

GLOBAL WELL-POSEDNESS OF THE DYNAMIC Φ_3^4 MODEL ON THE TORUS

JEAN-CHRISTOPHE MOURRAT, HENDRIK WEBER

ABSTRACT. We show global well-posedness of the dynamic Φ_3^4 model on the torus. This model is given by a non-linear stochastic PDE that can only be interpreted in a “renormalised” sense. A local well-posedness theory for this equation was recently developed by Hairer as well as Gubinelli, Perkowski, Imkeller and Catellier, Chouk. In the present article, we show that these solutions cannot blow up in finite time. Our method relies entirely on deterministic PDE arguments (such as embeddings of Besov spaces and interpolation), which are combined to derive energy inequalities.

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1. INTRODUCTION

The aim of this paper is to show global-in-time well-posedness for the stochastic quantisation equation on the three-dimensional torus. This model is formally given by the stochastic partial differential equation

$$(1.1) \quad \begin{cases} \partial_t X = \Delta X - X^3 + mX + \xi, & \text{on } \mathbb{R}_+ \times [-1, 1]^3, \\ X(0, \cdot) = X_0, \end{cases}$$

where ξ denotes a white noise over $\mathbb{R} \times [-1, 1]^3$, and m is a real parameter. Equation (1.1) describes the natural reversible dynamics for the Φ_3^4 quantum field theory, which is formally given by the expression

$$(1.2) \quad \mu \propto \exp \left(-2 \int_{[-1, 1]^3} \left[\frac{1}{2} |\nabla X|^2 + \frac{1}{4} X^4 - \frac{m}{2} X^2 \right] \right) \prod_{x \in [-1, 1]^3} dX(x).$$

Neither (1.1) nor (1.2) make sense as they stand. Due to the irregularity of the noise, solutions to (1.1) as well as realisations of the measure (1.2) should be distribution-valued, and the non-linear terms X^3 in (1.1) and X^4 in (1.2) have to be interpreted in a renormalised sense.

The construction of the measure (1.2) was a major result in the programme of constructive quantum field theory, accomplished in late the 60s and 70s [13, 8, 14, 10, 9]. The construction of the dynamics (1.1) was proposed in [32], but very little progress was made on this question until Hairer’s recent breakthrough results on *regularity structures*. Indeed, the construction of local-in-time solutions to (1.1) was one of the two principal applications of the theory of regularity structures presented in [23]. Hairer’s work triggered a lot of activity: Catellier and Chouk [5] were able to reproduce a similar local-in-time well-posedness result based on the notion of *paracontrolled distributions* put forward by Gubinelli, Imkeller and Perkowski in [18]. Yet another approach to obtain solutions for short times, based on Wilsonian *renormalisation group* analysis, was given by Kupianinen in [29]. Convergence of

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lattice approximations to (1.1) was shown in [25] and [33]. This was used in [25] to implement an argument in the spirit of Bourgain’s work on non-linear Schrödinger equations (see e.g. [3]) to show that for almost every initial datum with respect to the measure (1.2), solutions to (1.1) do not explode. This result relies on the analysis of the measure (1.2) performed in [4].

In this article, we present a global well-posedness theory for (1.1) based on the paracontrolled approach of [18, 5]. The emphasis is on ruling out the possibility of finite time blow-up. Our method relies solely on PDE arguments and energy inequalities, and shows that (1.1) is globally well posed for *every* initial datum in a suitable class, without making reference to the measure (1.2).

Every notion of solution relies heavily on the *subcriticality* of (1.1) in three dimensions. To explain this property, let us momentarily consider this equation over \mathbb{R}^d for an arbitrary $d \geq 1$. Formally rescaling the equation via

$$\hat{t} = \lambda^2 t, \quad \hat{x} = \lambda x, \quad \hat{\xi} = \lambda^{\frac{d+2}{2}} \xi, \quad \hat{X} = \lambda^{\frac{2-d}{2}} X, \quad \hat{m} = \lambda^2 m,$$

yields

$$(1.3) \quad \partial_t \hat{X} = \Delta \hat{X} - \lambda^{4-d} \hat{X}^3 + \hat{m} \hat{X} + \hat{\xi},$$

which suggests that for $d < 4$, the influence of the non-linear term should vanish as we consider smaller and smaller scales. This corresponds to the well-known fact that the Φ_d^4 theory is *superrenormalisable* in dimension $d < 4$.

Based on this observation, the first step in both the approach using regularity structures and the approach using paracontrolled distributions is the explicit construction of several terms in a perturbative expansion based on the solution of the *linear* stochastic heat equation

$$(1.4) \quad (\partial_t - \Delta)\dagger = \xi.$$

Throughout the article we adopt Hairer’s convention to denote the terms in this expansion by trees: here the symbol \dagger should be interpreted as a graph with a single vertex at the top which corresponds to the white noise, and with a line below corresponding to a convolution with the heat kernel. This graphical notation is extremely useful to keep track of a potentially large number of explicit stochastic objects.

The construction of these objects involves a renormalisation procedure, that is, the subtraction of several “infinite constants”. For example, the simplest stochastic objects constructed from \dagger are \mathbb{V} and \mathbb{V}^\dagger , which formally play the role of “ \dagger^2 ” and “ \dagger^3 ”. These objects are constructed by considering a regularised version \dagger_δ of \dagger , e.g. the solution obtained by replacing ξ with its convolution with a smoothing kernel on scale δ , and then taking the limits as δ tends to zero of

$$(1.5) \quad \dagger_\delta^2 - C_\delta \quad \text{and} \quad \dagger_\delta^3 - 3C_\delta \dagger_\delta,$$

for a suitable choice of diverging constant C_δ . The construction of these objects makes strong use of explicit representations of the covariances of \dagger and of its Gaussianity.

In both theories, the full non-linear system (1.1) is only treated in a second step. This step is completely deterministic, with the random terms constructed in the first step treated as an input. The solution X is sought in a space of functions whose small-scale behaviour is described in detail by the explicit stochastic objects. In both theories, this is implemented by replacing the scalar field X by a vector-valued function whose components correspond to the different “levels of regularity” of X . The scalar equation (1.1) then turns into a coupled system of equations. This point is at the heart of both methods. The approaches via regularity structures and via paracontrolled distributions then differ significantly. In the regularity

structures approach, a local description of the solution X in “real space” is given, whereas the paracontrolled approach uses tools from Fourier analysis. However, in both approaches, local-in-time solutions X are found by performing a Picard iteration for the system of equations interpreted in the mild sense. We stress that the renormalisation is completely treated at the level of the construction of the stochastic objects based on (1.4), and that no “infinite constants” appear in the deterministic analysis.

All approaches focus on the problems arising in the analysis of (1.1) on small scales, and devise a powerful method to deal with the so-called *ultra-violet divergences*. However, extra ingredients are necessary to obtain information on large scales. This already becomes apparent from the fact that the “good” sign of the term $-X^3$ is not used. The theories allow for the construction of solutions of (1.1) with the sign of the non-linear term reversed, and solutions of this modified equation are expected to blow up in finite time. Moreover, the scaling analysis above suggests that it is the non-linear term $-X^3$ which dominates the dynamics on large scales, so that it can no longer be treated as a perturbation.

In situations where the noise is less irregular, there are well-known tools available to obtain large scale information on non-linear equations such as (1.1). In fact, in the deterministic case $\xi = 0$, the non-linear term is known to have a strong damping effect, and the non-linear equation satisfies better bounds than the linearised version: for solutions of (1.1) with $\xi = 0$ (started with an L^∞ initial datum, say), a simple argument based on the comparison principle and the behaviour of the ODE $\dot{x} = -x^3 + mx$ yields L^∞ bounds on X which are independent of the initial datum. Other standard tools to extract information on the non-linear term involve a simple testing of X against itself or powers of itself. In this paper, we show how similar PDE arguments can be implemented in the context of the system of equations arising in the paracontrolled solution theory of (1.1).

1.1. Formal derivation of a system of equations. The obvious difficulty in developing a solution theory for (1.1) is the fact that the solution X will be a distribution, and that it is unclear how to interpret the non-linear expression $-X^3$. However, as we have explained in the previous section, on small scales X is expected to “behave like” the Gaussian process \dagger ; more precisely, we expect that $X - \dagger$ has better regularity than each of the terms separately. Moreover, the detailed knowledge of the covariance and the Gaussianity of \dagger can be used to define the “renormalised” products

$$(\dagger)^2 \rightsquigarrow \mathfrak{V} \quad \text{and} \quad (\dagger)^3 \rightsquigarrow \mathfrak{W},$$

via (1.5). In this section, we present a formal computation in the spirit of [5] to reorganise (1.1) into a system that we are able to solve, assuming that we can define the products of the explicit stochastic terms, even if they are distributions of low regularity. For the moment, we will ignore the “infinite constants” and manipulate the equation formally, adopting the following rules:

- Every term has a regularity exponent associated with it. We will say, for example, that the terms X and \dagger have regularity $(-\frac{1}{2})^-$, i.e. regularity $\frac{1}{2} - \varepsilon$ for ε arbitrarily small. All regularities are derived from the regularity of the white noise ξ , which is $(-\frac{5}{2})^-$.
- A function of regularity $\alpha_1 > 0$ can be multiplied with a distribution of regularity $\alpha_2 < 0$ if $\alpha_1 + \alpha_2 > 0$, resulting in a distribution of regularity α_2 .
- Convolution with the heat kernel of $\partial_t - \Delta$ increases the regularity by 2.
- Explicit stochastic objects can *always* be multiplied, irrespective of their regularity. The product of stochastic objects of regularity α_1 and α_2 has regularity $\min\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$.

In the next Section 1.2, we will then go back and give a precise meaning to these statements and discuss in particular how the last of these rules has to be interpreted. There, we will give a rigorous link between the system we derive formally in this section and the original system (1.1).

For illustration, we briefly show this calculation in the two-dimensional case $d = 2$, sketching a method introduced by Da Prato and Debussche in [7]. In dimension 2, the noise ξ has regularity $(-2)^-$, so both X and \dagger have regularity 0^- . According to the rules above, we cannot define X^3 directly (the regularity being negative), but we can define the square \mathfrak{V} and the cube \mathfrak{V}^3 of \dagger , both of which also have regularity 0^- . If we make the ansatz $X = \dagger + Y$, then Y solves

$$(1.6) \quad (\partial_t - \Delta)Y = -Y^3 - 3Y^2\dagger - 3Y\mathfrak{V} - \mathfrak{V} + m(\dagger + Y).$$

Convolution with the heat kernel increases regularity by 2, so that we expect Y to have regularity 2^- , which in turn allows to define all the products on the right hand side. Hence, we can solve (1.6), at least locally in time. We define the solution we seek, as a replacement for (1.1), to be $X := \dagger + Y$.

We now come back to our original problem, posed in three space dimensions. As stated above, in this case ξ has regularity $(-\frac{5}{2})^-$, so that X and \dagger have regularity $(-\frac{1}{2})^-$, \mathfrak{V} has regularity $(-1)^-$ and \mathfrak{V}^3 has regularity $(-\frac{3}{2})^-$. Therefore, the simple procedure leading to (1.6) does not suffice, as it would lead to Y being of regularity $(\frac{1}{2})^-$, which is not enough to define the products on the right-hand side of (1.6). The most irregular term we encounter in this approach, limiting the regularity of Y to $(\frac{1}{2})^-$, is the term \mathfrak{V}^3 , so we use it to define the next-order term in our expansion. We introduce \mathfrak{Y} , the solution of

$$(1.7) \quad (\partial_t - \Delta)\mathfrak{Y} = \mathfrak{V}^3,$$

which has regularity $(\frac{1}{2})^-$, and postulate an expansion of the form

$$(1.8) \quad X = \dagger - \mathfrak{Y} + u,$$

for some hopefully more regular u . Analogously to the two-dimensional case, we write the formal equation satisfied by u :

$$\begin{aligned} (\partial_t - \Delta)u &= -(u + \dagger - \mathfrak{Y})^3 + m(u + \dagger - \mathfrak{Y}) - \mathfrak{V} \\ &= -u^3 - 3(u - \mathfrak{Y})\mathfrak{V} + Q(u), \end{aligned}$$

where we introduced the notation

$$Q(u) = b_0 + b_1u + b_2u^2,$$

with

$$\begin{aligned} b_0 &= m(\dagger - \mathfrak{Y}) + (\mathfrak{Y})^3 - 3\dagger(\mathfrak{Y})^2, \\ b_1 &= m + 6\mathfrak{Y}\dagger - 3(\mathfrak{Y})^2, \\ b_2 &= -3\dagger + 3\mathfrak{Y}. \end{aligned}$$

All of these coefficients have regularity $(-\frac{1}{2})^-$. Since the regularity of \mathfrak{V} is $(-1)^-$, the regularity of u is expected to be 1^- , so that the product $u\mathfrak{V}$ is still ill-defined a priori.

In order to solve this problem, we use the notion of paraproducts, following [18]. Roughly speaking, the paraproduct of f and g , which we denote by $f \circledast g$, carries the high-frequency modes of g , modulated by the low-frequency modes of f . The product fg can be written

$$(1.9) \quad fg = f \circledast g + f \circledast g + f \circledast g,$$

where $f \circledast g$ carries the resonant interactions between f and g . The striking property of paraproducts is that, on the one hand, the quantities $f \circledast g$ and $f \circledast g$ are always

well-defined, and only the resonant term $f \circledast g$ can fail to be defined. But on the other hand, whenever the resonant term is well-defined, its regularity is given by the *sum* of regularities of f and g (as opposed to the minimum). We refer to the appendix for a more precise discussion, in particular Proposition A.7. We use (1.9) with $f = u - \Psi$ and $g = \mathbf{v}$, and decompose u into $v + w$ solving

$$(1.10) \quad (\partial_t - \Delta)v = -3(v + w - \Psi) \circledast \mathbf{v},$$

$$(1.11) \quad (\partial_t - \Delta)w = -(v + w)^3 - 3(v + w - \Psi) \circledast \mathbf{v} + Q(v + w),$$

where we write $\circledast = \circledast + \circledast$ for concision. The idea is that v carries the same local irregularity as u , while w should have better regularity, namely $(\frac{3}{2})^-$ instead of 1^- . The paraproduct in the right side of (1.10) contains the high-frequency modes of \mathbf{v} modulated by the low-frequency modes of $(v + w - \Psi)$. It is always well-defined and has regularity $(-1)^-$. The paraproduct $(v + w - \Psi) \circledast \mathbf{v}$ is also well-defined and has regularity $(-\frac{1}{2})^-$. It remains to consider the resonant term

$$(v + w - \Psi) \circledast \mathbf{v},$$

which cannot be made sense of classically (the criterion being the same as for the product of course, that is, the sum of regularities should be strictly positive). As was pointed out above, this term should have regularity given by the sum of the regularities of each term, that is, regularity $(-\frac{1}{2})^-$ in our case. Since w is expected to have regularity $(\frac{3}{2})^-$, the term $w \circledast \mathbf{v}$ can be made sense of classically. In extension of our rules, we postulate that we can define $\Psi \circledast \mathbf{v} =: \Psi \circledast \mathbf{v}$ as a distribution of regularity $(-\frac{1}{2})^{-1}$.

It remains to treat the term $v \circledast \mathbf{v}$. The key advantage of the decomposition using paraproducts lies in the following commutator estimates, which allow to rewrite this term using explicit graphical terms of low regularity and more regular objects involving v and w . As a first step, we denote by Υ the solution of

$$(1.12) \quad (\partial_t - \Delta)\Upsilon = \mathbf{v} \quad (\Upsilon(t=0) = 0),$$

that is,

$$(1.13) \quad \Upsilon(t) = \int_0^t e^{(t-s)\Delta} \mathbf{v}(s) ds.$$

We also write (1.10) in the mild form

$$v(t) = e^{t\Delta} v_0 - 3 \int_0^t e^{(t-s)\Delta} [(v + w - \Psi) \circledast \mathbf{v}](s) ds.$$

The behaviour of the heat kernel suggests that the local irregularity of v is that of $-3(v + w - \Psi) \circledast \Upsilon$. In other words, the difference

$$(1.14) \quad \text{com}_1(v, w)(t) := e^{t\Delta} v_0 - 3 \int_0^t e^{(t-s)\Delta} [(v + w - \Psi) \circledast \mathbf{v}](s) ds \\ + 3 [(v + w - \Psi) \circledast \Upsilon](t)$$

has better regularity than v itself. (Justifying this relies on Proposition A.15 and on suitable *time* regularity of v , w and Ψ .) We thus decompose $v \circledast \mathbf{v}$ into

$$v \circledast \mathbf{v} = -3 [(v + w - \Psi) \circledast \Upsilon] \circledast \mathbf{v} + \text{com}_1(v, w) \circledast \mathbf{v}.$$

The second of these terms is defined classically, and it only remains to control the first term. Recall that $(v + w - \Psi) \circledast \Upsilon$ carries the high-frequency modes of Υ , modulated by the low-frequency modes of $(v + w - \Psi)$. Hence, it is reasonable to expect $[(v + w - \Psi) \circledast \Upsilon] \circledast \mathbf{v}$ to have the same local irregularity as

$$(v + w - \Psi) \circledast \Upsilon,$$

where Ψ is a postulated version of the resonant term $\Psi \otimes \mathbf{v}$. To be more precise, the domain of the commutation operator

$$[\otimes, \otimes] : (f, g, h) \mapsto (f \otimes g) \otimes h - f(g \otimes h)$$

can be extended to cases for which the terms appearing in the definition are not well-defined separately (see Proposition A.9), so that

$$(1.15) \quad \text{com}_2(v + w) := [\otimes, \otimes](-3(v + w - \Psi), \Psi, \mathbf{v})$$

is well-defined. Our renormalisation rule is thus given by

$$-3[(v + w - \Psi) \otimes \Psi] \otimes \mathbf{v} \rightsquigarrow -3(v + w - \Psi)\Psi + \text{com}_2(v + w),$$

that is,

$$v \otimes \mathbf{v} \rightsquigarrow -3(v + w - \Psi)\Psi + \text{com}(v, w),$$

where

$$(1.16) \quad \text{com}(v, w) := \text{com}_1(v, w) \otimes \mathbf{v} + \text{com}_2(v + w).$$

To sum up, we are interested in solutions of the system

$$(1.17) \quad \begin{cases} (\partial_t - \Delta)v &= F(v + w), \\ (\partial_t - \Delta)w &= G(v, w), \end{cases}$$

where F and G are defined by

$$(1.18) \quad F(v + w) := -3(v + w - \Psi) \otimes \mathbf{v},$$

$$(1.19) \quad G(v, w) := -(v + w)^3 - 3\text{com}(v, w) \\ - 3w \otimes \mathbf{v} - 3(v + w - \Psi) \otimes \mathbf{v} + P(v + w),$$

with

$$(1.20) \quad P(v + w) = a_0 + a_1(v + w) + a_2(v + w)^2, \\ a_0 = b_0 - \Psi \otimes \mathbf{v} + 3\Psi\Psi \quad a_1 = b_1 + 9\Psi, \\ a_2 = b_2$$

with com defined by (1.16), (1.14) and (1.15).

1.2. Renormalised system. We now turn to giving a precise meaning to the discussion of the previous section. From now on, we refer to processes represented by diagrams as “the diagrams”. For such a process, we understand the notion of “being of regularity α ” as meaning that it belongs to $C([0, \infty), \mathcal{B}_\infty^\alpha)$. This definition would have to be modified for ξ and \mathbf{v} , which only make sense as space-time distributions, but we will not refer to these any longer. We refer the reader to Appendix A for the definition and some properties of the Besov spaces \mathcal{B}_p^α . These spaces are more commonly denoted by $\mathcal{B}_{p,q}^\alpha$, but since we do not make use of fine properties encoded by the second integrability index q , we will always set it equal to ∞ and drop it in the notation. For the graphical term Ψ , some additional information on time regularity will be needed. The regularity of v and w will also be measured in norms on the Besov scale, but we will vary the integrability index throughout the article. The content of this article is a global solution theory for the system (1.17), assuming that we control all of the graphical terms in these norms.

Before we pass to developing this theory, we briefly discuss in which way the system (1.17) can be linked to the original equation rigorously, and in particular in which sense the products (and resonant terms) of the graphical terms of low regularity should be interpreted. The diagrams entering our equations for v and w are

$$(1.21) \quad \mathbf{1}, \mathbf{v}, \Psi, \Psi\Psi, \Psi\Psi\Psi,$$

τ	\dagger	\mathbf{V}	Ψ	Ψ	Ψ	Ψ
α_τ	$-\frac{1}{2} - \varepsilon$	$-1 - \varepsilon$	$\frac{1}{2} - \varepsilon$	$-\varepsilon$	$-\frac{1}{2} - \varepsilon$	$-\varepsilon$

TABLE 1. The list of relevant diagrams, together with their regularity exponent, where $\varepsilon > 0$ is arbitrary.

as well as Ψ , which is defined as the solution of (1.12), that is, as a function of \mathbf{V} . These quantities, together with their regularity exponent, are summarized in Table 1.

The two remaining ambiguous terms in our formal derivation, namely $\dagger(\Psi)^2$ and $\Psi\dagger$, can be defined classically in terms of the more fundamental object Ψ . For $\Psi\dagger$, we can set

$$\Psi\dagger := \Psi \odot \dagger + \Psi.$$

As for $\dagger(\Psi)^2$, we only need to define $\dagger \odot (\Psi)^2$. This term can be formally decomposed into

$$2\dagger \odot [\Psi \odot \Psi] + \dagger \odot [\Psi \odot \Psi],$$

and only the first term is ill-defined. The commutator

$$[\odot, \odot](\Psi, \Psi, \dagger)$$

is well-defined, and we can thus set

$$\dagger \odot [\Psi \odot \Psi] := \Psi \Psi + [\odot, \odot](\Psi, \Psi, \dagger),$$

that is,

$$\dagger(\Psi)^2 := \dagger \odot (\Psi)^2 + \dagger \odot [\Psi \odot \Psi] + 2\Psi \Psi + 2[\odot, \odot](\Psi, \Psi, \dagger).$$

In this way, the coefficients a_0 , a_1 and a_2 appearing in (1.20) can be re-expressed as

$$a_0 = m(\dagger - \Psi) + (\Psi)^3 - 3 \left[\dagger \odot (\Psi)^2 + \dagger \odot [\Psi \odot \Psi] + 2\Psi \Psi + 2[\odot, \odot](\Psi, \Psi, \dagger) \right] - 9\Psi \Psi - 3\Psi,$$

$$a_1 = m + 6 \left[\Psi \odot \dagger + \Psi \right] - 3(\Psi)^2 + 9\Psi,$$

$$a_2 = -3\dagger + 3\Psi.$$

Throughout the article, we will never make use of the explicit form of these coefficients, but only that they are of regularity $(-\frac{1}{2})^-$.

A natural approach to construct the diagrams in (1.21) is via regularisation: if ξ is replaced by a smooth approximation ξ_δ , then these terms have a canonical interpretation: One can define $\tilde{\dagger}_\delta$ as the solution to (1.4) with ξ replaced by ξ_δ , $\tilde{\mathbf{V}}_\delta := \tilde{\dagger}_\delta^2$, $\tilde{\Psi}_\delta := \tilde{\dagger}_\delta^3$, and $\tilde{\Psi}_\delta$ and $\tilde{\Psi}_\delta$ as solutions of (1.12) and (1.7) with right hand sides $\tilde{\mathbf{V}}_\delta$ and $\tilde{\Psi}_\delta$. Furthermore, one can then define $\tilde{\Psi}_\delta = \tilde{\Psi}_\delta \odot \tilde{\dagger}_\delta$, $\tilde{\Psi}_\delta := \tilde{\Psi}_\delta \odot \tilde{\mathbf{V}}_\delta$ and $\tilde{\Psi}_\delta := \tilde{\Psi}_\delta \odot \tilde{\mathbf{V}}_\delta$. Finally, if $(\tilde{v}_\delta, \tilde{w}_\delta)$ solves (1.17), with diagrams interpreted in this way, then indeed, $\tilde{X}_\delta = \tilde{\dagger}_\delta - \tilde{\Psi}_\delta + \tilde{v}_\delta + \tilde{w}_\delta$ solves (1.1) (with ξ replaced by ξ_δ).

However, these ‘‘canonical’’ diagrams fail to converge as the regularisation parameter δ is sent to zero. Given their low regularity, this is not surprising. Yet, the first striking fact about renormalisation is that these terms *do* converge in the relevant spaces if they are modified in a rather mild way. Indeed, if we set

$$\dagger_\delta = \tilde{\dagger}_\delta, \quad \mathbf{V}_\delta = \tilde{\mathbf{V}}_\delta - C_\delta^{(1)}, \quad \Psi_\delta = \tilde{\Psi}_\delta - 3C_\delta^{(1)}\tilde{\dagger}_\delta,$$

for a suitable choice of diverging constant $C_\delta^{(1)}$, then define Υ_δ and Ψ_δ as solutions of (1.12) and (1.7) with right hand sides \mathbb{V}_δ and \mathbb{V}_δ , and finally

$$\Psi_{\delta} = \Upsilon_{\delta} \ominus \mathfrak{t}_{\delta}, \quad \Psi_{\delta} := \Upsilon_{\delta} \ominus \mathbb{V}_{\delta} - 3C_{\delta}^{(2)} \mathfrak{t}_{\delta}, \quad \Psi_{\delta} := \Upsilon_{\delta} \ominus \mathbb{V}_{\delta} - C_{\delta}^{(2)}.$$

for another choice of diverging constant $C_\delta^{(2)}$, then these terms converge to non-trivial limiting objects. This is shown in [5], and a very similar result is already contained in [23, Sec. 10]. We stress once more that these results rely heavily on explicit calculations involving variances of the terms involved, which allow to capture stochastic cancellations.

The second striking fact is that the "renormalisation" of these diagrams translates into a simple transformation of the original equation. Indeed, if (v_δ, w_δ) solves (1.17), with diagrams interpreted in the renormalised way, then $X_\delta = \mathfrak{t}_\delta - \Upsilon_\delta + v_\delta + w_\delta$ solves the identical equation (1.1), with ξ replaced by ξ_δ but with *renormalised massive term* $m_\delta := m + 3C_\delta^{(1)} - 9C_\delta^{(2)}$. Since the solution theory for (1.17) is stable under convergence of the diagrams, we can conclude that the solution X_δ to this renormalised equation does converge to a non-trivial limit, denoted by X , as δ tends to 0.

The fact that we have modified the equation we intended to solve may be disconcerting at first. That this modification is the "correct" one is ultimately justified by the fact that the solutions thus defined are indeed the physically relevant ones. In particular, these solutions arise as scaling limits of models of statistical mechanics near criticality. The connexion between renormalised fields and statistical mechanics has been studied at least since the 60s (see e.g. [15, 21, 16] and the references therein). We showed in [30] that the Φ_2^4 model can be obtained as the scaling limit of Ising-Kac models near criticality, as anticipated in [12]. Related results were obtained for the KPZ equation, first in [2] via a Cole-Hopf transformation, and then, following [22], in a series of works including [17, 11, 27, 20, 19, 26]. See also the survey articles [24, 6] for a summary of the work on the Φ^4 model with regularity structures.

1.3. Main result. Our aim is to show that such renormalised solutions of (1.1) are well-defined globally in time. We will not discuss further the convergence of the various diagrams, but only concentrate on the analysis of the deterministic system. Before we do so, we make a modification to the system (1.17). We give ourselves a (large) constant c , and consider instead the system

$$(1.22) \quad \begin{cases} (\partial_t - \Delta)v &= F(v+w) - cv, \\ (\partial_t - \Delta)w &= G(v,w) + cv, \end{cases}$$

with F and G as in (1.18) and (1.19) respectively, and with initial condition

$$(1.23) \quad v(0) = v_0, \quad w(0) = w_0.$$

Naturally, this modification changes the definitions of v and w , but we stress that it does *not* change the sum $v+w$, and therefore the final solution X . This can easily be seen on the level of the regularised solution (v_δ, w_δ) discussed in the previous section. Since (v, w) is the limit of the (v_δ, w_δ) , it follows that $v+w$ itself does not depend on the choice of c . Therefore, it is ultimately enough to show the existence of a constant c for which the system does not blow up. (For the same reason, the solution X depends on v_0 and w_0 only through the sum $v_0 + w_0$.)

Here is our main result.

Theorem 1.1 (Global existence). *Let $\varepsilon > 0$ be sufficiently small, let $\beta = \frac{1}{2} + 2\varepsilon$ and $\gamma = \frac{5}{4} + 2\varepsilon$. For every $K_0 > 0$, there exists $c_0 < \infty$ such that the following*

holds for every $c \geq c_0$. Let $T > 0$, let $\mathfrak{r}, \mathfrak{v}, \mathfrak{Y}, \mathfrak{Y}_\bullet, \mathfrak{Y}_\bullet^\bullet, \mathfrak{Y}_\bullet^\bullet^\bullet$ be any processes such that for every pair (τ, α_τ) as in Table 1, we have

$$(1.24) \quad \tau \in C([0, T], \mathcal{B}_\infty^{\alpha_\tau}), \quad \sup_{0 \leq t \leq T} \|\tau(t)\|_{\mathcal{B}_\infty^{\alpha_\tau}} \leq K_0,$$

as well as

$$\sup_{0 \leq s, t \leq T} \frac{\|\mathfrak{Y}(t) - \mathfrak{Y}(s)\|_{\mathcal{B}_\infty^{\frac{1}{4}-\varepsilon}}}{|t - s|^{\frac{1}{8}}} \leq K_0.$$

For every $(v_0, w_0) \in \mathcal{B}_6^\beta \times \mathcal{B}_2^\gamma$, there exists exactly one pair (v, w) in

$$\left(C([0, T], \mathcal{B}_6^\beta) \cap C^{\frac{1}{8}}([0, T], \mathcal{B}_6^{\beta-\frac{1}{4}}) \right) \times \left(C([0, T], \mathcal{B}_2^\gamma) \cap C^{\frac{1}{8}}([0, T], \mathcal{B}_2^{\gamma-\frac{1}{4}}) \right)$$

solving (1.22) with initial condition (1.23).

Remark 1.2. The notion of solution derived in [5] is closely related to (1.17), but slightly different: There, our ansatz

$$X = \mathfrak{r} - \mathfrak{Y} + v + w$$

is replaced by

$$X = \mathfrak{r} - \mathfrak{Y} + \Phi' \odot \mathfrak{Y} + \Phi^\sharp,$$

and a system of equations for Φ' and the remainder Φ^\sharp is solved. The term $\Phi' \odot \mathfrak{Y}$ in this decomposition corresponds to v up to a commutator term. Although these approaches are very similar, ours makes the equations solved by v and w more explicit.

Remark 1.3. As stated, this theorem might appear to make unnaturally strong restrictions on the choice of initial datum. Indeed, if $(v_0, w_0) \in \mathcal{B}_6^\beta \times \mathcal{B}_2^\gamma$ and if $\mathfrak{r}(0) = \mathfrak{Y}(0) = 0$, then the process X is started with $X_0 = v_0 + w_0 \in \mathcal{B}_6^\beta + \mathcal{B}_2^\gamma$. Given that $X(t)$ takes values in a distributional space of negative regularity (e.g. $\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}$) for all positive $t > 0$, this may appear to be an unreasonable assumption on X_0 . However, this apparent shortcoming can be easily fixed. First of all, our analysis does not rely on the convention to start the diagrams at 0, and other choices such as working with stationary processes would be possible. Second, it is possible to develop a local well-posedness theory for (1.17) for much less regular initial datum (in both [23] and [5], local well-posedness is shown for initial datum $X_0 \in \mathcal{B}_\infty^\alpha$ for any $\alpha > -\frac{2}{3}$). For these solutions, $(v(t), w(t))$ would have the required $\mathcal{B}_6^\beta \times \mathcal{B}_2^\gamma$ regularity for all $t > 0$, and we could use the solution at some small strictly positive time as initial datum in Theorem 1.1.

Remark 1.4. A similar analysis in the simpler two dimensional case was performed in [31]. There, we were able to push the analysis further and show global existence of solutions if the equation is posed on the full space \mathbb{R}^2 . The full-space setting is physically more relevant, but also more difficult to analyse, because the stochastic terms lack any decay at infinity, which mandates an analysis in weighted distribution spaces. Nevertheless, we expect that a solution theory in \mathbb{R}^3 is within reach of the methods presented here combined with those developed in [31].

Remark 1.5. Another very interesting extension of our result would be to obtain bounds which are uniform in time. For now, our final energy estimate, Theorem 6.1, is obtained through a Gronwall-type argument, and thus the constants in the resulting estimate grow exponentially in the time horizon. Obtaining bounds that hold uniformly over time would show the tightness of a Krylov-Bogolyubov scheme based on (1.1). Such bounds would provide an alternative construction of the Φ_3^4 measure (1.2), not appealing to correlation inequalities. Similar bounds for simpler

systems were derived, for example in [28], and we hope that the technique developed there can be combined with ours to yield such a result.

1.4. Organisation of the paper. We present a local existence and uniqueness result in Section 2. The main part of the article, which consists in deriving suitable a priori bounds on solutions to (1.22), is divided into several sections. In Section 3, we show that v can be controlled in terms of w ; in fact, we show that if the constant c is sufficiently large, then a suitable norm of v is controlled by a small multiple of a suitable norm of w . This allows us in effect to reduce the study of the system (1.22) to that of an equation involving w only. In Section 4, we control the time increments of w in terms of various norms, an ingredient made mandatory by the presence of the commutator term com_1 . The core testing argument is given in Section 5. As explained in more details there, the testing argument allows us to replace non-linear terms by linear ones. Once this is done, we can proceed via a Gronwall-type argument in Section 6 to finally obtain a self-contained a priori estimate. A moderate amount of post-processing is then performed in Section 7 to conclude the proof of Theorem 1.1.

2. LOCAL EXISTENCE AND UNIQUENESS

The aim of this section is to provide a local existence and uniqueness result for the system (1.22). The constant $c \in \mathbb{R}$ appearing there is introduced for reasons that will become clear in a later stage of the analysis, but plays no role for the results presented in this section.

We interpret the system (1.22) in the mild sense:

$$(2.1) \quad v(t) = e^{t(\Delta-c)}v_0 + \int_0^t e^{(t-s)(\Delta-c)}F(v(s) + w(s), s) ds,$$

$$(2.2) \quad w(t) = e^{t\Delta}w_0 + \int_0^t e^{(t-s)\Delta}[G(v(s), w(s), s) + cv(s)] ds.$$

A similar local theory was already presented in [5] in a slightly different formulation (see Remark 1.2). The main result of this section is the following.

Theorem 2.1. *Let $\varepsilon > 0$ be sufficiently small, $c \in \mathbb{R}$, $\beta = \frac{1}{2} + 2\varepsilon$, $\gamma = \frac{5}{4} + 2\varepsilon$, and let $\uparrow, \mathfrak{V}, \Psi, \Psi, \Psi, \Psi$ be any processes such that for every pair (τ, α_τ) as in Table 1, we have $\tau \in C([0, \infty), \mathcal{B}_\infty^{\alpha_\tau})$ and for every $T \geq 0$,*

$$(2.3) \quad \sup_{0 \leq t \leq T} \|\tau(t)\|_{\mathcal{B}_\infty^{\alpha_\tau}} \leq K_0(T)$$

as well as

$$(2.4) \quad \sup_{0 \leq s, t \leq T} \frac{\|\Psi(t) - \Psi(s)\|_{\mathcal{B}_\infty^{\frac{1}{4}-\varepsilon}}}{|t-s|^{\frac{1}{8}}} \leq K_0(T),$$

with $K_0(T) < \infty$. For every pair of initial conditions $(v_0, w_0) \in \mathcal{B}_6^\beta \times \mathcal{B}_2^\gamma$, there exists $T^* \in (0, \infty]$ such that the system (2.1)–(2.2) has a unique solution (v, w) in

$$\begin{aligned} \mathfrak{X}_{T^*} := & (C([0, T^*], \mathcal{B}_6^\beta) \cap C^{\frac{1}{8}}([0, T^*], \mathcal{B}_6^{\beta-\frac{1}{4}})) \\ & \times (C([0, T^*], \mathcal{B}_2^\gamma) \cap C^{\frac{1}{8}}([0, T^*], \mathcal{B}_2^{\gamma-\frac{1}{4}})). \end{aligned}$$

Moreover, this time T^* can be chosen maximal, in the sense that either $T^* = \infty$ or $\lim_{t \uparrow T^*} \|v(t)\|_{\mathcal{B}_6^\beta} \vee \|w(t)\|_{\mathcal{B}_2^\gamma} = \infty$.

We start by isolating a bound on the commutator com_1 defined in (1.14), which we will use again in subsequent sections. We introduce the difference operator

$$(2.5) \quad \delta_{st}f := f(t) - f(s).$$

Proposition 2.2 (First commutator estimate). *Let $\varepsilon > 0$, $\beta = \frac{1}{2} + 2\varepsilon$, $p \in [1, \infty]$ and $T > 0$. Under the assumption (2.3)–(2.4), we have for every $(v, w) \in \mathfrak{X}_T$ and $t \in [0, T)$,*

$$\begin{aligned} \|\text{com}_1(v, w)(t)\|_{\mathcal{B}_p^{1+2\varepsilon}} &\lesssim 1 + t^{-\frac{1}{2} + \frac{\beta}{2} - \varepsilon} \|v_0\|_{\mathcal{B}_p^\beta} \\ &\quad + \int_0^t \frac{1}{(t-s)^{1+2\varepsilon - \frac{\beta}{2}}} \left(\|v(s)\|_{\mathcal{B}_p^\beta} + \|w(s)\|_{\mathcal{B}_p^\beta} \right) ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{1+2\varepsilon}} \|\delta_{st}(v+w)\|_{L^p} ds, \end{aligned}$$

where the implicit multiplicative constant depends on ε , p , T and $K_0(T)$.

Proof. Recall the definition of com_1 in (1.14). The contribution of the initial condition v_0 is controlled via Proposition A.13:

$$\|e^{t\Delta}v_0\|_{\mathcal{B}_p^{1+2\varepsilon}} \lesssim t^{-\frac{1}{2} + \frac{\beta}{2} - \varepsilon} \|v_0\|_{\mathcal{B}_p^\beta}.$$

We now introduce the commutation operator

$$(2.6) \quad [e^{t\Delta}, \odot] : (f, g) \mapsto e^{t\Delta}(f \odot g) - f \odot (e^{t\Delta}g),$$

so that

$$(2.7) \quad \begin{aligned} &e^{(t-s)\Delta}[(v+w-\Psi) \odot \mathbf{V}](s) \\ &= (v+w-\Psi)(s) \odot [e^{(t-s)\Delta}\mathbf{V}(s)] + [e^{(t-s)\Delta}, \odot]((v+w-\Psi)(s), \mathbf{V}(s)). \end{aligned}$$

We start by estimating the last term in the sum above. The contribution of Ψ can be estimated using Proposition A.15:

$$\begin{aligned} \left\| \int_0^t [e^{(t-s)\Delta}, \odot] (\Psi(s), \mathbf{V}(s)) ds \right\|_{\mathcal{B}_p^{1+2\varepsilon}} &\lesssim \int_0^t \left\| [e^{(t-s)\Delta}, \odot] (\Psi(s), \mathbf{V}(s)) \right\|_{\mathcal{B}_p^{1+2\varepsilon}} ds \\ &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{4}+2\varepsilon}} ds \lesssim 1. \end{aligned}$$

By the same reasoning, we have

$$\left\| \int_0^t [e^{(t-s)\Delta}, \odot] ((v+w)(s), \mathbf{V}(s)) ds \right\|_{\mathcal{B}_p^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-s)^{\frac{2+3\varepsilon-\beta}{2}}} \|(v+w)(s)\|_{\mathcal{B}_p^\beta} ds.$$

We now turn to the first term in the right-hand side of (2.7), which we will combine with the last term in (1.14). Recalling (1.13), we observe that

$$\begin{aligned} &[(v+w-\Psi) \odot \mathbf{V}](t) - \int_0^t (v+w-\Psi)(s) \odot [e^{(t-s)\Delta}\mathbf{V}(s)] ds \\ &= \int_0^t [\delta_{st}(v+w-\Psi)] \odot [e^{(t-s)\Delta}\mathbf{V}(s)] ds. \end{aligned}$$

By Proposition A.7, the $\|\cdot\|_{\mathcal{B}_p^{1+2\varepsilon}}$ norm of the integral above is bounded by a constant times

$$\int_0^t \|\delta_{st}(v+w-\Psi)\|_{L^p} \|e^{(t-s)\Delta}\mathbf{V}(s)\|_{\mathcal{B}_\infty^{1+2\varepsilon}} ds \lesssim \int_0^t \frac{1}{(t-s)^{1+\frac{3\varepsilon}{2}}} \|\delta_{st}(v+w-\Psi)\|_{L^p} ds,$$

where we used Proposition A.13 and the fact that $\|\mathbf{V}(s)\|_{\mathcal{B}_\infty^{-1-\varepsilon}} \lesssim 1$ in the last step. By the assumption of Hölder regularity in time on Ψ (with exponent $\frac{1}{8}$), this last integral is bounded by a constant times

$$1 + \int_0^t \frac{1}{(t-s)^{1+\frac{3\varepsilon}{2}}} \|\delta_{st}(v+w)\|_{L^p} ds,$$

which completes the proof. \square

Proof of Theorem 2.1. We follow the usual strategy to first solve the system for some small but strictly positive $T \in (0, 1]$ using a Picard iteration. In a second step solutions are restarted iteratively to obtain maximal solutions.

We fix $\beta = \frac{1}{2} + 2\varepsilon$ and $\gamma = \frac{5}{4} + 2\varepsilon$. For every $T > 0$ and $M > 0$, we define the ball

$$\mathfrak{X}_{T,M} := \{(v, w) \in \mathfrak{X}_T : \|(v, w)\|_{\mathfrak{X}_T} \leq M\},$$

where

$$\|(v, w)\|_{\mathfrak{X}_T} := \max \left\{ \sup_{0 \leq t \leq T} \|v(t)\|_{\mathcal{B}_6^\beta}, \sup_{0 \leq s, t \leq T} \frac{\|v(t) - v(s)\|_{\mathcal{B}_6^{\beta-\frac{1}{4}}}}{|t-s|^{\frac{1}{8}}}, \right. \\ \left. \sup_{0 \leq t \leq T} \|w(t)\|_{\mathcal{B}_2^\gamma}, \sup_{0 \leq s, t \leq T} \frac{\|w(t) - w(s)\|_{\mathcal{B}_2^{\gamma-\frac{1}{4}}}}{|t-s|^{\frac{1}{8}}} \right\}.$$

Furthermore, we denote by Ψ the fixed point map, i.e. the mapping which associates to $(v, w) \in \mathfrak{X}_T$ the function $t \mapsto (\Psi^V[v, w], \Psi^W[v, w])(t)$, where

$$\Psi^V[v, w](t) = e^{t(\Delta-c)} v_0 + \int_0^t e^{(t-s)(\Delta-c)} F(v(s) + w(s), s) ds, \\ \Psi^W[v, w](t) = e^{t(\Delta-c)} w_0 + \int_0^t e^{(t-s)(\Delta-c)} G(v(s), w(s), s) ds.$$

We now show that for a suitable M and for T small enough, Ψ maps $\mathfrak{X}_{T,M}$ into itself. The core ingredients are the following bounds, which we formulate as a lemma.

Lemma 2.3. *There exists a constant C depending only on c and $K_0(1)$ (defined in the assumption of Theorem 2.1) such that the following holds. For every $M > 0$, $T \in (0, 1]$, $(v, w) \in \mathfrak{X}_{M,T}$ and $s \in [0, T]$, we have*

$$(2.8) \quad \|F(v(s) + w(s), s)\|_{\mathcal{B}_6^{-1-\varepsilon}} \leq C(M+1),$$

and the term G can be split into $G(v(s), w(s), s) = G_1(s) + G_2(s)$, with

$$(2.9) \quad \|G_1(s)\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} \leq C(1+M^3),$$

$$(2.10) \quad \|G_2(s)\|_{L^2} \leq C\left(1 + s^{-\frac{1}{4}} M\right).$$

The proof of this lemma is deferred to the end of the proof. We first use it to establish that Ψ maps $\mathfrak{X}_{T,M}$ into itself.

We start by deriving bounds on Ψ^V . Using Proposition A.13 and (2.8), we get that for every $t \leq T$,

$$\|\Psi^V[v, w](t)\|_{\mathcal{B}_6^\beta} \leq \|e^{t(\Delta-c)} v_0\|_{\mathcal{B}_6^\beta} + \int_0^t \frac{1}{(t-s)^{\frac{\beta+1+\varepsilon}{2}}} \|F(v(s) + w(s), s)\|_{\mathcal{B}_6^{-1-\varepsilon}} ds \\ \lesssim \|v_0\|_{\mathcal{B}_6^\beta} + t^{1-\frac{\beta+1+\varepsilon}{2}} (M+1).$$

To bound time differences, we make use of the identity

$$\begin{aligned} \Psi^V[v, w](t) - \Psi^V[v, w](s) &= (e^{(t-s)(\Delta-c)} - \text{Id})e^{s(\Delta-c)}v_0 \\ &\quad + (e^{(t-s)(\Delta-c)} - \text{Id}) \int_0^s e^{(s-r)(\Delta-c)} F(v(r) + w(r), r) dr \\ &\quad + \int_s^t e^{(t-r)(\Delta-c)} F(v(r) + w(r), r) dr, \end{aligned}$$

which holds for any $0 \leq s \leq t$. This allows us to write, using Proposition A.13 and (2.8) again,

$$\begin{aligned} &\|\Psi^V[v, w](t) - \Psi^V[v, w](s)\|_{\mathcal{B}_6^{\beta-\frac{1}{4}}} \\ &\lesssim (t-s)^{\frac{1}{8}} \|v_0\|_{\mathcal{B}_6^\beta} + (t-s)^{\frac{1}{8}} t^{1-\frac{\beta+1+\varepsilon}{2}} \sup_{0 \leq r \leq s} \|F(v(r) + w(r), r)\|_{\mathcal{B}_6^{-1-\varepsilon}} \\ &\quad + (t-s)^{\frac{1}{8}+1-\frac{\beta+1+\varepsilon}{2}} \sup_{s \leq r \leq t} \|F(v(r) + w(r), r)\|_{\mathcal{B}_6^{-1-\varepsilon}} \\ &\lesssim (t-s)^{\frac{1}{8}} (\|v_0\|_{\mathcal{B}_6^\beta} + t^{1-\frac{\beta+1+\varepsilon}{2}} (M+1)). \end{aligned}$$

The argument for Ψ^W is similar: replacing (2.8) by (2.9)–(2.10) and adjusting the exponents, we get

$$\begin{aligned} \|\Psi^W[v, w](t)\|_{\mathcal{B}_2^\gamma} &\leq \|e^{t(\Delta-c)}w_0\|_{\mathcal{B}_2^\gamma} + \int_0^t \frac{1}{(t-s)^{\frac{\gamma}{2}+\frac{1}{4}+\varepsilon}} \|G_1(s)\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{\frac{\gamma}{2}}} \|G_2(s)\|_{L^2} ds \\ &\lesssim \|w_0\|_{\mathcal{B}_6^\beta} + t^{1-(\frac{\gamma}{2}+\frac{1}{4}+\varepsilon)} (M^3+1) + t^{1-(\frac{\gamma}{2}+\frac{1}{4})} (1+M), \end{aligned}$$

and

$$\begin{aligned} (2.11) \quad &\|\Psi^W[v, w](t) - \Psi^W[v, w](s)\|_{\mathcal{B}_2^{\gamma-\frac{1}{4}}} \\ &\lesssim (t-s)^{\frac{1}{8}} \|w_0\|_{\mathcal{B}_2^\gamma} + (t-s)^{\frac{1}{8}} t^{1-(\frac{\gamma}{2}+\frac{1}{4}+\varepsilon)} \sup_{0 \leq r \leq s} \|G_1(r)\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} \\ &\quad + (t-s)^{\frac{1}{8}+1-(\frac{\gamma}{2}+\frac{1}{4}+\varepsilon)} \sup_{s \leq r \leq t} \|G_1(r)\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} \\ &\quad + (t-s)^{\frac{1}{8}} t^{1-(\frac{\gamma}{2}+\frac{1}{4})} \sup_{0 \leq r \leq s} r^{\frac{1}{4}} \|G_2(r)\|_{L^2} \\ &\quad + (t-s)^{\frac{1}{8}+1-(\frac{\gamma}{2}+\frac{1}{4})} \sup_{s \leq r \leq t} r^{\frac{1}{4}} \|G_2(r)\|_{L^2} \\ &\lesssim (t-s)^{\frac{1}{8}} (\|w_0\|_{\mathcal{B}_6^\beta} + t^{1-(\frac{\gamma}{2}+\frac{1}{4}+\varepsilon)} (M^3+1)). \end{aligned}$$

Note also that for $\varepsilon > 0$ sufficiently small, we have

$$\frac{\beta+1+\varepsilon}{2} \leq \frac{9}{10} \quad \text{and} \quad \frac{\gamma}{2} + \frac{1}{4} + \varepsilon \leq \frac{9}{10}.$$

Summarising, we conclude that there exists a constant C^* which depends only on $K_0(1)$ (whose value we momentarily want to remember) such that for all $T \leq 1$, $(v, w) \in \mathfrak{X}_{M,T}$ and $M \geq 1$, we have

$$\|(\Psi^V[v, w], \Psi^W[v, w])\|_{\mathfrak{X}_T} \leq C^* \max\{1, \|v_0\|_{\mathcal{B}_6^\beta}, \|w_0\|_{\mathcal{B}_2^\gamma}, T^{\frac{1}{10}} M^3\}.$$

Hence, if we choose $M = C^* \max\{1, \|v_0\|_{\mathcal{B}_6^\beta}, \|w_0\|_{\mathcal{B}_2^\gamma}\}$ and $T = (C^* M^2)^{-10}$, we can conclude that Ψ indeed maps $\mathfrak{X}_{M,T}$ into itself. The fact that it is also a contraction on this ball can be established with the same method and we omit the proof.

At this point, we can conclude that for every initial data (v_0, w_0) and every choice of processes τ satisfying (2.3), there exists a strictly positive time $0 < T_1 \leq 1$ such that (2.1)–(2.2) has a unique solution over $[0, T_1]$. Furthermore, any upper bound on $\|v_0\|_{\mathcal{B}_6^\beta}$, $\|w_0\|_{\mathcal{B}_2^\gamma}$ and $K_0(1)$ provides a lower bound on T_1 . Our argument also implies that $\|v(T_1)\|_{\mathcal{B}_6^\beta} < \infty$ and $\|w(T_1)\|_{\mathcal{B}_2^\gamma} < \infty$, which makes these functions admissible initial conditions to repeat the argument to obtain solutions on $[0, T_1 + T_2]$ for some strictly positive T_2 . By induction, we can extend solutions up to a maximal time $T^* = \sum_{k=1}^{\infty} T_k$. By the previous observation, the time T^* can only be finite if at least one of the quantities $\|v(t)\|_{\mathcal{B}_6^\beta}$, $\|w(t)\|_{\mathcal{B}_2^\gamma}$,

$$(2.12) \quad \sup_{t \leq r \leq t+1} \|\tau(r)\|_{\mathcal{B}_\infty^{\alpha_\tau}}, \quad \text{or} \quad \sup_{t \leq r_1, r_2 \leq t+1} \frac{\|\Psi(r_1) - \Psi(r_2)\|_{\mathcal{B}_\infty^{\frac{1}{4}-\varepsilon}}}{|r_1 - r_2|^{\frac{1}{8}}}$$

blows up as $t \uparrow T^*$. But by assumption, the quantities in (2.12) are bounded on any compact interval, which excludes their blowup.

There remains to argue about uniqueness of solutions to the system (2.1)–(2.2). This follows from the local contractivity of the fixed point map by classical arguments (see e.g. Step 3 of the proof of [31, Theorem 6.2]). \square

Proof of Lemma 2.3. According to the definition of F in (1.18) and Proposition A.7, we have (dropping the time argument s to lighten the notation)

$$\|F(v+w)\|_{\mathcal{B}_6^{-1-\varepsilon}} = 3\|(v+w - \Psi) \odot \mathbf{v}\|_{\mathcal{B}_6^{-1-\varepsilon}} \lesssim \|v+w - \Psi\|_{L^6} \|\mathbf{v}\|_{\mathcal{B}_\infty^{-1-\varepsilon}}.$$

The estimate (2.8) then follows from the fact that $\|v\|_{L^6}$ is controlled by $\|v\|_{\mathcal{B}_6^\beta}$, that according to Proposition A.2 and Remark A.3, $\|w\|_{L^6}$ is controlled by $\|w\|_{\mathcal{B}_2^\gamma}$ and that $\|\Psi\|_{L^6}$ is controlled by $\|\Psi\|_{\mathcal{B}_2^{\frac{1}{2}-3\varepsilon}}$.

In order to verify (2.9) and (2.10), we set (recalling the definition of G in (1.19))

$$\begin{aligned} G_1(s) &= -(v+w)^3 - 3w \odot \mathbf{v} - 3(v+w - \Psi) \odot \mathbf{v} + P(v+w), \\ G_2(s) &= -3\text{com}(v, w), \end{aligned}$$

where the polynomial P is defined in (1.20). We proceed by using the triangle inequality and bounding the terms on the right hand side of these expressions one by one. For G_1 , the least regular term is the term $a_2(v+w)^2$ arising in the polynomial P . We use Proposition A.7 to bound this term:

$$\begin{aligned} \|a_2(v+w)^2\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} &\lesssim \|a_2(v+w)^2\|_{\mathcal{B}_2^{-\frac{1}{2}-\varepsilon}} \\ &\lesssim \|a_2\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} \|(v+w)^2\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}} \\ &\lesssim \|a_2\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} (\|v\|_{\mathcal{B}_4^{\frac{1}{2}+2\varepsilon}}^2 + \|w\|_{\mathcal{B}_4^{\frac{1}{2}+2\varepsilon}}^2), \end{aligned}$$

which is bounded by $C(1+M^2)$. Indeed, this follows from the assumption that $(v, w) \in \mathfrak{X}_{M,T}$, the obvious comparison $\|v\|_{\mathcal{B}_4^{\frac{1}{2}+2\varepsilon}} \lesssim \|v\|_{\mathcal{B}_6^{\frac{1}{2}+2\varepsilon}}$ (recall that $\beta = \frac{1}{2} + 2\varepsilon$) and the fact that by Proposition A.2,

$$\|w\|_{\mathcal{B}_4^{\frac{1}{2}+2\varepsilon}} \lesssim \|w\|_{\mathcal{B}_2^{\frac{5}{4}+2\varepsilon}}.$$

(This is where our choice of the exponent γ is critical.)

For the remaining terms in the polynomial P , we observe that by Propositions A.7 and A.9,

$$(2.13) \quad \|a_0\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} \leq C \quad \text{and} \quad \|a_1\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} \leq C.$$

Hence, using Proposition A.7 again, we obtain

$$\|a_1(v+w) + a_0\|_{\mathcal{B}_2^{-\frac{1}{2}-\varepsilon}} \lesssim \|a_1\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} \|v+w\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}} + \|a_0\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} \leq C(1+M).$$

Another rather irregular term is that given by

$$\|3(v+w - \Psi) \odot \mathbf{V}\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} \lesssim (\|v\|_{\mathcal{B}_2^{\frac{1}{2}-\varepsilon}} + \|w\|_{\mathcal{B}_2^{\frac{1}{2}-\varepsilon}} + \|\Psi\|_{\mathcal{B}_2^{\frac{1}{2}-\varepsilon}}) \|\mathbf{V}\|_{\mathcal{B}_\infty^{-1-\varepsilon}},$$

where we used Proposition A.7 once more.

The remaining terms appearing in the definition of G_1 can be bounded in stronger norms. Indeed, we have

$$\|(v+w)^3\|_{L^2} \lesssim \|v\|_{L^6}^3 + \|w\|_{L^6}^3 \lesssim M^3,$$

where we used the fact that $\|v\|_{L^6} \lesssim \|v\|_{\mathcal{B}_6^\beta}$ as well as the embedding $\|w\|_{L^6} \lesssim \|w\|_{\mathcal{B}_2^\gamma}$ provided by Proposition A.2 and Remark A.3 (since $\gamma > 1$). The only term left to control is

$$\|3w \odot \mathbf{V}\|_{L^2} \lesssim \|w\|_{\mathcal{B}_2^{1+2\varepsilon}} \|\mathbf{V}\|_{\mathcal{B}_\infty^{-1-\varepsilon}} \lesssim M.$$

This completes the proof of (2.9).

We now turn to the proof of (2.10). We recall that according to (1.16), we have

$$\text{com}(v, w) = \text{com}_1(v, w) \odot \mathbf{V} + \text{com}_2(v, w),$$

and use Proposition A.7, Remark A.3 and Proposition 2.2 to write

$$\begin{aligned} \|\text{com}_1(v, w) \odot \mathbf{V}(s)\|_{L^2} &\lesssim \|\text{com}_1(v, w)(s)\|_{\mathcal{B}_2^{1+2\varepsilon}} \|\mathbf{V}(s)\|_{\mathcal{B}_\infty^{-1-\varepsilon}} \\ &\lesssim 1 + t^{-\frac{1}{4}} \|v_0\|_{\mathcal{B}_2^\beta} + \int_0^t \frac{1}{(t-s)^{\frac{3}{4}+\varepsilon}} \left(\|v(s)\|_{\mathcal{B}_2^\beta} + \|w(s)\|_{\mathcal{B}_2^\beta} \right) ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{1+2\varepsilon}} \|\delta_{st}(v+w)\|_{L^2} ds \\ &\lesssim 1 + t^{-\frac{1}{4}} M + t^{\frac{1}{4}-\varepsilon} M + t^{\frac{1}{8}-2\varepsilon} M, \end{aligned}$$

where in the last step we have used the assumption $(v, w) \in \mathfrak{X}_{M,T}$. Note in particular that we have made use of the control on the Hölder regularity in time of (v, w) in order to treat the last integral. For the second commutator term (defined in (1.15)), we use Proposition A.9 to obtain

$$\|\text{com}_2(v, w)\|_{L^2} \lesssim 1 + \|v+w\|_{\mathcal{B}_2^{3\varepsilon}} \lesssim 1 + M.$$

This completes the argument for (2.10). \square

3. A PRIORI ESTIMATE ON v

Before starting to derive a priori estimates for solutions of (1.22), we state some

Important conventions. We list a certain number of quantities that will remain fixed throughout the rest of the paper. We fix

$$(3.1) \quad \beta = \frac{1}{2} + 2\varepsilon, \quad \gamma = \frac{5}{4} + 2\varepsilon,$$

where $\varepsilon > 0$ will need to be chosen sufficiently small. We give ourselves $T_{\max} \in (0, \infty)$, and processes $\mathfrak{t}, \mathfrak{v}, \Psi, \Psi_1, \Psi_2, \Psi_3$ such that for every pair (τ, α_τ) as in Table 1 and for $\eta = \frac{1}{8}$, we have

$$\tau \in C([0, \infty), \mathcal{B}_\infty^{\alpha_\tau}), \quad \sup_{0 \leq t \leq T_{\max}} \|\tau(t)\|_{\mathcal{B}_\infty^{\alpha_\tau}} \leq K_0(T_{\max}),$$

as well as

$$\sup_{0 \leq s, t \leq T_{\max}} \frac{\|\Psi(t) - \Psi(s)\|_{\mathcal{B}_{\infty}^{\frac{1}{4}-\varepsilon}}}{|t-s|^{\frac{1}{8}}} \leq K_0(T_{\max}),$$

for some $K_0(T_{\max}) < \infty$. We also give ourselves an initial condition $(v_0, w_0) \in \mathcal{B}_6^{\beta} \times \mathcal{B}_2^{\gamma}$ and a solution (v, w) to (2.1)–(2.2) on a maximal time interval $[0, T^*)$, as given by Theorem 2.1. We write $T = T^* \wedge T_{\max}$.

(3.2) In the inequalities \lesssim of Sections 3 to 6,
the implicit constant does not depend on (v_0, w_0) .

Our final aim is to establish that if the constant c is sufficiently large (depending on $K_0(T_{\max})$), then

$$\lim_{t \uparrow T} \|v(t)\|_{\mathcal{B}_6^{\beta}} \vee \|w(t)\|_{\mathcal{B}_2^{\gamma}} < \infty.$$

In view of the maximality property of T^* , this shows that $T^* > T_{\max}$, as desired.

In this section, we focus on deriving an a priori estimate on v . This estimate becomes better as c increases. Roughly speaking, taking c sufficiently large will effectively enable us to reduce the study of the system (1.22) to an equation on w only. We recall our notation $\delta_{st}v = v(t) - v(s)$.

Theorem 3.1 (A priori estimate on v). *Let $\varepsilon > 0$, $\beta' \in (0, 1 - 2\varepsilon)$, $p \in (1, \infty)$,*

$$(3.3) \quad \sigma = \frac{\beta' + 1 + \varepsilon}{2} \quad \text{and} \quad \underline{c} = c - 1 - [\Gamma(1 - \sigma)]^{1/(1-\sigma)},$$

where Γ is Euler's Gamma function. For every $s \leq t \in [0, T]$,

$$(3.4) \quad \|v(t)\|_{\mathcal{B}_p^{\beta'}} \lesssim e^{-\underline{c}t} \|v_0\|_{\mathcal{B}_p^{\beta'}} + \int_0^t \frac{e^{-\underline{c}(t-u)}}{(t-u)^{\sigma}} (1 + \|w(u)\|_{L^p}) \, du,$$

$$(3.5) \quad \|\delta_{st}v\|_{L^p} \lesssim |t-s|^{\frac{\beta'-\varepsilon}{2}} \|v(s)\|_{\mathcal{B}_p^{\beta'}} + \int_s^t \frac{e^{-\underline{c}(t-u)}}{(t-u)^{\frac{1}{2}+\varepsilon}} (1 + \|w(u)\|_{L^p}) \, du,$$

where the implicit constants depend on ε, p, β' and $K_0(T_{\max})$, but not on $c \in \mathbb{R}$.

Remark 3.2. In view of the proof below and of Remarks A.3 and A.14, we also have

$$\|v(t)\|_{L^p} \lesssim e^{-\underline{c}t} \|v_0\|_{L^p} + \int_0^t \frac{e^{-\underline{c}(t-s)}}{(t-s)^{\frac{1}{2}+\varepsilon}} (1 + \|w(s)\|_{L^p}) \, ds.$$

Remark 3.3. It is also straightforward to see using Proposition A.13 that for any given $\eta \geq 0$, one can replace the term $\|v_0\|_{\mathcal{B}_p^{\beta'}}$ in (3.4) by $t^{-\frac{\eta}{2}} \|v_0\|_{\mathcal{B}_p^{\beta'-\eta}}$. (The implicit constant then depends on η .)

Proof of Theorem 3.1. By Proposition A.13, the first term in the right-hand side of (2.1) is estimated by

$$(3.6) \quad \|e^{t(\Delta-c)} v_0\|_{\mathcal{B}_p^{\beta'}} \lesssim e^{-ct} \|v_0\|_{\mathcal{B}_p^{\beta'}}.$$

As for the second term in the right-hand side of (2.1), by Proposition A.13,

$$\left\| \int_0^t e^{(t-s)(\Delta-c)} F(v+w, s) \, ds \right\|_{\mathcal{B}_p^{\beta'}} \lesssim \int_0^t \frac{e^{-c(t-s)}}{(t-s)^{\sigma}} \|F(v+w, s)\|_{\mathcal{B}_p^{-1-\varepsilon}} \, ds.$$

Recall the definition of F in (1.18). By Proposition A.7,

$$\|[(v+w - \Psi) \odot \mathbf{V}](s)\|_{\mathcal{B}_p^{-1-\varepsilon}} \lesssim \|(v+w - \Psi)(s)\|_{L^p} \lesssim \|v(s)\|_{\mathcal{B}_p^{\beta'}} + \|w(s)\|_{L^p} + 1.$$

Hence,

$$\|v(t)\|_{\mathcal{B}_p^{\beta'}} \lesssim e^{-ct} \|v_0\|_{\mathcal{B}_p^{\beta'}} + \int_0^t \frac{e^{-c(t-s)}}{(t-s)^{\sigma}} \left(1 + \|w(s)\|_{L^p} + \|v(s)\|_{\mathcal{B}_p^{\beta'}}\right) \, ds.$$

Inequality (3.4) then follows using the Gronwall-type Lemma 3.4 proved below.

We now turn to (3.5). By homogeneity in time of the equation, it suffices to show (3.5) for $s = 0$. By Remark A.3, we have $\|\cdot\|_{L^p} \lesssim \|\cdot\|_{\mathcal{B}_p^\varepsilon}$, so

$$\|v(t) - e^{t(\Delta-c)}v_0\|_{L^p} \lesssim \int_0^t \frac{e^{-c(t-s)}}{(t-s)^{\frac{1}{2}+\varepsilon}} \|F(v+w, s)\|_{\mathcal{B}_p^{-1-\varepsilon}} ds.$$

By Proposition A.13 and Remark A.3,

$$\|(1 - e^{t(\Delta-c)})v_0\|_{L^p} \lesssim t^{\frac{\beta'-\varepsilon}{2}} \|v_0\|_{\mathcal{B}_p^{\beta'}}.$$

Hence,

$$(3.7) \quad \|v(t) - v_0\|_{L^p} \lesssim t^{\frac{\beta'-\varepsilon}{2}} \|v_0\|_{\mathcal{B}_p^{\beta'}} + \int_0^t \frac{e^{-c(t-s)}}{(t-s)^{\frac{1}{2}+\varepsilon}} \|F(v+w, s)\|_{\mathcal{B}_p^{-1-\varepsilon}} ds.$$

By the same argument as above,

$$\|F(v+w, s)\|_{\mathcal{B}_p^{-1-\varepsilon}} \lesssim \|v(s)\|_{\mathcal{B}_p^\varepsilon} + \|w(s)\|_{L^p} + 1,$$

and by (3.4),

$$\|v(s)\|_{\mathcal{B}_p^\varepsilon} \lesssim e^{-cs} \|v_0\|_{\mathcal{B}_p^\varepsilon} + \int_0^s \frac{e^{-c(s-u)}}{(s-u)^{\frac{1}{2}+\varepsilon}} (1 + \|w(u)\|_{L^p}) du.$$

Inserting this estimate in (3.7), we are left estimating

$$\begin{aligned} & \int_0^t \frac{e^{-c(t-s)}}{(t-s)^{\frac{1}{2}+\varepsilon}} \int_0^s \frac{e^{-c(s-u)}}{(s-u)^{\frac{1}{2}+\varepsilon}} (1 + \|w(u)\|_{L^p}) du ds \\ & \leq \int_0^t e^{-c(t-u)} (1 + \|w(u)\|_{L^p}) \int_u^t \frac{1}{(t-s)^{\frac{1}{2}+\varepsilon} (s-u)^{\frac{1}{2}+\varepsilon}} ds du. \end{aligned}$$

The last integral is bounded by a constant times $(t-u)^{-2\varepsilon}$, so the proof is complete. \square

Lemma 3.4 (Gronwall-type lemma). *Let $0 < \sigma < 1$, $c \in \mathbb{R}$ and $k(s) = e^{-cs} s^{-\sigma} \mathbf{1}_{s>0}$. Assume that $f, g, h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are locally bounded measurable functions such that for every $t \geq 0$,*

$$f(t) \leq g(t) + \int_0^t k(t-s)(h(s) + f(s)) ds.$$

Then for every $t \geq 0$,

$$(3.8) \quad f(t) \leq g(t) + \int_0^t K(t-s)(g(s) + h(s)) ds,$$

where

$$K(s) = \frac{e^{-cs}}{s^\sigma} \sum_{n=0}^{+\infty} \frac{[\Gamma(1-\sigma)]^{n+1}}{\Gamma[(n+1)(1-\sigma)]} s^{n(1-\sigma)}.$$

Moreover,

$$(3.9) \quad \frac{1}{s} \log \left(\sum_{n=0}^{+\infty} \frac{[\Gamma(1-\sigma)]^{n+1}}{\Gamma[(n+1)(1-\sigma)]} s^{n(1-\sigma)} \right) \xrightarrow{s \rightarrow +\infty} [\Gamma(1-\sigma)]^{1/(1-\sigma)}.$$

Proof. Note that by iterating the hypothesis once,

$$\begin{aligned} f(t) & \leq g(t) + \int_0^t k(t-t_1) \left(h(t_1) + g(t_1) + \int_0^{t_1} k(t_1-t_2)h(t_2) dt_2 \right) dt_1 \\ & \quad + \int_{0 \leq t_2 \leq t_1 \leq t} k(t-t_1)k(t_1-t_2)f(t_2) dt_2 dt_1. \end{aligned}$$

We introduce some notation that will allow to iterate further. For every integer $n \geq 0$, we let

$$K^{(n)}(t_0, t_{n+1}) = \int_{t_0 \leq \dots \leq t_{n+1}} k(t_{n+1} - t_n) \cdots k(t_1 - t_0) dt_1 \cdots dt_n$$

(with $K^{(0)}(s, t) = k(t - s)$). By induction,

$$(3.10) \quad \begin{aligned} f(t) \leq g(t) &+ \left(\sum_{n=0}^{N-1} \int_0^t K^{(n)}(s, t) (g + h)(s) ds \right) \\ &+ \int_0^t K^{(N)}(s, t) (h(s) + f(s)) ds. \end{aligned}$$

The kernels satisfy

$$K^{(n)}(t_0, t_{n+1}) = e^{-c(t_{n+1} - t_0)} \int_{t_0 \leq \dots \leq t_{n+1}} (t_{n+1} - t_n)^{-\sigma} \cdots (t_1 - t_0)^{-\sigma} dt_1 \cdots dt_n.$$

A change of variables enables to rewrite the integral above as

$$\begin{aligned} &\int_{s_1 + \dots + s_n \leq t_{n+1} - t_0} s_1^{-\sigma} \cdots s_n^{-\sigma} (t_{n+1} - t_0 - s_1 - \dots - s_n)^{-\sigma} ds_1 \cdots ds_n \\ &= (t_{n+1} - t_0)^{n(1-\sigma) - \sigma} \int_{s_1 + \dots + s_n \leq 1} s_1^{-\sigma} \cdots s_n^{-\sigma} (1 - s_1 - \dots - s_n)^{-\sigma} ds_1 \cdots ds_n \end{aligned}$$

(the condition $s_i > 0$ is kept implicit). The latter integral is the beta function evaluated at $(1 - \sigma, \dots, 1 - \sigma)$, and is equal to

$$\frac{[\Gamma(1 - \sigma)]^{n+1}}{\Gamma[(n+1)(1 - \sigma)]}.$$

(In fact, one can check this by computing the L^1 norm of the n -fold convolution of the function $s \mapsto e^{-s} s^{-\sigma} \mathbf{1}_{s>0}$.) To sum up, we have shown that

$$K^{(n)}(s, t) = e^{-c(t-s)} (t-s)^{n(1-\sigma) - \sigma} \frac{[\Gamma(1 - \sigma)]^{n+1}}{\Gamma[(n+1)(1 - \sigma)]}.$$

This proves that the remainder term in (3.10) tends to 0 as N tends to infinity, and yields (3.8). In order to check (3.9), we use the fact that for $x \geq 1$,

$$\sum_{n=0}^{+\infty} \frac{x^{n(1-\sigma)}}{\Gamma[(n+1)(1-\sigma)]} \leq \sum_{n=0}^{+\infty} \frac{x^{\lfloor n(1-\sigma) \rfloor + 1}}{[n(1-\sigma)]!} \leq \left\lfloor \frac{1}{1-\sigma} + 1 \right\rfloor x e^x.$$

Since $\Gamma[(n+1)(1-\sigma)] = [(n+1)(1-\sigma)]^{-1} \Gamma[(n+1)(1-\sigma) + 1]$, this gives the upper bound for (3.9). Since we will never use the matching lower bound, we simply mention that it follows by evaluating the contribution of the summand indexed by n such that $n(1-\sigma) \simeq s[\Gamma(1-\sigma)]^{1/(1-\sigma)}$. \square

4. A PRIORI ESTIMATE ON $\delta_{st}w$

As was already apparent in Section 2, one difficulty in the analysis of the behaviour of solutions to (1.22) comes from the presence of the first commutator term com_1 in (1.16). Indeed, assessing the (finiteness and) spatial regularity of this term requires information on the *time* regularity of v , w and Ψ . Adequate information on the time regularity of v was obtained in Theorem 3.1, while the time regularity of Ψ is given. The purpose of this section is to derive a bound on $\|\delta_{st}w\|_{L^p}$ in terms of various norms of w . (Recall that $\delta_{st}w = w(t) - w(s)$.)

Theorem 4.1 (A priori estimate on $\delta_{st}w$). *Let $p \in [\frac{8}{7}, \frac{8}{3})$, and $\varepsilon > 0$ be sufficiently small. For every $s \leq t \in [0, T)$,*

$$(4.1) \quad \|\delta_{st}w\|_{L^p} \lesssim (t-s)^{\frac{1}{8}} \times \left[1 + \|v_0\|_{\mathcal{B}_{3p}^\beta}^3 + \|w(s)\|_{\mathcal{B}_p^\beta} + \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} du \right)^{\frac{1}{p}} \right. \\ \left. + \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}} + \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{1}{2}} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{3}{4}} \right],$$

where the implicit constant depends on ε , p , T_{\max} , $K_0(T_{\max})$ and c .

Before turning to the proof, we briefly explain why we choose to measure $\delta_{st}w$ with the quantities appearing on the right-hand side of (4.1). To begin with, the terms $\|w(s)\|_{\mathcal{B}_p^\beta}$ and $\left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}}$ are linear in their dependence in w (by “linear”, we simply mean that these quantities are 1-homogeneous, i.e. replacing w by λw for some $\lambda > 0$ changes the value of each of these quantities by a multiplicative factor λ); heuristically, their presence should not cause a blow-up. The other terms are non-linear in their dependence in w , and are thus more menacing. They appear because at this stage, we cannot make use of the fact that the leading term $-w^3$ in the definition of G has the right sign. In the next section, we test the equation for w against $|w|^{3p-4}w$ to benefit from this. This will give us control of $\|w(s)\|_{L^{3p}}$. In the case $p = \frac{4}{3}$ (when we test against w), we also gain control of $\|w(s)\|_{\mathcal{B}_2^1}$; in the case $p = 2$ (when we test against w^3), we gain control of $\|w^2(s)\|_{\mathcal{B}_2^1}$.

We introduce

$$\delta'_{st}w := w(t) - e^{(t-s)\Delta} w(s),$$

so that

$$\delta'_{st}w = \int_s^t e^{(t-u)\Delta} [G(v, w) + cv](u) du.$$

The core of the proof of Theorem 4.1 focuses on the estimation of the L^p norm of $\delta'_{st}w$. We then derive an estimate of $\|\delta_{st}w\|_{L^p}$ at the last step, which makes the term $\|w(s)\|_{\mathcal{B}_p^\beta}$ appear.

Recall the definition of G in (1.19) (see also (1.16)). There are several terms in G which require special attention: the cubic term $(v+w)^3$ has the highest degree. As was already said, for now we cannot make use of the “good” sign of this term, but only treat it as a “bad” term. This makes the cubic non-linearities in (4.1) appear. The estimation of $\text{com}_1(v, w)$ involves $\|\delta'_{st}w\|_{L^p}$ itself; we will derive an estimate of the form

$$\|\delta'_{st}w\|_{L^p} \lesssim (t-s)^{\frac{1}{8}} \left[\left(\sup_{u' \leq u \leq t} \frac{\|\delta'_{u'u}w\|_{L^p}}{|u-u'|^{\frac{1}{8}}} \right)^{1/2} (\dots) + \dots \right],$$

where \dots are quantities that do not involve $\delta'_{st}w$, so that an explicit estimate on $\|\delta'_{st}w\|_{L^p}$ follows. The term involving $w \odot v$ is the only term which requires to control derivatives of w of order higher than one. This is the reason for the appearance of the term $\left(\int_0^t \|w\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}}$ on the right-hand side of (4.1). Finally, the term $a_2(v+w)^2$ (the quadratic term in the polynomial $P(v+w)$ defined in (1.20)) involves controlling the spatial regularity of non-linear quantities of v and w . (Recall that a_2 is a distribution with spatial regularity of order $-\frac{1}{2} - \varepsilon$.) It is this term that causes the restriction $p < \frac{8}{3}$ and makes the last two terms in (4.1) appear. (We also

assume $p \leq 6$ in Lemma 4.4, but this is less essential.) We summarize this as

$$(4.2) \quad \delta'_{st} w = - \int_s^t e^{(t-u)\Delta} (v+w)^3(u) \, du$$

$$(4.3) \quad - 3 \int_s^t e^{(t-u)\Delta} [\mathbf{com}_1(v, w) \ominus \mathfrak{V}](u) \, du$$

$$(4.4) \quad - 3 \int_s^t e^{(t-u)\Delta} [w \ominus \mathfrak{V}](u) \, du$$

$$(4.5) \quad + \int_s^t e^{(t-u)\Delta} [a_2(v+w)^2](u) \, du$$

$$(4.6) \quad + \int_s^t e^{(t-u)\Delta} [\dots](u) \, du,$$

where $[\dots]$ stands for the easier terms left out. We provide bounds on the terms listed in (4.2)–(4.6) in the following lemmas.

Lemma 4.2. *Let $p \in (1, \infty)$ and $\varepsilon > 0$ be sufficiently small. For every $s \leq t \in [0, T)$,*

$$(4.7) \quad \left\| \int_s^t e^{(t-u)\Delta} (v+w)^3(u) \, du \right\|_{L^p} \lesssim (t-s)^{\frac{p-1}{p}} \left(1 + \|v_0\|_{L^{3p}}^{3p} + \int_0^t \|w(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{1}{p}},$$

where the implicit constant depends on ε , p , T_{\max} , $K_0(T_{\max})$ and c .

Proof. We start with the simple estimate

$$\begin{aligned} & \left\| \int_s^t e^{(t-u)\Delta} (v+w)^3(u) \, du \right\|_{L^p} \\ & \lesssim \int_s^t \|(v+w)^3(u)\|_{L^p} \, du \\ & \lesssim (t-s)^{\frac{p-1}{p}} \left(\int_0^t \|(v+w)(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{1}{p}}. \end{aligned}$$

We learn from Theorem 3.1 (in fact, Remark 3.2) that

$$\|v(u)\|_{L^{3p}} \lesssim \|v_0\|_{L^{3p}} + \int_0^u \frac{1}{(u-s)^\sigma} (1 + \|w(s)\|_{L^{3p}}) \, ds,$$

for $\sigma = \frac{1}{2} + \varepsilon$. We can focus on bounding

$$\int_0^t \left(\int_0^u \frac{1}{(u-s)^\sigma} (1 + \|w(s)\|_{L^{3p}}) \, ds \right)^{3p} \, du.$$

By Jensen's inequality, the quantity above is bounded by a constant times

$$\int_0^t \int_0^u \frac{1}{(u-s)^\sigma} \left(1 + \|w(s)\|_{L^{3p}}^{3p} \right) \, ds \, du \lesssim 1 + \int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds.$$

Summarizing, we obtain (4.7). \square

Lemma 4.3 (Estimating com_1). *Let $p \in (1, \infty)$ and $\varepsilon > 0$ be sufficiently small. For every $t \in [0, T)$,*

$$(4.8) \quad \begin{aligned} \|\text{com}_1(v, w)(t)\|_{\mathcal{B}_p^{1+2\varepsilon}} &\lesssim 1 + t^{-\frac{1-\beta}{2}-\varepsilon} \|v_0\|_{\mathcal{B}_p^\beta} \\ &\quad + \int_0^t \frac{1}{(t-s)^{1-\frac{\beta}{2}+2\varepsilon}} \|w(s)\|_{\mathcal{B}_p^\beta} \, ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{1+2\varepsilon}} \|\delta_{st}w\|_{L^p} \, ds, \end{aligned}$$

where the implicit constant depends on $\varepsilon, p, T_{\max}, K_0(T_{\max})$ and c .

Proof of Lemma 4.3. By Proposition 2.2,

$$\begin{aligned} \|\text{com}_1(v, w)(t)\|_{\mathcal{B}_p^{1+2\varepsilon}} &\lesssim \frac{1}{t^{\frac{1-\beta}{2}+\varepsilon}} \|v_0\|_{\mathcal{B}_p^\beta} + 1 \\ &\quad + \int_0^t \frac{1}{(t-s)^{1+2\varepsilon-\frac{\beta}{2}}} \left(\|v(s)\|_{\mathcal{B}_p^\beta} + \|w(s)\|_{\mathcal{B}_p^\beta} \right) \, ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{1+2\varepsilon}} \|\delta_{st}(v+w)\|_{L^p} \, ds. \end{aligned}$$

We now use the estimates of $\|v(s)\|_{\mathcal{B}_p^\beta}$ and $\|\delta_{st}v\|_{L^p}$ provided by Theorem 3.1. We start by estimating

$$(4.9) \quad \int_0^t \frac{1}{(t-s)^{1+2\varepsilon-\frac{\beta}{2}}} \|v(s)\|_{\mathcal{B}_p^\beta} \, ds$$

using (3.4), which takes the form of a sum of two terms. The first term is

$$\int_0^t \frac{1}{(t-s)^{1+2\varepsilon-\frac{\beta}{2}}} e^{-cs} \|v_0\|_{\mathcal{B}_p^\beta} \, ds \lesssim \|v_0\|_{\mathcal{B}_p^\beta}.$$

The second term of the upper bound for (4.9) is bounded by

$$\begin{aligned} &\int_0^t \frac{1}{(t-s)^{1+2\varepsilon-\frac{\beta}{2}}} \int_0^s \frac{1}{(s-u)^{\frac{\beta+1+\varepsilon}{2}}} (1 + \|w(u)\|_{L^p}) \, du \, ds \\ &\leq \int_0^t (1 + \|w(u)\|_{L^p}) \int_u^t \frac{1}{(t-s)^{1+2\varepsilon-\frac{\beta}{2}} (s-u)^{\frac{\beta+1+\varepsilon}{2}}} \, ds \, du, \end{aligned}$$

and the last integral is bounded by a constant times $(t-u)^{-\frac{1}{2}-\frac{5\varepsilon}{2}}$. Since for $\varepsilon > 0$ sufficiently small, $\frac{1}{2} + \frac{5\varepsilon}{2} \leq 1 - \frac{\beta}{2} + 2\varepsilon$, and $\|\cdot\|_{L^p} \lesssim \|\cdot\|_{\mathcal{B}_p^\beta}$, this term is bounded by the right-hand side of (4.8).

As for the term with $\|\delta_{st}v\|_{L^p}$, we have

$$\begin{aligned} &\int_0^t \frac{1}{(t-s)^{1+2\varepsilon}} \|\delta_{st}v\|_{L^p} \, ds \\ &\lesssim \int_0^t \frac{1}{(t-s)^{1+2\varepsilon}} \left(|t-s|^{\frac{\beta-\varepsilon}{2}} \|v(s)\|_{\mathcal{B}_p^\beta} + \int_0^s \frac{1}{(s-u)^{\frac{1}{2}+\varepsilon}} (1 + \|w(u)\|_{L^p}) \, du \, ds \right). \end{aligned}$$

The first term is (4.9) again (up to an irrelevant extra exponent $\varepsilon/2$), while by the same reasoning as above, the double integral is bounded by

$$\int_0^t \frac{1}{(t-u)^{\frac{1}{2}+3\varepsilon}} (1 + \|w(u)\|_{L^p}) \, du,$$

and this completes the proof. \square

Lemma 4.4. *Let $p \in [\frac{8}{7}, 6]$, and $\varepsilon > 0$ be small enough. For every $s \leq t \in [0, T)$,*

$$(4.10) \quad \begin{aligned} & \left\| \int_s^t e^{(t-u)\Delta} [\text{com}_1(v, w) \ominus \mathfrak{V}](u) \, du \right\|_{L^p} \\ & \lesssim (t-s)^{\frac{1}{4}} \left(1 + \|v_0\|_{\mathcal{B}_p^\beta} \right) \\ & \quad + (t-s)^{\frac{1}{8}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^\beta}^p \, du \right)^{\frac{1}{p}} \\ & \quad + (t-s)^{1-\frac{1}{6p}} \|w\|_{p,t}^{\frac{1}{2}} \left(\int_0^t \|w(u)\|_{L^p}^{3p} \, du \right)^{\frac{1}{6p}}, \end{aligned}$$

where $\|w\|_{p,t}$ is defined by

$$(4.11) \quad \|w\|_{p,t} := \sup_{u' \leq u \leq t} \frac{\|\delta'_{u'} w\|_{L^p}}{|u-u'|^{\frac{1}{8}}}.$$

The implicit constant in (4.10) depends on ε , p , T_{\max} , $K_0(T_{\max})$ and c .

Remark 4.5. By assumption, $w \in C^{\frac{1}{8}}([0, T), \mathcal{B}_2^{1+2\varepsilon})$. By Proposition A.2 and Remark A.3, the space $\mathcal{B}_2^{1+2\varepsilon}$ is continuously embedded in L^6 . As a consequence, the quantity $\|w\|_{p,t}$ is finite for every $p \leq 6$.

Proof. We start the proof of (4.10) by using Proposition A.7:

$$\left\| \int_s^t e^{(t-u)\Delta} [\text{com}_1(v, w) \ominus \mathfrak{V}](u) \, du \right\|_{L^p} \lesssim \int_s^t \|\text{com}_1(v, w)(u)\|_{\mathcal{B}_p^{1+2\varepsilon}} \, du.$$

By Lemma 4.3,

$$(4.12) \quad \begin{aligned} \|\text{com}_1(v, w)(u)\|_{\mathcal{B}_p^{1+2\varepsilon}} & \lesssim 1 + u^{-\frac{1-\beta}{2}-\varepsilon} \|v_0\|_{\mathcal{B}_p^\beta} \\ & \quad + \int_0^u \frac{1}{(u-u')^{1-\frac{\beta}{2}+2\varepsilon}} \|w(u')\|_{\mathcal{B}_p^\beta} \, du' \\ & \quad + \int_0^u \frac{1}{(u-u')^{1+2\varepsilon}} \|\delta_{u'} w\|_{L^p} \, du'. \end{aligned}$$

We can estimate the contribution of the first line above by

$$\int_s^t \left[1 + u^{-\frac{1-\beta}{2}-\varepsilon} \|v_0\|_{\mathcal{B}_p^\beta} \right] \, du \lesssim (t-s)^{1/4} \left(1 + \|v_0\|_{\mathcal{B}_p^\beta} \right).$$

As for the integral on the second line, since $p \geq \frac{8}{7}$ and $1 - \frac{\beta}{2} + 2\varepsilon < 1$ for ε sufficiently small, we can apply Hölder's and Jensen's inequalities to get

$$(4.13) \quad \begin{aligned} & \int_s^t \int_0^u \frac{1}{(u-u')^{1-\frac{\beta}{2}+2\varepsilon}} \|w(u')\|_{\mathcal{B}_p^\beta} \, du' \, du \\ & \lesssim (t-s)^{\frac{1}{8}} \left(\int_s^t \int_0^u \frac{1}{(u-u')^{1-\frac{\beta}{2}+2\varepsilon}} \|w(u')\|_{\mathcal{B}_p^\beta}^p \, du' \, du \right)^{\frac{1}{p}} \\ & \lesssim (t-s)^{\frac{1}{8}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^\beta}^p \, du \right)^{\frac{1}{p}}. \end{aligned}$$

We now analyse the more subtle term coming from (4.12):

$$(4.14) \quad \int_s^t \int_0^u \frac{1}{(u-u')^{1+\varepsilon}} \|\delta_{u'} w\|_{L^p} \, du' \, du.$$

To begin with, we replace $\delta_{u'u}w$ by $\delta'_{u'u}w$. The difference is estimated by Proposition A.13: for $\beta \in (1/4, 1)$,

$$\begin{aligned} \|\delta_{u'u}w\|_{L^p} - \|\delta'_{u'u}w\|_{L^p} &\leq \|(1 - e^{-(u-u')\Delta})w(u')\|_{L^p} \\ &\lesssim (u - u')^{\frac{1}{8}} \|w(u')\|_{\mathcal{B}_p^\beta}. \end{aligned}$$

Hence, the difference between (4.14) and the same expression with $\delta_{u'u}$ replaced by $\delta'_{u'u}$ is bounded by (using Hölder's and Jensen's inequalities and $p \geq \frac{8}{7}$)

$$\begin{aligned} &\int_s^t \int_0^u \frac{1}{(u-u')^{\frac{7}{8}+\varepsilon}} \|w(u')\|_{\mathcal{B}_p^\beta}^p \, du' \, du \\ &\lesssim (t-s)^{\frac{1}{8}} \left(\int_s^t \int_0^u \frac{1}{(u-u')^{\frac{7}{8}+\varepsilon}} \|w(u')\|_{\mathcal{B}_p^\beta}^p \, du' \, du \right)^{\frac{1}{p}} \\ &\lesssim (t-s)^{\frac{1}{8}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^\beta}^p \, du \right)^{\frac{1}{p}}. \end{aligned}$$

Note that this is the same error term as in (4.13). Moreover, by Remark A.14,

$$\|\delta'_{u'u}w\|_{L^p} \lesssim \|\delta'_{u'u}w\|_{L^p}^{1/2} \left(\|w(u)\|_{L^p}^{1/2} + \|w(u')\|_{L^p}^{1/2} \right).$$

Hence, the double integral in (4.14) with $\delta_{u'u}$ replaced by $\delta'_{u'u}$ is bounded by

$$(4.15) \quad \|w\|_{p,t}^{1/2} \int_s^t \int_0^u \frac{1}{(u-u')^{\frac{15}{16}+\varepsilon}} (\|w(u)\|_{L^p}^{1/2} + \|w(u')\|_{L^p}^{1/2}) \, du' \, du.$$

We have

$$\begin{aligned} &\int_s^t \int_0^u \frac{1}{(u-u')^{\frac{15}{16}+\varepsilon}} \|w(u)\|_{L^p}^{1/2} \, du' \, du \\ &\lesssim \int_s^t \|w(u)\|_{L^p}^{1/2} \, du \\ &\lesssim (t-s)^{1-\frac{1}{6p}} \left(\int_0^t \|w(u)\|_{L^p}^{3p} \, du \right)^{\frac{1}{6p}}, \end{aligned}$$

as well as

$$\begin{aligned} &\int_s^t \int_0^u \frac{1}{(u-u')^{\frac{15}{16}+\varepsilon}} \|w(u')\|_{L^p}^{1/2} \, du' \, du \\ &\lesssim (t-s)^{1-\frac{1}{6p}} \left(\int_s^t \int_0^u \frac{1}{(u-u')^{\frac{15}{16}+\varepsilon}} \|w(u')\|_{L^p}^{3p} \, du' \, du \right)^{\frac{1}{6p}} \\ &\lesssim (t-s)^{1-\frac{1}{6p}} \left(\int_0^t \|w(u)\|_{L^p}^{3p} \, du \right)^{\frac{1}{6p}}. \end{aligned}$$

Summarizing, we obtain (4.10). \square

The following lemma is the only place where we need to measure a derivative of index higher than 1 of w .

Lemma 4.6. *Let $p > 1$ and $\varepsilon > 0$ be small enough. For every $s \leq t \in [0, T)$,*

$$(4.16) \quad \left\| \int_s^t e^{(t-u)\Delta} [w \ominus \mathbf{v}](u) \, du \right\|_{L^p} \lesssim (t-s)^{\frac{p-1}{p}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p \, du \right)^{\frac{1}{p}},$$

where the implicit constant depends on ε , p and $K_0(T_{\max})$.

Proof. The estimate (4.16) follows easily by writing

$$\begin{aligned} & \left\| \int_s^t e^{(t-u)\Delta} [w \ominus \mathfrak{V}](u) \, du \right\|_{L^p} \\ & \lesssim \int_s^t \|w \ominus \mathfrak{V}\|_{L^p}(u) \, du \lesssim \int_s^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}} \, du \\ & \lesssim (t-s)^{\frac{p-1}{p}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p \, du \right)^{\frac{1}{p}}. \quad \square \end{aligned}$$

For the next lemma, we recall that a_2 is the coefficient in front of the quadratic term in P which was defined in (1.20), and that a_2 is a distribution with spatial regularity $-\frac{1}{2} - \varepsilon$ controlled uniformly in time. This is where the condition $p < 8/3$ enters.

Lemma 4.7. *Let $p \in (1, \frac{8}{3})$ and $\varepsilon > 0$ be small enough. For every $s \leq t \in [0, T)$,*

$$(4.17) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2(v+w)^2](u) \, du \right\|_{L^p} \lesssim (t-s)^{\frac{1}{8}} \\ \times \left[1 + \|v_0\|_{\mathcal{B}_{3p}^\beta}^3 + \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 \, du \right)^{\frac{1}{2}} + \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 \, du \right)^{\frac{3}{4}} \right],$$

where the implicit constant depends on ε , p , T_{\max} , $K_0(T_{\max})$ and c .

Proof. We start by bounding the term which is of highest order in w , using Remark A.3 and Propositions A.13 and A.7:

$$(4.18) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2 w^2](u) \, du \right\|_{L^p} \\ \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|a_2 w^2\|_{\mathcal{B}_p^{-\frac{1}{2}-\varepsilon}}(u) \, du \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|w^2(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \, du.$$

By Proposition A.4 and Remark A.3, we have uniformly over u

$$\|w^2(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \lesssim \|w^2(u)\|_{\mathcal{B}_2^1}^{\frac{1}{2}+2\varepsilon} \|w^2(u)\|_{L^r}^{\frac{1}{2}-2\varepsilon} = \|w^2(u)\|_{\mathcal{B}_2^1}^{\frac{1}{2}+2\varepsilon} \|w(u)\|_{L^{2r}}^{1-4\varepsilon},$$

as soon as $r \geq 1$ satisfies

$$\frac{1}{p} = \left(\frac{1}{2} + 2\varepsilon \right) \frac{1}{2} + \left(\frac{1}{2} - 2\varepsilon \right) \frac{1}{r}.$$

Note that for $p < 4$ and $\varepsilon > 0$ small enough (depending on p), such an r can always be found. It will however be most useful to choose r such that $2r \leq 3p$, in which case we need to impose

$$\frac{1}{p} \geq \left(\frac{1}{2} + 2\varepsilon \right) \frac{1}{2} + \left(\frac{1}{2} - 2\varepsilon \right) \frac{2}{3p}.$$

For $p < \frac{8}{3}$ and ε small enough, this condition is satisfied, and from now on we will assume that it holds. Then using $\|w(u)\|_{L^{2r}} \lesssim \|w(u)\|_{L^{3p}}$, we get

$$\|w^2(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \lesssim \|w^2(u)\|_{\mathcal{B}_2^1}^{\frac{1}{2}+2\varepsilon} \|w(u)\|_{L^{3p}}^{1-4\varepsilon}.$$

Plugging this estimate into (4.18) and applying Hölder's inequality, we get

$$\begin{aligned} & \left\| \int_s^t e^{(t-u)\Delta} [a_2 w^2](u) \, du \right\|_{L^p} \\ & \lesssim \left(\int_s^t \left(\frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \right)^{r'} \, du \right)^{\frac{1}{r'}} \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 \, du \right)^{\frac{1}{4}+\varepsilon} \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{1-4\varepsilon}{3p}} \\ & \lesssim (t-s)^{\frac{1}{r'} - (\frac{1}{4}+\varepsilon)} \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 \, du \right)^{\frac{1}{4}+\varepsilon} \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{1-4\varepsilon}{3p}}, \end{aligned}$$

as soon as

$$\frac{1}{r'} + \left(\frac{1}{4} + \varepsilon \right) + \frac{1-4\varepsilon}{3p} = 1 \quad \text{and} \quad r' \left(\frac{1}{4} + \varepsilon \right) < 1.$$

For any $p > 1$ and for $\varepsilon > 0$ small enough, the r' determined by the first condition is $\leq \frac{12}{5}$, which yields an exponent for $(t-s)$ which is $\geq \frac{1}{6} - \frac{\varepsilon}{2}$. By Young's inequality, we can conclude that for any $p > 1$ and for $\varepsilon > 0$ small enough,

$$(4.19) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2 w^2](u) \, du \right\|_{L^p} \lesssim (t-s)^{\frac{1}{8}} \left[1 + \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 \, du \right)^{\frac{1}{2}} + \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{1}{p}} \right].$$

We now turn to the term involving $a_2 v^2$. Arguing as in (4.18), we get

$$(4.20) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2 v^2](u) \, du \right\|_{L^p} \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|v^2(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \, du \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|v(u)\|_{\mathcal{B}_{2p}^{\frac{1}{2}+2\varepsilon}}^2 \, du.$$

Recall that by Theorem 3.1,

$$(4.21) \quad \|v(u)\|_{\mathcal{B}_{2p}^{\frac{1}{2}+2\varepsilon}} \lesssim \|v_0\|_{\mathcal{B}_{2p}^\beta} + \int_0^u \frac{1}{(u-u')^{\frac{3}{4}+\frac{3\varepsilon}{2}}} (1 + \|w(u')\|_{L^{2p}}) \, du'.$$

The term containing the initial condition contributes

$$(4.22) \quad \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|v_0\|_{\mathcal{B}_{2p}^\beta}^2 \, du \lesssim (t-s)^{\frac{3}{4}-\varepsilon} \|v_0\|_{\mathcal{B}_{2p}^\beta}^2.$$

The contribution of the second term in (4.21) to the integral on the right-hand side of (4.20) can be rewritten as

$$\int_s^t f(t-u) \left(\int_0^u g(u-u') h(u') \, du' \right)^2 \, du,$$

for

$$(4.23) \quad f(u) = \frac{1}{u^{\frac{1}{4}+\varepsilon}}, \quad g(u) = \frac{1}{u^{\frac{3}{4}+\frac{3\varepsilon}{2}}}, \quad h(u) = 1 + \|w(u)\|_{L^{2p}}.$$

Therefore, using Hölder's inequality in the first and Young's inequality in the second step we get

$$\begin{aligned} & \left\| \int_s^t e^{(t-u)\Delta} [a_2 v^2](u) \, du \right\|_{L^p} \\ & \lesssim \left(\int_s^t f(u)^{q_1} \, du \right)^{\frac{1}{q_1}} \left(\int_0^t \left| \int_0^u g(u-u') h(u') \, du' \right|^{2q_1'} \, du \right)^{\frac{1}{q_1'}} \\ & \lesssim \left(\int_s^t f(u)^{q_1} \, du \right)^{\frac{1}{q_1}} \left(\int_0^t g(u)^{q_2} \, du \right)^{\frac{2}{q_2}} \left(\int_0^t h(u)^{q_3} \, du \right)^{\frac{2}{q_3}}, \end{aligned}$$

where q_1' is the adjoint exponent of q_1 and $q_2, q_3 \in (1, \infty)$ satisfy $\frac{1}{q_2} + \frac{1}{q_3} = 1 + \frac{1}{2q_1'}$. We also impose q_1 and q_2 to be sufficiently small that the corresponding integrals are finite. That is, we impose

$$(4.24) \quad \frac{3}{2} = \frac{1}{2q_1} + \frac{1}{q_2} + \frac{1}{q_3}, \quad q_1 < \frac{4}{1+4\varepsilon} \quad \text{and} \quad q_2 < \frac{1}{\frac{3}{4} + \frac{3\varepsilon}{2}}.$$

Choosing $q_3 = 3p$ (for any $p > 1$), and $q_1 = \frac{2}{1+2\varepsilon}$ (which implies that the second condition in (4.24) is satisfied) one sees that the q_2 determined by the first condition in (4.24) satisfies $q_2 \leq \frac{12}{11}$ for any $p > 1$, which implies in turn that for $\varepsilon > 0$ small enough the third condition holds. Therefore, using $\|w\|_{L^{2p}} \lesssim \|w\|_{L^{3p}}$ we can summarise

$$(4.25) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2 v^2](u) \, du \right\|_{L^p} \lesssim (t-s)^{\frac{1}{2}} \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} \, du \right)^{\frac{2}{3p}}.$$

The term involving the product vw also requires some thought. As before, we write

$$(4.26) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2 vw](u) \, du \right\|_{L^p} \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|vw(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \, du.$$

It is convenient to split the integrability requirement on vw asymmetrically between the two factors. We use Proposition A.7 to write

$$(4.27) \quad \|vw(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \lesssim \|v(u)\|_{\mathcal{B}_{q_4}^{\frac{1}{2}+2\varepsilon}} \|w(u)\|_{\mathcal{B}_{q_5}^{\frac{1}{2}+2\varepsilon}},$$

where

$$\frac{1}{p} = \frac{1}{q_4} + \frac{1}{q_5} \quad \text{and} \quad q_5 = \frac{3}{1+2\varepsilon}.$$

In particular, we assume that $p \leq \frac{3}{1+2\varepsilon}$ (which, of course, is already implied by our earlier assumption $p < \frac{8}{3}$ if ε is small enough). We use Proposition A.2 to bound the term involving w :

$$\|w(u)\|_{\mathcal{B}_{q_5}^{\frac{1}{2}+2\varepsilon}} \lesssim \|w(u)\|_{\mathcal{B}_2^1}.$$

For the term involving v , we seek to use a norm with exponent of integrability $3p$. For $p \leq \frac{2}{1+2\varepsilon}$, this is automatic because then we have $q_4 \leq 3p$. For $\frac{2}{1+2\varepsilon} \leq p < \frac{3}{1+2\varepsilon}$ we resolve the lack of integrability by raising the differentiability index, using again Proposition A.2. This is summarised in the bound

$$\|v(u)\|_{\mathcal{B}_{q_4}^{\frac{1}{2}+2\varepsilon}} \lesssim \|v(u)\|_{\mathcal{B}_{3p}^{\frac{1}{2}+2\varepsilon+\sigma}}$$

for

$$\sigma = 3 \left(\frac{1}{3p} - \frac{1}{q_4} \right) \vee 0 = \left(1 + 2\varepsilon - \frac{2}{p} \right) \vee 0.$$

In particular, for $p < 3$ and $\varepsilon > 0$ small enough (depending on p), we have $\sigma < \frac{1}{3}$. Summarising these bounds in conjunction with the a priori bound on v from Theorem 3.1, we get

$$(4.28) \quad \left\| \int_s^t e^{(t-u)\Delta} [a_2 v w](u) \, du \right\|_{L^p} \lesssim \int_s^t f(t-u) \|v_0\|_{\mathcal{B}_{3p}^3} \|w(u)\|_{\mathcal{B}_2^1} \, du \\ + \int_s^t f(t-u) \left(\int_0^u g(u-u') h(u') \, du' \right) \|w(u)\|_{\mathcal{B}_2^1} \, du,$$

where, similarly to (4.23), we have set

$$f(u) = \frac{1}{u^{\frac{1}{4}+\varepsilon}}, \quad g(u) = \frac{1}{u^{\frac{3}{4}+\frac{\sigma}{2}+\frac{3\varepsilon}{2}}}, \quad h(u) = 1 + \|w(u)\|_{L^{3p}}.$$

(note that the functions f and h are the same as in (4.23), but that $\frac{\sigma}{2}$ has been added to the exponent of u in the definition g). We first bound the second term in the right-hand side of (4.28). By Hölder's inequality, this term is bounded by

$$\left(\int_s^t f(u)^{q_6} \, du \right)^{\frac{1}{q_6}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 \, du \right)^{\frac{1}{2}} \left(\int_0^t \left(\int_0^u g(u-u') h(u') \, du' \right)^{q_7} \, du \right)^{\frac{1}{q_7}},$$

where

$$(4.29) \quad 1 = \frac{1}{q_6} + \frac{1}{2} + \frac{1}{q_7}.$$

The integral involving f is finite as soon as

$$(4.30) \quad q_6 < \frac{1}{\frac{1}{4} + \varepsilon},$$

and in this case we can bound the first factor (up to a constant) by $(t-s)^{\frac{1}{q_6} - \frac{1}{4} - \varepsilon}$. Using Young's inequality, we bound the integral involving the convolution of g and h by

$$\left(\int_0^t \left(\int_0^u g(u-u') h(u') \, du' \right)^{q_7} \, du \right)^{\frac{1}{q_7}} \lesssim \left(\int_0^t g(r)^{q_8} \, dr \right)^{\frac{1}{q_8}} \left(\int_0^t h(r)^{3p} \, dr \right)^{\frac{1}{3p}},$$

for

$$(4.31) \quad 1 + \frac{1}{q_7} = \frac{1}{q_8} + \frac{1}{3p} \quad \text{and} \quad q_8 < \frac{1}{\frac{3}{4} + \frac{\sigma}{2} + \frac{3\varepsilon}{2}}.$$

Parameters q_6, q_7 , and q_8 satisfying the conditions (4.29)–(4.31) can always be found. Indeed, for $p < 2$ and ε small enough, we have $\sigma = 0$ and we can choose q_8 close to $\frac{4}{3}$, which even in the worst case $p = 1$ allows to chose q_7 close to 12. Plugging this into (4.29) yields a q_6 close to $\frac{12}{5}$, so that finally we obtain an exponent for $(t-s)$ which is as close as we want to $\frac{1}{6}$. For $2 \leq p \leq 3$, we use the crude bound $\sigma < \frac{1}{3}$ which (for ε small enough) allows for q_8 close to $\frac{12}{11}$. Again, for any $p \geq 2$, the first condition in (4.31) implies that for ε small we can choose q_7 close to 12, and we get the same control on the time regularity as before. To sum up, we get for any $p < 3$ and ε small enough that the second integral in the right-hand side of (4.28) is bounded by

$$(4.32) \quad (t-s)^{\frac{1}{8}} \left(\int_0^t \|w(r)\|_{\mathcal{B}_2^1}^2 \, dr \right)^{\frac{1}{2}} \left(1 + \int_0^t \|w(r)\|_{L^{3p}}^{3p} \, dr \right)^{\frac{1}{3p}} \\ \lesssim (t-s)^{\frac{1}{8}} \left[1 + \left(\int_0^t \|w(r)\|_{\mathcal{B}_2^1}^2 \, dr \right)^{\frac{3}{4}} + \left(\int_0^t \|w(r)\|_{L^{3p}}^{3p} \, dr \right)^{\frac{1}{p}} \right].$$

There remains to bound the first integral in the right-hand side of (4.28). By the Cauchy-Schwarz inequality, this integral is bounded by

$$\|v_0\|_{\mathcal{B}_{3p}^\beta} \left(\int_s^t f(t-u)^2 du \right)^{\frac{1}{2}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{1}{2}},$$

which, by Young's inequality, is bounded by

$$(t-s)^{\frac{1}{8}} \left[\|v_0\|_{\mathcal{B}_{3p}^\beta}^3 + \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{3}{4}} \right].$$

This concludes the estimation of (4.28). Combining this with (4.19), (4.22) and (4.25), the desired estimate (4.17) follows. \square

We now bound the terms which were not made explicit in (4.6).

Lemma 4.8. *Let $p \in (1, \frac{8}{3})$ and $\varepsilon > 0$ be small enough. For every $s \leq t \in [0, T)$,*

$$(4.33) \quad \int_s^t e^{(t-u)\Delta} [\dots](u) du \lesssim (t-s)^{\frac{p-1}{p} \wedge \frac{1}{5}} \\ \times \left[1 + \|v_0\|_{\mathcal{B}_p^\beta} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_{p^{1+2\varepsilon}}^p}^p du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{L^p}^{3p} du \right)^{\frac{1}{3p}} \right].$$

where the dots ... represent all the terms left out in (4.6) (spelled out explicitly in (4.34) below). The implicit constant depends on ε , p , T_{\max} , $K_0(T_{\max})$ and c .

Proof. We need to bound

$$(4.34) \quad \int_s^t e^{(t-u)\Delta} [-3\text{com}_2(v+w) - 3(v+w - \Psi) \odot \mathbf{v} + a_0 + a_1(v+w) + cv](u) du,$$

and we proceed by bounding these terms one by one.

To begin with, we show that

$$(4.35) \quad \left\| \int_s^t e^{-(t-u)\Delta} \text{com}_2(v+w)(u) du \right\|_{L^p} \\ \lesssim (t-s)^{\frac{p-1}{p}} \left[1 + \|v_0\|_{\mathcal{B}_p^\beta} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_{p^{1+2\varepsilon}}^p}^p du \right)^{\frac{1}{p}} \right].$$

Indeed, by Remark A.14 and Proposition A.9, the left-hand side above is bounded by

$$\int_s^t \|\text{com}_2(v+w)(u)\|_{L^p} du \\ \lesssim \int_s^t \|(v+w)(u)\|_{\mathcal{B}_p^{3\varepsilon}} du \\ \lesssim (t-s)^{\frac{p-1}{p}} \left[\left(\int_0^t \|v(u)\|_{\mathcal{B}_p^{3\varepsilon}}^p du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{3\varepsilon}}^p du \right)^{\frac{1}{p}} \right].$$

For the integral involving v , we apply Theorem 3.1 as before to obtain

$$\int_0^t \|v(u)\|_{\mathcal{B}_p^{3\varepsilon}}^p du \lesssim \|v_0\|_{\mathcal{B}_p^\beta}^p + \int_0^t \left(\int_0^u \frac{1}{(u-u')^{\frac{1+4\varepsilon}{2}}} (1 + \|w(u')\|_{L^p}) du' \right)^p du \\ \lesssim \|v_0\|_{\mathcal{B}_p^\beta}^p + \int_0^t (1 + \|w(u')\|_{L^p}^p) du',$$

where we have first used Jensen's inequality to move the p -th power inside the du' -integral, and then carried out the du integral. So (4.35) follows.

We now show that

$$(4.36) \quad \left\| \int_s^t e^{(t-u)\Delta} ((v+w) \otimes \mathbf{v})(u) du \right\|_{L^p} \\ \lesssim (t-s)^{\frac{p-1}{p}} \left[\|v_0\|_{\mathcal{B}_p^\beta} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}} + \left(\int_0^t (1 + \|w(u)\|_{L^p}^{3p}) du \right)^{\frac{1}{3p}} \right].$$

Indeed, on the one hand, by Proposition A.7,

$$\left\| \int_s^t e^{(t-u)\Delta} (w \otimes \mathbf{v})(u) du \right\|_{L^p} \lesssim \int_s^t \|(w \otimes \mathbf{v})(u)\|_{L^p} du \\ \lesssim \int_s^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}} du \\ \lesssim (t-s)^{\frac{p-1}{p}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}}.$$

On the other hand,

$$\left\| \int_s^t e^{(t-u)\Delta} (v \otimes \mathbf{v})(u) du \right\|_{L^p} \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|(v \otimes \mathbf{v})(u)\|_{\mathcal{B}_p^{-\frac{1}{2}}} du \\ \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|v(u)\|_{\mathcal{B}_p^{\frac{1}{2}+\varepsilon}} du.$$

We use Theorem 3.1 again to estimate $\|v(u)\|_{\mathcal{B}_p^{\frac{1}{2}+\varepsilon}}$. The contribution of the initial condition poses no difficulty (recall that $\beta \geq \frac{1}{2} + \varepsilon$). The contribution of the other term takes the form

$$\int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \left(\int_0^u \frac{1}{(u-u')^{\frac{3}{4}+\varepsilon}} (1 + \|w(u')\|_{L^p}) du' \right) du \\ \lesssim \left(\int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon} \frac{3p-1}{3p}} du \right)^{\frac{3p-1}{3p}} \left(\int_0^t \left(\int_0^u \frac{1}{(u-u')^{\frac{3}{4}+\varepsilon}} (1 + \|w(u')\|_{L^p}) du' \right)^{3p} du \right)^{\frac{1}{3p}} \\ \lesssim (t-s)^{\frac{3p-1}{3p} - \frac{1}{4} - \varepsilon} \left(\int_0^t (1 + \|w(u)\|_{L^p}^{3p}) du \right)^{\frac{1}{3p}}.$$

As before, we have used Jensen's inequality to move the power $3p$ inside the du' integral. Therefore, (4.36) follows, since for $1 < p < \frac{8}{3}$, we have $\frac{3p-1}{3p} - \frac{1}{4} \geq \frac{p-1}{p}$.

We also have

$$\left\| \int_s^t e^{(t-u)\Delta} (\Psi \otimes \mathbf{v} + a_0)(u) du \right\|_{L^p} \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+2\varepsilon}} \|(\Psi \otimes \mathbf{v} + a_0)(u)\|_{C^{-\frac{1}{2}-2\varepsilon}} du \\ \lesssim (t-s)^{\frac{3}{4}-2\varepsilon},$$

which is bounded by the right-hand side of (4.33), because for $p < \frac{8}{3}$, we have $\frac{p-1}{p} \leq \frac{5}{8} \leq \frac{3}{4} - 2\varepsilon$ for ε small enough.

Finally, we write

$$\left\| \int_s^t e^{(t-u)\Delta} [a_1(v+w) + cv](u) du \right\|_{L^p} \\ \lesssim \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \left(\|v(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} + \|w(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} \right) du.$$

For the term involving v , we have

$$\int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|v(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} du \lesssim (t-s)^{\frac{3p-1}{3p}-\frac{1}{4}-\varepsilon} \left(\int_0^t \|v(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}}^{3p} du \right)^{\frac{1}{3p}}.$$

Recall that the estimate given by Theorem 3.1 is a sum of two terms; the term involving v_0 poses no difficulty (since $\beta = \frac{1}{2} + 2\varepsilon$), while the other term contributes

$$\int_0^t \left(\int_0^u \frac{1}{(u-u')^{\frac{3}{4}+\frac{3\varepsilon}{2}}} (1 + \|w(u')\|_{L^p}) du' \right)^{3p} du \lesssim \int_0^t (1 + \|w(u')\|_{L^p})^{3p} du'.$$

By noting as before that for $p > 1$ and $\varepsilon > 0$ small enough, we have $\frac{3p-1}{3p} - \frac{1}{4} - \varepsilon < \frac{p-1}{p} \wedge \frac{5}{8} - \varepsilon$, we see that the term involving v is bounded by the right-hand side of (4.33). For the integral involving w , we write

$$\begin{aligned} & \int_s^t \frac{1}{(t-u)^{\frac{1}{4}+\varepsilon}} \|w(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}} du \\ & \lesssim (t-s)^{1-\frac{1}{q}-\frac{1}{4}-\varepsilon} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}}^q du \right)^{\frac{1}{q}} \\ & \lesssim (t-s)^{1-\frac{1}{q}-\frac{1}{4}-\varepsilon} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}}^{\left(\frac{1+4\varepsilon}{2+4\varepsilon}\right)q} \|w(u)\|_{L^p}^{\left(\frac{1}{2+4\varepsilon}\right)q} du \right)^{\frac{1}{q}} \\ & \lesssim (t-s)^{1-\frac{1}{q}-\frac{1}{4}-\varepsilon} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}}^p du \right)^{\left(\frac{1+4\varepsilon}{2+4\varepsilon}\right)\frac{1}{p}} \left(\int_0^t \|w(u)\|_{L^p}^{3p} du \right)^{\left(\frac{1}{2+4\varepsilon}\right)\frac{1}{3p}} \\ & \lesssim (t-s)^{1-\frac{1}{q}-\frac{1}{4}-\varepsilon} \left[\left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}}^p du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{L^p}^{3p} du \right)^{\frac{1}{3p}} \right], \end{aligned}$$

where in the first inequality we have set

$$q = 3p \left(\frac{1+2\varepsilon}{2+6\varepsilon} \right),$$

in the second step we have made use of the interpolation bound provided by Proposition A.4 and of Remark A.3; we have then applied Hölder's inequality with inverse exponents $\frac{3+12\varepsilon}{4+12\varepsilon}$ and $\frac{1}{4+12\varepsilon}$ in the third inequality. Note in particular that for every $p < \frac{4}{3}$ and ε small enough (depending on p), this choice of q ensures $1 - \frac{1}{q} - \frac{1}{4} - \varepsilon > \frac{p-1}{p}$, while for larger p (and ε small enough), it is bounded from below by $\frac{1}{5}$. So this term is bounded by the right-hand side of (4.33) as well. \square

Proof of Theorem 4.1. Combining the bounds we have derived in Lemmas 4.2–4.8, we obtain for $p \in [\frac{8}{7}, \frac{8}{3})$ and ε small enough that

$$\begin{aligned}
 & \|\delta'_{st}w\|_{L^p} \\
 & \lesssim (t-s)^{\frac{p-1}{p}} \left(1 + \|v_0\|_{L^{3p}}^{3p} + \int_0^t \|w(u)\|_{L^{3p}}^{3p} du \right)^{\frac{1}{p}} \\
 & + (t-s)^{\frac{1}{4}} \left(1 + \|v_0\|_{\mathcal{B}_p^\beta} \right) \\
 & + (t-s)^{\frac{1}{8}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^\beta}^p du \right)^{\frac{1}{p}} \\
 & + (t-s)^{1-\frac{1}{6p}} \|w\|_{p,t}^{\frac{1}{2}} \left(\int_0^t \|w(u)\|_{L^p}^{3p} du \right)^{\frac{1}{6p}} \\
 & + (t-s)^{\frac{p-1}{p}} \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}} \\
 & + (t-s)^{\frac{1}{8}} \left[1 + \|v_0\|_{\mathcal{B}_{3p}^\beta}^3 + \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{1}{2}} \right. \\
 & \quad \left. + \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{3}{4}} \right] \\
 & + (t-s)^{\frac{p-1}{p} \wedge \frac{1}{5}} \left[1 + \|v_0\|_{\mathcal{B}_p^\beta} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{L^p}^{3p} du \right)^{\frac{1}{3p}} \right],
 \end{aligned}$$

where we recall that $\|w\|_{p,t}$ is defined in (4.11), and that this quantity is finite by Remark 4.5. Using that $p \geq \frac{8}{7}$, the comparisons $\|\cdot\|_{L^{3p}} \lesssim \|\cdot\|_{\mathcal{B}_{3p}^\beta}$, $\|\cdot\|_{\mathcal{B}_p^\beta} \lesssim \|\cdot\|_{\mathcal{B}_{3p}^\beta}$ and $\|\cdot\|_{\mathcal{B}_p^\beta} \lesssim \|\cdot\|_{\mathcal{B}_p^{1+2\varepsilon}}$ together with the fact that

$$x \leq a + \sqrt{bx} \implies x \lesssim a + b,$$

we obtain

$$\begin{aligned}
 \frac{\|\delta'_{st}w\|_{L^p}}{(t-s)^{\frac{1}{8}}} & \lesssim 1 + \|v_0\|_{\mathcal{B}_{3p}^\beta}^3 + \left(\int_0^t \|w(u)\|_{L^{3p}}^{3p} du \right)^{\frac{1}{p}} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}} \\
 & + \left(\int_0^t \|w^2(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{1}{2}} + \left(\int_0^t \|w(u)\|_{\mathcal{B}_2^1}^2 du \right)^{\frac{3}{4}}.
 \end{aligned}$$

To conclude, we observe that by Proposition A.13, since $\beta \geq \frac{1}{4}$, we have

$$\left| \|\delta'_{st}w\|_{L^p} - \|\delta_{st}w\|_{L^p} \right| \leq \|(1 - e^{(t-s)\Delta})w(s)\|_{L^p} \lesssim (t-s)^{\frac{1}{8}} \|w(s)\|_{\mathcal{B}_p^\beta}. \quad \square$$

5. A NON-LINEAR TO LINEAR BOUND ON w

In this section, we test the equation for w against suitable powers of w . This will allow us to leverage on the “good” sign of the term $-w^3$ in the definition of G . We will rely on the results of the previous two sections: Section 3 allows us to neglect terms involving v and effectively reduce the analysis of the system (1.22) to that of a single equation on w ; while Section 4 provides us with sufficient information on the time regularity of w to allow us to handle the commutator term com_1 .

As was already observed, controlling the “bad term” $w \odot \mathfrak{V}$ requires information on the regularity of w beyond exponent 1. We cannot control this at the present stage of the analysis. We simply keep track of this term, and interpret the testing argument as allowing us to exchange a non-linear quantity in w for a linear one.

This “non-linear to linear” bound will be the starting point of a Gronwall argument in the next section.

Theorem 5.1 (A priori estimate on w). *Let $p \in \{\frac{4}{3}, 2\}$. There exists c_0 (depending on ε and $K_0(T_{\max})$) such that if $c \geq c_0$, then for every $t \in [0, T)$,*

$$(5.1) \quad \|w(t)\|_{L^{3p-2}}^{3p-2} + \int_0^t \left(\|\nabla w|^2 w^{3p-4}(s)\|_{L^1} + \|w(s)\|_{L^{3p}}^{3p} \right) ds \\ \lesssim 1 + \|w_0\|_{L^4}^{2p} + \|v_0\|_{\mathcal{B}_6^\beta}^{3p} + \left(\int_0^t \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 ds \right)^{\frac{p}{2}},$$

where the implicit constant depends on ε , T_{\max} , $K_0(T_{\max})$ and c .

The cases $p = \frac{4}{3}$ and $p = 2$ correspond to testing the equation for w against w and w^3 respectively. In the first case, we obtain a bound on the space-time L^4 norm of w , and also on the time integral of $\|\nabla w\|_{L^2}^2$, which is the square of the homogeneous part of the \mathcal{B}_2^1 norm of w . When $p = 2$, we obtain a bound on the space-time L^6 norm of w , as well as on the time integral of $\|\nabla(w^2)\|_{L^2}^2$, which is the homogeneous part of the \mathcal{B}_2^1 norm of w^2 .

In order to isolate the “good term” $-w^3$, we let \tilde{G} be such that

$$G(v, w) = -w^3 + \tilde{G}(v, w).$$

Proposition 5.2 (Testing against w^{3p-3}). *Let $p \in \{\frac{4}{3}, 2\}$. For every $t \in [0, T)$,*

$$(5.2) \quad \frac{1}{3p-2} \left(\|w(t)\|_{L^{3p-2}}^{3p-2} - \|w_0\|_{L^{3p-2}}^{3p-2} \right) + (3p-3) \int_0^t \|\nabla w|^2 w^{3p-4}(s)\|_{L^1} ds \\ + \int_0^t \|w(s)\|_{L^{3p}}^{3p} ds = \int_0^t \langle \tilde{G}(v, w) + cv, w^{3p-3} \rangle(s) ds.$$

The proof of Proposition 5.2 is very similar to that of [31, Proposition 6.8]. The main ingredient is the following lemma.

Lemma 5.3 (Time regularity of w). *We have*

$$w \in C([0, T], L^{12}) \cap C([0, T], \mathcal{B}_4^{\frac{1}{4}+\varepsilon}) \cap C^{\frac{1+\varepsilon}{2}}([0, T], L^{\frac{12}{5}})$$

and

$$\nabla w \in C([0, T], L^{\frac{12}{5}}).$$

Proof. For Hölder continuity, we use (2.11) with $\eta = \frac{1+\varepsilon}{2}$ to obtain that w is η -Hölder continuous as a function from $[0, T)$ to $\mathcal{B}_2^{\frac{1}{4}+\varepsilon}$. By Proposition A.2 and Remark A.3, the space $\mathcal{B}_2^{\frac{1}{4}+\varepsilon}$ is continuously embedded in $L^{\frac{12}{5}}$, which proves the claim. The same argument with $\eta = \frac{\varepsilon}{2}$ shows that w is a continuous function from $[0, T)$ to $\mathcal{B}_2^{\frac{5}{2}+\varepsilon}$. This space is continuously embedded in L^{12} , $\mathcal{B}_4^{\frac{1}{4}+\varepsilon}$ and $\mathcal{B}_{\frac{12}{5}}^{1+\varepsilon}$. This completes the proof of the statement concerning w . The statement concerning ∇w follows by Proposition A.5. \square

Proof of Proposition 5.2. Note first that by Lemma 5.3 and Hölder’s inequality, the quantities on the left-hand side of (5.2) are finite. (That the quantity on the right-hand side is finite will become apparent in the lemmas to follow.)

By classical arguments (see e.g. [31, Proposition 6.7]), w is a weak solution of (1.22), in the sense that for every $\phi \in C_{\text{per}}^\infty$,

$$\langle w(t), \phi \rangle - \langle w_0, \phi \rangle = \int_0^t [-\langle \nabla w(s), \nabla \phi \rangle + \langle [G(v, w) + cv](s), \phi \rangle] ds.$$

For the rest of the argument, we fix $p = 2$ for clarity (which corresponds to $p = 4$ in the notation of [31]). We proceed as in the proof of [31, Proposition 6.8]. We split the interval $[0, t]$ into a subdivision $0 = t_0 \leq \dots \leq t_n = t$. We first check that

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \nabla w(s), w^2(t_i) \nabla w(t_i) \rangle ds$$

converges to

$$\int_0^t \|\nabla w(s)\|^2 w^2(s) ds$$

as the subdivision gets finer and finer. This follows from the fact that $w \in C([0, T], L^{12})$ and $\nabla w \in C([0, T], L^{\frac{12}{5}})$, as given by Lemma 5.3. We postpone for a short while the verification of the fact that

$$(5.3) \quad \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle [G(v, w) + cv](s), w^3(t_i) \rangle ds \longrightarrow \int_0^t \langle G(v, w) + cv, w^3 \rangle(s) ds$$

as the subdivision gets finer and finer. The last point that needs to be verified (see [31, (6.16) and below]) is the fact that

$$\sum_{i=0}^{n-1} \langle w(t_{i+1}), (w(t_{i+1}) - w(t_i))(w^2(t_{i+1}) - w^2(t_i)) \rangle$$

tends to 0 as the subdivision gets finer and finer. The summand can be rewritten as

$$\begin{aligned} & \langle w(t_{i+1})(w(t_{i+1}) + w(t_i)), (w(t_{i+1}) - w(t_i))^2 \rangle \\ & \leq \|w(t_{i+1})(w(t_{i+1}) + w(t_i))\|_{L^6} \|w(t_{i+1}) - w(t_i)\|_{L^{\frac{6}{5}}}^2 \\ & \leq \|w(t_{i+1}) - w(t_i)\|_{L^{\frac{12}{5}}}^2 \sup_{0 \leq s \leq T} \|w(s)\|_{L^{12}}^2. \end{aligned}$$

The desired result then follows by Lemma 5.3.

We now come back to (5.3), which calls for a slightly longer argument. We denote by $C < \infty$ a constant which may depend on v and w in addition to ε , T_{\max} , $K_0(T_{\max})$ and c , and may vary from place to place. We decompose G into $G_0 + G_1 + G_2 + G_3$, where

$$\begin{aligned} G_0 &= a_1(v + w) + a_2(v + w)^2, \\ G_1 &= -w^3 - 3w^2v - 3wv^2 - v^3 - 3w \otimes \mathbf{v} - 3(v + w) \otimes \mathbf{v}, \\ G_2 &= -3\text{com}(v, w), \\ G_3 &= v \otimes \mathbf{v} + a_0. \end{aligned}$$

The definition of G_2 is that used in Lemma 2.3, hence

$$\|G_2(s)\|_{L^2} \leq C \left(1 + s^{-\frac{1}{4}}\right).$$

This and Lemma 5.3 thus ensure that

$$(5.4) \quad \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle G_2(s), w^3(t_i) \rangle ds \longrightarrow \int_0^t \langle G_2, w^3 \rangle(s) ds.$$

By Proposition A.7 and Remark A.3,

$$\|w \otimes \mathbf{v}(s)\|_{L^2} \lesssim \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}} \|\mathbf{v}(s)\|_{\mathcal{B}_\infty^{-1-\varepsilon}} \leq C,$$

and similarly,

$$\|w \otimes \mathbf{v}(s)\|_{L^2} \leq C.$$

By Lemma 5.3, $\|w(s)\|_{L^6}$ remains bounded, and so does $\|v(s)\|_{L^6}$ by the obvious comparison of norms $\|\cdot\|_{\mathcal{B}_6^{\frac{1}{2}+2\varepsilon}} \lesssim \|\cdot\|_{L^6}$. Hence, $\|G_1(s)\|_{L^2} \leq C$, and (5.4) holds with G_2 replaced by G_1 .

For G_3 , we observe that

$$\|v \odot \mathbf{V}(s)\|_{\mathcal{B}_6^{-\frac{1}{2}+\varepsilon}} \lesssim \|v(s)\|_{\mathcal{B}_6^{\frac{1}{2}+2\varepsilon}} \|\mathbf{V}(s)\|_{\mathcal{B}_\infty^{-1-\varepsilon}} \leq C.$$

Since $\|a_0(s)\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}} \lesssim 1$, we obtain

$$\|G_3(s)\|_{\mathcal{B}_6^{-\frac{1}{2}-\varepsilon}} \leq C.$$

By Proposition A.1, in order to show that (5.4) holds with G_2 replaced by G_3 , we thus need to assert that $w^3 \in C([0, T], \mathcal{B}_{6/5}^{\frac{1}{2}+\varepsilon})$. It follows from Lemma 5.3 and Proposition A.7 that $w^3 \in C([0, T], \mathcal{B}_{4/3}^{\frac{1}{2}+\varepsilon})$, which is better than needed.

We now turn to G_0 . Recall that both a_1 and a_2 are (locally) uniformly bounded in $\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}$. Hence, we will only discuss the term $a_2(v+w)^2$, the term $a_1(v+w)$ being only easier. Our goal is to show that

$$\langle a_2(v+w)^2(s), w^3(t_i) - w^3(s) \rangle$$

approaches 0 as $|t_i - s| \rightarrow 0$. Using Propositions A.1 and A.7, we observe that

$$\begin{aligned} \langle a_2(v+w)^2(s), w^3(t_i) - w^3(s) \rangle &= \langle a_2, (v+w)^2(s)(w^3(t_i) - w^3(s)) \rangle \\ &\lesssim \|(v+w)^2(s)(w^3(t_i) - w^3(s))\|_{\mathcal{B}_1^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \|(v+w)^2(s)\|_{\mathcal{B}_2^{\frac{1}{2}+\varepsilon}} \|w^3(t_i) - w^3(s)\|_{\mathcal{B}_2^{\frac{1}{2}+\varepsilon}}. \end{aligned}$$

Since $v, w \in C([0, T], \mathcal{B}_4^{\frac{1}{2}+\varepsilon})$, the first term is uniformly bounded. As for the second term, we use Proposition A.6 (in the crude form $\|u\|_{\mathcal{B}_2^{\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{L^2} + \|\nabla u\|_{L^2}$) to bound it by a constant times

$$\|w^3(t_i) - w^3(s)\|_{L^2} + \|w^2(t_i)\nabla w(t_i) - w^2(s)\nabla w(s)\|_{L^2}.$$

We then write

$$\begin{aligned} w^2(t_i)\nabla w(t_i) - w^2(s)\nabla w(s) &= w^2(t_i)(\nabla w(t_i) - \nabla w(s)) - \nabla w(s)(w^2(s) - w^2(t_i)), \end{aligned}$$

and conclude using Hölder's inequality and Lemma 5.3. \square

Similarly to (4.2)–(4.6), we now rewrite the right-hand side of (5.2) as

$$(5.5) \quad \int_0^t \langle \tilde{G}(v, w) + cv, w^{3p-3} \rangle(s) ds = - \int_0^t \langle 3w^2v + 3wv^2 + v^3, w^{3p-3} \rangle(s) ds$$

$$(5.6) \quad - 3 \int_0^t \langle \text{com}_1(v, w) \odot \mathbf{V}, w^{3p-3} \rangle(s) ds$$

$$(5.7) \quad - 3 \int_0^t \langle w \odot \mathbf{V}, w^{3p-3} \rangle(s) ds$$

$$(5.8) \quad + \int_0^t \langle a_2(v+w)^2, w^{3p-3} \rangle(s) ds$$

$$(5.9) \quad + \int_0^t \langle \dots, w^{3p-3} \rangle(s) ds.$$

We now proceed to estimate each of these terms. The first term has a cubic homogeneity. We need to control it with the contribution of the “good term” $-w^3$. This crucially relies on our ability to choose c sufficiently large (and it is the only place where we use this).

Lemma 5.4. *Let $p > 1$ and $\delta > 0$. For every c sufficiently large (depending on δ , ε , p and $K_0(T_{\max})$), and $t \in [0, T)$,*

$$\int_0^t \langle v^3 + 3v^2w + 3vw^2, w^{3p-3} \rangle (s) \, ds \leq \delta \left[\|v_0\|_{L^{3p}}^{3p} + \int_0^t \left(1 + \|w(s)\|_{L^{3p}}^{3p}\right) \, ds \right].$$

Proof. We start with the bound

$$\begin{aligned} & \int_0^t \langle v^3 + 3v^2w + 3vw^2, w^{3p-3} \rangle (s) \, ds \\ & \lesssim \int_0^t \left(\|v^3 w^{3p-3}\|_{L^1} + \|v^2 w^{3p-2}\|_{L^1} + \|v w^{3p-1}\|_{L^1} \right) (s) \, ds \\ & \leq \delta \int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds + C_\delta \int_0^t \|v(s)\|_{L^{3p}}^{3p} \, ds, \end{aligned}$$

which follows from Hölder's and Young's inequalities. Therefore, it is sufficient to bound the space-time L^{3p} -norm of v . By Theorem 3.1 (or rather Remark 3.2), we have

$$(5.10) \quad \|v(s)\|_{L^{3p}} \lesssim e^{-\underline{c}s} \|v_0\|_{L^{3p}} + \int_0^s \frac{e^{-\underline{c}(s-u)}}{(s-u)^{\frac{1}{2}+\varepsilon}} \left(1 + \|w(u)\|_{L^{3p}}\right) \, du,$$

where $\underline{c} = c - 1 - [\Gamma(\frac{1}{2} - \varepsilon)]^{\frac{1}{2}-\varepsilon}$. By Jensen's inequality, we have uniformly over $\underline{c} \geq 1$ and $s \geq 0$,

$$\left(\int_0^s \frac{e^{-\underline{c}(s-u)}}{(s-u)^{\frac{1}{2}+\varepsilon}} \left(1 + \|w(u)\|_{L^{3p}}\right) \, du \right)^{3p} \lesssim \int_0^s \frac{e^{-\underline{c}(s-u)}}{(s-u)^{\frac{1}{2}+\varepsilon}} \left(1 + \|w(u)\|_{L^{3p}}^{3p}\right) \, du.$$

Combining these estimates, we get

$$\begin{aligned} \int_0^t \|v(s)\|_{L^{3p}}^{3p} \, ds & \lesssim \int_0^t e^{-\underline{c}s} \|v_0\|_{L^{3p}}^{3p} \, ds + \int_0^t \int_0^s \frac{e^{-\underline{c}(s-u)}}{(s-u)^{\frac{1}{2}+\varepsilon}} \left(1 + \|w(u)\|_{L^{3p}}^{3p}\right) \, du \, ds \\ & \lesssim \frac{1}{\underline{c}} \|v_0\|_{L^{3p}}^{3p} + K(\underline{c}) \int_0^t \left(1 + \|w(s)\|_{L^{3p}}^{3p}\right) \, ds, \end{aligned}$$

where $K(\underline{c}) = \int_0^\infty \frac{e^{-\underline{c}s}}{s^{\frac{1}{2}+\varepsilon}} < \infty$. Since $K(\underline{c})$ can be made arbitrarily small by taking \underline{c} large enough, this completes the proof. \square

We now use the a priori estimate on $\delta_{st}w$ derived in the previous section to estimate the contribution of the first commutator term.

Lemma 5.5. *Let $p \in [\frac{8}{7}, \frac{8}{3})$ and $\varepsilon > 0$ be small enough. For every $\delta \in (0, 1]$ and $t \in [0, T)$,*

$$(5.11) \quad \left| \int_0^t \langle \text{com}_1(v, w) \ominus \mathfrak{V}, w^{3p-3} \rangle (s) \, ds \right| \lesssim \left(\int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds \right)^{\frac{p-1}{p}} \\ \times \left[1 + \|v_0\|_{\mathcal{B}_{3p}^\beta} + \left(\int_0^t \|w(s)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p \, ds \right)^{\frac{1}{p}} + \delta^{-1-2\varepsilon} \left(\int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds \right)^{\frac{1}{3p}} + \delta^{\frac{1}{8}-2\varepsilon} N(t) \right],$$

where

$$N(t) := \left(\int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds \right)^{\frac{1}{p}} + \left(\int_0^t \|w^2(s)\|_{\mathcal{B}_2^1}^2 \, ds \right)^{\frac{1}{2}} + \left(\int_0^t \|w(s)\|_{\mathcal{B}_2^1}^2 \, ds \right)^{\frac{3}{4}},$$

and where the implicit constant in (5.11) depends on ε , p , T_{\max} , $K_0(T_{\max})$ and c , but not on δ .

Proof. We start with the estimate (dropping the time variable in the notation)

$$\begin{aligned} |\langle [\mathbf{com}_1(v, w) \ominus \mathbf{v}], w^{3p-3} \rangle| &\leq \| \mathbf{com}_1(v, w) \ominus \mathbf{v} \|_{L^p} \| w^{3p-3} \|_{L^{\frac{p}{p-1}}} \\ &\lesssim \| \mathbf{com}_1(v, w) \|_{\mathcal{B}_p^{1+2\varepsilon}} \| w \|_{L^{3p}}^{3p-3}, \end{aligned}$$

where we used Proposition A.7 in the second step. Integrating this in time and applying Hölder's inequality, we get

$$\begin{aligned} &\left| \int_0^t \langle [\mathbf{com}_1(v, w) \ominus \mathbf{v}], w^{3p-3} \rangle ds \right| \\ &\lesssim \left(\int_0^t \| \mathbf{com}_1(v, w)(s) \|_{\mathcal{B}_p^{1+2\varepsilon}}^p ds \right)^{\frac{1}{p}} \left(\int_0^t \| w(s) \|_{L^{3p}}^{3p} ds \right)^{\frac{p-1}{p}}, \end{aligned}$$

so it remains to bound the integral involving the commutator. According to Lemma 4.3, for any fixed s , we have the bound

$$\begin{aligned} \| \mathbf{com}_1(v, w)(s) \|_{\mathcal{B}_p^{1+2\varepsilon}} &\lesssim 1 + t^{-\frac{1-\beta}{2}-\varepsilon} \| v_0 \|_{\mathcal{B}_p^\beta} \\ &\quad + \int_0^s \frac{1}{(s-u)^{\frac{1}{2}+3\varepsilon}} \| w(u) \|_{\mathcal{B}_p^{1-2\varepsilon}} du + \int_0^s \frac{1}{(s-u)^{1+2\varepsilon}} \| \delta_{us} w \|_{L^p} du. \end{aligned}$$

The contribution of $\| v_0 \|_{\mathcal{B}_p^\beta}$ is easily taken care of, since $\frac{1-\beta}{2} < \frac{3}{8} < \frac{1}{p}$. We calculate the L^p norm in time of the first integral, using the bound $\| \cdot \|_{\mathcal{B}_p^{1-2\varepsilon}} \lesssim \| \cdot \|_{\mathcal{B}_p^{1+2\varepsilon}}$:

$$\int_0^t \left(\int_0^s \frac{1}{(s-u)^{\frac{1}{2}+3\varepsilon}} \| w(u) \|_{\mathcal{B}_p^{1-2\varepsilon}} du \right)^p ds \lesssim \int_0^t \| w(s) \|_{\mathcal{B}_p^{1+2\varepsilon}}^p ds,$$

which gives the second term in the second line of (5.11). For the remaining integral, we need to make use of the time regularity bounds derived in Theorem 4.1. First, we write for any $\delta > 0$,

$$\int_0^{(s-\delta)\vee 0} \frac{1}{(s-u)^{1+2\varepsilon}} \| \delta_{us} w \|_{L^p} du \lesssim \frac{1}{\delta^{1+2\varepsilon}} \int_0^s (\| w(u) \|_{L^p} + \| w(s) \|_{L^p}) du,$$

which implies that

$$\int_0^t \left(\int_0^{(s-\delta)\vee 0} \frac{1}{(s-u)^{1+2\varepsilon}} \| \delta_{us} w \|_{L^p} du \right)^p ds \lesssim \frac{1}{\delta^{(1+2\varepsilon)p}} \int_0^t \| w(s) \|_{L^p}^p ds.$$

For the remaining integral, we use Theorem 4.1 to write

$$\begin{aligned} &\int_0^t \left(\int_{(s-\delta)\vee 0}^s \frac{1}{(s-u)^{1+2\varepsilon}} \| \delta_{us} w \|_{L^p} du \right)^p ds \\ &\lesssim \int_0^t \left(\int_{(s-\delta)\vee 0}^s \frac{1}{(s-u)^{\frac{7}{8}+2\varepsilon}} \left[\tilde{N}(t) + \| w(u) \|_{\mathcal{B}_p^\beta} \right] du \right)^p ds, \end{aligned}$$

where we have set

$$\begin{aligned} (5.12) \quad \tilde{N}(t) &:= 1 + \| v_0 \|_{\mathcal{B}_{3p}^\beta}^3 + \left(\int_0^t \| w(u) \|_{L^{3p}}^{3p} du \right)^{\frac{1}{p}} \\ &\quad + \left(\int_0^t \| w(u) \|_{\mathcal{B}_p^{1+2\varepsilon}}^p du \right)^{\frac{1}{p}} + \left(\int_0^t \| w^2(u) \|_{\mathcal{B}_2^1}^2 du \right)^{\frac{1}{2}} + \left(\int_0^t \| w(u) \|_{\mathcal{B}_2^1}^2 du \right)^{\frac{3}{4}}. \end{aligned}$$

Note that $\tilde{N}(t)$ does not depend on the variables of integration, and that

$$\int_0^t \left(\int_{(s-\delta)\vee 0}^s \frac{1}{(s-u)^{\frac{7}{8}+2\varepsilon}} du \right)^p ds \lesssim \delta^{p(\frac{1}{8}-2\varepsilon)} t.$$

Finally, by Jensen's inequality,

$$\begin{aligned} & \int_0^t \left(\int_{(s-\delta)\vee 0}^s \frac{1}{(s-u)^{\frac{7}{8}+2\varepsilon}} \|w(u)\|_{\mathcal{B}_p^\beta}^p du \right)^p ds \\ & \lesssim \int_0^t \int_{(s-\delta)\vee 0}^s \frac{1}{(s-u)^{\frac{7}{8}+2\varepsilon}} \|w(u)\|_{\mathcal{B}_p^\beta}^p du ds \lesssim \int_0^t \|w(u)\|_{\mathcal{B}_p^\beta}^p du. \end{aligned}$$

Since $\|w(u)\|_{\mathcal{B}_p^\beta} \leq \|w(u)\|_{\mathcal{B}_p^{1+2\varepsilon}}$, this completes the proof. \square

We now turn to the term involving $w \ominus \mathbf{V}$, which can only be controlled by a norm of w with regularity index above 1.

Lemma 5.6. *Let $p > 1$ and $\varepsilon > 0$. For every $t \in [0, T)$,*

$$\left| \int_0^t \langle w \ominus \mathbf{V}, w^{3p-3} \rangle(s) ds \right| \lesssim \left(\int_0^t \|w(s)\|_{L^{3p}}^{3p} ds \right)^{\frac{p-1}{p}} \left(\int_0^t \|w(s)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p ds \right)^{\frac{1}{p}}$$

where the implicit constant depends on ε , p , and $K_0(T_{\max})$.

Proof. This bound follows directly from Hölder's inequality and the bound

$$\|w \ominus \mathbf{V}(s)\|_{L^p} \lesssim \|w(s)\|_{\mathcal{B}_p^{1+2\varepsilon}}. \quad \square$$

The quadratic non-linearity is rather delicate to handle.

Lemma 5.7. *Let $p > 1$ and $\varepsilon > 0$ be sufficiently small. For every $\delta \in (0, 1]$, there exists $C_\delta < \infty$ (depending on δ , ε , p , T_{\max} , $K_0(T_{\max})$ and c) such that for every $t \in [0, T)$,*

$$(5.13) \quad \left| \int_0^t \langle a_2(v+w)^2, w^{3p-3} \rangle(s) ds \right| \leq C_\delta + \delta \left[\|v_0\|_{\mathcal{B}_p^{2\varepsilon}}^{3p} + \int_0^t \|\nabla w\|^2 w^{3p-4}(s)_{L^1} ds + \int_0^t \|w(s)\|_{L^{3p}}^{3p} ds \right].$$

Proof. We will use repeatedly the following version of Young's inequality: for every $\gamma_i > 0$ such that $\sum \gamma_i < 1$ and $\delta > 0$, there exists C_δ such that uniformly over $x_i \geq 0$,

$$(5.14) \quad \prod x_i^{\gamma_i} \leq C_\delta + \delta \sum x_i.$$

We treat the term of highest homogeneity in w first. Recall that a_2 is uniformly bounded in $\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}$. We write, using Propositions A.1 and A.6 (dropping the time variable in the notation),

$$(5.15) \quad \begin{aligned} \langle a_2 w^2, w^{3p-3} \rangle &= \langle a_2, w^{3p-1} \rangle \lesssim \|w^{3p-1}\|_{\mathcal{B}_1^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \left(\|w^{3p-1}\|_{L^1}^{\frac{1}{2}-\varepsilon} \|\nabla w\|_{L^1} w^{3p-2} \right)^{\frac{1}{2}+\varepsilon} + \|w^{3p-1}\|_{L^1}. \end{aligned}$$

An application of the Cauchy-Schwarz inequality yields the bound

$$\|\nabla w\|_{L^1} \|w^{3p-2}\|_{L^1} \leq \|\nabla w\|_{L^1}^2 \|w^{3p-4}\|_{L^1}^{\frac{1}{2}} \|w^{3p}\|_{L^1}^{\frac{1}{2}}.$$

Using that $\|w^{3p-1}\|_{L^1} \leq \|w\|_{L^{3p}}^{3p-1}$ and $\|w^{3p}\|_{L^1} \leq \|w\|_{L^{3p}}^{3p}$ and then Young's inequality, we deduce from (5.15) that

$$\langle a_2 w^2, w^{3p-3} \rangle \lesssim \|w\|_{L^{3p}}^{(3-\frac{3-2\varepsilon}{4})3p - (\frac{1-2\varepsilon}{2})} \|\nabla w\|_{L^1}^2 \|w^{3p-4}\|_{L^1}^{\frac{1+2\varepsilon}{4}} + \|w\|_{L^{3p}}^{3p-1}$$

Integrating in time, using Jensen's inequality and (5.14), we obtain the bound on the right-hand side of (5.13).

We next treat the term involving v^2 in the left side of (5.15). We use Propositions A.1 and A.7 to write, for any fixed time s (again dropping the time argument to lighten the notation)

$$(5.16) \quad \begin{aligned} \langle a_2 v^2, w^{3p-3} \rangle &= \langle a_2, w^{3p-3} v^2 \rangle \lesssim \|w^{3p-3} v^2\|_{\mathcal{B}_1^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \|w^{3p-3}\|_{\mathcal{B}_{q_1}^{\frac{1}{2}+\varepsilon}} \|v^2\|_{\mathcal{B}_{q_2}^{\frac{1}{2}+\varepsilon}}, \end{aligned}$$

where the exponents q_1 and q_2 satisfy $1 = \frac{1}{q_1} + \frac{1}{q_2}$ and will be specified below. The term involving w is bounded similarly to the previous step. Indeed, we appeal again to Proposition A.6 to write

$$(5.17) \quad \|w^{3p-3}\|_{\mathcal{B}_{q_1}^{\frac{1}{2}+\varepsilon}} \lesssim \left(\|\nabla w\|_{L^{q_1}} \|w^{3p-4}\|_{L^{q_1}}^{\frac{1}{2}+\varepsilon} \|w^{3p-3}\|_{L^{q_1}}^{\frac{1}{2}-\varepsilon} \right) + \|w^{3p-3}\|_{L^{q_1}}.$$

In order to control the L^{q_1} -norm of w^{3p-3} by the L^{3p} norm of w , we require

$$(5.18) \quad q_1 \leq \frac{p}{p-1} = 1 + \frac{1}{p-1},$$

which allows us to write $\|w^{3p-3}\|_{L^{q_1}} \lesssim \|w\|_{L^{3p}}^{3p-3}$ by Jensen's inequality. For the term involving the gradient of w , we write

$$\|\nabla w\|_{L^{q_1}} \|w^{3p-4}\|_{L^{q_1}} \leq \|\nabla w\|_{L^2} \|w^{\frac{3p-4}{2}}\|_{L^{q_3}}$$

where

$$(5.19) \quad \frac{1}{q_1} = \frac{1}{2} + \frac{1}{q_3}.$$

We assume $q_3 \leq \frac{6p}{3p-4}$, which amounts to

$$(5.20) \quad \frac{1}{q_1} \geq \frac{1}{2} + \frac{3p-4}{6p} = \frac{3p-2}{3p}, \quad \text{that is, } q_1 \leq \frac{3p}{3p-2} = 1 + \frac{2}{3p-2}.$$

For any $p > 1$, condition (5.20) implies (5.18), and therefore we can assume from now on that equality holds in (5.20). In this case, we get

$$\|w^{3p-3}\|_{\mathcal{B}_{\frac{3p}{3p-2}}^{\frac{1}{2}+\varepsilon}} \lesssim \|w\|_{L^{3p}}^{(\frac{3-2\varepsilon}{4})3p - (\frac{5-2\varepsilon}{2})} \|\nabla w\|_{L^1}^{2w^{3p-4}} \|\nabla w\|_{L^1}^{\frac{1+2\varepsilon}{4}} + \|w\|_{L^{3p}}^{3p-3}.$$

We now bound the term involving v^2 in (5.16), noting that for our choice of $q_1 = \frac{3p}{3p-2}$, we get $q_2 = \frac{3p}{2}$. We therefore have to bound

$$\|v^2\|_{\mathcal{B}_{\frac{3p}{2}}^{\frac{1}{2}+\varepsilon}} \lesssim \|v\|_{\mathcal{B}_{3p}^{\frac{1}{2}+\varepsilon}},$$

by Proposition A.7. For any fixed s , we can use Theorem 3.1 to bound this quantity by (recall that $\beta = \frac{1}{2} + 2\varepsilon$)

$$\|v(s)\|_{\mathcal{B}_{\frac{3p}{2}}^{\frac{1}{2}+\varepsilon}} \lesssim \|v_0\|_{\mathcal{B}_{3p}^\beta} + \int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) du.$$

Summarising these calculations, we get

$$(5.21) \quad \begin{aligned} &\int_0^t \langle a_2 v^2, w^{3p-3} \rangle(s) ds \\ &\lesssim \int_0^t \left[\|w(s)\|_{L^{3p}}^{(\frac{3-2\varepsilon}{4})3p - (\frac{5-2\varepsilon}{2})} \|\nabla w(s)\|_{L^1}^{2w^{3p-4}(s)} \|\nabla w(s)\|_{L^1}^{\frac{1+2\varepsilon}{4}} + \|w(s)\|_{L^{3p}}^{3p-3} \right] \\ &\quad \times \left[\|v_0\|_{\mathcal{B}_{3p}^\beta} + \int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) du \right]^2 ds. \end{aligned}$$

The contribution of $\|v_0\|_{\mathcal{B}_{3p}^\beta}$ can be handled with no difficulty using (5.14). We now leave it aside, and treat first the (easier) contribution coming from $\|w(s)\|_{L^{3p}}^{3p-3}$. Applying Hölder's and Jensen's inequalities, we get

$$\begin{aligned} & \int_0^t \|w(s)\|_{L^{3p}}^{3p-3} \left(\int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) \, du \right)^2 \, ds \\ & \lesssim \left(\int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds \right)^{\frac{3p-3}{3p}} \left[\int_0^t \left(\int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) \, du \right)^{3p} \, ds \right]^{\frac{2}{3p}} \\ & \lesssim \left(\int_0^t (1 + \|w(s)\|_{L^{3p}}^{3p}) \, ds \right)^{1-\frac{1}{3p}}. \end{aligned}$$

Now turning to the contribution of the first term of the first sum in (5.21) (still leaving $\|v_0\|_{\mathcal{B}_{3p}^\beta}$ aside, which has already been treated), we apply Hölder's inequality to bound it by

$$\begin{aligned} & \left(\int_0^t \|\nabla w\|^2 w^{3p-4}(s) \|_{L^1} \, ds \right)^{\frac{1+2\varepsilon}{4}} \left(\int_0^t \|w(s)\|_{L^{3p}}^{3p} \, ds \right)^{\frac{3-2\varepsilon}{4} - \frac{5-2\varepsilon}{6p}} \\ & \quad \times \left(\int_0^t \left(\int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) \, du \right)^{2q_4} \, ds \right)^{\frac{1}{q_4}}, \end{aligned}$$

where

$$\frac{1}{q_4} = \frac{5-2\varepsilon}{6p}.$$

By Jensen's inequality,

$$\begin{aligned} & \left(\int_0^t \left(\int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) \, du \right)^{2q_4} \, ds \right)^{\frac{1}{q_4}} \\ & \lesssim \left(\int_0^t 1 + \|w(s)\|_{L^{3p}}^{2q_4} \, ds \right)^{\frac{1}{q_4}}. \end{aligned}$$

This leads to the desired bound by another application of (5.14).

Finally, we move to the mixed term involving the product vw . As before, we start by writing for any fixed time (dropping the time argument, to lighten the notation)

$$\langle a_2 vw, w^{3p-3} \rangle \lesssim \|vw^{3p-2}\|_{\mathcal{B}_1^{\frac{1}{2}+\varepsilon}} \lesssim \|v\|_{\mathcal{B}_{q_5}^{\frac{1}{2}+\varepsilon}} \|w^{3p-2}\|_{\mathcal{B}_{q_6}^{\frac{1}{2}+\varepsilon}},$$

where

$$(5.22) \quad \frac{1}{q_5} + \frac{1}{q_6} = 1.$$

To bound the norm of w , we use Proposition A.6 to write

$$\|w^{3p-2}\|_{\mathcal{B}_{q_6}^{\frac{1}{2}+\varepsilon}} \lesssim \|w^{3p-2}\|_{L^{q_6}} + \|\nabla w\|_{L^{q_6}} \|w^{3p-3}\|_{L^{q_6}}^{\frac{1}{2}+\varepsilon} \|w^{3p-2}\|_{L^{q_6}}^{\frac{1}{2}-\varepsilon}.$$

Assuming that

$$(5.23) \quad q_6 \leq \frac{3p}{3p-2},$$

we can bound $\|w^{3p-2}\|_{L^{q_6}} \lesssim \|w^{3p-2}\|_{L^{3p}}^{3p-2}$. For the term involving the gradient, we write

$$\|\nabla w\|_{L^{q_6}} \|w^{3p-3}\|_{L^{q_6}} \lesssim \|\nabla w\|_{L^2} \|w^{\frac{3p-4}{2}}\|_{L^2} \|w^{\frac{3p-2}{2}}\|_{L^{q_7}},$$

provided that

$$(5.24) \quad \frac{1}{q_6} = \frac{1}{2} + \frac{1}{q_7}.$$

As before, we assume that $q_7 \leq \frac{6p}{3p-2}$, which allows to bound $\|w^{\frac{3p-2}{2}}\|_{L^{q_7}} \lesssim \|w\|_{L^{3p}}^{\frac{3p-2}{2}}$. With this constraint, (5.24) turns into

$$(5.25) \quad q_6 \leq \frac{3p}{3p-1}.$$

Noting that this condition implies our earlier condition (5.23), we can assume from now on that equality holds in (5.25), which by (5.22) implies that $q_5 = 3p$. We summarise these calculations in the estimate

$$\|w^{3p-2}\|_{\mathcal{B}_{\frac{3p}{3p-1}}^{\frac{1}{2}+\varepsilon}} \lesssim \|w\|_{L^{3p}}^{3p-2} + \|w\|_{L^{3p}}^{(\frac{1}{2}-\varepsilon)(3p-2)+(\frac{1}{2}+\varepsilon)(\frac{3p-2}{3})} \|\nabla w\|_{L^2} w^{\frac{3p-4}{2}}\|_{L^2}^{\frac{1}{2}+\varepsilon}.$$

For the term involving v , we get from Theorem 3.1 that

$$\begin{aligned} \int_0^t \|v(s)\|_{\mathcal{B}_{3p}^{\frac{3p}{2}+\varepsilon}}^{3p} ds &\lesssim \int_0^t \left(\|v_0\|_{\mathcal{B}_{3p}^{\frac{3p}{2}+\varepsilon}} + \int_0^s \frac{1}{(s-u)^{\frac{3}{4}+\varepsilon}} (1 + \|w(u)\|_{L^{3p}}) du \right)^{3p} ds \\ &\lesssim \|v_0\|_{\mathcal{B}_{3p}^\beta}^{3p} + \int_0^t (1 + \|w(s)\|_{L^{3p}}^{3p}) ds. \end{aligned}$$

The conclusion follows by integrating these bounds, using Jensen's inequality and applying (5.14). \square

Lemma 5.8. *Let $p > 1$ and $\varepsilon > 0$ be sufficiently small. For every $\delta \in (0, 1]$, there exists $C_\delta < \infty$ (depending on $\delta, \varepsilon, p, T_{\max}, K_0(T_{\max})$ and c) such that for every $t \in [0, T)$,*

$$\begin{aligned} \int_0^t \langle [\dots], w^{3p-3} \rangle (s) ds &\leq C_\delta \\ &+ \delta \left[\|v_0\|_{\mathcal{B}_{3p}^\beta}^{3p} + \int_0^t \|\nabla w\|^2 w^{3p-4}(s) \|_{L^1} ds + \int_0^t \|w(s)\|_{L^{3p}}^{3p} ds \right]. \end{aligned}$$

The dots ... represent all the terms left out in (5.9) (spelled out explicitly in (5.26) below).

Proof. We need to bound

$$(5.26) \quad \int_0^t \langle [-3\text{com}_2(v+w) - 3(v+w - \Psi) \odot \mathbf{v} + a_0 + a_1(v+w) + cv], w^{3p-3} \rangle (s) ds.$$

For the first term,

$$\begin{aligned} \langle \text{com}_2(v+w), w^{3p-3} \rangle (s) &\lesssim \|\text{com}_2(v+w)(s)\|_{L^p} \|w(s)\|_{L^{3p}}^{\frac{3p-3}{3p}} \\ &\lesssim \|(v+w)(s)\|_{\mathcal{B}_p^{3\varepsilon}} \|w(s)\|_{L^{3p}}^{\frac{3p-3}{3p}}, \end{aligned}$$

by Proposition A.9. Hence, the bound for this term follows from arguments very similar to those of the previous proof, and we leave out the details. For the second term in (5.26), Proposition A.1 gives

$$\langle (v+w - \Psi) \odot \mathbf{v}, w^{3p-3} \rangle (s) \lesssim \|(v+w - \Psi) \odot \mathbf{v}\|_{\mathcal{B}_p^{-\frac{1}{2}-2\varepsilon}} \|w^{3p-3}\|_{\mathcal{B}_{\frac{3p}{p-1}}^{\frac{1}{2}+2\varepsilon}}.$$

We have seen how to bound $\|w^{3p-3}\|_{\mathcal{B}_p^{\frac{1}{2}+2\varepsilon}}$ in the proof of the previous lemma, see (5.17). For the other term, by Proposition A.7,

$$\|(v + w - \Psi) \odot \mathbf{V}\|_{\mathcal{B}_p^{-\frac{1}{2}-2\varepsilon}} \lesssim \|v + w - \Psi\|_{\mathcal{B}_p^{\frac{1}{2}-\varepsilon}},$$

which we can bound to obtain the right-hand side of (5.26), as already discussed. The other terms in the left-hand side of (5.26) are only easier. \square

Proof of Theorem 5.1. Combining Proposition 5.2 with the bounds derived in Lemmas 5.4–5.8 (and with Young’s inequality and comparisons of norms), we obtain, for $p \in \{\frac{4}{3}, 2\}$, $\varepsilon > 0$ sufficiently small and every $\delta \in (0, 1]$, the existence of a constant $C_\delta < \infty$ depending on $\delta, \varepsilon, T_{\max}, K_0(T_{\max})$ and c such that

$$(5.27) \quad \begin{aligned} & \|w(t)\|_{L^{3p-2}}^{3p-2} + \int_0^t \left(\|\nabla w\|^2 w^{3p-4}(s) \|_{L^1} + \|w(s)\|_{L^{3p}}^{3p} \right) ds \\ & \leq C_\delta \left[1 + \|w_0\|_{L^{3p-2}}^{3p-2} + \int_0^t \|w(s)\|_{\mathcal{B}_p^{1+2\varepsilon}}^p ds \right] \\ & \quad + \delta \left[\|v_0\|_{\mathcal{B}_{3p}^\beta}^{3p} + \left(\int_0^t \|w^2(s)\|_{\mathcal{B}_2^1}^2 ds \right)^{\frac{p}{2}} + \left(\int_0^t \|w(s)\|_{\mathcal{B}_2^1}^2 ds \right)^{\frac{3p}{4}} \right]. \end{aligned}$$

Specialising this estimate to the case $p = \frac{4}{3}$ and choosing $\delta > 0$ sufficiently small allows to absorb the last term of the estimate. Indeed, by Proposition A.6, we have

$$(5.28) \quad \|w(s)\|_{\mathcal{B}_2^1}^2 \lesssim \|w(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2,$$

and thus

$$(5.29) \quad \begin{aligned} & \|w(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla w\|^2(s) \|_{L^1} + \|w(s)\|_{L^4}^4 \right) ds \\ & \lesssim 1 + \|w_0\|_{L^2}^2 + \|v_0\|_{\mathcal{B}_4^\beta}^4 + \int_0^t \|w(s)\|_{\mathcal{B}_4^{1+2\varepsilon}}^{\frac{4}{3}} ds + \left(\int_0^t \|w^2(s)\|_{\mathcal{B}_2^1}^2 ds \right)^{\frac{2}{3}}. \end{aligned}$$

Note that by (5.28), the right-hand side is also an upper bound for $\int_0^t \|w(s)\|_{\mathcal{B}_2^1}^2 ds$.

We now show that Theorem 5.1 is true for $p = 2$, that is,

$$(5.30) \quad \begin{aligned} & \|w(t)\|_{L^4}^4 + \int_0^t \left(\|w^2|\nabla w|^2(s) \|_{L^1} + \|w(s)\|_{L^6}^6 \right) ds \\ & \lesssim 1 + \|w_0\|_{L^4}^4 + \|v_0\|_{\mathcal{B}_6^\beta}^6 + \int_0^t \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 ds. \end{aligned}$$

In order to do so, we turn back to (5.27), this time with the choice $p = 2$. We just obtained an upper bound for the last integral in this estimate, which is now raised to the power $\frac{3}{2}$. We check that each term to the power $\frac{3}{2}$ in the right-hand side of (5.29) is bounded by a term in the right-hand side of (5.30), except for

$$\int_0^t \|w^2(s)\|_{\mathcal{B}_2^1}^2 ds,$$

which was already present in the right-hand side of (5.27) with $p = 2$. This term can be absorbed, choosing $\delta > 0$ sufficiently small and using Proposition A.6 again:

$$(5.31) \quad \|w^2(s)\|_{\mathcal{B}_2^1}^2 \lesssim \|w^2(s)\|_{L^2}^2 + \|w\nabla w(s)\|_{L^2}^2,$$

therefore (5.30) is proved. By (5.31), the right-hand side of (5.30) is also an upper bound for $\int_0^t \|w^2(s)\|_{\mathcal{B}_2^1}^2 ds$, and recall that the right-hand side of (5.29) is an upper

bound for $\int_0^t \|w(s)\|_{\mathcal{B}_2^1}^2 ds$. Using these estimates and (5.27) again for $p = \frac{4}{3}$ thus completes the proof of the theorem. \square

6. GRONWALL-TYPE ARGUMENT FOR w

For the first time, we are in a position to derive a self-contained a priori bound for a quantity involving v or w .

Theorem 6.1. *Let $c \geq c_0$ as given by Theorem 5.1. There exists a constant $C < \infty$ depending only on ε , T_{\max} , $K_0(T_{\max})$, c and on an upper bound on $\|v_0\|_{\mathcal{B}_6^\beta} \vee \|w_0\|_{\mathcal{B}_2^\gamma}$ such that*

$$\int_0^T \left(\|w(r)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + \|w(r)\|_{L^6}^6 + \|v(r)\|_{\mathcal{B}_6^\beta}^6 \right) dr \leq C.$$

From now on, we fix $c \geq c_0$ as given by Theorem 5.1. Throughout this section, we will use the short-hand notation

$$\mathcal{I}(s, t) = \left(\int_s^t \|w(r)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 dr \right)^{\frac{1}{2}}, \quad \mathcal{B}(s) = \|v(s)\|_{\mathcal{B}_6^\beta}^3 + \|w(s)\|_{L^4}^2.$$

We begin by rephrasing Theorem 5.1, allowing for more general starting time and dropping the first non-negative term on the left-hand side of (5.1):

$$(6.1) \quad \text{for } p \in \{\frac{4}{3}, 2\} \text{ and } s \leq t \in [0, T], \\ \int_s^t \left(\|\nabla w\|^2 w^{3p-4}(r) \|_{L^1} + \|w(r)\|_{L^{3p}}^{3p} \right) dr \lesssim 1 + \mathcal{I}^p(s, t) + \mathcal{B}^p(s),$$

where the implicit constant depends on ε , T_{\max} , $K_0(T_{\max})$ and c . We will mostly use (6.1) for $p = 2$. The only applications of the case $p = \frac{4}{3}$ are in the proofs of Lemmas 6.4 and 6.6.

We will use the mild form (2.2) of (1.22) once more to perform a Gronwall-type argument for \mathcal{I}^2 . The bound (6.1) plays a crucial role, because it allows us to replace “non-linear” quantities by “linear” ones. We start by proving the following proposition.

Proposition 6.2. *There exist $t_\star > 0$ and $C < \infty$ (both depending on ε , T_{\max} , $K_0(T_{\max})$ and $c \geq c_0$) such that for every $s \leq t \in [0, T]$ satisfying $t - s \leq 2t_\star$,*

$$(6.2) \quad \mathcal{I}^2(s, t) \leq C(t - s) \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + C(t - s)^{\frac{1}{3}} (1 + \mathcal{B}^2(s)).$$

As before, the proof of this proposition is split up into several steps. We use the mild formulation for w , Proposition A.13 and Remark A.3 to write

$$\mathcal{I}^2(s, t) \lesssim (t - s) \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + \sum_{j=1}^5 \int_s^t \mathcal{W}_j^2(s, r) dr,$$

where

$$(6.3) \quad \mathcal{W}_1(s, r) = \int_s^r \frac{1}{(r - \tau)^{\frac{1+2\varepsilon}{2}}} (\|w(\tau)\|_{L^6}^3 + \|v(\tau)\|_{L^6}^3) d\tau,$$

$$(6.4) \quad \mathcal{W}_2(s, r) = \int_s^r \frac{1}{(r - \tau)^{\frac{1+2\varepsilon}{2}}} \|\text{com}_1(v, w) \ominus \mathbf{V}\|_{L^2}(\tau) d\tau,$$

$$(6.5) \quad \mathcal{W}_3(s, r) = \int_s^r \frac{1}{(r - \tau)^{\frac{1+2\varepsilon}{2}}} \|w \ominus \mathbf{V}\|_{L^2}(\tau) d\tau,$$

$$(6.6) \quad \mathcal{W}_4(s, r) = \int_s^r \frac{1}{(r - \tau)^{\frac{3+6\varepsilon}{4}}} \|a_2(v + w)^2\|_{\mathcal{B}_2^{-\frac{1}{2}-\varepsilon}}(\tau) d\tau,$$

$$(6.7) \quad \mathcal{W}_5(s, r) = \int_s^r \frac{1}{(r - \tau)^{\frac{3+8\varepsilon}{4}}} \|\dots\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}}(\tau) d\tau.$$

We proceed by bounding these terms one by one.

Lemma 6.3. *For every $s \leq t \in [0, T)$,*

$$\int_s^t \mathcal{W}_1^2(s, r) \, dr \lesssim (t-s)^{1-2\varepsilon} (\mathcal{I}^2(s, t) + \mathcal{B}^2(s)),$$

where the implicit constant depends on ε , T_{\max} , $K_0(T_{\max})$ and c .

Proof. By Young's convolution inequality,

$$\begin{aligned} \int_s^t \mathcal{W}_1^2(s, r) \, dr &= \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{1+2\varepsilon}{2}}} (\|w(\tau)\|_{L^6}^3 + \|v(\tau)\|_{L^6}^3) \, d\tau \right)^2 \, dr \\ &\lesssim \left(\int_s^t \frac{1}{(t-r)^{\frac{1+2\varepsilon}{2}}} \, dr \right)^2 \int_s^t (\|w(r)\|_{L^6}^6 + \|v(r)\|_{L^6}^6) \, dr \\ &\lesssim (t-s)^{1-2\varepsilon} \int_s^t (\|w(r)\|_{L^6}^6 + \|v(r)\|_{L^6}^6) \, dr. \end{aligned}$$

According to (6.1), the integral involving w is bounded by

$$(6.8) \quad \int_s^t \|w(r)\|_{L^6}^6 \, dr \lesssim 1 + \mathcal{I}^2(s, t) + \mathcal{B}^2(s).$$

For the term involving v , we invoke Theorem 3.1 (in the form given by Remark 3.2) to get

$$\begin{aligned} \int_s^t \|v(r)\|_{L^6}^6 \, dr &\lesssim (t-s) \|v(s)\|_{L^6}^6 + \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{1}{2}+\varepsilon}} (1 + \|w(\tau)\|_{L^6}) \, d\tau \right)^6 \, dr \\ &\lesssim (t-s) \|v(s)\|_{L^6}^6 + (t-s)^{3-6\varepsilon} \int_s^t (1 + \|w(r)\|_{L^6}^6) \, dr. \end{aligned}$$

The claim thus follows by (6.8). \square

Lemma 6.4. *For every $s \leq t \in [0, T)$,*

$$\int_s^t \mathcal{W}_2^2(s, r) \, dr \lesssim (t-s)^{1-2\varepsilon} + (t-s)^{1+\beta-4\varepsilon} \|v(s)\|_{\mathcal{B}_2^\beta} + (t-s) \mathcal{I}^2(s, t),$$

where the implicit constant depends on ε , T_{\max} , $K_0(T_{\max})$ and c .

Proof. As before, we start by writing

$$\begin{aligned} \int_s^t \mathcal{W}_2^2(s, r) \, dr &\lesssim \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{1+2\varepsilon}{2}}} \|\mathbf{com}_1(v, w) \ominus \mathbf{v}\|_{L^2}(\tau) \, d\tau \right)^2 \, dr \\ &\lesssim (t-s)^{1-2\varepsilon} \int_s^t \|\mathbf{com}_1(v, w) \ominus \mathbf{v}\|_{L^2}^2(r) \, dr \\ &\lesssim (t-s)^{1-2\varepsilon} \int_s^t \|\mathbf{com}_1(v, w)(r)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 \, dr. \end{aligned}$$

According to Lemma 4.3 and the comparison $\|\cdot\|_{\mathcal{B}_2^\beta} \lesssim \|\cdot\|_{\mathcal{B}_2^{1+2\varepsilon}}$, we have for any r that

$$\begin{aligned} \|\mathbf{com}_1(v, w)(r)\|_{\mathcal{B}_2^{1+2\varepsilon}} &\lesssim 1 + (r-s)^{-\frac{1+2\varepsilon-\beta}{2}} \|v(s)\|_{\mathcal{B}_2^\beta} \\ &\quad + \int_s^r \frac{1}{(r-\tau)^{1-\frac{\beta}{2}+2\varepsilon}} \|w(\tau)\|_{\mathcal{B}_2^{1+2\varepsilon}} \, d\tau \\ &\quad + \int_s^r \frac{1}{(r-\tau)^{1+2\varepsilon}} \|\delta_{\tau r} w\|_{L^2} \, d\tau. \end{aligned}$$

We now derive bounds on the temporal L^2 norm of each of these terms separately. For the term involving $v(s)$, we have

$$\int_s^t (r-s)^{-(1+2\varepsilon-\beta)} \|v(s)\|_{\mathcal{B}_2^\beta}^2 dr \lesssim (t-s)^{\beta-2\varepsilon} \|v(s)\|_{\mathcal{B}_2^\beta}^2,$$

while for the term in the second line,

$$\int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{1-\frac{\beta}{2}+2\varepsilon}} \|w(\tau)\|_{\mathcal{B}_2^{1+2\varepsilon}} d\tau \right)^2 ds \lesssim (t-s)^{\beta-4\varepsilon} \int_s^t \|w(r)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 dr.$$

For the term involving $\delta_{\tau r} w$, Theorem 4.1 ensures that

$$\|\delta_{\tau r} w\|_{L^2} \lesssim (\tau-r)^{\frac{1}{8}} (\|w(r)\|_{\mathcal{B}_2^{1+2\varepsilon}} + \tilde{N}(s,t)),$$

where the remainder $\tilde{N}(s,t)$ (as in (5.12)) is given by

$$\begin{aligned} \tilde{N}(s,t) &= 1 + \|v(s)\|_{\mathcal{B}_6^\beta}^3 + \left(\int_s^t \|w(r')\|_{L^6}^6 dr' \right)^{\frac{1}{2}} \\ &\quad + \left(\int_s^t \|w(r')\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 dr' \right)^{\frac{1}{2}} + \left(\int_s^t \|w^2(r')\|_{\mathcal{B}_2^1}^2 dr' \right)^{\frac{1}{2}} + \left(\int_s^t \|w(r')\|_{\mathcal{B}_2^1}^2 dr' \right)^{\frac{3}{4}}. \end{aligned}$$

In particular, $\tilde{N}(s,t)$ does not depend on the variables τ and r appearing in the integration. Therefore, we can write

$$\begin{aligned} &\int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{1+2\varepsilon}} \|\delta_{\tau r} w\|_{L^2} d\tau \right)^2 dr \\ &\lesssim \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{7}{8}+2\varepsilon}} \|w(\tau)\|_{\mathcal{B}_2^{1+2\varepsilon}} d\tau \right)^2 dr + \tilde{N}^2(s,t) (t-s)^{\frac{5}{4}-4\varepsilon} \\ &\lesssim (t-s)^{\frac{1}{4}-4\varepsilon} \int_s^t \|w(r)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 dr + \tilde{N}^2(s,t) (t-s)^{\frac{5}{4}-4\varepsilon}. \end{aligned}$$

Now the claim follows observing that by (6.1) (applied both for $p=2$ and $p=\frac{4}{3}$) and Proposition A.6, we have $\tilde{N}^2(s,t) \lesssim 1 + \mathcal{I}^2(s,t) + \mathcal{B}^2(s)$. \square

Lemma 6.5. *For every $s \leq t \in [0, T)$,*

$$\int_s^t \mathcal{W}_3^2(s,r) dr \lesssim (t-s)^{1-2\varepsilon} \mathcal{I}^2(s,t),$$

where the implicit constant depends on ε and $K_0(T_{\max})$.

Proof. This bound follows easily by writing

$$\begin{aligned} &\int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{1+2\varepsilon}{2}}} \|w \ominus \mathbf{v}\|_{L^2(\tau)} d\tau \right)^2 dr \\ &\lesssim \left(\int_s^t \frac{1}{(t-r)^{\frac{1+2\varepsilon}{2}}} dr \right)^2 \int_s^t \|w \ominus \mathbf{v}\|_{L^2}^2(r) dr \\ &\lesssim (t-s)^{1-2\varepsilon} \int_s^t \|w\|_{\mathcal{B}_2^{1+2\varepsilon}}^2(r) dr. \end{aligned} \quad \square$$

Lemma 6.6. *Let $\varepsilon > 0$ be sufficiently small. For every $s \leq t \in [0, T)$,*

$$\int_s^t \mathcal{W}_4^2(s,r) dr \lesssim (t-s)^{\frac{1}{3}} (1 + \mathcal{I}^2(s,t) + \mathcal{B}^2(s)),$$

where the implicit constant depends on ε , T_{\max} , $K_0(T_{\max})$ and c .

Proof. The term \mathcal{W}_4 naturally splits into one term involving w^2 , one term involving v^2 , and a mixed term involving the product vw . To bound the terms involving w^2 and v^2 , we observe as before that

$$\begin{aligned} & \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3+6\varepsilon}{4}}} \|a_2(v^2 + w^2)\|_{\mathcal{B}_2^{-\frac{1}{2}-\varepsilon}}(\tau) \, d\tau \right)^2 \, dr \\ & \lesssim \left(\int_s^t \frac{1}{(t-r)^{\frac{3+6\varepsilon}{4}}} \, dr \right)^2 \int_s^t \|a_2(v^2 + w^2)(r)\|_{\mathcal{B}_2^{-\frac{1}{2}-\varepsilon}}^2 \, dr \\ & \lesssim (t-s)^{\frac{1}{2}-3\varepsilon} \int_s^t \|(v^2 + w^2)(r)\|_{\mathcal{B}_2^\beta}^2 \, dr, \end{aligned}$$

recalling that $\beta = \frac{1}{2} + 2\varepsilon$. The term involving w^2 can be bounded using Proposition A.6:

$$\int_s^t \|w^2(r)\|_{\mathcal{B}_2^\beta}^2 \, dr \lesssim \int_s^t (\|\nabla(w^2)(r)\|_{L^2}^2 + \|w(r)\|_{L^4}^4) \, dr \lesssim 1 + \mathcal{I}^2(s, t) + \mathcal{B}^2(s),$$

by (6.1). We use Proposition A.7 and Theorem 3.1 to control the term involving v^2 :

$$\begin{aligned} & \int_s^t \|v^2(r)\|_{\mathcal{B}_2^\beta}^2 \, dr \lesssim \int_s^t \|v(r)\|_{\mathcal{B}_4^\beta}^4 \, dr \\ & \lesssim (t-s) \|v(s)\|_{\mathcal{B}_4^\beta}^4 + \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3+6\varepsilon}{4}}} (1 + \|w(\tau)\|_{L^4}) \, d\tau \right)^4 \, dr \\ & \lesssim (t-s) \mathcal{B}^{\frac{4}{3}}(s) + (t-s)^{1-6\varepsilon} \int_s^t (1 + \|w(r)\|_{L^4}^4) \, dr \\ & \lesssim (t-s) \mathcal{B}^{\frac{4}{3}}(s) + (t-s)^{1-6\varepsilon} (1 + \mathcal{I}^{\frac{4}{3}}(s, t) + \mathcal{B}^{\frac{4}{3}}(s)), \end{aligned}$$

where we used (6.1) once more in the final step. Finally, we proceed to bound the term involving the product vw . We start by writing

$$\begin{aligned} & \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3+6\varepsilon}{4}}} \|a_2vw(\tau)\|_{\mathcal{B}_2^{-\frac{1}{2}-\varepsilon}} \, d\tau \right)^2 \, dr \\ & \lesssim (t-s)^{\frac{q_1-1}{q_1}} \left(\int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3+6\varepsilon}{4}}} \|vw(\tau)\|_{\mathcal{B}_2^\beta} \, d\tau \right)^{2q_1} \, dr \right)^{\frac{1}{q_1}}, \end{aligned}$$

for $q_1 > 1$ to be determined below. Now, reasoning as in (4.27) and the calculation that follows (specialised to $p = 2$), we obtain that for any τ ,

$$(6.9) \quad \|vw(\tau)\|_{\mathcal{B}_2^\beta} \lesssim \|v(\tau)\|_{\mathcal{B}_6^{\frac{1}{2}+4\varepsilon}} \|w(\tau)\|_{\mathcal{B}_2^1}.$$

Therefore, by Young's and Hölder's inequalities, for $q_1 < \frac{6}{5}$ (and $\varepsilon > 0$ small enough, depending on q_1), we have

$$\begin{aligned} & \left(\int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3+6\varepsilon}{4}}} \|vw(\tau)\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}} \, d\tau \right)^{2q_1} \, dr \right)^{\frac{1}{q_1}} \\ & \lesssim \left(\int_s^t \|vw(r)\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}}^{\frac{2}{3}} \, dr \right)^{\frac{2}{3}} \\ & \lesssim \left(\int_s^t \|v(r)\|_{\mathcal{B}_6^{\frac{1}{2}+4\varepsilon}}^6 \, dr \right)^{\frac{1}{3}} \int_s^t \|w(r)\|_{\mathcal{B}_2^1}^2 \, dr. \end{aligned}$$

Using (6.1) for $p = \frac{4}{3}$ and Proposition A.6, the integral involving w can be bounded by $1 + \mathcal{I}^{\frac{4}{3}}(s, t) + \mathcal{B}^{\frac{4}{3}}(s)$. For the integral involving v , by Theorem 3.1 and Remark 3.3, we have

$$\begin{aligned} & \int_s^t \|v(r)\|_{\mathcal{B}_6^{\frac{1}{2}+4\varepsilon}}^6 dr \\ & \lesssim (t-s)^{1-\varepsilon} \|v(s)\|_{\mathcal{B}_6^\beta}^6 + \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3}{4}+\frac{5\varepsilon}{2}}} (1 + \|w(\tau)\|_{L^6}) d\tau \right)^6 dr \\ & \lesssim (t-s)^{1-\varepsilon} \|v(s)\|_{\mathcal{B}_6^{\frac{1}{2}+4\varepsilon}}^6 + (t-s)^{6(\frac{1}{4}-\frac{5\varepsilon}{2})} \int_s^t (1 + \|w(r)\|_{L^6}^6) dr \\ & \lesssim (t-s)^{1-\varepsilon} (1 + \mathcal{I}^2(s, t) + \mathcal{B}^2(s)), \end{aligned}$$

where we used (6.1) in the last step. This completes the proof. \square

Lemma 6.7. *For every $s \leq t \in [0, T)$,*

$$\int_s^t \mathcal{W}_5^2(s, r) dr \lesssim (t-s)^{\frac{1}{2}-4\varepsilon} (1 + \mathcal{I}^2(s, t) + \mathcal{B}^2(s)),$$

where the implicit constant depends on ε , T_{\max} , $K_0(T_{\max})$ and c .

Proof. Recall that as before the dots in (6.7) represent the terms

$$\dots = -3\text{com}_2(v+w) - 3(v+w - \Psi) \otimes \mathbf{V} + a_0 + a_1(v+w) + cv.$$

It can easily be checked that

$$\| \dots \|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}}(\tau) \lesssim 1 + \|v(\tau)\|_{\mathcal{B}_2^\beta} + \|w(\tau)\|_{\mathcal{B}_2^\beta},$$

which implies that

$$\begin{aligned} & \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{3+8\varepsilon}{4}}} \| \dots \|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}}(\tau) d\tau \right)^2 ds \\ & \lesssim (t-s)^{\frac{1}{2}-4\varepsilon} \int_s^t \left(1 + \|v(r)\|_{\mathcal{B}_2^\beta}^2 + \|w(r)\|_{\mathcal{B}_2^\beta}^2 \right) dr. \end{aligned}$$

The integral involving w is obviously bounded by $\mathcal{I}^2(s, t)$. For the integral involving v , we write as before

$$\begin{aligned} \int_s^t \|v(r)\|_{\mathcal{B}_2^\beta}^2 dr & \lesssim (t-s) \|v(s)\|_{\mathcal{B}_2^\beta}^2 + \int_s^t \left(\int_s^r \frac{1}{(r-\tau)^{\frac{1+\varepsilon+\beta}{2}}} (1 + \|w(\tau)\|_{L^2}) d\tau \right)^2 ds \\ & \lesssim (t-s) \|v(s)\|_{\mathcal{B}_2^\beta}^2 + (t-s)^{1-\varepsilon-\beta} \int_s^t (1 + \|w(r)\|_{L^2}^2) dr, \end{aligned}$$

so that the desired estimate follows. \square

Proof of Proposition 6.2. Combining the estimates we have obtained in Lemmas 6.3 to 6.7, we get

$$\mathcal{I}^2(s, t) \leq C(t-s) \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + C(t-s)^{\frac{1}{3}} (1 + \mathcal{I}^2(s, t) + \mathcal{B}^2(s)).$$

Therefore, if $(t-s)$ is small enough to guarantee that

$$C(t-s)^{\frac{1}{3}} \leq \frac{1}{2},$$

we obtain the desired estimate (6.2). \square

Proof of Theorem 6.1. By Theorem 2.1, possibly after reducing $t_\star > 0$ and making it depend on an upper bound on $\|v_0\|_{\mathcal{B}_6^\beta} \vee \|w_0\|_{\mathcal{B}_2^\gamma}$, we have

$$\mathcal{I}^2(0, t_\star) + \int_0^{t_\star} \|w(r)\|_{L^6}^6 dr + \int_0^{t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr \leq C < \infty.$$

As a consequence, the desired bound follows by induction as soon as we establish that for every $k \in \mathbb{N}$,

$$\begin{aligned} & \mathcal{I}^2((k+1)t_\star, (k+2)t_\star) + \int_{(k+1)t_\star}^{(k+2)t_\star} \|w(r)\|_{L^6}^6 dr + \int_{(k+1)t_\star}^{(k+2)t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr \\ & \lesssim 1 + \mathcal{I}^2(kt_\star, (k+1)t_\star) + \int_{kt_\star}^{(k+1)t_\star} \|w(r)\|_{L^6}^6 dr + \int_{kt_\star}^{(k+1)t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr. \end{aligned}$$

In order to prove this, we may assume without loss of generality that $k = 0$.

We start by observing that by Theorem 3.1, for every $s \in [0, t_\star]$,

$$\begin{aligned} & \int_s^{2t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr \\ & \lesssim (2t_\star - s) \|v(s)\|_{\mathcal{B}_6^\beta}^6 + \int_s^{2t_\star} \left(\int_s^\tau \frac{1}{(\tau - \sigma)^{\frac{1+\varepsilon+\beta}{2}}} (1 + \|w(\sigma)\|_{L^6}) d\sigma \right)^6 d\tau \\ & \lesssim 1 + \|v(s)\|_{\mathcal{B}_6^\beta}^6 + \int_s^{2t_\star} \|w(r)\|_{L^6}^6 dr. \end{aligned}$$

Combining this estimate with (6.1) and Proposition 6.2, we obtain

$$\begin{aligned} & \mathcal{I}^2(s, 2t_\star) + \int_s^{2t_\star} \|w(r)\|_{L^6}^6 dr + \int_s^{2t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr \\ & \lesssim 1 + \mathcal{I}^2(s, 2t_\star) + \|v(s)\|_{\mathcal{B}_6^\beta}^6 + \|w(s)\|_{L^4}^4 \\ & \lesssim 1 + \|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + \|v(s)\|_{\mathcal{B}_6^\beta}^6 + \|w(s)\|_{L^6}^6. \end{aligned}$$

We average the resulting estimate over $s \in [0, t_\star]$, which yields

$$\begin{aligned} & \frac{1}{t_\star} \int_0^{t_\star} \left(\mathcal{I}(s, 2t_\star) + \int_s^{2t_\star} \|w(r)\|_{L^6}^6 dr + \int_s^{2t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr \right) ds \\ & \lesssim 1 + \frac{1}{t_\star} \int_0^{t_\star} \left(\|w(s)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + \|v(s)\|_{\mathcal{B}_6^\beta}^6 + \|w(s)\|_{L^6}^6 \right) ds. \end{aligned}$$

It only remains to observe that trivially,

$$\begin{aligned} & \frac{1}{t_\star} \int_0^{t_\star} \left(\mathcal{I}(s, 2t_\star) + \int_s^{2t_\star} \|w(r)\|_{L^6}^6 dr + \int_s^{2t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr \right) ds \\ & \geq \mathcal{I}(t_\star, 2t_\star) + \int_{t_\star}^{2t_\star} \|w(r)\|_{L^6}^6 dr + \int_{t_\star}^{2t_\star} \|v(r)\|_{\mathcal{B}_6^\beta}^6 dr, \end{aligned}$$

and we obtain the estimate we were seeking. \square

7. POSTPROCESSING

We fix $c \geq c_0$ as given by Theorem 5.1. We now depart from (3.2) and set the new convention that in the inequalities \lesssim of this section, the implicit constant may depend on ε , T_{\max} , $K_0(T_{\max})$, c **and on an upper bound on** $\|v_0\|_{\mathcal{B}_6^\beta} \vee \|w_0\|_{\mathcal{B}_2^\gamma}$.

Theorem 6.1 provides us with the bound

$$(7.1) \quad \int_0^T \left(\|w\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 + \|w\|_{L^6}^6 + \|v\|_{\mathcal{B}_6^\beta}^6 \right) (s) ds \lesssim 1.$$

Combining this with Theorem 5.1, we obtain

$$(7.2) \quad \sup_{0 \leq s \leq T} \|w(s)\|_{L^4} + \int_0^T \|\nabla(w^2)(s)\|_{L^2}^2 ds \lesssim 1.$$

This estimate and (5.31) also yield

$$(7.3) \quad \int_0^T \|w^2(s)\|_{\mathcal{B}_2^1}^2 ds \lesssim 1.$$

In order to invoke the local existence result, Theorem 2.1, and to exclude the possibility of finite-time blowup, we need to obtain a bound on stronger norms, namely on

$$\sup_{0 \leq t \leq T} \|v(t)\|_{\mathcal{B}_6^\beta} \quad \text{and} \quad \sup_{0 \leq t \leq T} \|w(t)\|_{\mathcal{B}_2^\gamma},$$

where we recall that $\beta = \frac{1}{2} + 2\varepsilon$ and $\gamma = \frac{5}{4} + 2\varepsilon$. The purpose of this section is to post-process the bounds (7.1)–(7.3) to get control on these quantities. The values of β and γ are fixed throughout the section.

The required bound on v follows easily.

Proposition 7.1. *We have*

$$(7.4) \quad \sup_{0 \leq t \leq T} \|v(t)\|_{\mathcal{B}_6^\beta} \lesssim 1.$$

For every $q < 12$, we also have

$$(7.5) \quad \int_0^T \|v(t)\|_{\mathcal{B}_9^\beta}^q dt \lesssim 1.$$

Proof. According to Theorem 3.1, we have for $\sigma = \frac{\beta+1+\varepsilon}{2} = \frac{3}{4} + \frac{3\varepsilon}{2}$ that

$$\begin{aligned} \|v(t)\|_{\mathcal{B}_6^\beta} &\lesssim \|v_0\|_{\mathcal{B}_6^\beta} + \int_0^t \frac{1}{(t-s)^\sigma} (1 + \|w(s)\|_{L^6}) ds \\ &\lesssim \|v_0\|_{\mathcal{B}_6^\beta} + \left(\int_0^t \frac{1}{(t-s)^{\frac{6}{5}\sigma} ds} \right)^{\frac{5}{6}} \left(\int_0^T (1 + \|w(s)\|_{L^6}^6) ds \right)^{\frac{1}{6}}. \end{aligned}$$

Estimate (7.4) then follows from (7.1) and the fact that $\frac{6}{5}\sigma < 1$ for $\varepsilon > 0$ sufficiently small. As for (7.5), we first use Proposition A.2 to bound $\|v(t)\|_{\mathcal{B}_9^\beta}$ by $\|v(t)\|_{\mathcal{B}_6^{\beta+\frac{1}{6}}}$, and then call Theorem 3.1 and Remark 3.3 to write

$$\|v(t)\|_{\mathcal{B}_6^{\beta+\frac{1}{6}}} \lesssim t^{-\frac{1}{12}} \|v_0\|_{\mathcal{B}_6^\beta} + \int_0^t \frac{1}{(t-s)^{\sigma+\frac{1}{12}}} (1 + \|w(s)\|_{L^6}) ds.$$

The function $t \mapsto t^{-\frac{1}{12}} \|v_0\|_{\mathcal{B}_6^\beta}$ is controlled in L^q in time for any $q < 12$. Using once more the L^6 bound on $s \mapsto \|w(s)\|_{L^6}$ provided by (7.1) and Young's inequality, we see that the convolution appearing in the second term is bounded in L^q for $q < \frac{2}{3\varepsilon}$. \square

We now prove the final estimate on w .

Proposition 7.2. *We have*

$$\sup_{0 \leq t \leq T} \|w(t)\|_{\mathcal{B}_2^\gamma} \lesssim 1.$$

We start by observing that as a consequence of Theorem 4.1 (for $p = 2$) and the bounds (7.1) and (7.3), we have the following result.

Corollary 7.3. *For every $s \in [0, T)$,*

$$\sup_{s \leq t \leq T} \frac{\|\delta_{s,t} w\|_{L^2}}{|t-s|^{\frac{1}{8}}} \lesssim 1 + \|w(s)\|_{\mathcal{B}_2^\beta}.$$

Proof of Proposition 7.2. We use the mild formulation of w one final time. As in (6.3)–(6.7), we use Proposition A.13 and Remark A.3 to write $w(t) = e^{\Delta t} w_0 + \sum_{j=1}^8 \mathcal{W}_j(t)$, where

$$\begin{aligned} \|\mathcal{W}_1(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2}}} \|w(r)\|_{L^6}^3 dr, \\ \|\mathcal{W}_2(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2}}} \|v(r)\|_{L^6}^3 dr, \\ \|\mathcal{W}_3(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2}}} \|\text{com}_1(v, w) \ominus \mathbf{V}(r)\|_{L^2} dr, \\ \|\mathcal{W}_4(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2}}} \|w \ominus \mathbf{V}\|_{L^2} dr, \\ \|\mathcal{W}_5(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2} + \frac{1}{4} + 2\varepsilon}} \|v^2(r)\|_{\mathcal{B}_2^{\frac{1}{2} + 2\varepsilon}} dr, \\ \|\mathcal{W}_6(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2} + \frac{1}{4} + 2\varepsilon}} \|vw(r)\|_{\mathcal{B}_2^{\frac{1}{2} + 2\varepsilon}} dr, \\ \|\mathcal{W}_7(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2} + \frac{1}{4} + 2\varepsilon}} \|w^2(r)\|_{\mathcal{B}_2^{\frac{1}{2} + 2\varepsilon}} dr, \\ \|\mathcal{W}_8(t)\|_{\mathcal{B}_2^\gamma} &\lesssim \int_0^t \frac{1}{(t-r)^{\frac{\gamma}{2} + \frac{1}{4} + 2\varepsilon}} \|\dots\|_{\mathcal{B}_2^{-\frac{1}{2} - 2\varepsilon}} dr. \end{aligned}$$

We have split the mild formulation into more terms than in (6.3)–(6.7); this will turn out to be convenient below. All of these bounds are of the form

$$\|\mathcal{W}_j(t)\|_{\mathcal{B}_2^\gamma} \lesssim \int_0^t \frac{1}{(t-r)^{\alpha_j}} F_j(r) dr,$$

for some non-negative function F_j , and either $\alpha_j = \frac{\gamma}{2} = \frac{5}{8} + \varepsilon$ (for $\mathcal{W}_1 - \mathcal{W}_4$) or $\alpha_j = \frac{\gamma}{2} + \frac{1}{4} + 2\varepsilon = \frac{7}{8} + 3\varepsilon$ (for $\mathcal{W}_5 - \mathcal{W}_8$). In each case, we will use Young's inequality to write

$$(7.6) \quad \left(\int_0^T \|\mathcal{W}_j(t)\|_{\mathcal{B}_2^\gamma}^q dt \right)^{\frac{1}{q}} \lesssim \left(\int_0^T \frac{1}{(T-r)^{\alpha_j \tilde{p}}} dr \right)^{\frac{1}{\tilde{p}}} \left(\int_0^T F_j(r)^p dr \right)^{\frac{1}{p}}$$

for $1 + \frac{1}{q} = \frac{1}{\tilde{p}} + \frac{1}{p}$. As usual, for $q = \infty$ the left-hand side of (7.6) should be interpreted as $\sup_{0 \leq t \leq T} \|\mathcal{W}_j(t)\|_{\mathcal{B}_2^\gamma}$. The first integral on the right-hand side is finite if and only if $\tilde{p}\alpha_j < 1$, which amounts to

$$(7.7) \quad \frac{1}{q} > \frac{1}{p} + \alpha_j - 1 = \begin{cases} \frac{1}{p} - \frac{3}{8} + \varepsilon & \text{for } \mathcal{W}_1 - \mathcal{W}_4, \\ \frac{1}{p} - \frac{1}{8} + 3\varepsilon & \text{for } \mathcal{W}_5 - \mathcal{W}_8. \end{cases}$$

We ultimately aim to set $q = \infty$ for all of the \mathcal{W}_j , which requires $p > \frac{8}{3-8\varepsilon}$ for $\mathcal{W}_1 - \mathcal{W}_4$, and the higher integrability $p > \frac{8}{1-24\varepsilon}$ for $\mathcal{W}_5 - \mathcal{W}_8$. We now proceed to derive bounds on the various quantities $\int_0^T F_j(r)^p dr$ to feed into (7.6)–(7.7). For several of the \mathcal{W}_j , namely \mathcal{W}_2 , \mathcal{W}_3 and \mathcal{W}_5 , we will be able to treat the case $q = \infty$ directly. For the other terms, this argument has to be iterated several times. For the reader's convenience, the exponents q that appear in this iteration are summarised in Table 2.

	\mathcal{W}_1	\mathcal{W}_2	\mathcal{W}_3	\mathcal{W}_4	\mathcal{W}_5	\mathcal{W}_6	\mathcal{W}_7	\mathcal{W}_8	w
1.)	8-	∞	∞	8-	∞	$\frac{24}{7}$ -	8-	8-	$\frac{24}{7}$ -
2.)	8-	∞	∞	8-	∞	$\frac{120}{11}$ -	8-	8-	8-
3.)	∞ -	∞	∞	∞	∞	∞	∞ -	∞ -	∞ -
4.)	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 2. The table displays the proved exponent of integrability in time of $\|\mathcal{W}_j\|_{\mathcal{B}_2^\gamma}$ and $\|w\|_{\mathcal{B}_2^\gamma}$ at each step of the iteration. A symbol “8-” denotes that any exponent below 8 can be reached, provided that ε is sufficiently small; the symbol “ ∞ -” denotes that any finite exponent can be reached provided that ε is sufficiently small.

For \mathcal{W}_2 , we use the bound on $\sup_{0 \leq t \leq T} \|v(t)\|_{\mathcal{B}_6^\beta}$ obtained in Proposition 7.1 (which implies in particular a bound on $\sup_{0 \leq t \leq T} \|v(t)\|_{L^6}$), i.e. we can choose $p = \infty$. We conclude that

$$\sup_{0 \leq t \leq T} \|\mathcal{W}_2(t)\|_{\mathcal{B}_2^\gamma} \lesssim 1.$$

For \mathcal{W}_3 , we proceed as in Lemma 6.4 and write

$$\begin{aligned} & \int_0^T \|\text{com}_1(v, w) \ominus \mathcal{V}(r)\|_{L^2}^p dr \lesssim \int_0^T \|\text{com}_1(v, w)\|_{\mathcal{B}_2^{1+2\varepsilon}}^p dr \\ & \lesssim 1 + \int_0^T r^{-(\frac{1+2\varepsilon-\beta}{2})p} dr \|v_0\|_{\mathcal{B}_2^\beta}^p + \int_0^T \left(\int_0^r \frac{1}{(r-\tau)^{1-\frac{\beta}{2}+2\varepsilon}} \|w(\tau)\|_{\mathcal{B}_2^{1+2\varepsilon}} d\tau \right)^p dr \\ (7.8) \quad & + \int_0^T \left(\int_0^r \frac{1}{(r-\tau)^{1+2\varepsilon}} \|\delta_{\tau r} w\|_{L^2} d\tau \right)^p dr. \end{aligned}$$

The first integral is finite for $(\frac{1+2\varepsilon-\beta}{2})p < 1$, which (up to taking ε sufficiently small) amounts to $p < 4$. The bound on $\int_0^T \|w(r)\|_{\mathcal{B}_2^{1+2\varepsilon}}^2 dr$ and Young’s inequality provide a bound on the second integral as soon as $1 + \frac{1}{p} > \frac{1}{2} + (1 - \frac{\beta}{2} + 2\varepsilon)$, i.e. for $p < 4$ (and ε sufficiently small). Finally, according to Corollary 7.3, the third integral is bounded by a constant times

$$\int_0^T \left(\int_0^r \frac{1}{(r-\tau)^{\frac{7}{8}+2\varepsilon}} (1 + \|w(\tau)\|_{\mathcal{B}_2^\beta} d\tau) \right)^p dr.$$

By Proposition A.4 and Remark A.3, we have

$$\|w(\tau)\|_{\mathcal{B}_2^\beta} \lesssim \|w(\tau)\|_{\mathcal{B}_2^{1+2\varepsilon}}^{\frac{\frac{1}{2}+2\varepsilon}{1+2\varepsilon}} \|w(\tau)\|_{L^4}^{\frac{1}{2+4\varepsilon}}.$$

Using the L^∞ -in-time bound on $\|w\|_{L^4}$ provided by (7.2) and the L^2 -in-time bound on $\|w\|_{\mathcal{B}_2^{1+2\varepsilon}}$ from (7.1), we can conclude that

$$\int_0^T \|w(\tau)\|_{\mathcal{B}_2^\beta}^{2\frac{1+2\varepsilon}{\frac{1}{2}+2\varepsilon}} d\tau \lesssim 1.$$

The convolution with the L^1 function $r \mapsto r^{-\frac{7}{8}-2\varepsilon}$ only improves the integrability. Therefore, we can conclude that for any exponent $p < 4$, all integrals on the right-hand side of (7.8) can be controlled, provided $\varepsilon > 0$ is small enough (depending on p). In particular, this covers p which are strictly larger than the threshold value $\frac{8}{3-8\varepsilon}$ which implies that

$$\sup_{0 \leq t \leq T} \|\mathcal{W}_3(t)\|_{\mathcal{B}_2^\gamma} \lesssim 1.$$

For \mathcal{W}_5 , we observe that by Proposition 7.1, we have an L^∞ -in-time bound on $\|v\|_{\mathcal{B}_6^\beta}$, which immediately yields

$$\sup_{0 \leq t \leq T} \|\mathcal{W}_5(t)\|_{\mathcal{B}_2^\gamma} \lesssim 1.$$

For the remaining terms \mathcal{W}_1 , \mathcal{W}_4 , \mathcal{W}_6 , \mathcal{W}_7 and \mathcal{W}_8 , we do not obtain an L^∞ -in-time bound immediately, but we have to improve the integrability iteratively. In order to control all terms, a maximum of four iterations is necessary.

For \mathcal{W}_1 , we use the L^2 -in-time bound on $\|w\|_{L^6}^3$ provided by (7.1), which by (7.7) implies that

$$(7.9) \quad \int_0^T \|\mathcal{W}_1(t)\|_{\mathcal{B}_2^\gamma}^q dt \lesssim 1 \quad \text{for } q < \frac{8}{1+8\varepsilon}.$$

The bound (7.9) is represented by the entry 8– in Table 2. In the same way, the L^2 -in-time bound on $\|w \ominus \mathbf{v}\|_{L^2} \lesssim \|w\|_{\mathcal{B}^{1+2\varepsilon}}$ yields

$$\int_0^T \|\mathcal{W}_4(t)\|_{\mathcal{B}_2^\gamma}^q dt \lesssim 1 \quad \text{for } q < \frac{8}{1+8\varepsilon}.$$

For \mathcal{W}_6 we use Proposition A.7 to write

$$(7.10) \quad \|vw\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}} \lesssim \|v\|_{\mathcal{B}_{p_1}^{\frac{1}{2}+2\varepsilon}} \|w\|_{\mathcal{B}_{p_2}^{\frac{1}{2}+2\varepsilon}},$$

for $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$. At this point (and only at this point), we make use of the improved integrability estimate (7.5) provided in Proposition 7.1. This allows us to choose $p_1 = 9$, which yields $p_2 = \frac{18}{7}$ (and in particular this exponent is < 3), and allows us to interpolate

$$\|w\|_{\mathcal{B}_{p_2}^{\frac{1}{2}+2\varepsilon}} \lesssim \|w\|_{\mathcal{B}_2^{1+2\varepsilon}}^{\frac{\frac{1}{2}+2\varepsilon}{1+2\varepsilon}} \|w\|_{L^6}^{\frac{1}{2+4\varepsilon}},$$

if $\varepsilon > 0$ is small enough. Hence, using the integrability of $\|v\|_{\mathcal{B}_9^{\frac{1}{2}+2\varepsilon}}$ provided by (7.5), we obtain an L^p -in-time control of $\|vw\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}}$ for

$$\frac{1}{p} > \frac{1}{12} + \frac{1}{2} \frac{\frac{1}{2}+2\varepsilon}{1+2\varepsilon} + \frac{1}{6} \frac{1}{2+4\varepsilon},$$

yielding

$$\int_0^T \|\mathcal{W}_6(t)\|_{\mathcal{B}_2^\gamma}^q dt \lesssim 1 \quad \text{for } q = \frac{24}{7}-,$$

where $q = \frac{24}{7}-$ means that any exponent $q < \frac{24}{7}$ can be reached by choosing $\varepsilon > 0$ small enough. For \mathcal{W}_7 , we observe that

$$\|w^2\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}} \lesssim \|w^2\|_{\mathcal{B}_2^1}^{\frac{1}{2}+2\varepsilon} \|w^2\|_{L^2}^{\frac{1}{2}-2\varepsilon} \lesssim \|w^2\|_{\mathcal{B}_2^1}^{\frac{1}{2}+2\varepsilon} \|w\|_{L^4}^{1-2\varepsilon}.$$

By interpolating between (7.3) and the L^∞ control on $\|w\|_{L^4}$ from (7.2), we can see that this term is controlled in L^p in time, for $p = \frac{4}{1+4\varepsilon}$. This yields

$$\int_0^T \|\mathcal{W}_7(t)\|_{\mathcal{B}_2^2}^q dt \lesssim 1 \quad \text{for } q < \frac{8}{1+32\varepsilon}.$$

We represent this fact by the symbol “8–” on the first line of Table 2. For \mathcal{W}_8 , we observe as in Lemma 6.7 that

$$\|\dots(r)\|_{\mathcal{B}_2^{-\frac{1}{2}-2\varepsilon}} \lesssim 1 + \|v(r)\|_{\mathcal{B}_2^\beta} + \|w(r)\|_{\mathcal{B}_2^\beta}.$$

The term $\|v(r)\|_{\mathcal{B}_2^\beta}$ is controlled uniformly in time by Proposition 7.1. For $\|w(r)\|_{\mathcal{B}_2^\beta}$, we use Proposition A.4 and Remark A.3 to get

$$\|w(r)\|_{\mathcal{B}_2^\beta} \lesssim \|w(r)\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}}^{\frac{1}{2}+2\varepsilon} \|w\|_{L^2}^{\frac{1}{2+4\varepsilon}}.$$

By (7.1)–(7.2), this quantity is in L^p in time for $p = 2\left(\frac{1+2\varepsilon}{\frac{1}{2}+2\varepsilon}\right)$. This yields a bound on

$$\int_0^T \|\mathcal{W}_8(t)\|_{\mathcal{B}_2^2}^q dt \lesssim 1 \quad \text{for } \frac{1}{q} > \frac{1+4\varepsilon}{4+8\varepsilon} - \frac{1}{8} + 3\varepsilon,$$

which is represented by 8– in Table 2, and completes our first round of estimates on $\mathcal{W}_1 - \mathcal{W}_8$.

The worst of all the bounds we obtained is that on \mathcal{W}_6 . We can thus conclude the first round of the iteration with the bound

$$(7.11) \quad \int_0^T \|w(t)\|_{\mathcal{B}_2^\gamma}^q dt \lesssim 1 \quad \text{for } q = \frac{24}{7} - .$$

In the second iteration, we use (7.11) to improve the bound on \mathcal{W}_6 . Indeed, by Proposition A.4 and Remark A.3,

$$\|w\|_{\mathcal{B}_{p_1}^{\frac{1}{2}+2\varepsilon}} \lesssim \|w\|_{\mathcal{B}_2^{\frac{5}{4}+2\varepsilon}}^\alpha \|w\|_{L^6}^{1-\alpha},$$

for

$$\alpha = \frac{\frac{1}{2}+2\varepsilon}{\frac{5}{4}+2\varepsilon} \quad \frac{1}{p_1} = \frac{\alpha}{2} + \frac{1-\alpha}{6}.$$

In particular, the exponent α is close to $\frac{2}{5}$, and the exponent p_1 close to $\frac{10}{3} > 3$ (for $\varepsilon > 0$ sufficiently small). Recalling (7.10), we obtain

$$\|vw\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}} \lesssim \|v\|_{\mathcal{B}_6^{\frac{1}{2}+2\varepsilon}}^\alpha \|w\|_{\mathcal{B}_2^{\frac{5}{4}+2\varepsilon}}^\alpha \|w\|_{L^6}^{1-\alpha}.$$

By Proposition 7.1, (7.11) and (7.1), this yields control on $\|vw\|_{\mathcal{B}_2^{\frac{1}{2}+2\varepsilon}}$ in L^p for

$$\frac{1}{p} = \frac{1}{\infty} + \alpha \frac{7+}{24} + (1-\alpha) \frac{1}{6} = \frac{13}{60} + ,$$

so that we obtain

$$\int_0^T \|\mathcal{W}_6(t)\|_{\mathcal{B}_2^2}^q dt \lesssim 1 \quad \text{for } q = \frac{120}{11} - .$$

The resulting estimate for w is iterated two more times to obtain the L^∞ control on all of the \mathcal{W}_j . We skip the straightforward details and only refer to Table 2 for the exponents obtained. \square

APPENDIX A. PRODUCTS AND PARAPRODUCTS IN BESOV SPACES

We denote by C_{per}^∞ the space of \mathbb{Z}^d -periodic infinitely differentiable functions. For $p \in [1, \infty]$, we write $L^p = L^p([-1, 1]^d, dx)$, with associated norm $\|\cdot\|_{L^p}$. We write $\langle \cdot, \cdot \rangle$ for the scalar product in L^2 . We denote by $\|\cdot\|_{\bar{L}^p}$ the norm of the space $L^p(\mathbb{R}^d, dx)$. For $u = (u_n)_{n \in I}$ with I a countable set, we write

$$\|u\|_{\ell^p} := \left(\sum_{n \in I} |u_n|^p \right)^{\frac{1}{p}},$$

with the usual interpretation as a supremum when $p = \infty$. We write $\mathcal{F}f$ or \hat{f} for the Fourier transform (and by \mathcal{F}^{-1} its inverse), which is well-defined for any Schwartz distribution f on \mathbb{R}^d , and reads, for $f \in L^1(\mathbb{R}^d)$,

$$\mathcal{F}f(\zeta) = \hat{f}(\zeta) = \int e^{-ix \cdot \zeta} f(x) dx.$$

A.1. Besov spaces. We recall briefly a construction of Besov spaces on the torus. Following [1, Proposition 2.10], there exist $\tilde{\chi}, \chi \in C_c^\infty$ taking values in $[0, 1]$ and such that

$$(A.1) \quad \text{Supp } \tilde{\chi} \subseteq B(0, 4/3),$$

$$(A.2) \quad \text{Supp } \chi \subseteq B(0, 8/3) \setminus B(0, 3/4),$$

$$(A.3) \quad \forall \zeta \in \mathbb{R}^d, \quad \tilde{\chi}(\zeta) + \sum_{k=0}^{+\infty} \chi(\zeta/2^k) = 1.$$

We use this partition of unity to decompose any function $f \in C_{\text{per}}^\infty$ as a sum of functions with localized spectrum. More precisely, we define

$$(A.4) \quad \chi_{-1} = \tilde{\chi}, \quad \chi_k = \chi(\cdot/2^k) \quad (k \geq 0),$$

and for $k \geq -1$ integer,

$$\delta_k f = \mathcal{F}^{-1} \left(\chi_k \hat{f} \right), \quad S_k f = \sum_{j < k} \delta_j f$$

(where the sum runs over $j \geq -1$), so that at least formally, $f = \sum \delta_k f$. We let

$$(A.5) \quad \eta_k = \mathcal{F}^{-1}(\chi_k), \quad \eta = \eta_0,$$

so that for $k \geq 0$, $\eta_k = 2^{kd} \eta(2^k \cdot)$, and for every k ,

$$(A.6) \quad \delta_k f = \eta_k \star f,$$

where \star denotes the convolution. For every $\alpha \in \mathbb{R}$, $p, q \in [1, +\infty]$ and $f \in C_{\text{per}}^\infty$, we define

$$(A.7) \quad \|f\|_{\mathcal{B}_{p,q}^\alpha} := \left\| \left(2^{\alpha k} \|\delta_k f\|_{L^p} \right)_{k \geq -1} \right\|_{\ell^q}.$$

It is easy to check that this quantity is finite (see [31, Lemma 3.2]). We define the Besov space $\mathcal{B}_{p,q}^\alpha$ as the completion of C_{per}^∞ with respect to this norm. Outside of this appendix, we use the shorthand notation $\mathcal{B}_p^\alpha := \mathcal{B}_{p,\infty}^\alpha$.

We first state a duality relation between Besov spaces, see [1, Proposition 2.76].

Proposition A.1 (Duality). *Denote by Let $\alpha \in \mathbb{R}$, and $p, q, p', q' \in [1, \infty]$ be such that*

$$(A.8) \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

The mapping $(f, g) \mapsto \langle f, g \rangle$ (defined for $f, g \in C_{\text{per}}^\infty$) can be extended to a continuous bilinear form on $\mathcal{B}_{p,q}^\alpha \times \mathcal{B}_{p',q'}^{-\alpha}$.

In particular, we can think of Besov spaces as being all embedded in the space of Schwartz distributions.

Clearly, $\mathcal{B}_{p_1, q_1}^\alpha$ is continuously embedded in $\mathcal{B}_{p_2, q_2}^\beta$ if $\beta \leq \alpha$, $p_2 \leq p_1$ and $q_2 \geq q_1$. We also have the following embeddings (cf. [31, Proposition 3.7]).

Proposition A.2 (Besov embedding). *Let $\alpha \leq \beta \in \mathbb{R}$ and $p \geq r \in [1, \infty]$ be such that*

$$\beta = \alpha + d \left(\frac{1}{r} - \frac{1}{p} \right).$$

There exists $C < \infty$ such that

$$\|f\|_{\mathcal{B}_{p, q}^\alpha} \leq C \|f\|_{\mathcal{B}_{r, q}^\beta}.$$

Remark A.3. By [31, Remarks 3.5 and 3.6], there exists $C < \infty$ such that

$$C^{-1} \|f\|_{\mathcal{B}_{p, \infty}^0} \leq \|f\|_{L^p} \leq C \|f\|_{\mathcal{B}_{p, 1}^0}.$$

An application of Hölder's inequality (see [31, Proposition 3.10]) yields the following interpolation result.

Proposition A.4 (Interpolation inequalities). *Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $p_0, q_0, p_1, q_1 \in [1, \infty]$ and $\nu \in [0, 1]$. Defining $\alpha = (1 - \nu)\alpha_0 + \nu\alpha_1$ and $p, q \in [1, \infty]$ such that*

$$\frac{1}{p} = \frac{1 - \nu}{p_0} + \frac{\nu}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \nu}{q_0} + \frac{\nu}{q_1},$$

we have

$$\|f\|_{\mathcal{B}_{p, q}^\alpha} \leq \|f\|_{\mathcal{B}_{p_0, q_0}^{1-\nu}}^{1-\nu} \|f\|_{\mathcal{B}_{p_1, q_1}^{\nu}}^{\nu}.$$

The effect of differentiating (in the sense of distributions) an element Besov space is described as follows (see e.g. [31, Proposition 3.8]).

Proposition A.5 (Effect of differentiating). *Let $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$. For every $i \in \{1, \dots, d\}$, the mapping $f \mapsto \partial_i f$ extends to a continuous linear map from $\mathcal{B}_{p, q}^\alpha$ to $\mathcal{B}_{p, q}^{\alpha-1}$.*

The following extends [31, Proposition 3.25] by allowing $\alpha = 1$ and arbitrary values of p .

Proposition A.6 (Estimate in terms of ∇f). *Let $\alpha \in (0, 1]$ and $p, q \in [1, \infty]$. When $\alpha = 1$, we also impose $q = \infty$. There exists $C < \infty$ such that*

$$C^{-1} \|f\|_{\mathcal{B}_{p, q}^\alpha} \leq \|f\|_{L^p}^{1-\alpha} \|\nabla f\|_{L^p}^\alpha + \|f\|_{L^p}.$$

Proof. We decompose the proof into two steps.

Step 1. We show the result for $\alpha \in (0, 1)$. By comparison of norms, it suffices to show the result for $q = 1$. We assume $p < \infty$, the case $p = \infty$ being similar. Let f be a smooth, one-periodic function. For $\ell \geq 0$, we define the projectors

$$\mathcal{P}_\ell f = \sum_{-1 \leq k < \ell} \delta_k f \quad \text{and} \quad \mathcal{P}_\ell^\perp f = \sum_{k \geq \ell} \delta_k f,$$

so that $f = \mathcal{P}_\ell f + \mathcal{P}_\ell^\perp f$, and by the triangle inequality,

$$\|f\|_{\mathcal{B}_{p, 1}^\alpha} \leq \|\mathcal{P}_\ell f\|_{\mathcal{B}_{p, 1}^\alpha} + \|\mathcal{P}_\ell^\perp f\|_{\mathcal{B}_{p, 1}^\alpha}.$$

For the first term, recalling (A.5) and (A.6), we have

$$(A.9) \quad \|\delta_k f\|_{L^p} = \|\eta_k \star f\|_{L^p} \leq \|\tilde{\eta}_k\|_{L^1} \|f\|_{L^p},$$

where we used Young's convolution inequality on the torus and set

$$(A.10) \quad \tilde{\eta}_k := \sum_{y \in (\mathbb{Z}^d)} \eta_k(\cdot + y).$$

Recall that $\eta_k = 2^{kd}\eta(2^k \cdot)$. By scaling and rapid decay to 0 at infinity of η , we have

$$(A.11) \quad \sup_{k \geq -1} \|\tilde{\eta}_k\|_{L^1} < \infty,$$

and thus

$$(A.12) \quad \|\mathcal{P}_\ell f\|_{\mathcal{B}_{p,1}^\alpha} = \sum_{-1 \leq k < \ell} 2^{k\alpha} \|\delta_k f\|_{L^p} \lesssim 2^{\ell\alpha} \|f\|_{L^p}.$$

On the other hand, using the fact that for $k \geq 0$, the function η_k has vanishing integral, we get

$$\begin{aligned} \|\mathcal{P}_\ell^\perp f\|_{\mathcal{B}_{p,1}^\alpha} &= \sum_{k \geq \ell} 2^{k\alpha} \|\delta_k f\|_{L^p} \\ &= \sum_{k \geq \ell} 2^{-k(1-\alpha)} \left(\int_{[-1,1]^d} \left| \int_{\mathbb{R}^d} 2^k \eta_k(y) (f(x-y) - f(x)) dy \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

By Hölder's inequality, the integral above is bounded by

$$\|2^k \cdot \|\eta_k\|_{L^1}^{p-1} \int_{[-1,1]^d} \int_{\mathbb{R}^d} |2^k y \eta_k(y)| \frac{|f(x-y) - f(x)|^p}{|y|^p} dx dy,$$

where we recall that $\|\cdot\|_{L^1}$ denotes the L^1 norm in the full space \mathbb{R}^d . For every $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \frac{|f(x-y) - f(x)|^p}{|y|^p} &= \frac{1}{|y|^p} \left| \int_0^1 \nabla f(x - ty) \cdot y dt \right|^p \\ &\leq \int_0^1 |\nabla f(x - ty)|^p dt. \end{aligned}$$

Therefore,

$$\int_{[-1,1]^d} \int_{\mathbb{R}^d} |2^k y \eta_k(y)| \frac{|f(x-y) - f(x)|^p}{|y|^p} dx dy \leq \|2^k \cdot \|\eta_k\|_{L^1} \|\nabla f\|_{L^p}^p.$$

Noting that $\|2^k \cdot \|\eta_k\|_{L^1}$ is finite and independent of $k \geq 0$ by scaling, we obtain

$$\|\mathcal{P}_\ell^\perp f\|_{\mathcal{B}_{p,1}^\alpha} \lesssim 2^{-\ell(1-\alpha)} \|\nabla f\|_{L^p},$$

so that uniformly over $\ell \geq 0$,

$$\|f\|_{\mathcal{B}_{p,1}^\alpha} \lesssim 2^{\ell\alpha} \|f\|_{L^p} + 2^{-\ell(1-\alpha)} \|\nabla f\|_{L^1}.$$

The result then follows by optimizing over ℓ .

Step 2. We show the result for $\alpha = 1$ and $q = \infty$. This is a minor modification of the arguments of the previous step. Indeed, we have

$$\|\mathcal{P}_0 f\|_{\mathcal{B}_{p,\infty}^1} = \|\delta_{-1} f\|_{L^p} \lesssim \|f\|_{L^p},$$

while

$$\|\mathcal{P}_0^\perp f\|_{\mathcal{B}_{p,\infty}^1} = \sup_{k \geq 0} 2^k \|\delta_k f\|_{L^p},$$

and we have seen that the latter is bounded by a constant times $\|\nabla f\|_{L^p}$, so the proof is complete. \square

A.2. Paraproducts. As in [18], the basis of our analysis rests on the regularity properties of paraproducts. For $f, g \in C_{\text{per}}^\infty$, we define the *paraproduct*

$$f \odot g = \sum_{j < k-1} \delta_j f \delta_k g = \sum_k S_{k-1} f \delta_k g,$$

and the *resonant term*

$$f \ominus g = \sum_{|j-k| \leq 1} \delta_j f \delta_k g.$$

We write $f \odot g = g \odot f$. At least formally, we have the *Bony decomposition*

$$fg = f \odot g + f \ominus g + f \odot g.$$

We will also use the symbols $\odot = \oplus + \ominus$, etc.

The most important estimates for our purpose are summarised in the following proposition (see [1, Theorems 2.82, 2.85 and Corollary 2.86] or [31, Theorem 3.17 and Corollaries 3.19 and 3.21]).

Proposition A.7 (paraproduct estimates). *Let $\alpha, \beta \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

- *If $\alpha + \beta > 0$, then the mapping $(f, g) \mapsto f \odot g$ extends to a continuous bilinear map from $\mathcal{B}_{p_1, q}^\alpha \times \mathcal{B}_{p_2, q}^\beta$ to $\mathcal{B}_{p, q}^{\alpha+\beta}$.*
- *The mapping $(f, g) \mapsto f \odot g$ extends to a continuous bilinear map from $L^{p_1} \times \mathcal{B}_{p_2, q}^\beta$ to $\mathcal{B}_{p, q}^\beta$.*
- *If $\alpha < 0$, then the mapping $(f, g) \mapsto f \odot g$ extends to a continuous bilinear map from $\mathcal{B}_{p_1, q}^\alpha \times \mathcal{B}_{p_2, q}^\beta$ to $\mathcal{B}_{p, q}^{\alpha+\beta}$.*
- *If $\alpha > 0$, then the mapping $(f, g) \mapsto fg$ extends to a continuous bilinear map from $\mathcal{B}_{p_1, q}^\alpha \times \mathcal{B}_{p_2, q}^\alpha$ to $\mathcal{B}_{p, q}^\alpha$.*
- *If $\alpha < 0 < \beta$ and $\alpha + \beta > 0$, then the mapping $(f, g) \mapsto fg$ extends to a continuous bilinear map from $\mathcal{B}_{p_1, q}^\alpha \times \mathcal{B}_{p_2, q}^\beta$ to $\mathcal{B}_{p, q}^\alpha$.*

Remark A.8. Although this will be sufficient for our purposes, we remark that for $\alpha > 0$, the extension of the product to a continuous map from $\mathcal{B}_{p_1, q}^\alpha \times \mathcal{B}_{p_2, q}^\alpha$ to $\mathcal{B}_{p, q}^\alpha$ is vastly suboptimal in its dependency on the parameter p . For instance, if $\alpha > \frac{d}{p}$, then $\mathcal{B}_{p, q}^\alpha$ is continuously embedded in L^∞ , and thus the space $\mathcal{B}_{p, q}^\alpha$ is in fact an algebra.

The next result is our first commutator estimate. It extends [18, Lemma 2.4] to more general Besov spaces.

Proposition A.9 (commutation between \odot and \ominus). *Let $\alpha < 1$, $\beta, \gamma \in \mathbb{R}$ and $p, p_1, p_2, p_3 \in [1, \infty]$ be such that*

$$\beta + \gamma < 0, \quad \alpha + \beta + \gamma > 0 \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

The mapping

$$(A.13) \quad [\odot, \ominus] : (f, g, h) \mapsto (f \odot g) \ominus h - f(g \ominus h)$$

extends to a continuous trilinear map from $\mathcal{B}_{p_1, \infty}^\alpha \times \mathcal{B}_{p_2, \infty}^\beta \times \mathcal{B}_{p_3, \infty}^\gamma$ to $\mathcal{B}_{p, \infty}^{\alpha+\beta+\gamma}$.

The proof of Proposition A.9 relies on the following two lemmas.

Lemma A.10. *For $f, g \in C^\infty$, define*

$$[\delta_k, f](g) = \delta_k(fg) - f \delta_k g.$$

Let $p, p_1, p_2 \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. There exists $C < \infty$ such that for every $k \geq 0$ and $f, g \in C_{\text{per}}^\infty$,

$$\|[\delta_k, f](g)\|_{L^p} \leq \frac{C}{2^k} \|\nabla f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

Proof. The proof is similar to that of [1, Lemma 2.97]. \square

Lemma A.11. For $f, g \in C^\infty$, define

$$(A.14) \quad [\delta_k, \odot](f, g) := \delta_k(f \odot g) - f(\delta_k g).$$

Let $p, p_1, p_2 \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$. There exists $C < \infty$ such that for every $f, g \in C_{\text{per}}^\infty$,

$$\|[\delta_k, \odot](f, g)\|_{L^p} \leq C 2^{-k(\alpha+\beta)} \|f\|_{\mathcal{B}_{p_1, \infty}^\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}.$$

Remark A.12. It would perhaps be more natural to define the commutator between δ_k and \odot as

$$(A.15) \quad \delta_k(f \odot g) - f \odot (\delta_k g)$$

(and similarly for (A.13)). However, the definition in (A.14) will be more convenient to work with in the proof of Proposition A.9 (besides matching the choice of [18]).

Proof. We decompose the proof into two steps, the first one being focused on deriving bounds for the quantity in (A.15).

Step 1. We show that

$$(A.16) \quad \|\delta_k(f \odot g) - f \odot (\delta_k g)\|_{L^p} \leq C 2^{-k(\alpha+\beta)} \|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}.$$

(The proof given now shows that (A.16) is also valid when $\alpha \leq 0$.) Note that

$$\delta_k(f \odot g) - f \odot (\delta_k g) = \sum_{i=0}^{+\infty} \delta_k(S_{i-1}f \delta_i g) - S_{i-1}f \delta_i \delta_k g.$$

The term $\delta_i \delta_k g = \delta_k \delta_i g$ vanishes unless $|i - k| \leq 1$. Moreover, for any h , the Fourier spectrum of $S_{i-1}f \delta_i h$ is contained in $2^i \mathcal{A}$, where \mathcal{A} is the annulus $B(0, 10/3) \setminus B(0, 1/12)$. Hence, $\delta_k(S_{i-1}f \delta_i h)$ vanishes unless $|i - k| \leq 5$, and

$$\delta_k(f \odot g) - f \odot (\delta_k g) = \sum_{|i-k| \leq 5} [\delta_k, S_{i-1}f](\delta_i g).$$

By Lemma A.10,

$$\|[\delta_k, S_{i-1}f](\delta_i g)\|_{L^p} \lesssim \frac{1}{2^k} \|\nabla S_{i-1}f\|_{L^{p_1}} \|\delta_i g\|_{L^{p_2}}.$$

Since we assume $\alpha < 1$, we have

$$\|\nabla S_{i-1}f\|_{L^{p_1}} \leq \sum_{j < i-1} \|\delta_j(\nabla f)\|_{L^{p_1}} \lesssim 2^{i(1-\alpha)} \|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}}.$$

Using also the fact that $\|\delta_i g\|_{L^{p_2}} \leq 2^{-i\beta} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}$, we arrive at

$$\|\delta_k(f \odot g) - f \odot (\delta_k g)\|_{L^p} \lesssim \sum_{|i-k| \leq 5} \frac{2^{i(1-\alpha-\beta)}}{2^k} \|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta},$$

which proves (A.16).

Step 2. Recall from Proposition A.5 that $\|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}} \lesssim \|f\|_{\mathcal{B}_{p_1, \infty}^\alpha}$. In order to conclude the proof, it thus suffices to show that

$$(A.17) \quad \|f \odot (\delta_k g)\|_{L^p} \lesssim 2^{-k(\alpha+\beta)} \|f\|_{\mathcal{B}_{p_1, \infty}^\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}.$$

We have

$$f \odot (\delta_k g) = \sum_{i,j: i \leq j+1} \delta_i \delta_k g \delta_j f.$$

As observed previously, $\delta_i \delta_k g$ vanishes unless $|i - k| \leq 1$. In this case, by writing δ_i as a convolution against η_i , applying Young's convolution inequality in the form of (A.9) and recalling (A.11), we obtain

$$\|\delta_i \delta_k g\|_{L^{p_1}} \lesssim \|\delta_k g\|_{L^{p_1}} \leq 2^{-k\beta} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}.$$

Since we also have $\|\delta_j f\|_{L^{p_2}} \leq 2^{-j\alpha} \|f\|_{\mathcal{B}_{p_2, \infty}^\alpha}$, we obtain

$$\|f \odot (\delta_k g)\|_{L^p} \lesssim 2^{-k\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta} \sum_{j \geq k-2} 2^{-j\alpha} \|f\|_{\mathcal{B}_{p_2, \infty}^\alpha},$$

and (A.17) follows since we assume that $\alpha > 0$. \square

Proof of Proposition A.9. Observe that

$$\begin{aligned} (f \odot g) \odot h &= \sum_{|k-k'| \leq 1} \delta_k (f \odot g) \delta_{k'} h \\ &= \sum_{i,k,k': |k-k'| \leq 1} \delta_k (\delta_i f \odot g) \delta_{k'} h. \end{aligned}$$

The Fourier spectrum of $\delta_i f \odot g$ is contained in $2^i \mathcal{A}$, where \mathcal{A} is the annulus $B(0, 20/3) \setminus B(0, 1/24)$. As a consequence, $\delta_k (\delta_i f \odot g)$ vanishes unless $|k - i| \leq 6$, and

$$\begin{aligned} (f \odot g) \odot h &= \sum_{|k-k'| \leq 1, i-k \leq 6} \delta_k (\delta_i f \odot g) \delta_{k'} h \\ (A.18) \quad &= \sum_{|k-k'| \leq 1, i-k \leq 6} \delta_i f \delta_k g \delta_{k'} h + \sum_{|k-k'| \leq 1, i-k \leq 6} [\delta_k, \odot] (\delta_i f, g) \delta_{k'} h. \end{aligned}$$

As a first step, we show that the $\mathcal{B}_{p, \infty}^{\alpha+\beta+\gamma}$ norm of the second sum is bounded by a constant times $\|f\|_{\mathcal{B}_{p_1, \infty}^\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta} \|h\|_{\mathcal{B}_{p_3, \infty}^\gamma}$. For each fixed k , the Fourier spectrum of

$$\mathbf{co}_k := \sum_{k', i: |k-k'| \leq 1, i-k \leq 6} [\delta_k, \odot] (\delta_i f, g) \delta_{k'} h$$

is contained in a ball whose radius grows proportionally to 2^k . By [1, Lemma 2.84] (or [31, Lemma 3.16]), and since $\alpha + \beta + \gamma > 0$, it thus suffices to show that

$$(A.19) \quad \left\| \left(2^{k(\alpha+\beta+\gamma)} \|\mathbf{co}_k\|_{L^p} \right)_{k \geq -1} \right\|_{\ell^\infty} \lesssim \|f\|_{\mathcal{B}_{p_1, \infty}^\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta} \|h\|_{\mathcal{B}_{p_3, \infty}^\gamma}.$$

We can rewrite \mathbf{co}_k as

$$\sum_{k': |k-k'| \leq 1} [\delta_k, \odot] \left(\sum_{i \leq k+6} \delta_i f, g \right) \delta_{k'} h.$$

By Lemma A.11 and Hölder's inequality, the L^p norm of \mathbf{co}_k is thus bounded by

$$\begin{aligned} &\sum_{k': |k-k'| \leq 1} 2^{-k(\alpha+\beta)} \left\| \sum_{i \leq k+6} \delta_i f \right\|_{\mathcal{B}_{p_1, \infty}^\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta} \|\delta_{k'} h\|_{L^{p_3}} \\ &\lesssim 2^{-k(\alpha+\beta+\gamma)} \|f\|_{\mathcal{B}_{p_1, \infty}^\alpha} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta} \|h\|_{\mathcal{B}_{p_3, \infty}^\gamma}, \end{aligned}$$

which proves (A.19).

Now that we have controlled the second sum in (A.18), we will argue that the first sum is close to $f(g \odot h)$. We observe that

$$f(g \odot h) = \sum_{i,k,k': |k-k'| \leq 1} \delta_i f \delta_k g \delta_{k'} h,$$

so the difference between the first sum in (A.18) and $f(g \odot h)$ is given by

$$\sum_{i,k,k': |k-k'| \leq 1, i-k > 6} \delta_i f \delta_k g \delta_{k'} h.$$

As above, in order to control the $\mathcal{B}_{p,\infty}^{\alpha+\beta+\gamma}$ norm of this term, we observe that for each i , the Fourier spectrum of

$$\mathbf{s}_i := \sum_{k,k': |k-k'| \leq 1, k < i-6} \delta_i f \delta_k g \delta_{k'} h$$

is contained in a ball whose radius grows proportionally to 2^i . Hence, it suffices to show that

$$(A.20) \quad \left\| \left(2^{i(\alpha+\beta+\gamma)} \|\mathbf{s}_i\|_{L^p} \right)_{i \geq -1} \right\|_{\ell^\infty} \lesssim \|f\|_{\mathcal{B}_{p_1,\infty}^\alpha} \|g\|_{\mathcal{B}_{p_2,\infty}^\beta} \|h\|_{\mathcal{B}_{p_3,\infty}^\gamma}.$$

By Hölder's inequality,

$$\begin{aligned} \|\mathbf{s}_i\|_{L^p} &\leq \sum_{k,k': |k-k'| \leq 1, k < i-6} \|\delta_i f\|_{L^{p_1}} \|\delta_k g\|_{L^{p_2}} \|\delta_{k'} h\|_{L^{p_3}} \\ &\leq 2^{-\alpha i} \|f\|_{\mathcal{B}_{p_1,\infty}^\alpha} \sum_{k,k': |k-k'| \leq 1, k < i-6} 2^{-k\beta-k'\gamma} \|g\|_{\mathcal{B}_{p_2,\infty}^\beta} \|h\|_{\mathcal{B}_{p_3,\infty}^\gamma} \\ &\leq 2^{-i(\alpha+\beta+\gamma)} \|f\|_{\mathcal{B}_{p_1,\infty}^\alpha} \|g\|_{\mathcal{B}_{p_2,\infty}^\beta} \|h\|_{\mathcal{B}_{p_3,\infty}^\gamma}, \end{aligned}$$

where we used the fact that $\beta+\gamma < 0$ in the last step. The proof is thus complete. \square

A.3. Heat flow. The next proposition quantifies the regularising effect of the heat flow, see e.g. [31, Propositions 3.11 and 3.12].

Proposition A.13 (Regularisation by heat flow). *Let $\alpha, \beta \in \mathbb{R}$ and $p, q \in [1, \infty]$.*

- *If $\alpha \geq \beta$, then there exists $C < \infty$ such that uniformly over $t > 0$,*

$$\|e^{t\Delta} f\|_{\mathcal{B}_{p,q}^\alpha} \leq C t^{\frac{\beta-\alpha}{2}} \|f\|_{\mathcal{B}_{p,q}^\beta}.$$

- *If $0 \leq \beta - \alpha \leq 2$, then there exists $C < \infty$ such that uniformly over $t \geq 0$,*

$$\|(1 - e^{t\Delta})f\|_{\mathcal{B}_{p,q}^\alpha} \leq C t^{\frac{\beta-\alpha}{2}} \|f\|_{\mathcal{B}_{p,q}^\beta}.$$

Remark A.14. We also have, for every $p \in [1, \infty]$ and $t \geq 0$,

$$\|e^{t\Delta} f\|_{L^p} \leq \|f\|_{L^p}.$$

Indeed, the heat kernel has unit L^1 norm, so the inequality above follows by Young's convolution inequality.

We now turn to our second commutator estimate, which extends Lemma 32 in the first arXiv version of [18] to more general Besov spaces (see also [5, Lemma 2.5]).

Proposition A.15 (commutation between $e^{t\Delta}$ and \odot). *Let $\alpha < 1$, $\beta \in \mathbb{R}$, $\gamma \geq \alpha + \beta$, and $p, p_1, p_2 \in [1, \infty]$ such that $1/p = 1/p_1 + 1/p_2$. For every $t \geq 0$, define*

$$[e^{t\Delta}, \odot] : (f, g) \mapsto e^{t\Delta}(f \odot g) - f \odot (e^{t\Delta}g).$$

There exists $C < \infty$ such that uniformly over $t > 0$,

$$\|[e^{t\Delta}, \odot](f, g)\|_{\mathcal{B}_{p,\infty}^\gamma} \leq C t^{\frac{\alpha+\beta-\gamma}{2}} \|f\|_{\mathcal{B}_{p_1,\infty}^\alpha} \|g\|_{\mathcal{B}_{p_2,\infty}^\beta}.$$

Proof. We will actually show that

$$\| [e^{t\Delta}, \odot](f, g) \|_{\mathcal{B}_{p, \infty}^\gamma} \leq Ct^{\frac{\alpha+\beta-\gamma}{2}} \|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}.$$

Since $\|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}} \lesssim \|f\|_{\mathcal{B}_{p_1, \infty}^\alpha}$ by Proposition A.5, this implies the proposition.

We decompose $[e^{t\Delta}, \odot](f, g)$ into $\sum_{k=0}^{+\infty} \mathbf{h}_k$, where

$$\mathbf{h}_k := e^{t\Delta}(S_{k-1}f \delta_k g) - S_{k-1}f \delta_k(e^{t\Delta}g).$$

The Fourier spectrum of \mathbf{h}_k is contained in $2^k \mathcal{A}$, where we recall that \mathcal{A} is the annulus $B(0, 10/3) \setminus B(0, 1/12)$. By [1, Lemma 2.84] (or [31, Lemma 3.16]), it thus suffices to show that

$$\left\| (2^{k\gamma} \|\mathbf{h}_k\|_{L^p})_{k \geq 0} \right\|_{\ell^\infty} \lesssim t^{\frac{\alpha+\beta-\gamma}{2}} \|\nabla f\|_{\mathcal{B}_{p_1, \infty}^{\alpha-1}} \|g\|_{\mathcal{B}_{p_2, \infty}^\beta}.$$

Let $\phi \in C_c^\infty$ be supported on an annulus and such that $\phi = 1$ on \mathcal{A} , and let

$$G_{k,t} = \mathcal{F}^{-1} \left(\phi(2^{-k} \cdot) e^{-t|\cdot|^2} \right).$$

Any function h whose Fourier spectrum lies in $2^k \mathcal{A}$ satisfies

$$e^{t\Delta}h = G_{k,t} \star h.$$

In particular,

$$\mathbf{h}_k = G_{k,t} \star (S_{k-1}f \delta_k g) - S_{k-1}f (G_{k,t} \star \delta_k g),$$

that is,

$$\mathbf{h}_k(x) = \int G_{k,t}(y) \delta_k g(x-y) (S_{k-1}f(x) - S_{k-1}f(x-y)) dy.$$

We can rewrite the difference of $S_{k-1}f$ at two points in terms of its gradient:

$$S_{k-1}f(x) - S_{k-1}f(x-y) = - \int_0^1 \nabla S_{k-1}f(x-sy) \cdot y ds,$$

so that

$$\mathbf{h}_k(x) = \int_0^1 \int \delta_k g(x-y) \tilde{G}_{k,t}(y) \cdot \nabla S_{k-1}f(x-sy) dy ds,$$

where $\tilde{G}_{k,t}(y) := y G_{k,t}(y)$. Let us denote the inner integral above by $\mathbf{h}_{k,s}(x)$. We now show that

$$(A.21) \quad \|\mathbf{h}_k\|_{L^p} \lesssim \|\tilde{G}_{k,t}\|_{L^1} \|\nabla S_{k-1}f\|_{L^{p_1}} \|\delta_k g\|_{L^{p_2}}.$$

We will in fact show that (A.21) holds with $\mathbf{h}_{k,s}$ in place of \mathbf{h}_k , uniformly over s . (This inequality is a minor variant of Young's and Hölder's inequalities; in particular, it does not depend on the specific properties of the functions involved, and the implicit multiplicative constant would be 1 if all functions were real-valued.) We first observe that by Hölder's inequality,

$$\mathbf{h}_{k,s}(x) \lesssim \|\tilde{G}_{k,t}\|_{L^1}^{1-\frac{1}{p}} \left(\int |\tilde{G}_{k,t}(y)| |\delta_k g(x-y)|^p |\nabla S_{k-1}f(x-sy)|^p dy \right)^{\frac{1}{p}}.$$

As a consequence,

$$\|\mathbf{h}_{k,s}\|_{L^p}^p \lesssim \|\tilde{G}_{k,t}\|_{L^1}^{p-1} \iint |\tilde{G}_{k,t}(y)| |\delta_k g(x-y)|^p |\nabla S_{k-1}f(x-sy)|^p dy dx.$$

By Hölder's inequality,

$$\int |\nabla S_{k-1}f(x-sy)|^p |\delta_k g(x-y)|^p dx \leq \|\nabla S_{k-1}f\|_{L^{p_1}}^p \|\delta_k g\|_{L^{p_2}}^p,$$

and we obtain (A.21).

The remaining step consists in uncovering the size of $\|\tilde{G}_{k,t}\|_{L^1}$ in terms of k and t . By symmetry, it suffices to study the L^1 norm of the function $y \mapsto y_1 \tilde{G}_{k,t}(y)$. Up to a factor i , this function is the inverse Fourier transform of

$$\zeta \mapsto \partial_1 \left(\phi(2^{-k}\zeta) e^{-t|\zeta|^2} \right) = (2^{-k} \partial_1 \phi(2^{-k}\zeta) - 2\zeta_1 t \phi(2^{-k}\zeta)) e^{-t|\zeta|^2}.$$

We learn from the proof of [1, Lemma 2.4] (or that of [31, Lemma 2.10]) that for every $\tilde{\phi} \in C^\infty$ with support in an annulus, there exists $c > 0$ such that

$$\left\| \mathcal{F}^{-1} \left(\tilde{\phi}(2^{-k} \cdot) e^{-t|\cdot|^2} \right) \right\|_{L^1} \lesssim e^{-ct2^{2k}}.$$

As a consequence, there exists $c > 0$ such that

$$\|\tilde{G}_{k,t}\|_{L^1} \lesssim 2^{-k} (1 + t2^{2k}) e^{-ct2^{2k}}.$$

Combining with (A.21), we get

$$\|\mathbf{h}_k\|_{L^p} \lesssim 2^{-k} (1 + t2^{2k}) e^{-ct2^{2k}} \|\nabla S_{k-1} f\|_{L^{p_1}} \|\delta_k g\|_{L^{p_2}}.$$

By definition of the Besov norm, we have $\|\delta_k g\|_{L^{p_2}} \leq 2^{-k\beta} \|g\|_{\mathcal{B}_{p_2,\infty}^\beta}$. Since we assume $\alpha < 1$, we also have $\|\nabla S_{k-1} f\|_{L^{p_1}} \lesssim 2^{k(1-\alpha)} \|\nabla f\|_{\mathcal{B}_{p_1,\infty}^{\alpha-1}}$, and thus

$$\begin{aligned} 2^{k\gamma} \|\mathbf{h}_k\|_{L^p} &\lesssim 2^{k(\gamma-\alpha-\beta)} (1 + t2^{2k}) e^{-ct2^{2k}} \|\nabla S_{k-1} f\|_{L^{p_1}} \|\delta_k g\|_{L^{p_2}} \\ &\lesssim t^{\frac{\alpha+\beta-\gamma}{2}} \left[(t2^{2k})^{\frac{\gamma-\alpha-\beta}{2}} (1 + t2^{2k}) e^{-ct2^{2k}} \right] \|\nabla S_{k-1} f\|_{L^{p_1}} \|\delta_k g\|_{L^{p_2}}. \end{aligned}$$

The term between square brackets is uniformly bounded, so the proof is complete. \square

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(Jean-Christophe Mourrat) ECOLE NORMALE SUPÉRIEURE DE LYON, CNRS, LYON, FRANCE
E-mail address: `jean-christophe.mourrat@ens-lyon.fr`

(Hendrik Weber) UNIVERSITY OF WARWICK, COVENTRY, UNITED KINGDOM
E-mail address: `hendrik.weber@warwick.ac.uk`