RIGIDITY OF ACTIONS ON PRESYMPLECTIC MANIFOLDS

PHILIPPE MONNIER

ABSTRACT. We prove the rigidity of Hamiltonian or presymplectic actions of a compact semisimple Lie algebra on a presymplectic manifold of constant rank in the local and global case. The proof uses an abstract normal form theorem we had stated in a previous work, based on an iterative process of Nash-Moser type. In order to use correctly this abstract theorem, we need to construct a new smoothing operator for differential forms and multivector fields which preserves the Hamiltonian formalism associated to the presymplectic structure.

1. Introduction

The symplectic manifolds (M, ω) represent a nice framework to study many physical systems ; the manifold M corresponds to the phase space, ω is a closed non degenerate 2-form on M, and the dynamics of the system is given by the vector field X which satisfies $i_X\omega = dH$ where H, the energy, is a given smooth function defined on M. This vector field is called the Hamiltonian vector field associated to H. Because ω is nondegenerate, X always exists and is unique.

The constrained Hamiltonian systems (electrodynamics, gravitation) have been studied initially by Dirac in [7]. These systems arise for instance from degenerate Lagrangian systems, i.e. the Legendre transform is not a diffeomorphism (for instance for Maxwell system). It appeared in this case that the presymplectic formalism could be more relevant. More precisely, the constraints are described by a submanifold of a symplectic manifold ; the restriction of the symplectic form to this submanifold may be degenerate which gives a presymplectic manifold. In this case, if the function H is given, the relation $i_X\omega = dH$ may not have solutions and, if it has solutions they may not be unique.

The geometrical framework to describe the constrained systems has been studied by several authors, see [3], [7], [11], [12], [17] and [27].

In this paper, we study the Hamiltonian actions and presymplectic actions of a Lie group on a presymplectic manifold. We prove the rigidity of the Hamiltonian and presymplectic actions of a compact semisimple Lie group on a presymplectic manifold of constant rank. This result is proved in the local and global case. More precisely we prove (see Theorems 4.2 and 4.3) :

Take two Hamiltonian actions or two presymplectic actions of a compact semisimple Lie algebra on a presymplectic manifold of constant rank. If they are sufficiently close, then they are equivalent.

Such a result already exists in symplectic geometry (see for instance [21]). In this case, it is just an application of Moser's path method and averaging. This method does not work so much in the presymplectic case because the two form is degenerate. We had to face this difficulty in [21] where we proved a similar result for Poisson manifolds. In order to prove this result, we had stated a theorem of abstract normal form which was a generalization of a first tentative in [22]. This abstract theorem can in particular be used in order to prove the smooth linearization theorem of Conn ([5]), the smooth Levi decomposition ([22]) for smooth Poisson structures and the rigidity of Hamiltonian actions of a Poisson-Lie group on a Poisson manifold ([8]). Moreover, it has been recently used, in a slightly modified version, in the context of generalized complex geometry, see [2]. The proof of this abstract normal form theorem was an iterative process based on Newton's method and the use of Smoothing operators in order to correct the loss of differentiability. This "Nash-Moser type" method has been used in Poisson geometry for instance in [5], [18] and [22].

In this present paper we prove the rigidity theorem of Hamiltonian actions and presymplectic actions on presymplectic manifolds using again this abstract normal form theorem. We recall in the Appendix the statement of this theorem and the SCI-formalism (for the local case) and the CI-formalism (for the global case) in which this theorem can be used. Roughly speaking, this formalism corresponds to Fréchet spaces on which there is a smoothing operator. More precisely, a CI-space is a sequence of Banach spaces $(\mathcal{H}_k, \| \|_k)$ (with $k \in \mathbb{Z}_+$) such that if $k \leq k'$ then $\mathcal{H}_{k'} \subset \mathcal{H}_k$ and $\|f\|_k \leq \|f\|_{k'}$ for every f in $\mathcal{H}_{k'}$. Moreover, for all real number $t > 1$, there exists a smoothing operator $S_t : \mathcal{H}_0 \longrightarrow \mathcal{H}_\infty = \bigcap_{k=0}^{\infty} \mathcal{H}_k$ which satisfies for every $f \in \mathcal{H}$,

$$\|S_t(f)\|_{p+s} \leq C_{p,s} t^s \|f\|_p \tag{1.1}$$

$$\|f - S_t(f)\|_p \leq C_{p,s} t^{-s} \|f\|_{p+s} \tag{1.2}$$

where $C_{p,s}$ is a positive constant which depends only on p and s. A classical example of CI-space is given by the differentiable functions on a compact manifold or on a compact domain of $\mathbb{R}^n$. The usual definition of the smoothing operator can be found in [5], [23], [24] or [25].

Grosso modo, the way we use the abstract normal form theorem and the SCI and CI formalism follows more or less the same ideas as in the Poisson case in [21]. Nevertheless, there is a great technical difference. Indeed, in the presymplectic case, if we take the "classical" definition of the smoothing operator of vector fields, and apply it to a Hamiltonian or presymplectic vector field, we don't know if we still obtain a Hamiltonian or presymplectic vector field. It is precisely the difficulty of this work.

In fact, in the local case, it is not hard to prove that if the presymplectic form is of constant rank then the smoothing operator defined for instance in [23] transforms an Hamiltonian or presymplectic vector field to an Hamiltonian or presymplectic vector field. The proof of the rigidity theorem of Hamiltonian actions in the local case is then a direct application of the abstract normal form theorem, see Theorem 4.2.

But in the global case, on a compact manifold, it is not so easy. The main difficulty of this present work is to construct a new smoothing operator which transforms an Hamiltonian or presymplectic vector field to another Hamiltonian or presymplectic vector field. The construction we give here is inspired by the regularization operator defined by de Rham ([6]) on differential forms. This regularization operator is generally used in a different way related to L_p-topology (see [6], [9] and [10]) and, to our knowledge, the properties (1.1) and (1.2) don't appear in the literature. Therefore, we prove these two properties for the de Rham regularization operator in Section 6.2. The de Rham construction can be extended to the multivector fields but unfortunately this operator does not preserve, a priori, the Hamiltonian and presymplectic feature. We then adapt in Section 6.3 the definition of de Rham to our context of presymplectic manifold of constant rank and construct a new global smoothing operator on the spaces of differential forms and multivector fields. This smoothing operator satisfies the properties (1.1) and (1.2) and transforms an Hamiltonian or presymplectic vector field to an Hamiltonian or presymplectic vector field. We then have the good tool to use correctly the abstract normal form theorem and prove the rigidity of Hamiltonian actions and presymplectic actions on a compact presymplectic manifold of constant rank ; see Theorem 4.3.

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2. Presymplectic manifolds

In this section, we recall briefly the basic definitions about the presymplectic manifolds.

A *presymplectic* structure on a smooth manifold M is a smooth closed two-form ω on M. If we suppose, in addition, that ω is non-degenerate, then it is a symplectic form.

We denote by $\mathfrak{X}(M)$ the vector space of smooth vector fields on M. Recall that the *kernel* of ω, is defined by

$$\ker(\omega) = \{X \in \mathfrak{X}(M)\,;\, i_X\omega = 0\}\,, \tag{2.1}$$

where $i_X\omega$ is the contraction of ω by X. The *rank* of ω at a point x of M is the codimension of $\ker\omega_x$ in T_xM.

By the Stefan-Sussmann Theorem, it is easy to check that $\ker(\omega)$ is a completely integrable involutive singular distribution which gives the null singular foliation. Of course, it the rank of ω is constant then we have a regular foliation.

A vector field X on a presymplectic manifold (M,ω) is said *presymplectic* if $L_X\omega = 0$ (the Lie derivative of ω by X), i.e. the 1-form $i_X\omega$ is closed. In this case, the flow of X preserves the form ω. We denote by $\mathcal{S}(M,\omega)$ the vector space of presymplectic vector fields. Note that it is a Lie algebra with respect with the classical bracket of vector fields : the bracket of two presymplectic vector fields is still presymplectic (it is Hamiltonian, in fact).

Let us define the homomorphism of vector bundles $\omega^\flat : TM \longrightarrow T^*M$ by $\omega^\flat(v) = \omega_x(v,.)$ (for $x \in M$ and $v \in T_xM$). The rank of ω at the point x is then the rank of the linear map $\omega^\flat_x$. As we do not require that ω is non-degenerate, $\omega^\flat$ is not an isomorphism. Therefore, the notion of Hamiltonian vector field is slightly more delicate than in the symplectic case.

The vector field X is called *Hamiltonian* if there exists a smooth function f on M which satisfies $i_X\omega = -df$. Of course, any Hamiltonian vector field is presymplectic. It is easy to check that if X and Y are Hamiltonian, then $[X, Y]$ is Hamiltonian too. We denote by $\mathcal{H}(M,\omega)$ the set of Hamiltonian vector fields on (M,ω). It is then a Lie subalgebra of $\mathfrak{X}(M)$.

Remark 2.1. Some authors impose in the definition of a presymplectic structure the additional condition that ω must have a constant rank. In this paper, we do not require this condition in our definition even if our work concerns the presymplectic structures of constant rank.

Remark 2.2. Note that if f is a smooth function, there may not exist a corresponding Hamiltonian vector field X_f. Moreover, when X_f exists, it is defined up to elements of the kernel of ω.

Therefore, the Poisson bracket, as the one defined in the symplectic case, is not well defined here. Nevertheless, if we denote by $\mathcal{C}^\infty_h(M)$ the vector space of smooth functions f such that there exists a vector field X satisfying $i_X\omega = -df$, we can define the Poisson bracket on $\mathcal{C}^\infty_h(M)$ in the following classical way. If the functions f and g correspond to the Hamiltonian vector fields X_f and X_g then

$$\{f, g\} = \omega(X_f, X_g)\,.$$

It is easy to check that it is a Poisson bracket.

Example 2.3. If (M,ω) is a symplectic manifold and N a submanifold of M then $(N, i_N^*\omega)$ is a presymplectic manifold where $i_N : N \longrightarrow M$ is the injection. A priori, $i_N^*\omega$ is not nondegenerate any more.

For instance : $M = \mathbb{R}^{2q}$, $\omega = \sum_{i=1}^q dx_i \wedge dx_{q+i}$ and $N = \mathbb{S}^{2q-1}$.

If the rank of $i_N^*\omega$ is constant on N one says that $(N, i_N^*\omega)$ is a presymplectic submanifold of constant rank (see [19]). Such submanifolds play an important role in the reduction of symplectic manifolds (see for instance [16]). For instance, isotropic, coisotropic, Lagrangian and symplectic submanifolds of a symplectic manifold are of constant rank.

Conversely, it is easy to show that a presymplectic manifold (N,ω_N) can be embedded in a symplectic manifold (M,ω_M) ; $\omega_N = j^*\omega_M$ where $j : N \longrightarrow M$ is the embedding. Moreover, if ω_N is of constant rank, M, ω_M and j can be constructed in such a way that N is a coistropic submanifold of M (see [13], [19]).

Finally, let us recall the Darboux Theorem for presymplectic forms of constant rank (see for instance [16]).

Theorem 2.4. *If (M,ω) is a n-dimensional presymplectic manifold of constant rank $2q$ (with $2q \le n$) then, for any point x of M, there exist local coordinates $(x_1, \ldots, x_n)$ around x in which, ω is expressed by*

$$\omega = \sum_{i=1}^q dx_i \wedge dx_{q+i}\,. \tag{2.2}$$

3. Hamiltonian and presymplectic actions on a presymplectic manifold

Let (M, ω) be a presymplectic manifold and consider a smooth left action of a connected Lie group G on M, $\Phi : G \times M \longrightarrow M$, $(g, x) \longmapsto \Phi(g, x) = \Phi_g(x)$. We suppose that this action preserves ω, i.e. for all $g \in G$,

$$\Phi_g^* \omega = \omega \,. \tag{3.1}$$

Such an action is called a *presymplectic action*.

We denote by $\mathfrak{g}$ the Lie algebra of G. For every $\xi \in \mathfrak{g}$ we define the fundamental vector field ξ_M by

$$(\xi_M)_x = \frac{d}{dt}_{t=0} \Phi(exp(-t\xi), x)\,, \quad \text{for all } \; x \in M\,. \tag{3.2}$$

The map $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ defined by $\rho(\xi) = \xi_M$ is then a Lie algebras homomorphism. A *fix point* of the action is an element x of M such that $\rho(\xi)_x = 0$ for all $\xi \in \mathfrak{g}$.

The condition (3.1) can be written as

$$L_{\xi_M} \omega = 0\,, \quad \text{for all } \; \xi \in \mathfrak{g}\,, \tag{3.3}$$

i.e. all the vector fields ξ_M are presymplectic. A presymplectic action of $\mathfrak{g}$ (or G) on M is given by a Lie algebras homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ such that $\rho(\xi)$ is a presymplectic vector field for all $\xi \in \mathfrak{g}$, i.e. it is a Lie homomorphism $\rho : \mathfrak{g} \longrightarrow \mathcal{S}(M, \omega)$.

An *Hamiltonian action* of $\mathfrak{g}$ (or G) on M is a Lie algebras homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ such that $\rho(\xi)$ is a Hamiltonian vector field for all $\xi \in \mathfrak{g}$, i.e. it is a Lie homomorphism $\rho : \mathfrak{g} \longrightarrow \mathcal{H}(M, \omega)$. In this case, there exists a smooth map $\mu : M \longrightarrow \mathfrak{g}^*$, called *moment map* of the action, which satisfies $i_{\rho(\xi)}\omega = -d\mu^\xi$ for all ξ in $\mathfrak{g}$, where μ^ξ is the smooth function from M to $\mathbb{R}$ defined by $\mu^\xi(x) = \langle \mu(x), \xi \rangle$ for all x in M.

If we assume in addition that the moment map μ is equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$, we say that the action is *strongly Hamiltonian* and the moment map μ is *strong* (or *equivariant*).

Remark 3.1. It is not hard to check that the condition of equivariance is equivalent to

$$\omega\big(\rho(\xi), \rho(\zeta)\big) = \mu^{[\xi, \zeta]}\,, \tag{3.4}$$

for all ξ and ζ in $\mathfrak{g}$.

With the notation of Remark 2.2, it is equivalent to $\mu^{[\xi,\zeta]} = \{\mu^\xi, \mu^\zeta\}$.

Example 3.2. If a Lie group G acts on a manifold M and if α is a G-invariant 1-form on M then $(M, d\alpha)$ is clearly a presymplectic manifold and the action of G on M is strongly Hamiltonian with moment map μ defined by

$$\mu^\xi = i_{\xi_M}\theta \quad \text{for all} \quad \xi \in \mathfrak{g}\,.$$

Example 3.3. If we consider a Hamiltonian action of the Lie group G on a symplectic manifold (M, ω), with equivariant moment map $\mu : M \longrightarrow \mathfrak{g}^*$, and a G-invariant submanifold N of M, then we get a strongly Hamiltonian action of G on the induced presymplectic manifold (N, ω_N), with moment map μ_N which is the restriction of μ to N.

A particular important case is when $N = \mu^{-1}(\xi)$ where ξ is a regular value of the moment map μ (see [20]).

Remark 3.4. As in the symplectic case (see for instance [16], chapter IV) if ρ is a presymplectic action and $\mathfrak{g} = [\mathfrak{g}; \mathfrak{g}]$ (the derived ideal) then ρ is in fact Hamiltonian. It is a consequence of the fact that the bracket of two presymplectic vector fields is Hamiltonian.

Remark 3.5. The moment map of an Hamiltonian action is not unique. If μ and μ' are two moment maps for the same Hamiltonian action ρ, then $\mu - \mu'$ is constant on each connected component. If the manifold M is connected, which is reasonable, the moment map of an Hamiltonian action is defined up to a constant in $\mathfrak{g}^*$.

If we assume in addition that the action is strongly Hamiltonian, then this constant is an element of $[\mathfrak{g}, \mathfrak{g}]^\circ \subset \mathfrak{g}^*$. Therefore, if $H^1(\mathfrak{g}; \mathbb{R}) = \{0\}$ (Chevalley cohomology group) then the equivariant moment map is unique. It is, in particular, the case when $\mathfrak{g}$ is semisimple.

On the other side, two Hamiltonian actions ρ_1 and ρ_2 have a same moment map (modulo a constant) if and only if $\rho_1 - \rho_2$ takes values in $\ker \omega$.

4. Rigidity of Hamiltonian and presymplectic actions

In this section, we state the main results of this paper :

Take two Hamiltonian (or presymplectic) actions of a compact semisimple Lie group on a presymplectic manifold (M, ω) of constant rank. If they are sufficiently close, then they are equivalent.

We will give two versions of this result.

The local case : We can assume that we work in a neighbourhood of the origin 0 in $\mathbb{R}^n$. We assume that 0 is a fix point for one of the two actions and that the rank of the presymplectic form ω is constant in a neighbourhood of 0.

The global case : We assume that the manifold M is compact and that the rank of the presymplectic form ω is constant on M.

4.1. Equivalence of actions.

Two Hamiltonian or presymplectic actions ρ_1 and ρ_2 are *equivalent* if there exists a smooth diffeomorphism φ which preserves ω (i.e. $\varphi^*\omega = \omega$) such that

$$\varphi_* \rho_1(\xi) = \rho_2(\xi) \quad \text{for all } \xi \in \mathfrak{g}\,. \tag{4.1}$$

In the Hamiltonian case, if μ_1 is a moment map for ρ_1, one can check that $\mu_1 \circ \varphi^{-1}$ is a moment map for ρ_2. Moreover, if we know that ρ_1 is strongly Hamiltonian, i.e. μ_1 is equivariant, then $\mu_1 \circ \varphi^{-1}$ is also equivariant and ρ_2 is strongly Hamiltonian.

In this definition, φ is a diffeomorphism of M in the global case and a local diffeomorphism in the neighbourhood of a chosen point of the manifold in the local situation.

If the diffeomorphism is not smooth but only of class C^k, then we can talk about C^k-equivalence.

4.2. **Topology on the space of actions.** In this section, we explain what we mean by *two close actions*.

We first recall the classical definitions of the norms of differentiability, C^p-norms, for sections of vector bundles in the local and global case. All these definitions can be found, for instance, in [1].

If $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, $F(x) = (F^1(x), \ldots, F^k(x))$, is a smooth map, z in $\mathbb{R}^n$ and K a compact subset of $\mathbb{R}^n$, then we can define for any integer p the C^p-norms

$$\|F(z)\|_p = \max_{\substack{i=1,\ldots,k \\ |\alpha| \leq p}} \left| \frac{\partial^{|\alpha|} F^i}{\partial x_\alpha}(z) \right| \quad \text{and} \quad \|F\|_{p,K} = \sup_{x \in K} \|F(x)\|_p$$

In the same way, one can define norms for differential forms, vector fields, multi-vector fields and smooth sections of a vector bundle over an open subset of $\mathbb{R}^n$.

Now, consider a paracompact n-dimensional smooth manifold M and a vector bundle $\pi : E \longrightarrow M$ over M of rank k. Take a trivializing atlas $\{(V_i\,,\varphi_i)\}_{i \in I}$ of M. If θ is a smooth section of the vector bundle it is defined, via the trivialization, by smooth applications $\theta_i : V_i' \longrightarrow \mathbb{R}^k$ where the $V_i' = \varphi_i(V_i)$ are open subsets of $\mathbb{R}^n$.

Take a locally finite trivializing open covering $\{U_i\}_{i \in I}$ such that $\overline{U_i} \subset V_i$ is compact and a partition of unity $\{\alpha_i\}_{i \in I}$ subordinate to this covering.

If θ is a smooth section and $x \in M$, we can define for any integer p,

$$\|\theta_x\|_p = \sum_{i \in I_x} \alpha_i(x) \|\theta_i(\varphi_i(x))\|_p$$

where $I_x = \{i \in I\,;\, x \in U_i\}$. Moreover, if $K \subset M$ is compact, then we define

$$\|\theta\|_{p,K} = \sup_{z \in K} \|\theta_z\|_p$$

In particular, if M is compact, we get well defined norms.

The following result will be used to pass from local to global estimates and can be found, for instance, in [1].

Lemma 4.1. *With the same notation as above, for every $i \in I$, there exist positive real numbers A_i and B_i such that for every section θ and all $x \in U_i$, we have*

$$A_i \|\theta_i(\varphi_i(x))\|_p \leq \|\theta_x\|_p \leq B_i \|\theta_i(\varphi_i(x))\|_p\,.$$

Finally, we define the norms for actions.

In the global case, we suppose that the manifold M is compact. We can denote by $Hom_{Lie}\big(\mathfrak{g}; \mathfrak{X}(M)\big)$ the vector space of homomorphisms of Lie algebras form $\mathfrak{g}$

to $\mathfrak{X}(M)$ and define on it a structure of Fréchet space. If $\xi_1, \ldots, \xi_m$ is a basis of $\mathfrak{g}$ then, for any $\rho \in Hom_{Lie}\big(\mathfrak{g}; \mathfrak{X}(M)\big)$ and every integer k, we have

$$\|\rho\|_k = \max_{i=1,\ldots,m} \|\rho(\xi_i)\|_k \,, \tag{4.2}$$

where $\|\rho(\xi_i)\|_k$ is the C^k-norm on $\mathfrak{X}(M)$

In the same way, in the local case, we denote by B_r (for every positive real number $r > 0$) the open ball in $\mathbb{R}^n$ of radius r and of center 0. If ρ is a homomorphisms of Lie algebras form $\mathfrak{g}$ to $\mathfrak{X}(B_r)$ then we put

$$\|\rho\|_{k,r} = \max_{i=1,\ldots,m} \|\rho(\xi_i)\|_{k,\overline{B}_r} \,. \tag{4.3}$$

4.3. **The Rigidity Theorems.** Now, we give the main theorems of this paper. In the local situation, we state the following rigidity result.

Theorem 4.2. *Take a presymplectic form ω on $\mathbb{R}^n$ and suppose that its rank is constant in a neighbourhood U of 0 containing the open ball B_R of radius $R > 0$. Consider a Hamiltonian action (respectively, a presymplectic action) $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(U)$ of a compact type semisimple Lie algebra $\mathfrak{g}$ on (U, ω) for which 0 is a fix point.*

There exist a positive integer l and two positive real numbers α and β (with $\beta < 1 < \alpha$) such that :

if $\sigma : \mathfrak{g} \longrightarrow \mathfrak{X}(U)$ is another Hamiltonian action of $\mathfrak{g}$ on (U, ω) (respectively, a presymplectic action) which satisfies

$$\|\rho - \sigma\|_{2l-1,R} \leq \alpha \quad \text{and} \quad \|\rho - \sigma\|_{l,R} \leq \beta$$

then there exists a diffeomorphism φ of class C^k (for $k \in \mathbb{N} \cup \{\infty\}$, $k \geq l$) on the closed ball $\overline{B}_{R/2}$ such that $\varphi_*\sigma = \rho$ and $\varphi^*\omega = \omega$.

Proof. The proof of this theorem uses the abstract normal form Theorem 7.1 given in the Appendix. Let us explicit how it works.

Let us prove first the Hamiltonian case. The SCI-space $\mathcal{T}$ is defined by the vector spaces $\mathcal{T}_{k,r}$ of Lie homomorphisms from $\mathfrak{g}$ to the Lie algebra $\mathfrak{X}_k(\overline{B}_r)$ of C^k-vector fields on the closed ball of radius r and center 0 in $\mathbb{R}^n$. It is indeed a SCI-space when it is equipped with the norms defined by (4.3) and the smoothing operators : for every $\sigma \in \mathcal{T}$ and $\xi \in \mathfrak{g}$ we have $\big(S_t\sigma\big)(\xi) = S_t\big(\sigma(\xi)\big)$ where the smoothing operator of a vector field is the one defined by in [5] or [23] (see Section 6.1).

The SCI-subset $\mathcal{S}$ of $\mathcal{T}$ consists on the Hamiltonian actions, i.e. Lie homomorphisms from $\mathfrak{g}$ to $\mathcal{H}_k(\overline{B}_r, \omega)$ of C^k-Hamiltonian vector fields on the closed ball $\overline{B}_r$.

The origin $\mathsf{f_O}$ is ρ and we put $\mathcal{F} = \mathcal{N} = \{0\}$, and $\pi = 0$ (the estimate (7.7) is then obvious).

The SCI-group $\mathcal{G}$ is given by the local diffeomorphisms on $(\mathbb{R}^n, 0)$ and $\mathcal{G}_0$ is the closed subgroup of $\mathcal{G}$ of local diffeomorphisms which preserve the closed form ω. The SCI-group $\mathcal{G}$ acts on the SCI-space $\mathcal{T}$ by push-forward : if $\varphi \in \mathcal{G}$ and $\sigma \in \mathcal{T}$ then $\varphi_*\sigma$ is defined by $\big(\varphi_*\sigma\big)(\xi) = \varphi_*\big(\sigma(\xi)\big)$ for any ξ in $\mathfrak{g}$ (the push-forward of the vector field $\sigma(\xi)$ by the diffeomorphism φ). This action is a SCI-action (see Lemma 5.3).

The SCI-space $\mathcal{H}$ is given by the spaces $\mathcal{H}_k(\overline{B}_r, \omega)$ of local Hamiltonian vector fields. The norms and the smoothing operators are the same as for $\mathcal{T}$ but we have to check that the smoothing operators of an element of $\mathcal{H}$ is still in $\mathcal{H}$. Since the closed form ω has a constant rank around 0, this property is satisfied, according to Proposition 6.3. Note that if we do not impose this condition on the rank of ω, this property is not obvious and is maybe false.

The map $\Phi : \mathcal{H} \longrightarrow \mathcal{G}_0$ is defined for any $X \in \mathcal{H}$ by $\Phi(X) = \phi_X^1$, the time 1 flow of X.

Finally, the map $\mathrm{H} : \mathcal{S} \longrightarrow \mathcal{H}$ is defined by $\mathrm{H}(\sigma) = h_0(\rho - \sigma)$, where h_0 is the homotopy operator of the Chevalley-Eilenberg complex defined in Section 5.1 and the estimate (7.8) is a direct consequence of Lemma 5.1.

The inequality (7.9) is a direct consequence of Lemma B.3 in [21] (p. 1172). Moreover, the inequalities (7.10) is proved in Lemma 5.4 and (7.11) is given by Lemma 5.5.

With the assumption of Theorem 4.2, the technical Theorem 7.1 says that there exists a local diffeomorphism ψ which preserves the presymplectic form ω such that $\psi_*\sigma = \rho$.

To prove the presymplectic case, we just replace the spaces $\mathcal{H}_k(\overline{B}_r, \omega)$ by the spaces $\mathcal{S}_k(\overline{B}_r, \omega)$ of C^k presymplectic vector fields on $\overline{B}_r$. □

In the global case, we prove the following.

Theorem 4.3. *Take a compact presymplectic manifold (M, ω) and suppose that the rank of ω is constant on M. Consider a Hamiltonian action (respectively, a presymplectic action) $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ of a compact type semisimple Lie algebra $\mathfrak{g}$ on (M, ω).*

There exist a positive integer l and two positive real numbers α and β (with $\beta < 1 < \alpha$) such that :

if $\sigma : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ is another Hamiltonian action of $\mathfrak{g}$ on (M, ω) (respectively, a presymplectic action) which satisfies

$$\|\rho - \sigma\|_{2l-1} \leq \alpha \quad \text{and} \quad \|\rho - \sigma\|_l \leq \beta$$

then there exists a diffeomorphism φ of class C^k (for $k \in \mathbb{N} \cup \{\infty\}$, $k \geq l$) on M such that $\varphi_*\sigma = \rho$ and $\varphi^*\omega = \omega$.

Proof. This theorem is a consequence of Theorem 7.4 given in the Appendix. The definitions of all the objects $\mathcal{T}$, $\mathcal{G}$, $\mathcal{H}$, H etc., are exactly the same as in the local case, considering a compact manifold M instead of local closed ball $\overline{B}_r$.

It is clear that $\mathcal{T}$ is a CI-space. The smoothing operator taken for $\mathcal{T}$ can be the one defined for instance in [14] or [24] or the one defined in Section 6.

In order to prove that the action of the group of diffeomorphisms of M on the vector fields on M by push-forward is of CI-type at the identity, we just have to reproduce the proof of Lemma 5.3 and use the definition of the global norms (Section 4.2).

In comparison with the local case, the only difficulty concerns the CI-space $\mathcal{H}$ given by the Hamiltonian or presymplectic vector fields on M. It is a subspace of the space of vector fields but if we take the classical smoothing operator defined in [14] or [24] it is not clear that $\mathcal{H}$ is stable by this smoothing operator. A priori we don't know if the smoothing of a Hamiltonian or presymplectic vector field is still Hamiltonian or presymplectic !

In Section 6.3, we define another smoothing operator on the spaces of differential forms and on the space of vector fields and we prove that $\mathcal{H}$ equipped with this new smoothing operator is indeed a CI-space (see Proposition 6.16).

The inequalities (7.21), (7.22) and (7.23) can be proved using the local case and the definition of the global norms. □

5. Technical ingredients

In this section, we give some technical results which allow us to use the abstract normal form theorems 7.1 and 7.4 in order to prove Theorem 4.2 and Theorem 4.3.

5.1. A Chevalley-Eilenberg complex associated to an Hamiltonian or presymplectic action.

Suppose that (M,ω) is a presymplectic manifold where M is either a compact manifold or a closed ball of radius r and center 0 in $\mathbb{R}^n$. Recall that $\mathcal{H}(M,\omega)$ denotes the vector space of Hamiltonian vector fields and $\mathcal{S}(M,\omega)$ the vector space of presymplectic vector fields.

If $\rho : \mathfrak{g} \longrightarrow \mathcal{H}(M,\omega)$ is an Hamiltonian action of the Lie algebra $\mathfrak{g}$ on (M,ω) then it is easy to check that we define a representation $\mathcal{R}$ of $\mathfrak{g}$ on $\mathcal{H}(M,\omega)$ by : if $\xi \in \mathfrak{g}$ and X is a Hamiltonian vector field,

$$\mathcal{R}_\xi(X) = [\rho(\xi), X]\,.$$

Indeed, if $i_X\omega = -df$ where f is a smooth function on M, then an easy computation shows that $i_{[\rho(\xi),X]}\omega = -d\big(\rho(\xi).f\big)$, where $\rho(\xi).f$ is the smooth function defined by $\rho(\xi).f(z) = df_z(\rho(\xi)_z)$ for any z in M.

In the same way, if $\rho : \mathfrak{g} \longrightarrow \mathcal{S}(M,\omega)$ is presymplectic action of $\mathfrak{g}$ on (M,ω) then the same formula defines a representation $\mathcal{R}$ of $\mathfrak{g}$ on $\mathcal{S}(M,\omega)$ (if $di_X\omega = 0$ then $di_{[\rho(\xi),X]}\omega = 0$).

Now, we denote $\mathcal{E} = \mathcal{H}(M,\omega)$ or $\mathcal{S}(M,\omega)$ and we consider an Hamiltonian or presymplectic action ρ. The representation $\mathcal{R}$ of the Lie algebra $\mathfrak{g}$ on $\mathcal{E}$ allows us to define the corresponding Chevalley-Eilenberg complex (see [4]).

The p-cochains ($p \in \mathbb{N}$) are the alternating p-linear maps from $\mathfrak{g}$ to $\mathcal{E}$ and form the vector space $C^p\big(\mathfrak{g},\mathcal{E}\big) = Hom\big(\wedge^p \mathfrak{g},\mathcal{E}\big)$ with the differential denoted by δ_p.

In this paper, we only need δ_0 and δ_1 :

$$\mathcal{E} \xrightarrow{\delta_0} C^1(\mathfrak{g},\mathcal{E}) \xrightarrow{\delta_1} C^2(\mathfrak{g},\mathcal{E})$$

$$\delta_0(X)(\xi_1) = \mathcal{R}_{\xi_1}(X), \quad X \in \mathcal{E}$$

$$\delta_1(\sigma)(\xi_1\wedge\xi_2) = \mathcal{R}_{\xi_1}(\sigma(\xi_2)) - \mathcal{R}_{\xi_2}(\sigma(\xi_1)) - \sigma([\xi_1,\xi_2]), \quad \sigma \in C^1(\mathfrak{g},\mathcal{E})$$

where $\xi_1,\xi_2 \in \mathfrak{g}$.

The same arguments as developed for instance in [5], [22] and [21] allow to prove the following lemma.

Lemma 5.1. *If we suppose that the Lie algebra $\mathfrak{g}$ is semisimple of compact type then, in the Chevalley-Eilenberg complex associated to ρ:*

$$\mathcal{E} \xrightarrow{\delta_0} C^1(\mathfrak{g},\mathcal{E}) \xrightarrow{\delta_1} C^2(\mathfrak{g},\mathcal{E})$$

there exists a chain of homotopy operators

$$\mathcal{E} \xleftarrow[h_0]{} C^1(\mathfrak{g},\mathcal{E}) \xleftarrow[h_1]{} C^2(\mathfrak{g},\mathcal{E})$$

such that

$$\delta_0 \circ h_0 + h_1 \circ \delta_1 = id_{C^1(\mathfrak{g},\mathcal{E})} \,. \tag{5.1}$$

Moreover, there exists an integer s such that for each k, there exists a real constant $C_k > 0$ such that

$$\|h_j(\sigma)\|_{k,M} \leq C_k \|\sigma\|_{k+s,M}, \quad j = 0, 1 \tag{5.2}$$

for all $\sigma \in C^{j+1}(\mathfrak{g},\mathcal{E})$

In the local case ($M = \overline{B}_r$), the constant C_k does not depend on the radius r.

5.2. **Technical lemmas.** We give here some technical properties in the local case (in a neighbourhood of 0 in $\mathbb{R}^n$) but they are still valid in the global case, on a compact manifold M. We get the same estimates, deleting the radius parameters in the norms.

Lemma 5.2. *Let $\varphi = Id + \chi$ and $\psi = Id + \zeta$ (with $\chi(0) = \zeta(0) = 0$) be two local diffeomorphisms of $(\mathbb{R}^n, 0)$. Take two real numbers $r > 0$ and $0 < \eta < 1$. There exists a real number $\alpha > 0$ such that, if $\|\chi\|_{1,r} < \alpha\eta$ then*

$$\|\psi^{-1} - \varphi^{-1}\|_{k,r(1-\eta)} \leq \|\psi - \varphi\|_{k,r}\big(1 + P_k(\|\chi\|_{k+1,r}, \|\zeta\|_{k+1,r})\big)\,,$$

where P_k is a polynomial function of two variables with vanishing constant term, positive coefficients, and which is independent of ψ and φ.

Proof. We can write $\psi^{-1} - \varphi^{-1} = \big(\psi^{-1}\circ\varphi - \psi^{-1}\circ\psi\big)\circ\varphi^{-1}$. Therefore, using Lemmas 3.2 and 3.4 of [22] and Lemma 3.2 of [5], we can find a positive real number α such that if $\|\chi\|_{1,r} < \alpha\eta$ then

$$\|\psi^{-1} - \varphi^{-1}\|_{k,r(1-\eta)} \leq \|\psi^{-1}\circ\varphi - \psi^{-1}\circ\psi\|_{k,r(1-\eta/2)}\big(1 + Q_k(\|\chi\|_{k,r(1-\eta/2)})\big)\,, \tag{5.3}$$

where Q_k is a polynomial function with positive coefficients independent of φ and ψ.

Now, if we denote $\xi = \varphi - \psi$, if x is in the ball $\overline{B}_{r(1-\eta/2)}$, we write

$$\psi^{-1}\big(\varphi(x)\big) - \psi^{-1}\big(\psi(x)\big) = \int_0^1 \frac{d}{dt}\psi^{-1}\big(\psi(x) + t(\xi(x))\big)dt\,. \tag{5.4}$$

Once again, Lemma 3.2 of [22] gives

$$\|\psi^{-1}\circ\varphi - \psi^{-1}\circ\psi\|_{k,r(1-\eta/2)} \leq \|\xi\|_{k,r(1-\eta/2)} \tag{5.5}$$

$$+ \quad \|\xi\|_{k,r(1-\eta/2)}\|\psi^{-1} - Id\|_{k+1,r(1-\eta/2)}\big(1 + R_k(\|\chi\|_{k,r(1-\eta/2)}, \|\xi\|_{k,r(1-\eta/2)})\big)\,,$$

where R_k is a polynomial function with positive coefficients and vanishing constant term. Finally, Lemma 3.4 of [22] gives the result. □

Lemma 5.3. *The left action of the SCI-group of local diffeomorphisms of $(\mathbb{R}^n,0)$ on the SCI-space of vector fields on $(\mathbb{R}^n,0)$ by push-forward is an SCI-action. More precisely, there is a positive constant c such that for any integer $k \geq 1$, there are polynomials Q_k, R_k and T_k with positive coefficients, such that if $\varphi = Id + \chi$ and $\psi = \varphi + \zeta$ are two local diffeomorphisms and X is a vector field then*

$$\begin{aligned} \|\varphi_* X\|_{2k-1,r'} &\leq \|X\|_{2k-1,r}\big(1+\|\chi\|_{k+1,r}Q_k(\|\chi\|_{k+1,r})\big) \\ &\quad +\|\chi\|_{2k,r}\|X\|_{k,r}R_k(\|\chi\|_{k+1,r}) \end{aligned} \tag{5.6}$$

where $r' = (1-c\|\chi\|_{1,r})r$; and also

$$\|\psi_* X - \varphi_* X\|_{k,r''} \leq \|X\|_{k+2,r}\|\zeta\|_{k+2,r}T_k(\|\chi\|_{k+2,r},\|\zeta\|_{k+2,r})\,. \tag{5.7}$$

where $r'' = (1-c(\|\chi\|_{1,r}+\|\zeta\|_{1,r}))r$.

Proof. For every x, we have by definition $\big(\varphi_* X\big)_x = d\varphi(X)(\varphi^{-1}(x))$.

If $\varphi = Id + \chi$, then using Lemmas 3.3 and 3.4 of [22] and the Leibniz rule of derivation, we get easily the estimate (5.6).

Now, for every x in a neighbourhood of 0 in $\mathbb{R}^n$, we have

$$\begin{aligned} \big(\psi_* X\big)_x - \big(\varphi_* X\big)_x &= \int_0^1 \frac{d}{dt}\Big(d\varphi(X)\big(\varphi^{-1}(x)+t(\psi^{-1}(x)-\varphi^{-1}(x))\big)\Big)dt \\ &\quad + d\zeta(X)\big(\psi^{-1}(x)\big)\,. \end{aligned} \tag{5.8}$$

Therefore, using Lemmas 3.2 and 3.4 of [22] and Lemma 5.2 above, we get the estimate (5.7). □

Remark that a consequence of (5.6) is

$$\|(Id+\chi)_* X\|_{k,r'} \leq \|X\|_{k,r}\big(1+\|\chi\|_{k+1,r}P(\|\chi\|_{k+1,r})\big)\,, \tag{5.9}$$

where P is a polynomial with positive coefficients.

In the following lemma, for a vector field X we use the notation ϕ_X^t to indicate the flow of X and ϕ_X for ϕ_X^1, the time 1 flow of X.

Lemma 5.4. *Take two real numbers $r > 0$ and $0 < \eta < 1$. There exists a positive constant α such that if X_1 and X_2 are two vector fields on the closed ball $\overline{B}_r$ with $\|X_1\|_{1,r} < \alpha\eta$ and $\|X_2\|_{1,r} < \alpha\eta$, if $\sigma : \mathfrak{g} \longrightarrow \mathcal{X}(\overline{B}_r)$ is a Lie algebras homomorphism, then we have*

$$\begin{aligned} \big\|\big(\phi_{X_1}\big)_*\sigma - \big(\phi_{X_2}\big)_*\sigma\big\|_{k,r(1-\eta)} &\leq \|X_1-X_2\|_{k+1,r}\|\sigma\|_{k+1,r}\times P_k\big(\|X_1\|_{k,r},\|X_2\|_{k,r}\big) \\ &\quad +\|\sigma\|_{k+3,r}R_k\big(\|X_1\|_{k+3,r},\|X_2\|_{k+3,r}\big) \end{aligned} \tag{5.10}$$

where P_k and R_k are polynomials with positive coefficients (independent of X_1, X_2 and σ), and R_k is a polynomial of two variables which contains only terms of degree greater or equal to 2.

Proof. For any $\xi \in \mathfrak{g}$ we have

$$\phi_{X_1 *}\sigma(\xi) - \phi_{X_2 *}\sigma(\xi) = \int_0^1 \Big(\frac{d}{dt}\big(\phi^t_{X_1 *}\sigma(\xi)\big) - \frac{d}{dt}\big(\phi^t_{X_2 *}\sigma(\xi)\big)\Big) dt \tag{5.11}$$

$$= \int_0^1 \Big(\phi^t_{X_1 *}[\sigma(\xi), X_1] - \phi^t_{X_2 *}[\sigma(\xi), X_2]\Big) dt \tag{5.12}$$

$$= \int_0^1 \Big(\phi^t_{X_1 *}[\sigma(\xi), X_1 - X_2]\Big) dt + \int_0^1 \Big(\phi^t_{X_1 *}[\sigma(\xi), X_2] - \phi^t_{X_2 *}[\sigma(\xi), X_2]\Big) dt \tag{5.13}$$

According to (5.9) we can write

$$\|\phi^t_{X_1 *}[\sigma(\xi), X_1 - X_2]\|_{k, r(1-\eta)} \leq \|[\sigma(\xi), X_1 - X_2]\|_{k,\varrho}\big(1 + P(\|\phi^t_{X_1} - Id\|_{k+1,\varrho})\big) \tag{5.14}$$

where P is a polynomial with vanishing constant term and $r(1-\eta) < \varrho < r$. Now, Lemma B.3 of [21] gives

$$\|\phi^t_{X_1 *}[\sigma(\xi), X_1 - X_2]\|_{k, r(1-\eta)} \leq \|X_1 - X_2\|_{k+1,r}\|\sigma\|_{k+1,r}\big(1 + Q(\|X_1\|_{k+1,r})\big) \tag{5.15}$$

On the other hand, (5.7) gives

$$\|\phi^t_{X_1 *}[\sigma(\xi), X_2] - \phi^t_{X_2 *}[\sigma(\xi), X_2]\|_{k, r(1-\eta)} \leq \|[\sigma(\xi), X_2]\|_{k+2,\varrho} \times \|\phi^t_{X_1} - \phi^t_{X_2}\|_{k+2,\varrho} T\big(\|\phi^t_{X_1} - Id\|_{k+2,\varrho}, \|\phi^t_{X_2} - Id\|_{k+2,\varrho}\big) \tag{5.16}$$

which gives, using Lemma B.3 of [21],

$$\|\phi^t_{X_1 *}[\sigma(\xi), X_2] - \phi^t_{X_2 *}[\sigma(\xi), X_2]\|_{k, r(1-\eta)} \leq \|\sigma\|_{k+3,r} R(\|X_1\|_{k+2,r}, \|X_2\|_{k+3,r}) \tag{5.17}$$

where R is a polynomial of two variables which contains only terms of degree greater or equal to 2. We then obtain the lemma.

□

In the following lemma, we consider a Hamiltonian action (respectively a presymplectic action) $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(U)$ of the Lie algebra $\mathfrak{g}$ on an open neighbourhood of 0 in $\mathbb{R}^n$. We have the differentials δ_0 and δ_1 of the Chevalley-Eilenberg complex associated to ρ and the homotopy operators h_0 and h_1 (see Section 5.1). The integer s below is the loss of differentiability which appears in Lemma 5.1.

Now, we take another Hamiltonian action (respectively a presymplectic action) $\sigma : \mathfrak{g} \longrightarrow \mathfrak{X}(U)$ and put X the Hamiltonian vector field (respectively the presymplectic vector field) $X = h_0(\rho - \sigma)$. Finally, we denote by ϕ^t the flow of X and $\phi = \phi^1$ the time 1 flow of X. We know that the flow of X preserves the presymplectic form ω.

Lemma 5.5. *Consider two real numbers $r > 0$ and $0 < \eta < 1$. There exists a positive constant α such that, with the notations given above, if $\|\sigma - \rho\|_{s+1,r} < \alpha\eta$ then, we have for any integer k,*

$$\|\phi_* \sigma - \rho\|_{k, r(1-\eta)} \leq \|\sigma - \rho\|^2_{k+s+1,r} P_k\big(\|\sigma - \rho\|_{k+s+1,r}\big)\,, \tag{5.18}$$

where P_k is a polynomial function with positive coefficients, independent of σ and ρ.

Proof. If $\xi \in \mathfrak{g}$, we can write

$$\phi_*\sigma(\xi) - \rho(\xi) = \phi_*\sigma(\xi) - \phi_*\rho(\xi) + \phi_*\rho(\xi) - \rho(\xi)\,. \tag{5.19}$$

We first notice that

$$\phi_*\rho(\xi) - \rho(\xi) = \int_0^1 \frac{d}{dt}\phi_*^t\rho(\xi)dt = \int_0^1 \phi_*^t[\rho(\xi), X]dt = \int_0^1 \phi_*^t\delta_0(X)(\xi)dt\,. \tag{5.20}$$

Then, recalling that $X = h_0(\rho - \sigma)$, the relation (5.1) gives

$$\phi_*\rho(\xi) - \rho(\xi) = \int_0^1 \phi_*^t\big((\rho-\sigma)(\xi) - h_1\circ\delta_1(\rho-\sigma)(\xi)\big)dt\,, \tag{5.21}$$

which gives

$$\begin{aligned}\phi_*\sigma(\xi) - \rho(\xi) \quad &= \quad \int_0^1 \Big(\phi_*^t(\rho-\sigma)(\xi) - \phi_*(\rho-\sigma)(\xi)\Big)dt \\ &\quad - \int_0^1 \phi_*^t h_1\circ\delta_1(\rho-\sigma)(\xi)dt\,.\end{aligned} \tag{5.22}$$

Now, we write

$$\phi_*^t(\rho-\sigma)(\xi) - \phi_*(\rho-\sigma)(\xi) = \int_0^t \frac{d}{d\tau}\phi_*^\tau\big((\rho-\sigma)(\xi)\big)d\tau = \int_0^t \phi_*^\tau[(\rho-\sigma)(\xi), X]d\tau \tag{5.23}$$

and using (5.9) and Lemma B.3 of [21] we get

$$\begin{aligned}\|\phi_*\sigma - \rho\|_{k,r(1-\eta)} \quad &\leq \quad \|\sigma-\rho\|_{k+1,r}\|X\|_{k+1,r}Q(\|X\|_{k+1,r}) \\ &\quad +\|h_1\circ\delta_1(\rho-\sigma)\|_{k,r}R(\|X\|_{k+1,r})\end{aligned} \tag{5.24}$$

where Q and R are polynomial functions with positive coefficients.

By definition and Lemma 5.1 we have

$$\|X\|_{k+1,r} = \|h_0(\rho-\sigma)\|_{k+1,r} \leq C\|\rho-\sigma\|_{k+s+1,r}\,, \tag{5.25}$$

where C is a positive constant. Moreover, an obvious computation shows that if ξ_1 and ξ_2 are in $\mathfrak{g}$ then

$$\delta_1(\rho-\sigma)(\xi_1\wedge\xi_2) = \big[(\rho-\sigma)(\xi_1), (\rho-\sigma)(\xi_2)\big]\,, \tag{5.26}$$

It implies the following estimate

$$\|h_1\circ\delta_1(\rho-\sigma)\|_{k,r} \leq C\|\delta_1(\rho-\sigma)\|_{k+s,r} \leq \tilde{C}\|\rho-\sigma\|^2_{k+s+1,r}\,, \tag{5.27}$$

where $\tilde{C} > 0$. The estimate of the lemma follows. □

6. Smoothing operators

In order to use the SCI and CI formalism, we need to construct a notion of smoothing operator on the space of vector fields and differential forms on a closed ball in $\mathbb{R}^n$ for the local case, or on a compact manifold for the global case.

Such operators have already been defined for instance in [5], [23], [24] and [25]. We will see that in the local case these operators can be used in our situation but, for the global case, we will have to define another smoothing operator which will have the properties adapted to our problem.

6.1. **Smoothing operator in the local case.** We follow the construction of smoothing operators given by Conn ([5]) and Moser ([23]). Recall that for all positive real number r, we denote by B_r (resp. $\overline{B}_r$) the open ball (resp. closed ball) of radius r and of center 0 in $\mathbb{R}^n$. For an integer $l \geq 1$ we consider a smooth function $\chi_l : \mathbb{R}^n \longrightarrow \mathbb{R}$ whose support is included in the unit closed ball $\overline{B}_1$ and which satisfies

$$\int_{\mathbb{R}^n} \chi_l(x)dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} x^\alpha \chi_l(x)dx = 0 \quad \text{for } 1 \leq |\alpha| \leq l \tag{6.1}$$

If R and r are two radii with $0 < r < R \leq 1$, we fix $t > 1/(R-r)$ and define the *smoothing operator*

$$S_t : \mathcal{C}^\infty(\overline{B}_R) \longrightarrow \mathcal{C}^\infty(\overline{B}_r)$$

by

$$S_t(f)(x) = \int_{\mathbb{R}^n} t^n \chi_l\big(t(x-y)\big) f(y)dy\,. \tag{6.2}$$

This operator depends on l and t which can be chosen in a convenient way. One can check (see [23]) the two important properties.

Proposition 6.1. *If $p \geq 0$ and $s \geq 0$ are integers, there exists a positive real number C_s such that for all f is in $\mathcal{C}^\infty(\overline{B}_R)$ we have*

$$\|S_t(f)\|_{p+s,\overline{B}_r} \leq C_s t^s \|f\|_{p,\overline{B}_R} \tag{6.3}$$

$$\|f - S_t(f)\|_{p,\overline{B}_r} \leq C_s t^{-s} \|f\|_{p+s,\overline{B}_R} \quad \text{for } s \leq l \tag{6.4}$$

Note that in this proposition, the constants C_s depend also on l.

In the same way, we can define the smoothing operator for the differential forms $S_t : \Omega^k(\overline{B}_R) \longrightarrow \Omega^k(\overline{B}_r)$. If $\theta = \displaystyle\sum_I \theta_I dx_I$ ($I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \ldots < i_k \leq n$) is a k-differential form, then $S_t(\theta) = \sum_I S_t(\theta_I)dx_I$.

We also get the smoothing operator for vector fields $S_t : \mathfrak{X}(\overline{B}_R) \longrightarrow \mathfrak{X}(\overline{B}_r)$. If $X = \displaystyle\sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ is a vector field then $S_t(X) = \sum_{i=1}^n S_t(X_i)\frac{\partial}{\partial x_i}$. It can be easily generalized to k-vector fields.

Using the definitions of norms for vector fields and differential forms, we can show the same two estimates as in Proposition 6.1 for these smoothing operators.

Remark 6.2. By the definition of the smoothing operator, we get that for every smooth function f,

$$S_t\Big(\frac{\partial f}{\partial x_k}\Big) = \frac{\partial S_t(f)}{\partial x_k}\,.$$

Therefore, for every differential form θ, we can also write $S_t(d\theta) = d\big(S_t\theta\big)$.

Now, we have the following property about the smoothing of Hamiltonian vector fields which is very important in our situation.

Proposition 6.3. *Let $(x_1,\dots,x_n)$ be coordinates on an open set U of $\mathbb{R}^n$ and $\omega=\sum_{i=1}^q dx_i\wedge dx_{q+i}$ where $q\le n/2$. If X is a smooth vector field on U, then we have*

$$i_{S_t(X)}\omega = S_t\big(i_X\omega\big)\,.$$

In particular, if X is Hamiltonian or presymplectic, then $S_t(X)$ is still Hamiltonian or presymplectic.

Proof. We can write in coordinates $X=\sum_{i=1}^n X_i\frac{\partial}{\partial x_i}$ and $S_t(X)=\sum_{i=1}^n S_t(X_i)\frac{\partial}{\partial x_i}$. We then have

$$S_t\big(i_X\omega\big) = S_t\big(-\sum_{i=1}^q X_{q+i}dx_i+\sum_{i=1}^q X_i dx_{q+i}\big) \tag{6.5}$$

$$= -\sum_{i=1}^q S_t(X_{q+i})dx_i+\sum_{i=1}^q S_t(X_i)dx_{q+i} \tag{6.6}$$

$$= i_{S_t(X)}\omega\,. \tag{6.7}$$

The end of the proposition is then a consequence of Remark 6.2. □

6.2. **Smoothing operator in the global case.** There is a construction of a global smoothing operator on a compact manifold described by J. Nash in [24] which consists in embedding the compact manifold in an Euclidian vector space, using the smoothing operator on this vector space and then restricting to the compact manifold. This construction has two drawbacks. First, it is not clear if this smoothing operator defined on the differential forms is compatible with the exterior derivation. Secondly, and it is really a problem for the subject we are interested in, if we suppose that M is equipped with a presymplectic structure, it is not clear if the smoothing of an Hamiltonian or presymlectic vector fields is still Hamiltonian or presymplectic.

In this section, we define a smoothing operator on vector fields and differential forms which is compatible with the exterior derivative of differential forms. In fact, we follow the classical construction of the regularization operator on differential forms of de Rham (see [6]). Some properties of this operator, expressed in terms of L_p-norms or Sobolev norms, can be found for instance in [9] and [10]. Nevertheless, the way we need to use this smoothing operator in this paper is slightly different from the motivations of [6] and [10]. Indeed, we work with the norms of C^k differentiability and we essentially need the two estimates of Proposition 6.1. To our knowledge, these estimates don't appear in the literature, that is why we prove them in this paper, even if this smoothing operator is not exactly well adapted to our situation (see Remark 6.10).

We first recall the construction of the regularization operator on an open set of $\mathbb{R}^n$. Let us define the radial smooth diffeomorphism $h : B_1 \longrightarrow \mathbb{R}^n$ by

$$h(x)=\frac{\rho(\|x\|)}{\|x\|}x$$

where $\rho : [0;1[\longrightarrow [0;+\infty[$ is a smooth strictly increasing function defined by

$$\rho(\tau)=\begin{cases} \tau & \text{if} \quad \tau<1/3 \\ exp\big(\frac{1}{1-\tau^2}\big) & \text{if} \quad \tau\ge 2/3 \end{cases}$$

and $\rho(\tau) \geq \tau$ for any $\tau \geq 0$. Note that for all $y \in \mathbb{R}^n$, we have

$$h^{-1}(y) = \frac{\rho^{-1}(\|y\|)}{\|y\|} y \,.$$

If $v \in \mathbb{R}^n$, the smooth diffeomorphism $\sigma_v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by

$$\sigma_v(x) = \begin{cases} h^{-1}\big(h(x) + v\big) & \text{if} \quad \|x\| < 1 \\ x & \text{if} \quad \|x\| \geq 1 \end{cases}$$

We then get a smooth map $\sigma : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma(v, x) = \sigma_v(x)$. Note that if $x \in B_1$ is fixed, then the map $\sigma^x : \mathbb{R}^n \longrightarrow B_1$ defined by $\sigma^x(v) = \sigma(v, x)$ is a diffeomorphism.

Now, take an integer $l \geq 1$, a real number $t > 1$ and a function χ_l as in the previous section 6.1.

If V is an open set of $\mathbb{R}^n$ with $\overline{B}_1 \subset V$, we define for each integer k the *regularization operator* (see [6] and [10]) $R_t : \Omega^k(V) \longrightarrow \Omega^k(V)$ by

$$R_t\theta = \int_{\mathbb{R}^n} \big(\sigma_v^* \theta\big)\, t^n\, \chi_l(tv) dv \,. \tag{6.8}$$

Remark 6.4. If θ is a smooth k-differential form on V, we clearly have $R_t\theta = \theta$ on $V \setminus \overline{B}_1$.

Lemma 6.5. *Take a real number r such that $0 < r < 1$. If $p \geq 0$ and $s \geq 0$ are integers, there exists a positive constant C_{ps} such that for all $\theta \in \Omega^k(V)$ and for all x in $\overline{B}_r$, we have*

$$\|(R_t\theta)_x\|_{p+s} \leq C_{ps} t^s \|\theta\|_{p, \overline{B}_1} \tag{6.9}$$

$$\|(\theta - R_t\theta)_x\|_p \leq C_{ps} t^{-s} \|\theta\|_{p+s, \overline{B}_1} \quad \text{for } s \leq l \tag{6.10}$$

Proof. We follow more or less the same idea of proof as in [23].

We suppose that $\theta = f\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ where f is a smooth function on V. We introduce the set

$$\mathcal{J} = \{(j_1, \ldots, j_k) \in \mathbb{N}^k \,;\, 1 \leq j_1 < j_2 < \ldots < j_k \leq n\}$$

For $x \in \overline{B}_r$, we can write $(R_t\theta)_x = \displaystyle\sum_{J \in \mathcal{J}} g_J(x) dx_{j_1} \wedge \ldots \wedge dx_{j_k}$ with

$$g_J(x) = \int_{\mathbb{R}^n} f(\sigma(v, x)) \Delta_J(v, x) t^n \chi_l(tv) dv$$

where $\Delta_J(v, x)$ is a polynomial in the $\frac{\partial \sigma^\alpha}{\partial x_\beta}(v, x)$ (we denote $\sigma = (\sigma^1, \ldots, \sigma^n)$).

We give here the proof of the estimate (6.9) in the case $s = 1$ because it is more convenient but the general case works in the same way. Recall that for any x fixed in B_1, σ^x is a diffeomorphism from $\mathbb{R}^n$ onto B_1. By definition of σ, we have

$$w = \sigma^x(v) \Longleftrightarrow v = (\sigma^x)^{-1}(w) = h(w) - h(x)$$

which allows us to write

$$g_J(x) = \int_{B_1} f(w) \Delta_J\big((\sigma^x)^{-1}(w), x\big) t^n \chi_l\big(t(\sigma^x)^{-1}(w)\big) \big|Jac\, h(w)\big| dw \,.$$

where $Jac\, h(w)$ is the Jacobian of h at w.

Let $r' = \rho^{-1}\big(\rho(r) + 1/t\big)$, we have $0 < r < r' < 1$. Note that the support of the function $w \longmapsto \chi_l\big(t(\sigma^x)^{-1}(w)\big)$ is included in $\sigma^x(\overline{B}_{1/t})$. If $v \in \overline{B}_{1/t}$ then $\|h(x)+v\| \leq \rho(r) + 1/t$ and

$$\|h^{-1}\big(h(x)+v\big)\| = \rho^{-1}\big(\|h(x)+v\|\big) \leq r' .$$

Therefore, $\sigma^x(\overline{B}_{1/t})$ is included is the closed ball $\overline{B}_{r'}$ (included in B_1).

For $i \in \{1, \ldots, n\}$, we then have

$$\Big|\frac{\partial g_J}{\partial x_i}(x)\Big| \leq \|f\|_{0,\overline{B}_1} \int_{\overline{B}_{r'}} \Big|\frac{\partial}{\partial x_i}\Big[\Delta_J\big((\sigma^x)^{-1}(w), x\big) t^n \chi_l\big(t(\sigma^x)^{-1}(w)\big)\Big] Jac\, h(w)\Big| dw\,.$$

We first write

$$\frac{\partial}{\partial x_i}\Delta_J\big((\sigma^x)^{-1}(w), x\big) = -\sum_{\alpha=1}^{n} \frac{\partial \Delta_J}{\partial v_\alpha}((\sigma^x)^{-1}(w), x)\frac{\partial h^\alpha}{\partial x_i}(x) + \frac{\partial \Delta_J}{\partial x_i}((\sigma^x)^{-1}(w), x)$$

where $h = (h^1, \ldots, h^n)$.

On the other hand, we have

$$\frac{\partial}{\partial x_i}\chi_l\big(t((\sigma^x)^{-1}(w))\big) = -t\sum_{\alpha=1}^{n}\frac{\partial \chi_l}{\partial x_\alpha}\big(t((\sigma^x)^{-1}(w)\big)\frac{\partial h^\alpha}{\partial x_i}(x)\,.$$

Therefore, applying back the change of variable $v = (\sigma^x)^{-1}(w)$, we can write

$$\begin{aligned}\Big|\frac{\partial g_J}{\partial x_i}(x)\Big| \quad &\leq \quad \|f\|_{0,\overline{B}_1}\int_K \Big|\sum_{\alpha=1}^{n}\frac{\partial \Delta_J}{\partial v_\alpha}(v,x)\frac{\partial h^\alpha}{\partial x_i}(x) + \frac{\partial \Delta_J}{\partial x_i}(v,x)\Big| t^n \chi_l(tv)dv \\ &\qquad + t\|f\|_{0,\overline{B}_1}\int_K t^n\Big|\Delta_J(v,x)\sum_{\alpha=1}^{n}\frac{\partial\chi_l}{\partial x_\alpha}(tv)\frac{\partial h^\alpha}{\partial x_i}(x)\Big| dv\,. \end{aligned} \tag{6.11}$$

where $K = (\sigma^x)^{-1}\big(\overline{B}_{r'}\big)$.

Note that the support of the function $t \longmapsto \chi_l(tv)$ is included in $\overline{B}_{1/t}$ which is itself included in K. Therefore, the two integrals in the previous estimate can be computed on $\overline{B}_{1/t}$.

Using the variable $v' = tv$ in the integrals, we get

$$\Big|\frac{\partial g_J}{\partial x_i}(x)\Big| \leq C_1\|f\|_{0,\overline{B}_1} + C_2 t\|f\|_{0,\overline{B}_1}$$

where the constants C_1 and C_2 depend on $\|\Delta_J\|_{1,\overline{B}_1\times\overline{B}_r}$ (i.e. on $\|\sigma\|_{2,\overline{B}_1\times\overline{B}_r}$), $\|h\|_{1,\overline{B}_r}$ and χ_l. Since $t > 1$, we get the result (6.9) for $s = 1$. The case $s \geq 1$ works in the same way.

Now, we prove the estimate (6.10). In the same way as above, we have for $x \in \overline{B}_r$,

$$(R_t\theta)_x = \sum_{J\in\mathcal{J}}\Big(\int_{\mathbb{R}^n} G_J(v,x)\chi_l(v)dv\Big)dx_{j_1}\wedge\ldots\wedge dx_{j_k}$$

with

$$G_J(v,x) = f\big(\sigma(v/t,x)\big)\Delta_J(v/t,x)\,.$$

Now, we write the Taylor expansion at the order $s-1$ (with $s \leq l$) of the function $v \mapsto G_J(v,x)$ at 0. We denote by $T_J(v,x)$ the remainder term of this Taylor series.

Note that $\Delta_J(0,x)$ equals 0 if $J \neq (i_1,\dots,i_k)$ and equals 1 if $J=(i_1,\dots,i_k)$. According to the assumption (6.1), we get

$$(\theta - R_t\theta)_x = \int_{\mathbb{R}^n} T_J(v,x)\chi_l(v)dv\,.$$

We have

$$T_J(v,x) = \int_0^1 \frac{(1-\tau)^{s-1}}{(s-1)!} D_v^{(s)} G_{J(\tau v/t,x)}(v/t,0)^{[s]}\,d\tau\,.$$

Therefore, noting that $\sigma(\overline{B}_1 \times \overline{B}_r) \subset \overline{B}_1$, we get

$$|(\theta - R_t\theta)_x| \leq \frac{A}{t^s}\|f\|_{s,\overline{B}_1}$$

where A is a positive constant depending on $\|\sigma\|_{s+1,\overline{B}_1\times\overline{B}_r}$.

□

Now, let us define the smoothing operator in the global case. Suppose that M is a smooth compact manifold. We can find (see for instance [15]) an atlas $\{(V_i,\varphi_i)\}_{i=1,\dots,m}$ of M, a small real number $\varepsilon > 0$ and a covering $\{U_i\}_{i=1,\dots,m}$ of M by open sets such that $\overline{U_i} \subset V_i$ and

$$\varphi_i(U_i) \subset \overline{B}_{1-\varepsilon} \subset \overline{B}_1 \subset \varphi_i(V_i)\,.$$

For each $i \in \{1,\dots,m\}$ we define the operator $S_{t,i} : \Omega^k(M) \longrightarrow \Omega^k(M)$ by

$$\big(S_{t,i}\theta\big)_x = \begin{cases} \theta_x & \text{if} \quad x \in M \setminus V_i \\ \big(\varphi_i^* \circ R_t \circ (\varphi_i^{-1})^*\theta_{|V_i}\big)_x & \text{if} \quad x \in V_i \end{cases} \tag{6.12}$$

Note that, by Remark 6.4 and the definition of the atlas, $S_{t,i}\theta$ is well defined on the manifold M.

Now, we consider a partition of the unity $\{\alpha_i\}_{i=1,\dots,m}$ subordinate to the covering $\{U_i\}_{i=1,\dots,m}$ and then define the norms $\|\ \|_p$ as in Section 4.2.

Each operator $S_{t,i}$ satisfies the following property.

Lemma 6.6. *If $p \geq 0$ and $s \geq 0$ are integers, there exists a positive constant C_{ips} such that for all x in U_i and θ in $\Omega^k(M)$, we have*

$$\|(S_{t,i}\theta)_x\|_{p+s} \leq C_{ips}t^s\|\theta\|_{p,M} \tag{6.13}$$

$$\|(\theta - S_{t,i}\theta)_x\|_p \leq C_{ips}t^{-s}\|\theta\|_{p+s,M} \quad \text{for } s \leq l \tag{6.14}$$

Proof. Let θ in $\Omega^k(M)$ and x in U_i.

We first check the estimate (6.13). We have

$$\|(S_{t,i}\theta)_x\|_{p+s} = \|\big(\varphi_i^* \circ R_t \circ {\varphi_i^{-1}}^* \theta_{|V_i}\big)_x\|_{p+s} \tag{6.15}$$

$$= \sum_{j=1}^{m} \alpha_j(x) \|{\varphi_j^{-1}}^* \big(\varphi_i^* \circ R_t \circ {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_j(x)}\|_{p+s} \tag{6.16}$$

$$= \sum_{j=1}^{m} \alpha_j(x) \|(\varphi_i \circ \varphi_j^{-1})^* \big(R_t \circ {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_j(x)}\|_{p+s} \tag{6.17}$$

If x belongs to the support of α_j we have $x \in U_i \cap U_j$ and $\varphi_j(x) \in \overline{B}_1$. Consequently, we get

$$\|(\varphi_i \circ \varphi_j^{-1})^* \big(R_t \circ {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_j(x)}\|_{p+s} \leq M_{i,j} \|\big(R_t \circ {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_i(x)}\|_{p+s}$$

where $M_{i,j}$ is a positive constant, which depends on $\|\varphi_i \circ \varphi_j^{-1}\|_{p+s+1,\overline{B}_1}$.

Now, since $x \in U_i$ we have $\varphi_i(x) \in \overline{B}_{1-\varepsilon}$. Lemma 6.5 gives then

$$\|(\varphi_i \circ \varphi_j^{-1})^* \big(R_t \circ {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_j(x)}\|_{p+s} \leq M_{ij} C_{ps} t^s \|{\varphi_i^{-1}}^* \theta_{|V_i}\|_{p,\overline{B}_1} \tag{6.18}$$

Finally, we get the estimate (6.13) using Lemma 4.1.

In the same way, we can write

$$\|(\theta - S_{t,i}\theta)_x\|_p = \sum_{j=1}^{m} \alpha_j(x) \|(\varphi_i \circ \varphi_j^{-1})^* \big({\varphi_i^{-1}}^* \theta_{|V_i} - R_t {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_j(x)}\|_p \tag{6.19}$$

$$\leq \sum_{j=1}^{m} \alpha_j(x) M_{ij} \|\big({\varphi_i^{-1}}^* \theta_{|V_i} - R_t {\varphi_i^{-1}}^* \theta_{|V_i}\big)_{\varphi_i(x)}\|_p \tag{6.20}$$

$$\leq \sum_{j=1}^{m} \alpha_j(x) M_{ij} C_{ps} t^{-s} \|{\varphi_i^{-1}}^* \theta_{|V_i}\|_{p+s,\overline{B}_1} \tag{6.21}$$

We then obtain the estimate (6.14) by Lemma 4.1. □

Remark 6.7. A direct consequence of this lemma, when $s = 0$, is the following. There exists a positive constant C_{ip} such that, for every k-form θ on M, we have

$$\|S_{t,i}\theta\|_{p,M} \leq C_{ip} \|\theta\|_{p,M} \tag{6.22}$$

$$\|\theta - S_{t,i}\theta\|_p \leq C_{ip} \|\theta\|_{p+s,M} \quad \text{for } s \leq l \tag{6.23}$$

Indeed, if $x \in U_i$ we have $\|(S_{t,i}\theta)_x\|_p \leq C_{ip0} \|\theta\|_{p,M}$ and if x is not in U_i then $S_{t,i}\theta_x = \theta_x$. Taking $C_{ip} = Max\,(1, C_{ip0})$, we get the first estimate. The second estimate is obtained in the same way.

Finally, the global *smoothing operator* $S_t : \Omega^k(M) \longrightarrow \Omega^k(M)$ is defined by

$$S_t = S_{t,1} \circ \ldots \circ S_{t,m}\,. \tag{6.24}$$

We prove now the equivalent of Lemma 6.1 in the global case.

Proposition 6.8. *If $p \geq 0$ and $s \geq 0$ are integers, there exists a positive constant C_{ps} such that for all θ in $\Omega^k(M)$ we have*

$$\|S_t\theta\|_{p+s,M} \leq C_{ps} t^s \|\theta\|_{p,M} \tag{6.25}$$

$$\|\theta - S_t\theta\|_{p,M} \leq C_{ps} t^{-s} \|\theta\|_{p+s,M} \quad \text{for } s \leq l \tag{6.26}$$

Proof. Let θ in $\Omega^k(M)$ and $x \in M$. We use the covering $\{U_i\}_{i=1,\ldots,m}$ of M. We can write $x \in U_{i_1} \cap \ldots \cap U_{i_q}$ with $i_1 < i_2 < \ldots < i_q$.

By the definition of the $S_{t,i}$, we have

$$S_t\theta_x = (S_{t,1} \circ \ldots \circ S_{t,m}\theta)_x = S_{t,i_1}\big(S_{t,i_1+1} \circ \ldots \circ S_{t,m}\theta\big)_x$$

By Lemma 6.6 we get

$$\|S_t\theta_x\|_{p+s} \leq C_{i_1,p,s} t^s \|S_{t,i_1+1} \circ \ldots \circ S_{t,m}\theta\|_{p,M}$$

Now, the remark 6.7 gives the estimate (6.46). The estimate (6.47) can be proven in the same way. □

By the definition of the $S_{t,i}$ and S_t, the following important property is obvious.

Proposition 6.9. *If θ is a k-differential form on M then we have $d\big(S_t\theta\big) = S_t\big(d\theta\big)$.*

In the same way as for differential forms, we can define the regularization operator for the vector fields on an open set V of $\mathbb{R}^n$ containing the closed ball $\overline{B}_1$, $R_t : \mathfrak{X}(V) \longrightarrow \mathfrak{X}(V)$ by

$$R_t X = \int_{\mathbb{R}^n} \big({\sigma_v}^{-1}{}_* X\big)\, t^n \chi_l(tv) dv\,. \tag{6.27}$$

Using the same kind of proof, we get the same estimates as in Lemma 6.5

The construction we made in (6.12) and (6.24) holds in this case with

$$\big(S_{t,i}X\big)_x = \begin{cases} X_x & \text{if} \quad x \in M \setminus V_i \\ \big((\varphi_i^{-1})_* \circ R_t \circ \varphi_{i*} X_{|V_i}\big)_x & \text{if} \quad x \in V_i \end{cases} \tag{6.28}$$

and it gives the definition of the smoothing operator for vector fields on the manifold M, which satisfies the same properties as in Proposition 6.8. This construction can be generalized, with the same properties, to the space of multivectors $\mathfrak{X}^k(M)$.

Remark 6.10. This smoothing operator may be useful in some situations dealing with differential forms and vector fields but, unfortunately, it presents an important failure in our case. Indeed, if X is an Hamiltonian or presymplectic vector field with respect to the presymplectic form ω, then the vector field $S_t(X)$ is not necessarily Hamiltonian or presymplectic.

Therefore, keeping the same idea of construction, we need to correct a bit the definition (6.8) replacing σ_v by Hamiltonian diffeomorphisms.

6.3. **Smoothing operator associated to a presymplectic form.** In this section we give a construction of a global smoothing operator on a compact presymplectic manifold of constant rank, with the classical estimates like in Proposition 6.8, and which will satisfy the following property : *the smoothing of an Hamiltonian (or presymplectic) vector field is still Hamiltonian (or presymplectic).*

In this section, for any real number $\varrho > 0$ we denote by $\overline{D}_\varrho$ the closed ball $[\varrho;\varrho]^n$ of $\mathbb{R}^n$. In the same way as in the previous section, we first define the regularization operator on an open set V of $\mathbb{R}^n$ which contains the closed ball $\overline{D}_1$. We fix coordinates $(x_1,\ldots,x_n)$ on $\mathbb{R}^n$ and we consider the presymplectic form (with $2q \leq n$)

$$\omega_0 = \sum_{i=1}^{q} dx_i \wedge dx_{q+i}\,.$$

We fix two real numbers $r \in]0\,;1[$ and $\varepsilon > 0$ such that $r-\varepsilon > 0$. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth positive function such that $h(\tau) = 0$ if $|\tau| \geq r$, $h(\tau) > 0$ if $|\tau| < r$ and $h(\tau) \leq \varepsilon/2$ for every $\tau \in \mathbb{R}$. Moreover, we suppose that $h(\tau) \geq \alpha > 0$ if $|\tau| < r - \varepsilon/2$ (where $\alpha > 0$).

Now we define the following vector fields, for $i \in \{1,\ldots,q\}$ and $j \in \{2q+1,\ldots,n\}$

$$\left\{\begin{array}{ccc} Z_i & = & h(x_{q+i})\frac{\partial}{\partial x_i} \\ Z_{q+i} & = & h(x_i)\frac{\partial}{\partial x_{q+i}} \\ Z_j & = & h(x_j)\frac{\partial}{\partial x_j} \end{array}\right. \tag{6.29}$$

Note that these vector fields are Hamiltonian and then preserve the presymplectic form ($L_{Z_k}\omega_0 = 0$). Moreover, they have a compact support and they pairwise commute except Z_i and Z_{q+i} for $1 \leq i \leq q$.

We denote by $\phi_\tau^{(k)}$ the flows of the vector fields Z_k. For $i \in \{1,\ldots,q\}$, we have

$$\phi_\tau^{(i)}(x) = (x_1,\ldots,x_{i-1},x_i+\tau h(x_{q+i}),x_{i+1},\ldots,x_n) \tag{6.30}$$

$$\phi_\tau^{(q+i)}(x) = (x_1,\ldots,x_{q+i-1},x_{q+i}+\tau h(x_i),x_{q+i+1},\ldots,x_n) \tag{6.31}$$

For $j \in \{2q+1,\ldots,n\}$, the flow of Z_j is of the form

$$\phi_\tau^{(j)}(x) = (x_1,\ldots,x_{j-1},F(\tau,x_j),x_{j+1},\ldots,x_n) \tag{6.32}$$

where $F(\tau,x_j) = x_j + \int_0^\tau h\big(F(u,x_j)\big)du$.

Now, we define the smooth map $\Phi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$\Phi(v,x) = \phi_{v_n}^{(n)} \circ \ldots \circ \phi_{v_1}^{(1)}(x)\,. \tag{6.33}$$

If $v \in \mathbb{R}^n$ is fixed, we denote by $\Phi_v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the smooth diffeomorphism defined by $\Phi_v(x) = \Phi(v,x)$.

Note that we have $\Phi_v(x) = x$ for every $x \in V \setminus D_r$.

Take an integer $l \geq 1$, a real number $t > 1$ and a function χ_l as in the previous sections 6.1 and 6.2.

Following the idea of Section 6.2, we define for each integer k the *regularization operator* $R_t^{(\omega_0)} : \Omega^k(V) \longrightarrow \Omega^k(V)$ by

$$R_t^{(\omega_0)}\theta = \int_{\mathbb{R}^n} \left(\Phi_v^*\theta\right) t^n \, \chi_l(tv) dv \,. \tag{6.34}$$

We have the equivalent of Lemma 6.5 for this regularization operator.

Lemma 6.11. *If $p \geq 0$ and $s \geq 0$ are integers, there exists a positive constant C_{ps} such that for all $\theta \in \Omega^k(V)$ and for all x in $\overline{D}_{r-\varepsilon}$, we have*

$$\|(R_t^{(\omega_0)}\theta)_x\|_{p+s} \quad \leq \quad C_{ps} t^s \|\theta\|_{p,\overline{D}_r} \tag{6.35}$$

$$\|(\theta - R_t^{(\omega_0)}\theta)_x\|_p \quad \leq \quad C_{ps} t^{-s} \|\theta\|_{p+s,\overline{D}_r} \quad \text{for } s \leq l \tag{6.36}$$

Proof. The proof of this lemma (and in particular of the estimate (6.36) is the same as for Lemma 6.5, replacing σ by Φ. For the estimate (6.35), we give here some precisions.

We suppose that $\theta = f \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ where f is a smooth function on V. For x in $\overline{D}_{r-\varepsilon}$, we write (with the notations of the proof of Lemma 6.5) $(R_t^{(\omega_0)}\theta)_x = \sum_{J\in\mathcal{J}} g_J(x) dx_{j_1} \wedge \ldots \wedge dx_{j_k}$ with

$$g_J(x) = \int_{\mathbb{R}^n} f(\Phi(v,x))\Delta_J(v,x) t^n \chi_l(tv) dv \tag{6.37}$$

where $\Delta_J(v,x)$ is a polynomial in the terms $\dfrac{\partial \Phi_\alpha}{\partial x_\beta}(v,x)$.

We check that for $x \in \overline{D}_{r-\varepsilon}$, the map Φ^x defined by $\Phi^x(v) = \Phi(v,x)$ is a diffeomorphism from D_1 onto its image. Indeed, we have (for $i \in \{1,\ldots,q\}$ and $j \in \{2q+1,\ldots,n\}$)

$$w = \Phi^x(v) \Longleftrightarrow \left\{ \begin{array}{ccc} w_i & = & x_i + v_i h(x_{q+i}) \\ w_{q+i} & = & x_{q+i} + v_{q+i} h(w_i) \\ w_j & = & F(v_j, x_j) \end{array} \right. \tag{6.38}$$

Since $x \in \overline{D}_{r-\varepsilon}$, we have $h(x_{q+i}) \geq \alpha > 0$. In the same way, since $v \in D_1$, we have $|w_i| < r - \varepsilon + \varepsilon/2 = r - \varepsilon/2$ so $h(w_i) \geq \alpha > 0$. We deduce that

$$v_i = \frac{w_i - x_i}{h(x_{q+i})} \qquad \text{and} \qquad v_{q+i} = \frac{w_{q+i} - x_{q+i}}{h(w_i)} \,. \tag{6.39}$$

Now, if $\|x\| \leq r - \varepsilon$ and $|\tau| \leq 1$ then we have $|F(\tau, x_j)| \leq r - \varepsilon/2$ which gives $h\big(F(\tau,x_j)\big) \geq \alpha > 0$ for all $\tau \in [0;1]$.

Since $\frac{\partial}{\partial \tau} F(\tau, x_j) = h\big(F(\tau,x_j)\big)$, $F(0,x_j) = x_j$ and $w_j = F(v_j, x_j)$, we have

$$v_j = \int_{x_j}^{w_j} \frac{du}{h(u)} \tag{6.40}$$

We deduce that Φ^x is a diffeomorphism from D_1 onto its image and the relations (6.39) and (6.40) give the expression of $(\Phi^x)^{-1}$.

Note that the support of the function $v \longmapsto \chi_l(tv)$ is included in $\overline{D}_{1/t} \subset D_1$ therefore, the integral of (6.37) can be taken on $\overline{D}_{1/t}$.

Now, we apply the change of variable $w = \Phi^x(v)$. For every $x \in \overline{D}_{r-\varepsilon}$ we have $\Phi^x(\overline{D}_{1/t}) \subset \overline{D}_{r-\varepsilon+\varepsilon/2t}$ and moreover, the support of the function $w \longmapsto \chi_l\Big(t\big(\Phi^x\big)^{-1}(w)\Big)$ is included in $\Phi^x(\overline{D}_{1/t})$. We then get

(6.41)

$$g_J(x) = \int_{\overline{D}_{r-\varepsilon+\varepsilon/2t}} f(w)\Delta_J\big(\big(\Phi^x\big)^{-1}(w), x\big)t^n\chi_l\big(t\big(\Phi^x\big)^{-1}(w)\big)\big|Jac\,\big(\Phi^x\big)^{-1}(w)\big|dw\,,$$

where $Jac\,\big(\Phi^x\big)^{-1}(w)$ is the Jacobian of $\big(\Phi^x\big)^{-1}$ at w, i.e.

$$Jac\,\big(\Phi^x\big)^{-1}(w) = \frac{1}{h(x_{q+1})}\cdots\frac{1}{h(x_{2q})}\frac{1}{h(w_1)}\cdots\frac{1}{h(w_q)}\frac{1}{h(w_{2q+1})}\cdots\frac{1}{h(w_n)}\,. \tag{6.42}$$

In the same way as in the proof of Lemma 6.5 we then compute $\frac{\partial g_J}{\partial x_i}$ and apply back the change of variable $v = (\Phi^x)^{-1}(w)$ for the estimate of $|\frac{\partial g_J}{\partial x_i}(x)|$.

We note that the support of the function $\chi_l(tv)$ is included in $\overline{D}_{1/t}$ which is itself included in $(\Phi^x)^{-1}(\overline{D}_{r-\varepsilon+\varepsilon/2t})$.

We then get an estimate of type

$$\big|\frac{\partial g_J}{\partial x_i}(x)\big| \leq \|f\|_{0,\overline{D}_r}\int_{\overline{D}_{1/t}}\mathcal{F}_1(v,x)dv + \|f\|_{0,\overline{D}_1}t\int_{\overline{D}_{1/t}}\mathcal{F}_2(v,x)dv\,, \tag{6.43}$$

where $\mathcal{F}_1(v,x)$ and $\mathcal{F}_2(v,x)$ depend on Δ_J, $(\Phi^x)^{-1}$, $\Phi^x(v)$, χ_l and t.

Finally, as in Lemma 6.5, we use the variable $v' = tv$ and we get (6.35) where the constant depends on $\|\Delta\|_{1,\overline{D}_1\times\overline{D}_r}$, α, ε, $\sup_{\tau\in\mathbb{R}}|h'(\tau)|$ and χ_l.

The proof of (6.36) is exactly the same as in Lemma 6.5. □

By definition of $R_t^{(\omega_0)}$, the following proposition is clear.

Proposition 6.12. *For all $\theta \in \Omega^k(V)$, we have $d\big(R_t^{(\omega_0)}\theta\big) = R_t^{(\omega_0)}d\theta$.*

Now, we construct the smoothing operator associated to a presymplectic form on a manifold in the same way as in Section 6.2. More precisely, we consider a compact manifold M with a presymplectic form ω of constant rank $2q$. We can find an atlas $\{(V_i,\varphi_i)\}_{i=1,\dots,m}$ of M, positive real numbers r and ε, and a covering $\{U_i\}_{i=1,\dots,m}$ of M such that

$$\varphi_i(U_i) \subset \overline{D}_{r-\varepsilon} \subset \overline{D}_r \subset \varphi_i(V_i)\,, \tag{6.44}$$

and

$$\big(\varphi_i^{-1}\big)^*\omega_{|V_i} = \omega_0 = \sum_{i=1}^{q} dx_i \wedge dx_{q+i}\,. \tag{6.45}$$

The construction of the operators $S_{t,i}^{(\omega)}$ and $S_t^{(\omega)}$ is exactly the same as in Section 6.2 (see the definitions (6.12) and (6.24)) replacing R_t by $R_t^{(\omega_0)}$. The same proof gives the following estimates.

Proposition 6.13. *If $p \geq 0$ and $s \geq 0$ are integers, there exists a positive constant C_{ps} such that for all θ in $\Omega^k(M)$ we have*

$$\|S_t^{(\omega)}\theta\|_{p+s,M} \leq C_{ps}t^s\|\theta\|_{p,M} \tag{6.46}$$

$$\|\theta - S_t^{(\omega)}\theta\|_{p,M} \leq C_{ps}t^{-s}\|\theta\|_{p+s,M} \quad \text{for } s \leq l \tag{6.47}$$

By the definition we still have the property

Proposition 6.14. *If θ is a k-differential form on M then we have for every $i \in \{1, \ldots, m\}$, $d\big(S_{t,i}^{(\omega)}\theta\big) = S_{t,i}^{(\omega)}\big(d\theta\big)$ and then $d\big(S_t^{(\omega)}\theta\big) = S_t^{(\omega)}\big(d\theta\big)$.*

In the same way, we define the smoothing operator on the space of vector fields (and multivector fields). The regularization operator for the vector fields on an open set V of $\mathbb{R}^n$ containing the closed ball $\overline{D}_1$, $R_t^{(\omega_0)} : \mathfrak{X}(V) \longrightarrow \mathfrak{X}(V)$ is defined by

$$R_t^{(\omega_0)}X = \int_{\mathbb{R}^n} \Big(\big(\Phi_v^{-1}\big)_* X\Big)\, t^n \chi_l(tv) dv\,. \tag{6.48}$$

We can then construct the smoothing operator $S_t^{(\omega)}$ on the space of vector fields on a compact manifold in the same way as for differential forms and we can prove in the same way the same estimates as in Lemma 6.13.

Finally, we have to prove that the smoothing operator $S_t^{(\omega)}$ transforms a presymplectic or Hamiltonian vector field to a presymplectic or Hamiltonian vector field.

Lemma 6.15. *If X is a smooth vector field on an open set V of $\mathbb{R}^n$ containing the closed ball $\overline{D}_1$, then we have*

$$i_{R_t^{(\omega_0)}X}\omega_0 = R_t^{(\omega_0)}\big(i_X\omega_0\big)\,.$$

In particular, if X is a presymplectic (or Hamiltonian) vector field, then $R_t^{(\omega_0)}X$ is presymplectic (or Hamiltonian) too.

Proof. We denote $\tilde{X} = \Phi_v^{-1}{}_* X$ and $Y = R_t^{(\omega_0)}X$. We can write for any $x \in V$

$$X = \sum_{j=1}^n X_j(x)\frac{\partial}{\partial x_j}\,, \quad \tilde{X} = \sum_{j=1}^n \tilde{X}_j(v,x)\frac{\partial}{\partial x_j} \quad \text{and} \quad Y = \sum_{j=1}^n Y_j(x)\frac{\partial}{\partial x_j} \tag{6.49}$$

with

$$Y_j(x) = \int_{\mathbb{R}^n} \tilde{X}_j(v,x) t^n \chi_l(tv) dv\,. \tag{6.50}$$

We have

$$i_Y\omega_0 = -\sum_{i=1}^{q} Y_{q+i}dx_i + \sum_{i=1}^{q} Y_i dx_{q+i} \tag{6.51}$$

$$= \int_{\mathbb{R}^n} \big[-\sum_{i=1}^{q} \tilde{X}_{q+i}dx_i + \sum_{i=1}^{q} \tilde{X}_i dx_{q+i}\big] t^n \chi(tv)dv \tag{6.52}$$

$$= \int_{\mathbb{R}^n} \big(i_{\tilde{X}}\omega_0\big) t^n \chi(tv)dv\,. \tag{6.53}$$

On the other hand, we can write

$$R_t^{(\omega_0)}\big(i_X\omega_0\big) = \int_{\mathbb{R}^n} \Phi_v^*(i_X\omega_0)t^n\chi(tv)dv \tag{6.54}$$

$$= \int_{\mathbb{R}^n} \Big(i_{(\Phi_v^{-1})_*X}\Phi_v^*\omega_0)\Big)t^n\chi(tv)dv \tag{6.55}$$

$$= \int_{\mathbb{R}^n} (i_{\tilde{X}}\omega_0)t^n\chi(tv)dv \tag{6.56}$$

because the diffeomorphism Φ_v preserves ω_0.

Now, since $R_t^{(\omega_0)}$ commutes with the differential d, it transforms a presymplectic (or Hamiltonian) vector field to a presymplectic (or Hamiltonian) vector field. □

Proposition 6.16. *If X is a Hamiltonian vector field on the presymplectic manifold (M,ω), then we have*

$$i_{S_t^{(\omega)}(X)}\omega = S_t^{(\omega)}\big(i_X\omega\big)\,.$$

In particular, if X is a presymplectic (or Hamiltonian) vector field, then $S_t^{(\omega)}X$ is presymplectic (or Hamiltonian) too.

Proof. Recall that we have the atlas $\{(V_j,\varphi_j)\}_{j=1,\ldots,m}$ of M given in (6.45). We show that for each $j\in\{1,\ldots,m\}$,

$$i_{S_{t,j}^{(\omega)}X}\omega = S_{t,j}^{(\omega)}\big(i_X\omega\big)\,, \tag{6.57}$$

which will give the result, by definition of $S_t^{(\omega)}$.

By the definition of $S_{t,j}^{(\omega)}$, we just have to show that this relation is true on V_j. If we restrict on V_j, we have using Lemma 6.15,

$$i_{S_{t,j}^{(\omega)}X}\omega = i_{\varphi_{j*}^{-1}R_t^{(\omega_0)}\big(\varphi_{j*}X_{|V_j}\big)}\varphi_j^*\omega_0 \tag{6.58}$$

$$= \varphi_j^*\Big(i_{R_t^{(\omega_0)}\varphi_{j*}X_{|V_j}}\omega_0\Big) \tag{6.59}$$

$$= \varphi_j^*\Big(R_t^{(\omega_0)}\big(i_{\varphi_{j*}X_{|V_j}}\omega_0\big)\Big) \tag{6.60}$$

$$= \varphi_j^*R_t^{(\omega_0)}\big(\varphi_j^{-1*}i_X\omega\big) \tag{6.61}$$

$$= S_t^{(\omega)}\big(i_X\omega\big)\,. \tag{6.62}$$

The end of the proof is a consequence of the compatibility of $S_{t,j}^{(\omega)}$ with the differential d of differential forms. □

7. Appendix : Abstract normal form theorem

In this section, we recall a normal form theorem we had proved in [21] as an improvement of the work we had started in [22]. The motivation was to give an abstract theorem which could be used in order to prove normal form results, essentially in the local case but it works in the global case (it is even easier in this case). The proof of this theorem is based on a fast converging iterative process inspired by the Newton method and uses some techniques of the Nash-Moser type. The initial applications of this theorem was the proof of a Levi decomposition for smooth Poisson structures ([22]) and the rigidity of Hamiltonian actions on Poisson manifolds ([21]). This theorem has been also used in a slightly different version in [2] in order to study the local classification of generalized complex structures.

7.1. **The local case.** We just recall briefly the formalism about SCI-spaces. The interested reader can find more details and examples in [21].

An *SCI-space* $\mathcal{H}$ is a collection of Banach spaces $(\mathcal{H}_{k,r}, \| \|_{k,r})$ with $0 < r \leq 1$ and $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ (r is called the *radius* parameter, k is called the *smoothness parameter*) with some natural conditions about radius restriction and smoothness restriction (see [21] for details). Moreover, we impose the existence of a *smoothing operator* for each r, which depends continuously on r. More precisely, for each $0 < r \leq 1$ and each $t > 1$ there is a linear map, called the *smoothing operator*,

$$S_r(t) : \mathcal{H}_{0,r} \longrightarrow \mathcal{H}_{\infty,r} = \bigcap_{k=0}^{\infty} \mathcal{H}_{k,r} \tag{7.1}$$

which satisfies the following inequalities: for any p and s in $\mathbb{Z}_+$, we have

$$\|S_\rho(t) f\|_{p+s,r} \leq C_{r,p,s} t^s \|f\|_{p,r} \tag{7.2}$$

$$\|f - S_r(t) f\|_{p,r} \leq C_{r,p,s} t^{-s} \|f\|_{p+s,r} \tag{7.3}$$

where $C_{r,p,s}$ is a positive constant (which does not depend on f nor on t) and which is continuous with respect to r.

Note that we can generalize the definition of the smoothing operator assuming that $S(t) : \mathcal{H}_{0,r} \longrightarrow \mathcal{H}_{\infty,r'}$ where $r' \leq r$. We can also accept a shift by r (which is a fixed positive integer) : t^{s+r} and t^{-s+r}. All the results given below still work in this situation. If the two conditions (7.2) and (7.3) are satisfied only for $s \leq l$ where l is an integer which can be fixed as large as we want, the abstract normal forms theorem still works in the same way.

A natural example of SCI-space (which is at the origin of this definition) is given by the differentiable functions defined in a neighbourhood of 0 in $\mathbb{R}^n$. The smoothing operators are classics (see [23], [24] or [25]). More generally, the spaces of differential forms, vector fields or multivector fields in a neighbourhood of 0 in $\mathbb{R}^n$ are examples of SCI-spaces.

An important consequence of the smoothing operator estimates (7.2) and (7.3) is the interpolation inequality : if $l \leq m \leq n$ are positive integers and $r > 0$ a real number, there exists a positive constant C such that

$$(\|f\|_{m,r})^{n-l} \leq C (\|f\|_{l,r})^{n-m} (\|f\|_{n,r})^{m-l},$$

for all f in $\mathcal{H}$.

A *subspace* (respectively, a *subset*) of an SCI-space $\mathcal{H}$ is a collection $\mathcal{V}$ of subspaces $\mathcal{V}_{k,r}$ (respectively, subsets) of $\mathcal{H}_{k,r}$, which are invariant under the inclusion and radius restriction maps of $\mathcal{H}$. We do not impose that the subspaces or subsets are invariant by the smoothing operator.

An *SCI-group* $\mathcal{G}$ consists of elements ϕ which are written as a (formal) sum

$$\phi = Id + \chi, \tag{7.4}$$

where χ belongs to an SCI-space $\mathcal{W}$, together with scaled group laws, see [21]. The neutral element of $\mathcal{G}$ is $Id = Id + 0$ and we impose estimates for the composition of elements and the inversion.

A natural example of SCI-group is given by the local diffeomorphisms on $(\mathbb{R}^n, 0)$. This definition has been motivated principally by this example.

We will say that a linear left action of an SCI-group $\mathcal{G}$ on an SCI-space $\mathcal{H}$ is *SCI* if the usual axioms of a left group action modulo appropriate restrictions of radii (so we have *scaled action laws*) are satisfied. Moreover, there is a positive integer γ and a positive constant c such that for any integer k there exist polynomial functions with positive coefficients Q, R and T such that, for each $\phi = Id + \chi$ and $\psi = Id + \xi$ in $\mathcal{G}$ and $f \in \mathcal{H}$ we have

$$\begin{aligned} \|\phi \cdot f\|_{2k-1,\rho'} \quad \leq \quad & \|f\|_{2k-1,\rho}\big(1 + \|\chi\|_{k+\gamma,\rho} Q(\|\chi\|_{k+\gamma,\rho})\big) \\ & + \|\chi\|_{2k-1+\gamma,\rho}\|f\|_{k,\rho} R(\|\chi\|_{k+\gamma,\rho}) \end{aligned} \tag{7.5}$$

where $\rho' = (1 - c\|\chi\|_{1,r})\rho$; and also

$$\|\psi \cdot f - \phi \cdot f\|_{k,\rho''} \leq \|\xi - \chi\|_{k+\gamma,\rho}\|f\|_{k+\gamma,\rho} T(\|\chi\|_{k+\gamma,\rho}, \|\xi\|_{k+\gamma,\rho}) \tag{7.6}$$

where $\rho'' = (1 - c(\|\chi\|_{1,r} + \|\xi - \chi\|_{1,r}))\rho$.

Of course, we can define in the same way the notion of *linear right SCI-action*.

In [21], the SCI-action we worked with was the right-action of the SCI-group of local diffeomorphisms of $(\mathbb{R}^n, 0)$ on differentiable functions on $(\mathbb{R}^n, 0)$: $\phi \cdot f = f \circ \phi$.

In the following theorem, the notation $Poly(\|f\|_{k,r})$ denotes a polynomial term in $\|f\|_{k,r}$ where the polynomial has positive coefficients and does not depend on f (it may depend on k and on r continuously) ; the notation $Poly_{(p)}(\|f\|_{k,r})$ (where p is a strictly positive integer) denotes a polynomial term in $\|f\|_{k,r}$ where the polynomial has positive coefficients and does not depend on f (it may depend on k and on r continuously) and *which contains terms of degree greater or equal to* p.

The following theorem is an affine version of a general abstract normal form theorem (see [21], pages 1158 and 1160).

Theorem 7.1. ([21]) *Let $\mathcal{T}$ be a SCI-space, $\mathcal{F}$ a subspace of $\mathcal{T}$, and $\mathcal{S}$ a subset of $\mathcal{T}$. Take an element $\mathsf{f_O}$ in $\mathcal{S}$ that will be considered as the origin in $\mathcal{T}$.*

Denote $\mathcal{N} = \mathcal{F} \cap \mathcal{S}$. Assume that there is a projection $\pi : \mathcal{T} \longrightarrow \mathcal{F}$ (compatible with restriction and inclusion maps) such that for every f in $\mathcal{T}_{k,r}$, the element $\zeta(f) = f - \pi(f)$ satisfies

$$\|\zeta(f) - \mathsf{f_O}\|_{k,r} \leq \|f - \mathsf{f_O}\|_{k,r} Poly(\|f - \mathsf{f_O}\|_{[(k+1)/2],r}) \tag{7.7}$$

for all $k \in \mathbb{N}$ (or at least for all k sufficiently large), where $[\ ]$ is the integer part.

Let $\mathcal{G}$ be an SCI-group acting on $\mathcal{T}$ by a linear left SCI-action and let $\mathcal{G}^0$ be a closed subgroup of $\mathcal{G}$ formed by elements preserving $\mathcal{S}$.

Let $\mathcal{H}$ be a SCI-space and assume that there exist maps $\mathrm{H} : \mathcal{S} \longrightarrow \mathcal{H}$ and $\Phi : \mathcal{H} \longrightarrow \mathcal{G}^0$ and an integer $s \in \mathbb{N}$ such that for every $0 < r \leq 1$, every f in $\mathcal{S}$ and g in $\mathcal{H}$, and for all k in $\mathbb{N}$ (or at least for all k sufficiently large) we have the three properties :

$$\begin{aligned}\|\mathrm{H}(f)\|_{k,r} \quad \leq \quad & \|\zeta(f)-\mathsf{f_O}\|_{k+s,r} Poly(\|f-\mathsf{f_O}\|_{[(k+1)/2]+s,r}) \\ & +\|f-\mathsf{f_O}\|_{k+s,r}\|\zeta(f)-\mathsf{f_O}\|_{[(k+1)/2]+s,r} Poly(\|f-\mathsf{f_O}\|_{[(k+1)/2]+s,r}) ,\end{aligned} \tag{7.8}$$

$$\|\Phi(g)-Id\|_{k,r'} \leq \|g\|_{k+s,r} Poly(\|g\|_{[(k+1)/2]+s,r}) \tag{7.9}$$

and

$$\begin{aligned}\|\Phi(g_1)\,.f-\Phi(g_2)\,.f\|_{k,r'} \quad \leq \quad & \|g_1-g_2\|_{k+s,r}\|f\|_{k+s,r} Poly(\|g_1\|_{k+s,r},\|g_2\|_{k+s,r}) \\ & +\|f\|_{k+s,r} Poly_{(2)}(\|g_1\|_{k+s,r},\|g_2\|_{k+s,r})\end{aligned} \tag{7.10}$$

if $r' \leq r(1-c\|g\|_{2,r})$ in (7.9) and $r' \leq r(1-c\|g_1\|_{2,r})$ and $r' \leq r(1-c\|g_2\|_{2,r})$ in (7.10).

Finally, for every f in $\mathcal{S}$ denote $\phi_f = Id + \chi_f = \Phi\big(\mathrm{H}(f)\big) \in \mathcal{G}^0$ and assume that there is a positive real number δ such that we have the inequality

$$\|\zeta(\phi_f\,.\,f)-\mathsf{f_O}\|_{k,r'} \leq \|\zeta(f)-\mathsf{f_O}\|_{k+s,r}^{1+\delta} Q(\|f-\mathsf{f_O}\|_{k+s,r},\|\chi_f\|_{k+s,r},\|\zeta(f)-\mathsf{f_O}\|_{k+s,r},\|f-\mathsf{f_O}\|_{k,r})\,. \tag{7.11}$$

(if $r' \leq r(1-c\|\chi_f\|_{1,r})$) where Q is a polynomial of four variables and whose degree in the first variable does not depend on k and with positive coefficients.

Then there exist $l \in \mathbb{N}$ and two positive constants α and β ($\beta < 1 < \alpha$) with the following property: for all $p \in \mathbb{N} \cup \{\infty\}, p \geq l$, and for all $f \in \mathcal{S}_{2p-1,R}$ with $\|f-\mathsf{f_O}\|_{2l-1,R} < \alpha$ and $\|\zeta(f)-\mathsf{f_O}\|_{l,R} < \beta$, there exists $\psi \in \mathcal{G}^0_{p,R/2}$ such that $\psi \cdot f - \mathsf{f_O} \in \mathcal{N}_{p,R/2}$.

Remark 7.2. Of course, this theorem still works if we have a linear *right* SCI-action.

Remark 7.3. In [21] we imposed that $\mathcal{F}$ is an SCI-subspace of $\mathcal{T}$, i.e. it is invariant by the smoothing operator. In fact, we don't need this condition because in the proof of the theorem we only apply the smoothing operator in $\mathcal{H}$. To the elements of $\mathcal{T}$ we only apply the interpolation inequality.

7.2. **The global case.** A similar normal forms theorem can be stated in order to deal with normal forms problems in global situations. It is somehow easier than the local case because we do not have to deal with the radius parameter which has no sense in the global case. Note that in [21] we did not give so many details on the global case, principally because it works more or less in the same way as in the local case. In this present paper we give the formalism and the normal form theorem which can be used to prove the global rigidity theorem stated in [21].

A *CI-type space* is a sequence of Banach spaces $(\mathcal{H}_k, \|\,\|_k)$ (with $k \in \mathbb{Z}_+$) such that if $k \leq k'$ then $\mathcal{H}_{k'} \subset \mathcal{H}_k$ and $\|f\|_k \leq \|f\|_{k'}$ for every f in $\mathcal{H}_{k'}$. Moreover, we impose that there exists a family of *smoothing operators* $S_t : \mathcal{H}_0 \longrightarrow \mathcal{H}_\infty = \bigcap_{k=0}^{\infty} \mathcal{H}_k$,

($t \in]1;+\infty[$), which satisfies for every $f \in \mathcal{H}$,

$$\|S_t(f)\|_{p+s} \leq C_{p,s} t^s \|f\|_p \tag{7.12}$$

$$\|f - S_t(f)\|_p \leq C_{p,s} t^{-s} \|f\|_{p+s} \tag{7.13}$$

where $C_{p,s}$ is a positive constant which depends only on p and s.

Note that the vector space $\mathcal{H}_\infty$ with the increasing sequence of norms is *a tame Fréchet space* (see [14]) or a *$\mathcal{L}$-object* (see [26]).

A classical example of CI-type space is given by all the spaces $\mathcal{C}^p(M)$ of C^p-functions on a compact manifold M. More generally, if $E \longrightarrow M$ is a vector bundle over a compact manifold M, then the spaces of C^p-sections $\Gamma^p(E)$ form a CI-type space.

A topological group $\mathcal{G}$ is said of *CI-type at the Identity* if there is a neighbourhood $\mathcal{U}$ of the neutral element (called identity and denoted by Id) in $\mathcal{G}$ which is homeomorphic to a CI-type space $\mathcal{W}$ (if $\phi \in \mathcal{U}$, we write $\phi = Id + \chi$, with $\chi \in \mathcal{W}$) in which we have the following property :

There exists a real number $\eta > 0$ such that for each integer k there exist polynomial functions with positive coefficients P, Q_1, Q_2, R_1 and R_2 such that for every $\phi = Id + \chi$ and $\psi = Id + \xi$ in $\mathcal{U}$ with $\|\chi\|_1 < \eta$ and $\|\xi\|_1 < \eta$, the elements ϕ^{-1} and $\phi \circ \psi$ are in $\mathcal{U}$ and we have the estimates

$$\|\phi^{-1} - Id\|_k \leq \|\chi\|_k P(\|\chi\|_k) \ , \tag{7.14}$$

$$\|\phi \circ \psi - \phi\|_k \leq \|\xi\|_k Q_1(\|\xi\|_k) + \|\chi\|_{k+1} \|\xi\|_k Q_2(\|\xi\|_k) \, . \tag{7.15}$$

and

$$\|\phi \circ \psi - Id\|_k \leq \|\xi\|_k R_1(\|\xi\|_k) + \|\chi\|_k \big(1 + \|\xi\|_k R_2(\|\xi\|_k)\big) \, . \tag{7.16}$$

An example is given by the group $\mathcal{D}(M)$ of diffeomorphisms of a compact manifold M. The three inequalities can be obtained in the same way as for local diffeomorphisms (see [5], [21] and [22]). This group is actually a tame Fréchet Lie group and a description of the homeomorphism between a neighbourhood of the identity and a CI-type space, using geodesics of a Riemannian metric, can be found for instance in [14] and [26].

Finally, if $\mathcal{G}$ is a topological group of CI-type at the identity, we say that a left-action of $\mathcal{G}$ on a CI-type space $\mathcal{H}$ is *of CI-type at the identity* if $\mathcal{G}$ admits a neighbourhood $\mathcal{U}$ of the identity (for which $\mathcal{G}$ is of CI-type at the identity) which satisfies the following property :

There exist a real number $\eta > 0$ such that for every integer k there exist polynomial functions with positive coefficient Q, R and T such that for each $\phi = Id + \chi$ and $\psi = Id + \xi$ in $\mathcal{G}$ and $f \in \mathcal{H}$ with $\|\chi\|_1 < \eta$ and $\|\xi\|_1 < \eta$ we have the two inequalities

$$\begin{aligned} \|\phi \cdot f\|_{2k-1} \leq \; & \|f\|_{2k-1} \big(1 + \|\chi\|_{k+\gamma} Q(\|\chi\|_{k+\gamma})\big) \\ & + \|\chi\|_{2k-1+\gamma} \|f\|_k R(\|\chi\|_{k+\gamma}) \, , \end{aligned} \tag{7.17}$$

$$\|\psi \cdot f - \phi \cdot f\|_k \leq \|\xi - \chi\|_{k+\gamma} \|f\|_{k+\gamma} T(\|\chi\|_{k+\gamma}, \|\xi\|_{k+\gamma}) \, , \tag{7.18}$$

where $\gamma \geq 0$ is an integer independent of k, f, ϕ and ψ.

An example of right-action of CI-type at the identity is given by the action of the group of diffeomorphisms $\mathcal{D}(M)$ of a compact manifold M on the CI-space $\mathcal{C}(M)$ of differentiable functions on M. This is the action we considered in [21]. The proof of the two inequalities above is the same as in the local case (see [5] and [22]) using the definition of the norms in Section 4.2.

Now, we give the equivalent of Theorem 7.1 in the global case. The notations are the same as in the local case. The proof is exactly the same (see [21]), we just have to delete all the radius parameters.

Theorem 7.4. *Let $\mathcal{T}$ be a CI-type space, $\mathcal{F}$ a subspace of $\mathcal{T}$, and $\mathcal{S}$ a subset of $\mathcal{T}$. Take an element $\mathsf{f_O}$ in $\mathcal{S}$ that will be considered as the origin in $\mathcal{T}$.*

Denote $\mathcal{N} = \mathcal{F} \cap \mathcal{S}$. Assume that there is a projection $\pi : \mathcal{T} \longrightarrow \mathcal{F}$ such that for every f in $\mathcal{T}_k$, the element $\zeta(f) = f - \pi(f)$ satisfies

$$\|\zeta(f) - \mathsf{f_O}\|_k \leq \|f - \mathsf{f_O}\|_k Poly(\|f - \mathsf{f_O}\|_{[(k+1)/2]}) \tag{7.19}$$

for all $k \in \mathbb{N}$ (or at least for all k sufficiently large), where $[\ ]$ is the integer part.

Let $\mathcal{G}$ be a topological group of CI-type at the identity acting on $\mathcal{T}$ and suppose that this action is of CI-type at the identity. Let $\mathcal{G}^0$ be a closed subgroup of $\mathcal{G}$ formed by elements preserving $\mathcal{S}$.

Let $\mathcal{H}$ be a CI-type space and assume that there exist maps $\mathrm{H} : \mathcal{S} \longrightarrow \mathcal{H}$ and $\Phi : \mathcal{H} \longrightarrow \mathcal{G}^0$ with $\Phi(0) = Id$, and an integer $s \in \mathbb{N}$ such that for every f in $\mathcal{S}$ and g in $\mathcal{H}$, and for all k in $\mathbb{N}$ (or at least for all k sufficiently large) we have the three properties :

$$\begin{aligned}\|\mathrm{H}(f)\|_k \quad \leq \quad & \|\zeta(f) - \mathsf{f_O}\|_{k+s} Poly(\|f - \mathsf{f_O}\|_{[(k+1)/2]+s}) \\ & + \|f - \mathsf{f_O}\|_{k+s} \|\zeta(f) - \mathsf{f_O}\|_{[(k+1)/2]+s} Poly(\|f - \mathsf{f_O}\|_{[(k+1)/2]+s}) \ ,\end{aligned} \tag{7.20}$$

$$\|\Phi(g) - Id\|_k \leq \|g\|_{k+s} Poly(\|g\|_{[(k+1)/2]+s}) \tag{7.21}$$

and

$$\begin{aligned}\|\Phi(g_1)\,.f - \Phi(g_2)\,.f\|_k \quad \leq \quad & \|g_1 - g_2\|_{k+s} \|f\|_{k+s} Poly(\|g_1\|_{k+s}, \|g_2\|_{k+s}) \\ & + \|f\|_{k+s} Poly_{(2)}(\|g_1\|_{k+s}, \|g_2\|_{k+s})\,.\end{aligned} \tag{7.22}$$

Finally, for every f in $\mathcal{S}$ denote $\phi_f = Id + \chi_f = \Phi\big(\mathrm{H}(f)\big) \in \mathcal{G}^0$ and assume that there is a positive real number δ such that we have the inequality

$$\|\zeta(\phi_f\,.\,f) - \mathsf{f_O}\|_k \leq \|\zeta(f) - \mathsf{f_O}\|_{k+s}^{1+\delta} Q(\|f - \mathsf{f_O}\|_{k+s}, \|\chi_f\|_{k+s}, \|\zeta(f) - \mathsf{f_O}\|_{k+s}, \|f - \mathsf{f_O}\|_k)\,. \tag{7.23}$$

where Q is a polynomial of four variables and whose degree in the first variable does not depend on k and with positive coefficients.

Then there exist $l \in \mathbb{N}$ and two positive constants α and β ($\beta < 1 < \alpha$) with the following property:

for all $p \in \mathbb{N} \cup \{\infty\}, p \geq l$, and for all $f \in \mathcal{S}_{2p-1}$ with $\|f - \mathsf{f_O}\|_{2l-1} < \alpha$ and $\|\zeta(f) - \mathsf{f_O}\|_l < \beta$, there exists $\psi \in \mathcal{G}^0_p$ such that $\psi \cdot f - \mathsf{f_O} \in \mathcal{N}_p$.

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Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France

E-mail address: philippe.monnier@math.univ-toulouse.fr