) and (c) in Lemma 3.3 hold, the other conditions hold naturally. For all $x,y,z,w\in\mathfrak{g}$, let $f=\omega^\sharp(z)$ and $g=\omega^\sharp(w)$, we have

$$\langle \mathrm{d}l_3(x,y,f),g\rangle=-\langle l_3(x,y,\omega^\sharp(z)),w\rangle=-\langle L^*_{x,y}\omega^\sharp(z),w\rangle=\langle\omega^\sharp(z),\{x,y,w\}\rangle=\omega(z,\{x,y,w\}),$$

and

$$\langle l_3(x,y,\mathrm{d}f),g\rangle=\langle l_3(x,y,z),\omega^\sharp(w)\rangle=\omega(w,[x,y,z]).$$

By (25), we deduce that $\mathrm{d}l_3(x,y,f)=l_3(x,y,\mathrm{d}f)$, i.e. Condition (a) hold. Furthermore, also by (25), we have

$$\begin{array}{lcl}\langle l_3(\mathrm{d}f,g,x),y\rangle &=& \langle L^*_{x,z}\omega^\sharp(w),y\rangle=-\omega(w,\{x,z,y\})=-\omega(y,[x,z,w]),\\ \langle l_3(f,\mathrm{d}g,x),y\rangle &=& \langle L^*_{w,x}\omega^\sharp(z),y\rangle=-\omega(z,\{w,x,y\})=-\omega(y,[w,x,z]).\end{array}$$

Therefore, we have $l_3(\mathrm{d}f,g,x)=l_3(f,\mathrm{d}g,x)$, which implies that Condition (b) hold. The proof is finished. ■

References

[1] M. Ammar and N. Poncin, Coalgebraic Approach to the Loday Infinity Category, Stem Differential for $2n$-ary Graded and Homotopy Algebras, *Ann. Inst. Fourier (Grenoble).* 60 (1) (2010), 355-387.

[2] J. Baez and A. S. Crans, Higher-Dimensional Algebra VI: Lie 2-Algebras, *Theory and Appl. Categ.* 12 (2004), 492-528.

[3] J. Bagger and N. Lambert, Three-algebras and N=6 Chern-Simons gauge theories. *Phys. Rev. D* 79 (2009), no. 2, 025002, 8 pp.

[4] C. Bai, L. Guo and Y. Sheng, Bialgebras, classical Yang-Baxter equation and Manin triple for 3-Lie algebras, in preparation.

[5] R. Bai, L. Guo, J. Li and Y. Wu, Rota-Baxter 3-Lie algebras, *J. Math. Phys.* **54** (2013), 064504, 14pp.

[6] J. M. Casas, M. Insua and M. Ladra, Poincaré-Birkhoff-Witt theorem for Leibniz n-algebras. *J. Symbolic Comput.* 42 (2007), no. 11-12, 1052-1065.

[7] J. M. Casas, E. Khmaladze, and M. Ladra, On solvability and nilpotency of Leibniz n-algebras. *Comm. Algebra* 34 (2006), no. 8, 2769-2780.

[8] J. M. Casas, E. Khmaladze, and M. Ladra, Crossed modules for Leibniz n-algebras. *Forum Math.* 20 (2008), no. 5, 841-858.

[9] J. M. Casas, E. Khmaladze, and M. Ladra, Higher Hopf formula for homology of Leibniz n-algebras. *J. Pure Appl. Algebra* 214 (2010), no. 6, 797-808.

[10] S. Cherkis and C. Sämann, Multiple M2-branes and generalized 3-Lie algebras. Phys. Rev. D 78 (2008), no. 6, 066019, 11 pp.

[11] J. A. de Azcarraga and J. M. Izquierdo, n-ary algebras: a review with applications, *J. Phys. A: Math. Theor.* 43 (2010), 293001.

[12] A. Fialowski and M. Penkava, Deformation theory of infinity algebras. *J. Algebra* 255 (2002), no. 1, 59-88.

[13] J. Figueroa-O′Farrill, Deformations of 3-algebras. *J. Math. Phys.* 50 (2009), no. 11, 113514, 27 pp.

[14] V. T. Filippov, n-Lie algebras, *Sib. Mat. Zh.* 26 (1985) 126-140.

[15] P. Gautheron, Some remarks concerning Nambu mechanics, *Lett. Math. Phys.*, 37 (1996) 103-116.

[16] J. Gomis, D. Rodríguez-Gómez, M. Van Raamsdonk and H. Verlinde, Supersymmetric Yang-Mills theory from Lorentzian three-algebras. *J. High Energy Phys.* 2008, no. 8, 094, 18 pp.

[17] P. Ho, R. Hou and Y. Matsuo, Lie 3-algebra and multiple M_2-branes, *J. High Energy Phys.* 2008, no. 6, 020, 30 pp.

[18] Sh. M. Kasymov, On a theory of n-Lie algebras. (Russian) *Algebra i Logika* 26 (1987), no. 3, 277-297, 398.

[19] M. Livernet, Homologie des algèbres stables de matrices sur une A_∞-algèbre, *C. R. Acad. Sci. Paris Sér. I Math.*, 329(2) (1999), 113-116.

[20] Y. Nambu, Generalized Hamiltonian dynamics, *Phys. Rev. D* 7 (1973) 2405-2412.

[21] M. Penkava, L-infinity algebras and their cohomology, arXiv:q-alg/9512014.

[22] G. Papadopoulos, M2-branes, 3-Lie algebras and Plucker relations, *J. High Energy Phys.* 2008, no. 5, 054, 9 pp.

[23] Y. Sheng and Z. Liu, Leibniz 2-algebras and twisted Courant algebroids. *Comm. Algebra* 41 (2013), no. 5, 1929–1953.

[24] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, *J. Pure Appl. Algebra* 38 (1985), 313-322.

[25] L. Takhtajan, On foundation of the generalized Nambu mechanics, *Comm. Math. Phys.* 160 (1994) 295-315.

[26] K. Uchino, Derived brackets and sh Leibniz algebras, *J. Pure Appl. Algebra*, 215 (2011) 1102-1111.

[27] M. Van Raamsdonk, Comments on the Bagger-Lambert theory and multiple M2-branes. *J. High Energy Phys.* 2008, no. 5, 105, 9 pp.