BOUNDED NEGATIVITY AND SYMPLECTIC 4-MANIFOLDS

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Abstract. Let (M,ω) be a symplectic manifold of negative Kodaira dimension. Let C be an ω-symplectic curve, J-holomorphic for some J tamed by ω. Then $[C]^2$ is bounded below by a constant depending only on ω. Related bounded negativity problems for other structures are also briefly discussed. In particular, the symplectic result implies the bounded negativity conjecture for complex projective surfaces with Kodaira dimension $\kappa=-\infty$.

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1. Introduction

On any oriented closed smooth 4-manifold M there is a skew-symmetric pairing

$$\begin{array}{ccc} H^2(M,\mathbb{Z})\times H^2(M,\mathbb{Z}) & \to & H^4(M,\mathbb{Z})\cong\mathbb{Z} \\ (\alpha,\beta) & \mapsto & \alpha\cdot\beta=\langle\alpha\cup\beta,[M]\rangle \end{array}$$

where $[M]$ is the fundamental class of M. This map is called the *intersection form* on M. By Poincaré duality, this can also be viewed as a map on $H_2(M,\mathbb{Z})$.

The intersection form has the following geometric interpretation when viewed as a map on homology: Given two generic representatives of the classes $A,B\in H_2(M,\mathbb{Z})$, $A\cdot B$ is the signed sum, corresponding to the given orientation on M, of the index of their intersection points. This is generically a finite sum. $A^2=A\cdot A$ is called the self-intersection number of a curve in the class A.

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This note is concerned with the following symplectic bounded negativity question: Given a symplectic manifold (M, ω), is there a lower bound on the self-intersection numbers for ω-symplectic curves?

An *ω-symplectic curve* C is any curve which can be made J-holomorphic for some almost complex structure J tamed by ω, i.e. $\omega(\cdot, J\cdot)$ is positive definite. In particular, this occurs if and only if all self-intersections of C are locally positive. Section 2 gives an overview of this and related questions. Originally this question was considered in the algebraic category.

While this question is generally rather hard to approach, it is possible to make some progress when M is irrational ruled or rational. In this setting, a number of factors come into play:

- A full understanding of which classes are representable by symplectic forms ([15],[19], [23]),
- A rather detailed understanding of the interplay between diffeomorphisms of M and their action on homology ([12],[15]),
- The structure of M as a symplectic sum, and
- The stability results in [6].

Together these results will give a proof the following result.

Theorem 1.1. *Let M be a manifold of Kodaira dimension $-\infty$ and ω a symplectic form on M. Let $A \in H_2(M, \mathbb{Z})$ be representable by an ω-symplectic curve. Then there exists a positive constant $N(M, \omega)$, depending only on ω, such that*

$$A^2 \geq -N(M, \omega).$$

This result, together with Lemma 2.9, implies the following.

Lemma 1.2. *Let X be a smooth projective surface over $\mathbb{C}$ birationally equivalent to a geometrically ruled surface. There exists a positive constant $b(X)$ bounding the self-intersection of reduced irreducible curves on X, i.e.*

$$[C]^2 \geq -b(X)$$

for every reduced irreducible curve $C \subset X$.

This result is well-known when X is minimal, see the discussion following Conj. 2.2. Lemma 2.9 gives a bound on the value for $b(X)$ in terms of $N(M, \omega)$.

In some cases a more precise lower bound can be given.

Lemma 1.3. *Let M be a ruled manifold and ω a symplectic form on M. Let $A \in H_2(M, \mathbb{Z})$ be representable by an ω-symplectic curve.*

(1) *$M \simeq S^2 \times \Sigma_h$, $h \geq 0$: Fix a ruling on M. Let ω_1 be the symplectic area of the fiber, ω_2 the symplectic area of a section. Then*

$$A^2 > -2\frac{\omega_2}{\omega_1}.$$

(2) $M \simeq S^2\tilde{\times}\Sigma_h$, $h \geq 1$: *Fix a ruling on* M. *Let* ω_1 *be the symplectic area of the fiber,* ω_2 *the symplectic area of a section. Then*

$$A^2 > -2\frac{\omega_2}{\omega_1} - 1.$$

(3) $M \simeq (S^2 \times \Sigma_h)\#k\overline{\mathbb{C}P^2}$, $h \geq 0$: *Assume that the symplectic canonical class* K_ω *of the symplectic form* ω *is given by* $K_\omega = K_{st}$. *Fix a ruling on the trivial minimal model* $S^2 \times \Sigma_h$ *of* M *and let* $\tilde{\omega}$ *be the symplectic form obtained from* ω *on* $S^2 \times \Sigma_h$ *under blow-down. If* $[\tilde{\omega}] = \omega_1 S + \omega_2 F$, *then*

$$A^2 > -2\frac{\omega_2}{\omega_1} - k.$$

The symplectic canonical class K_ω associated to the symplectic form ω is defined as the first Chern class of the cotangent bundle for any almost complex structure J tamed by ω:

$$K_\omega = c_1(T^*M, J) = -c_1(TM, J).$$

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2. Bounded Negativity Conjectures

This section gives an overview of questions related to bounded negativity. It is natural to ask such a question in a variety of different categories, placing different structures on the geometric objects: Given an additional structure on M, such as an (almost) complex structure or a symplectic structure, is there a bound on A^2 for classes representable by curves compatible with this additional structure?

Smooth Category: Let M be smooth, is there a lower bound on which classes can be represented by smooth curves? This has a classical answer:

Lemma 2.1 ([24]; [9], Prop 1.2.3)**.** *On a closed, oriented, smooth 4-manifold* M *every* $A \in H_2(M, \mathbb{Z})$ *can be represented by an embedded surface.*

Thus if $b^- > 0$, then there is no lower bound in the smooth category. If $b^- = 0$, then Donaldson [3] showed that M must have intersection form $\oplus m\langle 1\rangle$.

Algebraic Category: In the algebraic context this question appears in [11], [1], [2], and others (for a brief history see [2]) and is generally stated as a conjecture. The classic bounded negativity conjecture is stated as follows:

Conjecture 2.2 (Conj. 3.3.1, [1])**.** Let X be a smooth projective surface in characteristic zero. There exists a positive constant $b(X)$ bounding the self-intersection of reduced irreducible curves on X, i.e.

$$[C]^2 \geq -b(X)$$

for every reduced irreducible curve $C \subset X$.

Note that the condition on the characteristic cannot be weakened, see [11] for a counterexample due to Kóllar.

This note uses symplectic techniques to prove a part of this conjecture. For this reason, the base field will always be $\mathbb{C}$ and thus consider smooth complex projective surfaces. It is known that Conj. 2.2 holds in a number of cases. The following is a partial list:

- For some $m > 0$ the class $-mK_X$ is effective, for example on toric surfaces and minimal surfaces with Kodaira dimension 0 ([11], [1], [2]),
- $-K_X$ is nef [1],
- X is rational with $K_X^2 > 0$ [11],
- X is a S^2-bundle, an abelian or hyperelliptic surface or an elliptic surface of Kodaira dimension 1 and Euler number 0 [2].

In [1] a number of variations on this conjecture are stated, in particular a weak version which considers bounds on curves of a fixed genus.

Almost Complex Category: Let (M, J) be an almost complex manifold. Consider connected curves in M which can be represented as J-holomorphic embeddings off a finite set of points. At each of the non-embedded points the curve will have locally positive self-intersection.

Question 2.3. Let (M, J) be a smooth almost complex 4-manifold. Does there exist a positive constant $b(M, J)$ bounding the self-intersection of connected J-holomorphic curves C on M, i.e.

$$[C]^2 \geq -b(M, J)?$$

Restricting the almost complex structures in this question to Kähler complex structures relates this question to Conj. 2.2. Note however that there may be many canonical classes K_M admitting no almost complex structures which are Kähler (or even tamed by some symplectic form), for example on non-minimal ruled surfaces, see [15].

Symplectic Category: Let (M, ω) be a symplectic 4-manifold and $C \subset M$ an ω-symplectic submanifold. This means there is some surface Σ and an immersion $i : \Sigma \to M$ such that $i^*\omega$ is symplectic on Σ. Note that such an immersed surface need not be locally positive at its points of self-intersection (i.e. the points at which i is not injective). Such a surface must satisfy $[\omega] \cdot [C] > 0$.

Question 2.4. Let (M, ω) be a symplectic 4-manifold. Does there exist a positive constant $N_s(M, \omega)$ such that

$$C^2 \geq -N_s(M, \omega)$$

for all connected ω-symplectic submanifolds C?

The following shows that the condition $[\omega] \cdot [C] > 0$ is sufficient to ensure the existence of a submanifold in the class $[C]$.

Lemma 2.5 ([13])**.** *Let (M, ω) be a symplectic 4-manifold and $A \in H_2(M, \mathbb{Z})$ any class with $[\omega] \cdot A > 0$. Then A is represented by a connected immersed ω-symplectic submanifold.*

This allows us to completely answer question 2.4:

Lemma 2.6. *Let (M, ω) be a symplectic 4-manifold and $A \in H_2(M, \mathbb{Z})$ any class with $[\omega] \cdot A > 0$ and $A^2 < 0$. Then there are ω-symplectic submanifolds of arbitrarily large negative self-intersection.*

Notice that due to the possible presence of points of locally negative self-intersection on the immersed submanifolds, such submanifolds are not related to the objects considered in the algebraic or complex setting. This is the reason for restricting to ω-symplectic curves in the following.

Let (M, ω) be a smooth symplectic 4-manifold and let $C \subset M$ be a *connected ω-symplectic curve*. This means that C is a ω-symplectic submanifold and there is some almost complex structure J tamed by ω making C J-holomorphic. In particular, this implies that C is the image of a J-holomorphic map which is an embedding except at a finite number of points.

Question 2.7. Let (M, ω) be a symplectic 4-manifold. Does there exist a positive constant $N(M, \omega)$ depending on the symplectic form ω such that

$$C^2 \geq -N(M, \omega)$$

for all connected ω-symplectic curves C?

Note that $N(M, \omega)$ is invariant under rescaling of ω by a positive real number. A symplectic manifold (M, ω) is said to have *symplectic bounded negativity* if it admits such a positive constant.

A consequence of Lemma 2.6 is that any ω-symplectic submanifold representing the class $A \in H_2(M, \mathbb{Z})$ with $[\omega] \cdot A > 0$ and $A^2 < -N(M, \omega)$ must have points of locally negative self-intersection (and in particular cannot be embedded).

In the symplectic setting Question 2.7 has not been extensively studied. Some results are known, for example every minimal symplectic manifold of Kodaira dimension 0 has symplectic bounded negativity by [14] and adjunction. Moreover, for spheres in irrational ruled manifolds, the following was shown in [6].

Lemma 2.8. *Let (M, ω) be an irrational ruled manifold with $b^- = k$. Assume that $A \in H_2(M, \mathbb{Z})$ is representable by an ω-symplectic curve with genus 0. Then*

$$A^2 \geq -k.$$

Relations among the Categories: Let X be a smooth projective surface over $\mathbb{C}$. Then there exists an embedding of X into a complex projective space $\mathbb{C}P^n$ endowing X with a complex structure J. Note that in [11] a surface is viewed as already being embedded in some projective space. Under this embedding, the set of reduced irreducible curves in X is in bijection with the set of J-holomorphic curves on (X, J).

Moreover, (X, J) always admits a Kähler structure ω, hence the set $\mathcal{S}_J$ of symplectic structures tamed by J is non-empty. Furthermore, with respect to the symplectic manifold (X, ω) for any $\omega \in \mathcal{S}_J$, any reduced irreducible curve on X from Conj. 2.2 becomes a ω-symplectic curve in Question 2.7 via the complex structure J.

Thus there is the following connection:

$$\begin{array}{ccccc} X & \rightarrow & (X, J, \omega) & \rightarrow & (X, \omega) \\ \text{reduced, irreducible} & \rightarrow & \text{holomorphic} & \rightarrow & \omega-\text{symplectic} \end{array}$$

This provides a connection between Conj. 2.2 and Question 2.7. The following Lemma relates the answers.

Lemma 2.9. *Let X be a complex projective surface. Let J be the complex structure on X obtained via an embedding ϕ into $\mathbb{C}P^n$ and $\mathcal{S}_J$ the set of symplectic structures ω tamed by J. If (X, ω) satisfies symplectic bounded negativity for some symplectic structure $\omega \in \mathcal{S}_J$, then Conj. 2.2 holds. In this case*

$$b(X) \leq \inf_{\phi} \inf_{\omega \in \mathcal{S}_J} N(X, \omega).$$

Note that by the Kodaira embedding theorem and the scaling invariance of $N(, M, \omega)$, it would suffice to consider only integral forms $\omega \in \mathcal{S}_J$.

Corollary 2.10. *Let X be a complex projective surface. If X has symplectic bounded negativity for each $\omega \in \mathcal{S}_X = \{\omega \in \Omega^2(X) \mid \omega$ a symplectic form on $X\}$, then Conj. 2.2 holds.*

The converse is not necessarily true, even when allowing J to be almost complex, as symplectic curves can intersect negatively, while holomorphic curves cannot. For example, on $(S^2 \times \Sigma_h)\#4\overline{\mathbb{C}P^2}$ let E_i denote the classes of the exceptional divisors and $A = E_1 - E_2 - E_3 - E_4$ the class of an embedded symplectic -4-sphere obtained via blow-up of 3 points on E_1. Then $A \cdot E_1 = -1$ and hence there is no almost complex structure taming the symplectic structure such that both E_1 and A are represented by embedded pseudoholomorphic curves while both can be represented by embedded symplectic curves for some ω.

Lemma 2.11. *Let (M, ω) be a symplectic 4-manifold. Let $\mathcal{J}_\omega$ be the set of all almost complex structures on M tamed by ω. Then*

$$\sup_{J \in \mathcal{J}_\omega} b(M, J) < \infty \quad \Leftrightarrow \quad N(M, \omega) < \infty.$$

Proof. The key point is this: On (M, ω, J) with $J \in \mathcal{J}_\omega$, each J-holomorphic curve is also a ω-symplectic curve. On the other hand, every ω-symplectic curve is J-holomorphic for some $J \in \mathcal{J}_\omega$ but not necessarily all such J.

□

If Question 2.7 has a negative answer for some $\omega \in \mathcal{S}_M$, then a further interesting question is the following.

Question 2.12. Let M be a manifold with $\mathcal{S}_M \neq \emptyset$. Do there exist symplectic forms $\omega, \omega_\infty \in \mathcal{S}_M$ such that $N(M,\omega) < \infty$ but $N(M,\omega_\infty)$ unbounded? If so, can $[\omega] = [\omega_\infty]$?

Remark: Thus far the focus has been on negativity of curves. What of bounded positivity? In the symplectic setting, it follows from a result of Donaldson [4] that on any symplectic manifold (M,ω) if the Poincaré dual of $A \in H_2(M,\mathbb{Z})$ is sufficiently close to $[\omega] \in H^2(M,\mathbb{R})$, then kA is representable by an embedded ω-symplectic curve for any sufficiently large $k \in \mathbb{N}$. As $[\omega]^2 > 0$, this implies that $A^2 > 0$ and thus there can be no upper bound.

The discussions above show, that if no bound exists in the symplectic setting, no such bound exists for any almost complex structure tamed by some symplectic form and also for any smooth projective surface over $\mathbb{C}$.

In the smooth case, only manifolds with $b^+ = 0$ satisfy bounded positivity.

3. Geometrically Ruled Manifolds

Let M be a geometrically ruled manifold over a closed surface Σ_h of genus $h \geq 0$. In particular, fix an orientation on M and fix a ruling

$$S^2 \hookrightarrow M \to \Sigma_h.$$

Further fix an orientation on the fibers. Denote by $S \in H_2(M,\mathbb{Z})$ the class of a section and by $F \in H_2(M,\mathbb{Z})$ the class of the fiber.

The total space M is diffeomorphic to either a trivial bundle $S^2 \times \Sigma_h$ or the non-trivial bundle $S^2 \tilde{\times} \Sigma_h$. Denote the standard canonical classes on M by $K_{st} = -2S + (2h-2)F$ and $K_{st} = -2S + (2h-1)F$ respectively.

The symplectic cone $\mathcal{C}_M \subset H^2(M,\mathbb{R})$ of classes representable by symplectic forms has been determined in [16], [19], see also [23], [15]:

$$\mathcal{C}_M = \{\alpha \in H^2(M,\mathbb{R}) \mid \alpha^2 > 0\}.$$

$\mathcal{C}_M$ decomposes as a disjoint union of cones $\mathcal{C}_{M,K}$, where each cone contains only symplectic classes with symplectic canonical class $K_\omega = K$. In the minimal case, it was shown in [16] that there are only two possible symplectic canonical classes: $\pm K_{st}$. With regard to the standard canonical class the cone $\mathcal{C}_{M,K_{st}}$ is given by

$$\mathcal{C}_{M,K_{st}} = \{\alpha \in H^2(M,\mathbb{R}) \mid \alpha^2 > 0,\ \alpha \cdot F > 0\}.$$

Moreover, each class $\alpha \in \mathcal{C}_{M,K_{st}}$ contains a symplectic form ω compatible with the ruling, i.e. such that ω is non-degenerate along the fibers. It is shown in [16] (see also [23], [12]) that every symplectic form $\tilde{\omega}$ cohomologous to ω is diffeomorphic to it. In fact, this diffeomorphism can be chosen to be identity on homology. This implies the following lemma.

Lemma 3.1. *Let (M,ω) be a symplectic geometrically ruled manifold. Then it has symplectic bounded negativity if and only if $(M,\tilde{\omega})$ has symplectic bounded negativity for any symplectic form $\tilde{\omega}$ with $[\tilde{\omega}] = k[\omega]$ for any $k \in \mathbb{R}^+$. In particular,*

$$N(M,\omega) = N(M,\tilde{\omega}).$$

Let $D(M) \subset Aut(H^2(M,\mathbb{Z}))$ be the subgroup generated by Diff(M) on integral cohomology. If $h \geq 1$, then it is shown in [7] that an automorphism lies in $D(M)$ if and only if it preserves F up to a sign. In particular, $-\mathrm{Id} \in D(M)$. When $M \simeq S^2 \times S^2$, $-\mathrm{Id} \in D(M)$ still holds. Finally, when $M \simeq S^2 \tilde{\times} S^2$, then a result of Wall [25] shows that $D(M) = Aut(H^2(M,\mathbb{Z}))$. Moreover, the main result of [16], states that if $b^+(M) = 1$, then $D(M)$ acts transitively on the set of symplectic canonical classes.

Combining these results we obtain the following useful lemma:

Lemma 3.2. *Let M be a geometrically ruled manifold. Let*

$$\mathcal{S}_{M,F} = \{\omega \in \Omega^2(M) \mid \omega \text{ is compatible with the ruling, } K_\omega = K_{st}\}.$$

If bounded negativity holds for each $\omega \in \mathcal{S}_{M,F}$, then bounded negativity holds for any symplectic form on M.

Proof. Assume first that M is minimal. Let ω be a symplectic form on M with $K_\omega = -K_{st}$. Then use the diffeomorphism covered by $-\mathrm{Id}$ to switch to some symplectic form $\tilde{\omega}$ with standard canonical class. Note that for any class $A \in H_2(M,\mathbb{Z})$ representable by a ω-symplectic curve the effect of the diffeomorphism is $A \mapsto -A$ which preserves A^2 while $-A$ admits an $\tilde{\omega}$-symplectic representative.

Now switch from $\tilde{\omega}$ to a diffeomorphic and cohomologous symplectic form compatible with the ruling. The map induced by the diffeomorphism may change $-A$ to some other class, but it will again preserve the square and again be representable by a symplectic curve with respect to the new symplectic form.

By assumption, for this symplectic form, bounded negativity holds. As the square of the class A has been preserved throughout, the result follows.

Now let $M \simeq S^2 \tilde{\times} S^2$. Use a diffeomorphism to map ω to $\tilde{\omega}$ with $K_{\tilde{\omega}} = K_{st}$ and then a further diffeomorphism to obtain a symplectic form compatible with the ruling and cohomologous to $\tilde{\omega}$. As before, the proof is now complete. □

We shall first provide a homological classification of those classes of negative square with the potential to admit symplectic representatives.

Lemma 3.3. *Let (M,ω) be a symplectic geometrically ruled manifold and assume the symplectic structure is chosen such that $K_w = K_{st}$ and ω is compatible with the ruling. Assume that $A \in H_2(M,\mathbb{Z})$ such that $A^2 < 0$ and A is representable by an ω-symplectic curve. Then $A = S - kF$ with $k > 0$.*

Proof. **Case 1:** Assume that M is the trivial bundle with $h \geq 1$. In this case, we can basically repeat the calculation in Thm 4.11, [26]. Consider a class $A = aS + bF$ on $M = S^2 \times \Sigma_h$ for $h \geq 1$. Assume that the class A is represented by a ω-symplectic curve C and further assume that the curve admits only transverse self-intersections as singularities (see [20], [21]). Then the adjunction equality

$$A^2 + K_{st}A + 2 - 2g = 2\delta$$

holds, where δ denotes the number of transverse self-intersections. Smoothing these singular points and projecting to the base surface, it follows from Kneser, that

$$g + \delta - 1 \geq a(h-1).$$

Therefore, combining these two estimates, one obtains

$$2ab - 2b + 2(h-1)a = 2(g + \delta - 1) \geq 2(h-1)a$$

and thus

$$b(a-1) \geq 0.$$

Assuming that $A^2 = 2ab < 0$, it then follows that $a = 1$ and $b < 0$. Hence the only classes with negative square which can admit symplectic representatives are $S - kF$ for $k > 0$.

Case 2: Assume now that M is the non-trivial bundle with $h \geq 1$. The intersection behavior of S and F is

$$S^2 = 1, \quad F^2 = 0, \quad S \cdot F = 1.$$

A similar calculation employing the adjunction formula as above for a class $A = aS + bF$ with $A^2 = a(a+2b) < 0$ leads to the inequality $(a-1)(a+2b) \geq 0$. This implies that $a = 1$ and $a + 2b < 0$, which again implies that $A = S - kF$ with $k > 0$.

Case 3: Let $M \simeq S^2 \times S^2$. As in the proof of Lemma 3.3, the adjunction formula provides the equality

$$2ab - 2a - 2b + 2 - 2g = 2\delta$$

which can be rewritten as

$$ab - a - b \geq -1.$$

The assumption $A^2 < 0$ implies that $a \neq 0 \neq b$. Therefore, this inequality implies

$$a(b-1) \geq 0 \text{ or } b(a-1) \geq 0$$

from which the result follows as in the first case in Lemma 3.3. Note that only the class $S - kF$ can be symplectic with regard to the given ruling.

Case 4: Now let $M \simeq S^2 \tilde{\times} S^2 \simeq \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Assume that $A = aS + bF$ with $A^2 = a(a+2b) < 0$. To be compatible with the ruling it must have $a > 0$. The rest of the calculation using the adjunction formula is identical to the $h \geq 1$ case. □

This result coincides nicely with Prop. 3.14 in [17]. Note that the curves in Lemma 3.3 are explicitly excluded in Prop 3.14 when $h \geq 1$. Moreover, when $h = 0$ it holds that

$$(K_{st} + A)^2 > 0.$$

With these results we are now ready to prove that symplectic bounded negativity holds for geometrically ruled manifolds. This is presumably known to experts, see [22], we state the result in the context of bounded negativity and for completeness.

Lemma 3.4. *Let M be diffeomorphic to $S^2 \times \Sigma_h$. Let ω be a symplectic structure on M in the class $[\omega] = \omega_1 S + \omega_2 F$ and A represented by an ω-symplectic curve. Then (M, ω) has symplectic bounded negativity and*

$$A^2 > -2\frac{\omega_2}{\omega_1}.$$

Proof. We may assume that $A^2 < 0$. Then $A = S - kF$ and $A^2 = -2k$ by Lemma 3.3.

Consider a symplectic form ω in the class $[\omega] = \omega_1 S + \omega_2 F$ with $\omega_1, \omega_2 > 0$, i.e. $[\omega] \in \mathcal{C}_{M,K_{st}}$. Then for a curve in the class A to be symplectic it must hold that $[\omega] \cdot A > 0$ and thus

$$-\omega_1 k + \omega_2 > 0.$$

If $[\omega] \in \mathcal{C}_{M,K_{st}}$, then the result follows from the calculation above. If $[\omega] \in \mathcal{C}_{M,-K_{st}}$, then Lemma 3.2 proves the existence of a lower bound for A^2. More precisely, use the diffeomorphism covered by $-\mathrm{Id}$. This has the effect of mapping $[\omega] \mapsto -[\omega]$. This preserves the fraction $\frac{\omega_2}{\omega_1}$. □

Corollary 3.5. *If $\frac{\omega_2}{\omega_1} < 1$, then (M, ω) admits no negative ω-symplectic curves.*

Proof. If $A^2 < 0$, then $A^2 = -2k$. However, under the given assumption Lemma 3.4 implies that $A^2 > -2$. □

Now let M be diffeomorphic to $S^2 \tilde{\times} \Sigma_h$. Consider a symplectic form ω in the class $[\omega] = \omega_1 S + \omega_2 F$ with standard canonical class $K_{st} = -2S + (2h - 1)F$. If A is such that $A^2 < 0$, then Lemma 3.3 and the condition $[\omega] \cdot A > 0$ imply that

$$\omega_1 + \omega_2 - k\omega_1 > 0.$$

This further implies that for any $A \in H_2(M, \mathbb{Z})$

$$A^2 \geq 1 - 2k > 1 - 2\frac{\omega_2 + \omega_1}{\omega_1} = -1 - 2\frac{\omega_2}{\omega_1}.$$

Arguments using diffeomorphisms as in the proof of Lemma 3.4 lead to the following:

Lemma 3.6. *Let M be diffeomorphic to $S^2 \tilde{\times} \Sigma_h$. Let ω be a symplectic structure on M in the class $[\omega] = \omega_1 S + \omega_2 F$ and A represented by an ω-symplectic curve. Then (M, ω) has symplectic bounded negativity.*

A precise estimate for A^2 can be given as well. In the $h \geq 1$ case, this has been calculated above.

The $h = 0$ case has the peculiarity of being the only non-minimal example in the current discussion. As such, depending on the symplectic canonical class, the class of the symplectic form can have negative coefficients in the given basis with respect to the given ruling. The precise lower bound can be determined by considering an orbit of a symplectic class under $D(M)$: Let ϕ_1 be the automorphism of $H^2(M, \mathbb{Z})$ that fixes S and sends F to $2S - F$. Let ϕ_2 be the automorphism which sends S to $-S$ and F to $-2S + F$. Note that these maps are their own inverses. Then

$$\begin{array}{ccc} \omega_1 S + \omega_2 F & \overset{\phi_1}{\longleftrightarrow} & \omega_1 S + (2\omega_1 - \omega_2)F \\ \updownarrow \phi_2 & & \updownarrow \phi_2 \\ -\omega_1 S + (-2\omega_1 + \omega_2)F & \overset{\phi_1}{\longleftrightarrow} & -\omega_1 S - \omega_2 F \end{array}$$

Therefore, given $[\omega] = \omega_1 S + \omega_2 F$, if $\pm(\omega_1 S + \omega_2 F) \in \mathcal{C}_{M,K_{st}}$, then

$$A^2 > -1 - 2\frac{\omega_2}{\omega_1}.$$

If $\pm(\omega_1 S + (2\omega_1 - \omega_2)F) \in \mathcal{C}_{M,K_{st}}$, then

$$A^2 > -5 + 2\frac{\omega_2}{\omega_1}.$$

Note that in the second case, $\omega_1 \cdot \omega_2 < 0$, so the lower bound continues to be negative.

This proves the following lemma.

Lemma 3.7. *Let M be diffeomorphic to $S^2 \tilde{\times} \Sigma_h$. Let ω be a symplectic structure on M in the class $[\omega] = \omega_1 S + \omega_2 F$ and A represented by an ω-symplectic curve. If $h = 0$, assume further that $K_\omega = K_{st}$. Then*

$$A^2 > -1 - 2\frac{\omega_2}{\omega_1}.$$

4. Ruled Manifolds

In this section M will be the blow-up of a geometrically ruled manifold. Note that after blowing up, the two manifolds $(S^2 \times \Sigma_h)\#\overline{\mathbb{C}P^2}$ and $(S^2 \tilde{\times} \Sigma_h)\#\overline{\mathbb{C}P^2}$ are diffeomorphic. In this case always consider $(S^2 \times \Sigma_h)\#\overline{\mathbb{C}P^2}$ with a given fixed ruling on $S^2 \times \Sigma_h$ and fixed orientations. The standard basis of $H_2(M, \mathbb{Z})$ is given by $\{S, F, E_1, ..., E_k\}$ and denote by $K_{st} = -2S + (2h - 2)F + \sum E_i$ the standard canonical class.

If M is non-minimal, let

$$\mathcal{E} = \{A \in H_2(M, \mathbb{Z}) \mid A \text{ is represented by a smooth sphere, } A^2 = -1\}.$$

The symplectic cone $\mathcal{C}_M \subset H^2(M, \mathbb{R})$ of classes representable by symplectic forms has been determined in [19], [16], see also [23], [16]:

$$\mathcal{C}_M = \{\alpha \in H^2(M, \mathbb{R}) \mid \alpha^2 > 0,\ \alpha \cdot e \neq 0\ \forall e \in \mathcal{E}\}.$$

$\mathcal{C}_M$ decomposes as a disjoint union of cones $\mathcal{C}_{M,K}$, where each cone contains only symplectic classes with $K_\omega = K$. When $h \geq 1$, one has with regard to the standard canonical class that

$$\mathcal{C}_{M,K_{st}} = \{\alpha \in H^2(M,\mathbb{R}) \mid \alpha^2 > 0,\ \alpha \cdot F > 0,\ \alpha \cdot E_i > 0,\ \alpha \cdot (F - E_i) > 0\}.$$

4.1. **Positive Genus Base.** Assume that $M = (S^2 \times \Sigma_h)\#k\overline{\mathbb{C}P^2}$ with $h \geq 1$, i.e. M is non-minimal irrational ruled. The case of curves of genus 0 has been stated in Lemma 2.8. Moreover, it is clear that no curves of genus $g < h$ exist in M. One may thus assume that $g \geq h \geq 1$.

In the geometrically ruled case the symplectic form was always assumed to be compatible with the ruling. In the current setting the "correct" symplectic forms to consider are as follows: Let $\omega \in \mathcal{S}_M$ be such that (M,ω) can be decomposed as a symplectic sum of k copies of $((S^2 \times S^2)\#\overline{\mathbb{C}P^2}, \omega_i)$ along distinct fibers in $(S^2 \times \Sigma_h, \omega_0)$ and that $K_\omega = K_{st}$. Denote the collection of such symplectic forms by $\mathcal{S}_{M,D}$.

The symplectic sum of two symplectic 4-manifolds (M_1,ω_1) and (M_2,ω_2) along the co-dimension 2 submanifolds $V_i \subset M_i$, where V_1 is diffeomorphic to V_2 and $[V_1]^2 + [V_2]^2 = 0$, is the smooth manifold

$$M = M_1\#_V M_2 = (M_1\backslash NV_1) \cup_\phi (M_2\backslash NV_2)$$

obtained by removing a tubular neighborhood NV_i of each V_i from M_i and gluing the two open manifolds along their boundary by an orientation reversing diffeomorphism ϕ together with the symplectic form ω obtained from the ω_i. This operation is easily seen to produce a smooth manifold, that this is also a symplectic surgery was shown in [8] and [18], see also [10].

Lemma 4.1. *Let M be non-minimal irrational ruled. Consider the map*

$$C : \mathcal{S}_{M,D} \to \mathcal{C}_{M,K_{st}}$$

given by $\omega \mapsto [\omega]$. Then this map is a surjection.

Proof. Let $\alpha \in \mathcal{C}_{M,K_{st}}$. Write $\alpha = a_1 S + a_2 F + \sum_{i=1}^k e_i E_i$ in the standard basis of $(S^2 \times \Sigma_h)\#k\overline{\mathbb{C}P^2}$. It must be shown that this class contains a symplectic form obtained from the symplectic sum of $(S^2 \times \Sigma_h, \omega_0)$ with k copies of $((S^2 \times S^2)\#\overline{\mathbb{C}P^2}, \omega_i)$ along disjoint fibers.

Observe that α must satisfy the following: $\alpha^2 > 0$, $\alpha \cdot E_i > 0$ and $\alpha \cdot (F - E_i) > 0$. These imply the inequalities

$$2a_1a_2 - \sum e_i^2 > 0 \text{ and } a_1 > -e_i > 0.$$

Let $\alpha_0 \in H^2(S^2 \times \Sigma_h, \mathbb{R})$ be $\alpha_0 = a_1 S + a_{20} F$. Similarly, let $\alpha_i \in H^2((S^2 \times S^2)\#\overline{\mathbb{C}P^2}, \mathbb{R})$ be $\alpha_i = a_1 S + a_{2i} F + e_i E_i$ for $i \in \{1, .., k\}$. To ensure that all of these classes are representable by symplectic forms, the following conditions must be satisfied:

- $\alpha_0^2 > 0$: This implies that $a_{20} > 0$.

- $\alpha_i^2 > 0$: This implies that $2a_1a_{2i} - e_i^2 > 0$ and in particular that $a_{2i} > \frac{e_i^2}{2a_1} > 0$.
- $\alpha_i \cdot E_i > 0$ and $\alpha \cdot (F - E_i) > 0$: These imply that $a_1 > -e_i > 0$.

Choose the symplectic form ω_0 to be compatible with the ruling on $S^2 \times \Sigma_h$ and each ω_i on $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ such that they stem from a symplectic form on the trivial minimal model of M compatible with that ruling.

The symplectic sum will produce a symplectic form ω, compatible with the ruling in the sense described above, in the class (see [8])

$$a_1 S + \left(\sum_{i=0}^{k} a_{2i}\right) F + \sum_{i=1}^{k} e_i E_i.$$

Thus, in order to get a symplectic form in the class α, the following must hold:

$$\sum_{i=0}^{k} a_{2i} = a_2. \tag{1}$$

The classes α_i must now be chosen appropriately. Note that a_1 and e_i are already fixed by α and in this way automatically satisfy all of the constraints placed on e_i and a_1 in both settings. It remains to carefully choose the a_{2i}: Let $a_2 = \frac{\sum_i e_i^2}{2a_1} + \epsilon + \delta$ for some choice of $\epsilon, \delta > 0$. For $i \neq 0$, choose $a_{2i} = \frac{e_i^2}{2a_1} + \epsilon_i$ for some $\epsilon_i \ll 1$. Let $a_{20} = \delta$ and $\epsilon = \sum_i \epsilon_i$. Then

$$\sum_{i=0}^{k} a_{2i} = \delta + \sum_i \frac{e_i^2}{2a_1} + \sum_i \epsilon_i = a_2.$$

With this choice, each of the α_i is representable by a symplectic form, compatible with the base ruling, such that one can perform the symplectic sum along disjoint fibers of $S^2 \times \Sigma_h$ to obtain a symplectic form in the class α as needed. □

We now begin a series of reductions which will ultimately reduce the problem to showing symplectic bounded negativity for $\omega \in \mathcal{S}_{M,D}$ and certain curve configurations. Let M be a non-minimal irrational ruled manifold diffeomorphic to $(S^2 \times \Sigma_h)\#k\overline{\mathbb{C}P^2}$.

(1) The main result in [16] shows that given $(M, \tilde{\omega}, \tilde{C})$, $\tilde{C}$ an $\tilde{\omega}$-symplectic curve, then there is a diffeomorphism taking $\tilde{\omega}$ to some symplectic form ω with $K_\omega = K_{st}$. This maps $\tilde{C}$ to some ω-symplectic curve C while preserving the self-intersection number.
(2) Let C be a ω-symplectic curve representing $A \in H_2(M, \mathbb{Z})$. If necessary, perturb C such that it has only transverse self-intersections (see [20], [21]). The adjunction formula ensures there are at most finitely many of these. Perform a small blow-up of each of these. Thus obtain a smooth symplectic curve C_b with respect to a symplectic form

ω_b in $M\#l\overline{\mathbb{C}P^2}$ with $[\omega_b] = [\omega] + \sum_{i=1}^{l} e_i E_i$ and $[C_b] = A - 2\sum_{i=1}^{l} E_i$. Note that $[\omega_b] \cdot [C_b] > 0$ and, by construction, $e_i \ll 1$.

The main stability result in [6] (see also Cor. 3.5 therein and Lemma 4.11, [15]) implies that there exists a $\tilde{\omega}_b$-symplectic curve $\tilde{C}_b$ in the class $[C_b]$ for any symplectic form $\tilde{\omega}_b$ cohomologous to ω_b. Choose $\tilde{\omega}_b \in \mathcal{S}_{M\#l\overline{\mathbb{C}P^2},D}$ by Lemma 4.1.

Now blow-down the l exceptional spheres to obtain a curve $\tilde{C} \subset M$, symplectic with respect to a symplectic form $\tilde{\omega}$. Ideally this curve will satisfy the following conditions:

- $[\tilde{C}] = A$ and $[\tilde{\omega}] = [\omega]$
- Under the symplectic sum decomposition given by $\tilde{\omega}$, all transverse self-intersections of $\tilde{C}$ as well as all genus is carried by components lying in $S^2 \times \Sigma_h$.
- The curve C has only disjoint spheres in $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ after the cut. Assume that all of these spheres intersect at least one of the two exceptional spheres in the classes E and $F - E$.

The first property is ensured by the construction above as is the first part of the second property. If necessary, to ensure the remaining properties hold, blow-down the remaining exceptional spheres and perform a small blow-up on each of the resulting singularities on the blown-down curve. This will occur at the cost of a change in the class $[\omega]$, namely the coefficients of the exceptional spheres will become smaller. The symplectic form thus obtained will still lie in $\mathcal{S}_{M,D}$.

Theorem 4.2. *Let (M, ω) be a non-minimal irrational ruled manifold diffeomorphic to $(S^2 \times \Sigma_h)\#k\overline{\mathbb{C}P^2}$ endowed with a symplectic structure ω such that $K_\omega = K_{st}$. Let $A \in H_2(M, \mathbb{Z})$ be representable by an ω-symplectic curve C. Let $[\omega] = \omega_1 S + \omega_2 F + \sum_i e_i E_i \in \mathcal{C}_{M,K_{st}}$. Then*

$$A^2 > -2\frac{\omega_2}{\omega_1} - k.$$

Proof. Without loss of generality, assume a configuration $(M, \tilde{\omega}, \tilde{C})$ as described above. For convenience, we now drop the tilde.

Let $[C] = A = aS + bF + \sum_{i=1}^{k} c_i E_i$. Assume that all $c_i \neq 0$, otherwise blow-down E_i. Observe that the assumption $g > 0$ and $K_\omega = K_{st}$ implies that $A \cdot E_i > 0$ and $A \cdot (F - E_i) \geq 0$ by positivity of intersection of pseudo-holomorphic curves. Therefore

$$a \geq -c_i > 0.$$

The symplectic cut decomposes the curve C into components lying in $S^2 \times \Sigma_h$ and in each of the $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$. We consider each case separately.

$\mathbf{S^2} \times \mathbf{\Sigma_h}$: Let $A_{S^2 \times \Sigma_h} = \sum_i A_i$ with $A_i = a_i S + b_i F$. Observe that each curve component C_i with $[C_i] = A_i$ intersects the fiber F along which the sums are performed locally positively as the original curve C was connected.

Hence $a_i > 0$. Moreover,

$$A_i \cdot A_j = a_i b_j + a_j b_i \geq 0$$

for all $i \neq j$.

Assume that some A_i has $A_i^2 < 0$. Then it follows from Lemma 3.3 that $A_i = S - k_i F$ with $k_i > 0$. Then

$$A_i \cdot A_j = b_j - a_j k \geq 0$$

implies that at most one curve component C_i can have $b_i < 0$ and hence non-positive square. Thus by Lemma 3.4, with regard to the symplectic form ω_0 obtained in the cut, it follows that

$$A^2_{S^2\times\Sigma_h} > -2\frac{\omega_{20}}{\omega_1}.$$

Observe that $\omega_1 = [\omega_0] \cdot F = [\omega] \cdot F$ while the coefficients of F differ.

$(\mathbf{S^2} \times \mathbf{S^2})\#\overline{\mathbb{C}\mathbf{P^2}}$: Now consider the curve components in $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$. Let $A_{(S^2\times S^2)\#\overline{\mathbb{C}P^2}} = \sum A_i$ with $A_i = a_i S + b_i F + c_i E$. Here S and F are both represented by spheres, we continue to distinguish the fiber class as the class along which the symplectic sum is performed.

Note that it is still true that $a_i \geq -c_i > 0$. The curve component C_i corresponding to the class A_i is obtained in the symplectic cut from the curve C. The original curve C intersected both E and $F - E$ locally positively. Hence this also holds for each C_i.

Moreover,

$$A_i \cdot A_j = a_i b_j + a_j b_i - c_i c_j = 0.$$

Assume that there is some class A_i with $b_i \leq 0$. Then this inequality implies that A_i is the only class with this property. Hence there is at most one class $A_0 = a_0 S + b_0 F + c_0 E$ with $b_0 \leq 0$.

Case 1: Assume first that $b_0 < 0$. The curve representing A_0 is a blow-up of a symplectic curve $\tilde{C}$ in $S^2 \times S^2$. Then our assumptions imply that $[\tilde{C}]^2 < 0$ and Lemma 3.3 shows that $a = 1$. Let $\tilde{\omega}_l$ be the symplectic form on $S^2 \times S^2$ obtained by blowing down $((S^2 \times S^2)\#\overline{\mathbb{C}P^2}, \omega_l)$ which in turn was obtained from the symplectic sum decomposition. Denote the class of $\tilde{\omega}_l$ by $[\tilde{\omega}_l] = \omega_1 S + \omega_{2l} F$. Then $[\tilde{\omega}_l] \cdot [\tilde{C}] > 0$ implies again that

$$[\tilde{C}]^2 > -2\frac{\omega_{2l}}{\omega_1}.$$

Recall that $a_0 \geq -c_0 > 0$, which in this case implies that $c_0 = -1$. Therefore it follows that

$$A_0^2 = [\tilde{C}]^2 - 1 > -2\frac{\omega_{2l}}{\omega_1} - 1.$$

Case 2: Consider now the remaining components $A_i = a_i S + b_i F + c_i E$ with $b_i \geq 0$. Recall that $(S^2 \times S^2)\#\overline{\mathbb{C}P^2} \simeq \mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. For the standard basis $\{H, E_1, E_2\}$ of $H_2(\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}, \mathbb{Z})$ we have the coordinate change formulas

$$S = H - E_1, \quad F = H - E_2, \quad E = H - E_1 - E_2.$$

Hence $A_i = a_iS + b_iF + c_iE = (a_i + b_i + c_i)H - (b_i + c_i)E_1 - (a_i + c_i)E_2$. Observe that $a_i + b_i + c_i \geq 0$.

Case 2.1: If $a_i + b_i + c_i > 0$, then according to Prop 4.5, [26], the class A_i must be given as $A_i = H - E_1 - E_2 = E$. However, this violates $a_i > 0$. Therefore no such curve can occur.

Case 2.2: If $a_i + b_i + c_i = 0$, then this implies that $b_i = 0$ and $a_i + c_i = 0$. Thus $A_i = a_i(S - E)$, a multiple of the class of an exceptional curve. For this class to be represented by an embedded sphere, it follows from the adjunction formula that $a_i = 1$. Thus $A_i = S - E$ and $A_i^2 = -1$.

Notice that either $A_i = S - E$ or $A_i = S - k_iF - E$ can occur, but not both as their pairwise intersection is negative. These are the only two classes of negative self-intersection that can occur in $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ in this symplectic cut construction.

In summary: Given $M \simeq (S^2 \times \Sigma_h)\#k\overline{\mathbb{C}P^2}$, a symplectic form ω on M allowing for a symplectic sum decomposition of M with $[\omega] = \omega_1 S + \omega_2 F + \sum_i e_i E_i$, and a curve in the class A lying in the special position as described above, it follows that:

(1) On $(S^2 \times \Sigma_h, \omega_{S^2\times\Sigma_h})$ with $[\omega_{S^2\times\Sigma_h}] = \omega_1 S + \omega_{20} F$ the class $A_{S^2\times\Sigma_h}$ satisfies the lower bound

$$A^2_{S^2\times\Sigma_h} > -2\frac{\omega_{20}}{\omega_1}.$$

(2) On $((S^2 \times S^2)\#\overline{\mathbb{C}P^2}, \omega_l)$ with $[\omega_l] = \omega_1 S + \omega_{2l} F + e_l E_l$ the class $A_{(S^2\times S^2)\#\overline{\mathbb{C}P^2}}$ satisfies the lower bound

$$A^2_{(S^2\times S^2)\#\overline{\mathbb{C}P^2}} > -2\frac{\omega_{2l}}{\omega_1} - 1.$$

(3) Due to Eq. 1, we have

$$\omega_{20} + \sum_{l=1}^{k} \omega_{2l} = \omega_2$$

Squares of curves in the symplectic sum are additive, see [5], and hence

$$A^2 = A^2_{S^2\times\Sigma_h} + \sum_{l=1}^{k} A^2_{(S^2\times S^2)\#\overline{\mathbb{C}P^2}} >$$

$$> -2\frac{\omega_{20}}{\omega_1} + \sum_{l=1}^{k}\left(-2\frac{\omega_{2l}}{\omega_1} - 1\right) = -2\frac{\omega_2}{\omega_1} - k$$

□

Let (M, ω, C) be given. The construction reduced the problem as follows:

$$(M, \omega, C) \xrightarrow{\phi} (M, \omega_{st}, C_{st}) \to (M, \omega_{ss}, C_{ss})$$

where the first arrow is a diffeomorphism such that $K_{\omega_{st}} = K_{st}$ and $\omega_{ss} \in \mathcal{S}_{M,D}$. Note that $[\omega_{st}]$ and $[\omega_{ss}]$ have the same S and F coefficients. Moreover, the whole construction preserved $[C]^2$. We have shown that for any (M, ω_{st}) the lower bound on the self-intersection is given by $-2\frac{\omega_2}{\omega_1} - k$. This in particular implies that M has symplectic bounded negativity for any symplectic form $\omega \in \mathcal{S}_M$.

Using reflections along -1-spheres in M a lower bound can be explicitly determined for any $\omega \in \mathcal{S}_M$. It takes the following form:

$$A^2 \geq -2\frac{\beta}{\alpha} - \tau(K_\omega) - k$$

where α and β are the coefficients of S and F respectively of the symplectic form obtained by blowing down the ω-symplectic exceptional curves which lead to the trivial minimal model. The term $\tau(K_\omega)$ is positive and depends only on the symplectic canonical class K_ω (which is of course determined completely by ω).

4.2. **Spherical Base.** The rational case will be discussed in two parts: First, consider only curves with $g > 1$. Once a suitable version of Lemma 4.1 has been proven, the argument is mostly the same as in the irrational case. Note that the condition that $g > 0$ implies that $A \cdot E \geq 0$ and $A \cdot (F - E) \geq 0$ continue to hold, these played a central role in the arguments to prove Theorem 4.2.

In the second part spherical symplectic curves will be discussed. Here the estimates on the intersection behavior with exceptional curves no longer needs to hold, the Gromov limit of the exceptional curves may have the spherical symplectic curve as a component and hence negative intersections are possible. This forms the core of the argument, showing that there are no curves with $A \cdot E < 0$ that contribute to the decomposition.

Lemma 4.3. *Let M be rational. Consider the map*

$$C : \mathcal{S}_{M,D} \to \mathcal{C}_{M,K_{st}}$$

given by $\omega \mapsto [\omega]$. Then this map is a surjection.

Proof. In contrast to the irrational ruled case, here the set of exceptional curves can be very complicated. Thus the symplectic cone of classes representable by symplectic forms with canonical class K_{st} can only be generally stated as

$$\mathcal{C}_{M,K_{st}} = \{\alpha \in H^2(M, \mathbb{R}) \mid \alpha^2 > 0,\ \alpha \cdot E > 0 \text{ for all } E \in \mathcal{E}_{K_{st}}\}$$

where $\mathcal{E}_{K_{st}} \subset \mathcal{E}$ is the set of classes E with $K_{st} \cdot E = -1$.

Let $\alpha \in \mathcal{C}_{M,K_{st}}$. Note that when $M \simeq (S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ the classes $E_i, F - E_i, S - E_i \in \mathcal{C}_{M,K_{st}}$ and thus α is positive on each of these.

Let $\alpha = \omega_1 S + \omega_2 F + \sum e_i E_i$. Choose the classes α_i as in the proof of Lemma 4.1, noting that the conditions imposed on α_i on each $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ are precisely the positivity conditions on the classes $E, F - E, S - E$, where the condition from $S-E$ is automatically satisfied as the F coefficient and the E coefficient are given by ω_1 and the corresponding e_i.

□

It is interesting to note, that once the α_i satisfy the homological constraints as given in the proof, due to the way the exceptional curves are constructed in the symplectic sum, the symplectic form produced in the symplectic sum evaluates positively on every $E \in \mathcal{E}_{K_{st}}$. For example, the exceptional class $S+F-E_1-E_2-E_3-E_4$ in $(S^2\times S^2)\#4\overline{\mathbb{C}P^2}$ is produced from the diagonal class $S+F$ in $S^2\times S^2$ combined with the exceptional spheres $S-E_i$ in each $((S^2\times S^2)\#\overline{\mathbb{C}P^2}$ summand. The class $S+F$ is symplectic, so any ω produced in this fashion will automatically evaluate positively on this exceptional class.

4.2.1. *Positive Genus Curves.* With this result, the proof of the following theorem is identical to the proof of Theorem 4.2.

Theorem 4.4. *Let (M,ω) be a rational ruled manifold diffeomorphic to $(S^2\times S^2)\#k\overline{\mathbb{C}P^2}$ endowed with a symplectic structure ω such that $K_\omega = K_{st}$. Let $A\in H_2(M,\mathbb{Z})$ be representable by an ω-symplectic curve C with $g(C)>0$. Let $[\omega]=\omega_1 S+\omega_2 F+\sum_i e_iE_i \in \mathcal{C}_{M,K_{st}}$. Then*

$$A^2 > -2\frac{\omega_2}{\omega_1} - k.$$

4.2.2. *Spherical Curves.* The key point in this section is to show that in the symplectic cut in the proof of Theorem 4.2 there cannot appear classes $A\in H_2((S^2\times S^2)\#\overline{\mathbb{C}P^2},\mathbb{Z})$ with $A\cdot E<0$ or $A\cdot(F-E)<0$. If $A^2\geq 0$, then this follows from standard arguments using pseudoholomorphic curve techniques.

When $A^2<0$, the arguments in the proof of Theorem 4.2 relied crucially on the two estimates $A\cdot E>0$ and $A\cdot(F-E)\geq 0$. These same arguments, assuming these two inequalities, continue to hold in the spherical case and one would obtain the same estimate as before for the class A.

Assume now that $A^2<0$ where $A=aS+bF+cE$ and $a>0$. If $A\cdot E<0$, then $c>0$ and thus $A\cdot(F-E)=a+c>0$. On the other hand, if $a+c<0$, then $c<0$ and so $A\cdot E>0$. Thus the case $A\cdot E<0$ and $A\cdot(F-e)<0$ does not occur.

Case 1: Assume that $A\cdot E<0$ but $A\cdot(F-e)\geq 0$. Then $a,c>0$. As A is represented by an embedded sphere in this construction, adjunction implies that

$$2ab-c^2-2a-2b-c+2=0$$

which can be rewritten as

$$2b(a-1)=c^2+c+2a-2>0.$$

Thus $b>0$. Now switch from $(S^2\times S^2)\#\overline{\mathbb{C}P^2}$ to $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. Then $A=(a+b+c)H-(a+c)E_1-(b+c)E_2$ where $a+b+c>0$. Again Prop. 4.5,

[26], implies that $A = H - E_1 - E_2 = E$ and thus $a = 0$, violating $a > 0$. Hence such a curve does not appear.

Case 2: Now let $A \cdot E > 0$ but $A \cdot (F - E) < 0$. Thus $c < 0 < a$ and $a + c < 0$.

Case 2.1: Assume first that $b > 0$. If $a + b + c > 0$, then again $A = E$ contradicting $a > 0$. If $a + b + c \leq 0$, then both $A \cdot E_1 = a + c < 0$ and $A \cdot E_2 = b + c < 0$. However, this is impossible due to Theorem 3.4, [26], which we restate in terms of the current setting: Let J be any almost complex structure taming the symplectic structure ω on $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. Then at least two of the classes $\{E_1, E_2, H - E_1 - E_2\}$ are represented by smooth J-holomorphic embedded spheres.

This would imply that A must intersect those two exceptional spheres non-negatively, which is in contradiction to the calculations above.

Case 2.2: Assume now that $b < 0$. Blow-down along E to get a curve C_b in $S^2 \times S^2$ in the class $A_b = aS + bF$ with $A^2 = 2ab < 0$. Thus by Lemma 3.3, $A = S - kF$. Observe that by adjunction this is an embedded sphere, thus any blow-up of this curve would have intersection 1 with the exceptional curve E. However, by assumption $1 = a < -c$. Thus such a curve cannot appear.

Case 2.3: Let $b = 0$. Then $A = aS + cE$ and $A^2 = -c^2$. The adjunction formula shows that

$$\underbrace{-(c^2 + c)}_{\leq 0} \underbrace{-2a}_{<0} + 2 = 0$$

and thus $a = -c = 1$ in violation of $a + c < 0$. Hence this case also does not occur.

Therefore we may assume that every component in $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ satisfies $A \cdot E > 0$ and $A \cdot (F - E) \geq 0$, thus reducing the calculation to precisely that which can be found in the proof of Theorem 4.2. This proves the following theorem.

Theorem 4.5. *Let (M, ω) be a rational ruled manifold diffeomorphic to $(S^2 \times S^2)\#k\overline{\mathbb{C}P^2}$ endowed with a symplectic structure ω such that $K_\omega = K_{st}$. Let $A \in H_2(M, \mathbb{Z})$ be representable by an ω-symplectic curve C with $g(C) = 0$. Let $[\omega] = \omega_1 S + \omega_2 F + \sum_i e_i E_i \in \mathcal{C}_{M,K_{st}}$. Then*

$$A^2 > -2\frac{\omega_2}{\omega_1} - k.$$

As in the irrational ruled case, the result for general (M, ω) can be obtained via a diffeomorphism taking ω to some symplectic form with standard canonical class.

This completes the proof of Theorem 1.1.

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