Obstacles to mathematization in introductory physics

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Abstract

Recent studies have demonstrated that although physics students are generally successful executing mathematical procedures in context, they struggle with the use of mathematical concepts for sense making. University physics instructors often note that their students struggle with basic algebraic reasoning, a foundation on which more advanced mathematical thinking rests. However, little systematic research has been done to measure and categorize difficulties in this population. This paper describes a large-scale study (N > 600) designed to investigate trends in student reasoning with ratio and proportion, quantification, and symbolizing within the calculus-based introductory physics course. Although the assessment items require mathematical reasoning typically taught at the middle school level in mathematics courses, we find success rates of about 50% among calculus-based physics students. For many of these students, numerical complexity and physical context interferes with basic arithmetic reasoning. We argue that the algebraic thinking of physicists stems from an idiosyncratic cognitive blend, and is not addressed in prerequisite algebra courses. We suggest that for most students to understand and to adopt the mathematical thinking characteristic of physics, the community of physics instructors and education researchers must explore how to effectively make mathematization a more explicit part of the curriculum.

I. Introduction

Mathematizing in physics involves translating between the physical world and the symbolic world in an effort to understand how things work.[1, 2] Specific skills include representing concepts symbolically, defining problems quantitatively, and verifying that solutions make sense. Physicists develop and communicate ideas through the shared meanings they have built around these strong connections between mathematics and physics.

Arithmetic and algebraic reasoning are cornerstones of mathematization in an introductory physics course. Although students in a calculus-based course will have successfully completed prerequisite algebra courses, experienced instructors recognize that even their well-prepared students commonly struggle with algebraic decision making in physics. Whether it is the naïve but common association of negative acceleration with decreasing speed, or attributing the $F_{net} = 0$ condition to an absence of forces acting on an object, basic mathematization seems to pose significant challenges to students throughout the introductory physics curriculum.

Physics curricula typically rely on flexibility with algebraic reasoning that is special to physics and deeply embedded in the discipline.[3, 4] While many researchers have focused on student use of mathematics at the level of pre-calculus and above in the context of physics, little research has focused specifically on the challenges students face thinking arithmetically in a calculus-based physics course. This paper extends work into this area by assessing reasoning competence with basic mathematical concepts, and identifying specific reasoning difficulties, in a large population of mathematically well-prepared university physics students. We have administered written questions involving ratio and proportion in introductory calculus-based physics courses to investigate facility in the mathematical cognitive domains of generalized structural reasoning, quantification, and symbolizing. Results suggest that standard instruction fails to build student capacity for the basic mathematical thinking fundamental to the discipline, and serves to quantify just how limited a standard, large-enrollment physics course is in promoting mathematization.

The following specific questions have guided our research:

1. To what extent are mathematically well-prepared engineering students able to reason successfully with ratio and proportion after one semester of calculus-based physics? (generalized structural reasoning)

2. How robust is students' arithmetic reasoning across physical contexts and levels of numerical complexity? (quantification)

3. After one semester of calculus-based introductory physics, how well do students reason with variable quantities in an arithmetically simple situation?" (symbolizing)

We approach both the design of assessment questions and analysis of student responses from a cognitive blending framework, [5, 6] treating the mathematics and physics as a single thinking space. The following section describes how overall productive mathematical thinking in physics relies on generalized structural reasoning, quantification and symbolizing, summarizes relevant prior research on student learning in each of these three areas, and then describes the cognitive blending framework. Sections III and IV present our research methods and results. Section V revisits our three research questions in light of these results; we find that generalized structural reasoning, quantification, and symbolizing present substantial obstacles to the development of introductory physics students' mathematization, and that surface features of problem context and numerical complexity can interfere with the reasoning of even well-prepared students. Section VI concludes with a discussion of implications for research and instruction.

II. Background and prior research

A substantial and diverse body of literature from mathematics and physics education research documents student and expert use of mathematics in physics at the level of pre-calculus and above [7-17] (for an upper-division summary see Caballero, Wilcox, Doughty and Pollock [18]). Caballero *et al.* characterize the research body associated with mathematical reasoning in upper-division physics courses using two broad categories: 1) *macroscopic*, or whole class studies focused on uncovering student difficulties, and 2) *microscopic,* typically theory-driven studies, focused on in-situ interviews with small groups of students. We agree with Caballero *et al.*'s assertion that a more complete understanding emerges from connecting these two approaches.

We extend the distinction articulated in Caballero *et al.* to include research on mathematical sensemaking in introductory physics. For example, Tuminaro,[19] in his microscopic study of students in algebra-based introductory physics courses, points out that if students do not expect conceptual knowledge of mathematics to connect to their work in solving physics problems, then they are likely to frame their problem-solving activities in terms of plug-and-chug manipulations or intuitive sense-making that is primarily qualitative. He concludes that for these students, sense making is not part of calculating. Hull, Kuo, Gupta and Elby [20] have seen similar results in their microscopic study. Our study connects to these reported outcomes.

Fewer than 3% of the students who take the introductory course go on to major in physics, so the majority of introductory level students are not amongst the sample populations of upper-division studies. An accurate, quantitative sense of the extent to which introductory level students struggle with mathematical sensemaking will help develop a more complete understanding of student difficulties with sensemaking. A carefully constructed large-N study of calculus-based introductory physics students is a compelling method to document trends that characterize the entire population. The work reported on in this paper can be considered a macroscopic study with theoretical underpinnings, intended to uncover the extent to which students struggle with algebraic ideas that are foundational in introductory physics.

Just as with upper division courses, sensemaking at the introductory level involves creative mathematical thinking. In contrast to procedural use of math, generating algebraic descriptions of physical events and systems requires students to try approaches without knowing whether or not they will work, which in turn requires courage and tolerance of failure. Students must learn to check whether or not the mathematics that they generate makes sense, and how to iterate toward better solutions. We view the three mathematical cognitive domains of structural reasoning, quantification and symbolizing as building blocks for this productive type of mathematical thinking, and as a foundation for the sophisticated algebraic and calculus reasoning ubiquitous beyond first year physics. Below we describe each of these areas, summarizing prior work and illustrating how difficulties might impact physics learning.

A. Generalized structural reasoning

In order to connect mathematics to physical phenomena, physicists draw on generalized structural reasoning. A generalized mathematical structure (*e.g.* a $1/r^2$ force), when recognized, can guide thinking in a new context. While common practice for physicists, this way of thinking is novel for students (even students who, in the above example, may have solved many inverse square problems using Newton's Universal Law of Gravitation in one course and Coulomb's law in another.)

We view ratio as one of the most important general mathematical structures in introductory physics. Student reasoning about ratio and proportion was examined in the early physics education work of Arons, Karplus, and others,[21-24] as well as extensively in mathematics education research.[25-28] Thompson [29] describes proportional reasoning as interconnected skills that are context-dependent, claiming that proportional reasoning "…appears in various guises in different contexts and different levels of sophistication." The assessment items reported on in this paper stem from a larger project focused on delineating and assessing specific skills that make up proportional reasoning in physics.[30]

In his microscopic study of 3rd semester engineering students at a highly selective university, Sherin [14] reports on the opacity of the mathematical structures underlying the kinematics equations. He posed the following task to mathematically well-prepared students from a third-semester introductory calculus-based physics course: *"Imagine that we've got a pile of sand and that, each second, R grams of sand are added to the pile. Initially, the pile has P grams in it. Write an expression for the mass of the sand after t seconds."* In clinical interviews, all students were able to generate a correct expression for the mass as a function of time. None, however, recognized that the arithmetic progression for the sand pile mass matched that of the velocity function $v(t)$ for a motion with constant acceleration. The interview subjects could not explain why the "correction" to v_o should be at, and seemed perplexed to be asked to consider such a simple question about sand.

Rebello, Cui, Bennett, and Zollman [13] report on students' capacity to generalize reasoning and methods learned in trigonometry and calculus to end-of-chapter textbook problems in physics. Calculus-based physics students were asked to solve physics problems involving simple integration or differentiation similar to problems they had already solved in homework. While the students were able to execute the required calculus procedures when prompted, they were largely unsuccessful at setting up and solving problems that required them to select appropriate calculus tools and adapt them to fit a physical situation. The researchers also surveyed algebra-based students before and after instruction and found little evidence that students would spontaneously generalize trigonometry from math to physics; students lacked flexibility with the prerequisite mathematics. Schoenfield describes math students' *belief systems* (*i.e.* expectations) as rigid due to their emphasis on the context-specific nature of problem-solving approaches (*e.g.,* they use deductive argumentation in geometry proofs but not in other contexts).[31]

We associate the observed lack of spontaneous generalizing to rigidity in students' systems of belief surrounding mathematics. In particular, student beliefs may not allow for, at least implicitly, a spontaneous use of mathematical reasoning unprompted outside of math class. We find it productive to think in terms of Hammer's resource framework,[32] and Wittman's coordinated set of resources.[33] Rebello's students do not activate calculus or trigonometry resources in the physics context unless they are explicitly prompted to do so, nor do Sherin's even when they are prompted by analogy.

Each of these examples describes a reasoning structure that experts readily activate in a variety of contexts. We think of the reasoning structure as the activation of a coordinated set of resources, which in turn require a robust set of individual resources to be readily accessible. In the case of Sherin's (very strong) physics students, activation of their coordinated arithmetic progression set was perhaps context dependent, and sand was a context that activated the reasoning. In the absence of activities that challenge students to broaden this coordinated set to include physics quantities, however, they did not generalize the reasoning.

B. Quantification

Concepts associated with quantitative operational definitions are the building blocks of physics, and in our introductory course students encounter ~10^2 new quantities. A physical quantity involves a number, an associated unit, and sometimes a direction. Physicists commonly use this association to guide their own thinking about how quantities are related, and even to formulate new quantities. In order to begin to quantify efficiently in physics, students must have a conceptual facility with numbers of all kinds, including positive and negative numbers, as well as the use of units. As more abstract units are introduced, the interpretation becomes more challenging. (Consider, for example, a compound unit such as m/s^2, which combines fundamental units of length and time in a complicated way. It's not evident to the untrained eye what meaning "s^2" has in the denominator.) Facility with number and unit is supported by the development of conceptual understanding of the arithmetic involved in combining quantities. Introductory physics introduces a new challenge with vector quantities and the specific algebra they obey.

Mastery of number may seem trivial and well outside the domain of college physics. But *mastering number in the context of physical quantities* is not the same as mastering number in math class, where units are rarely involved; in physics units carry a deep meaning. This particular use is in contrast with the most common uses of numbers in everyday life, which include categorizing ("What's behind door number 1?"), ordering ("My amp goes up to 11!") and defining thresholds (a blood pressure of 120/80 means don't worry). Physicists function almost entirely at the interval level of measurement, where the quantity itself (the numeric value and its associated unit) carries important information. For example, a steady speed of 25 mph immediately conveys that the vehicle will travel 25 miles in one hour, while students may think about 25 mph simply as "slower than I want to drive."

Researchers in mathematics education have identified *quantification* as a significant challenge to students who are learning to mathematize. Thompson, who has researched and written extensively on this topic over the past two decades, defines quantification to be " the process of conceptualizing a mathematical object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship … with its unit." He considers quantification to be "a root of mathematical thinking", and argues that learners develop their mathematics from reasoning about quantities. [29, 34]

The complexity of numeric values may produce cognitive strain for students as they learn new physics. Decimals and fractions are much more common than whole numbers for measured physical quantities, while whole numbers are more common when students learn algebra in school. Large and small numbers, ubiquitous in physics, also pose difficulty and are not frequently used in algebra courses. We explore the effect of numerical complexity in this study.

Physics involves compound quantities that result from multiplying and dividing other quantities (e.g., momentum). While the arithmetic procedures involved in creating new quantities is not necessarily challenging, deciding when and why this arithmetic makes sense can be difficult for students.(34) To interpret and understand ratios or products can require conceptualizing multiplication and division. Many compound quantities are rates of change, which require a conceptual understanding not only of ratios but also of differences. Conservation principles, which require a conceptual understanding of summation, are a common motivation for the development of a new quantity. Finally, some quantities, such as electric flux, combine quantities already poorly understood (electric field, area).

Vergnaud argues that multiplication, division, fractions, ratios, proportions, linear functions, dimensional analysis and vector spaces are not mathematically independent, and should be included in a domain he names *multiplicative structures*.(35) Tuminaro reports on student difficulties conceptualizing arithmetic and the simplest multiplicative structures in physics.(19) In addition, unlike working with pure numbers in math class, adding, subtracting, multiplying, and dividing physical quantities involves special constraints. Physicists recognize that addition and subtraction can be carried out only with like quantities expressed in like units, and that multiplying and dividing can create a quantity completely different from either of the constituents. Students have little or no experience reasoning about multiplicative structures with physics quantities from prerequisite math classes, but are expected to reason this way in physics. For example, a student might be expected to recognize from context whether the product of a force and a distance yields a torque or a work. The research described in this report explores how physics contexts might pose challenges to student's arithmetic reasoning.

C. Symbolizing

Mathematics education researchers have been grappling with learning theories related to symbolizing and communicating for decades.(36) The context dependence of symbol use in physics is nuanced, and often not part of students' mathematics preparation. Below we explore roles that fundamental symbols – signed numbers, zero, the equals sign, and variables – play in the development of physics concepts.

Signed quantities and zero take on a variety of roles in physics. For example, the quantity "−5m" could indicate a position or a change in position, depending on the context. In the first case the negative sign tells us on which side of an arbitrary reference an object is located; in the second, the direction in which the object has moved. The same symbols thus quantify different ideas, with different names (position and displacement). Similarly, the positive sign in "+5 N•m" could mean that the work done on a system contributed to an increase mechanical energy, or that the exerted torque contributed to initiating rotation in a particular direction. Physicists also use positive and negative numbers to quantify opposites (e.g., opposite types of electric charge), with zero representing a balance rather than an absence.

Students may lack the flexible understanding of the equals sign required in physics. High school and college students commonly use the equals sign inappropriately as they solve equations or evaluate expressions in algebra and calculus.(37, 38) While students can often interpret an equals sign as a prompt for calculation (e.g., $v = 3$ m/s × 5 s); far fewer understand it to be the relational symbol of mathematical equivalence (e.g., $(\boldsymbol{F}_{\text{a on b}} + \boldsymbol{F}_{\text{c on b}})/m_{\text{b}} = \boldsymbol{a}_{\text{b}}$).

Mathematics education researchers have extensively studied student use of variables, and have identified persistent difficulties. These difficulties can be compounded in physics contexts: as with signed numbers and zeros, physicists use symbols in distinct ways that may confuse students. At the beginning of many mathematics textbooks there are lists of letters that are to be considered variables *(x, y, z)* and of letters to be considered constants *(a, b, c)*. Typically, an expression will contain only a single variable; the task at hand is usually to solve for that variable.[39] In contrast, physicists are more fluid in their use of letter symbols, and the same letter might be a constant in one problem, and a variable in another. Even a physical constant can be considered as a variable under certain conditions. For example, students may be asked to "find the value of *g* on this planet," or even to "take the derivative with respect to *h-bar*."

Cohen and Kanim [40], building on early work by Clement, Lochhead, and Monk [38] administered the "students-and-professors" question to probe student ability to convert a natural language sentence into a mathematical expression. Students were asked to write an equation, using *S* for the number of students and *P* for the number of professors, to represent the statement, "There are six times as many students as professors at this university." Clement *et al*. report that students taking calculus-based introductory physics found this task challenging, commonly placing the number 6 on the wrong side of the equation. Cohen and Kanim explored this "reversal error" in greater detail by changing sentence structure and the choice of symbols, and found that about two-fifths of students making the error seemed to be performing a word-order translation of the sentence (referred to as *syntactic translation)*, while most of the remaining students seemed to be treating the symbols *S* and *P* as units or labels, rather than variables.

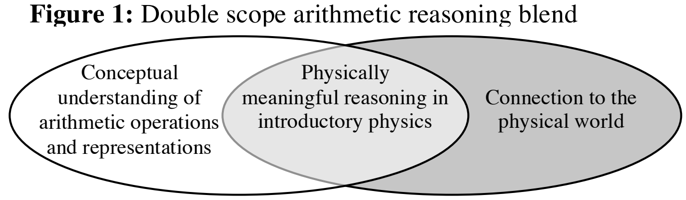
The equations typically encountered in physics courses are far more symbol rich than the equation involved in the students-and-professors question. In a macroscopic study at the University of Illinois, Torigoe and Gladding posed isomorphic questions on final exams in the introductory physics course for engineers.[41] In one member of a question pair, quantities were represented by numbers, while its partner used only symbols. Differences in success rates of up to 65% were observed. Student success on exam questions that relied on accurate manipulation of uniquely symbolic representations correlated to course grades, with the strongest correlation occurring for the bottom quartile of the students. Students who reason poorly when faced with a barrage of symbolic quantities are more likely to do poorly in an introductory physics course.

The results of Torigoe and Gladding suggest that seemingly small changes in the way quantities are represented in the statement of a physics problem can lead to very different levels of student success. Our research uncovers how representing just a single physical quantity with a variable can pose an obstacle to student's arithmetic reasoning.

D. Theoretical Framework: Cognitive Blending

The theoretical framework of cognitive blending [5, 6] supports our view that continuous interdependence of thinking about the mathematical and physical worlds is necessary for expert problem solving in physics. Figure 1 illustrates a *double scope arithmetic reasoning blend*, in which two distinct domains of thinking are merged to form a new cognitive space that is optimally suited for productive work.

Figure 1: Double scope arithmetic reasoning blend



Prior work in this area presents a theoretical framework that spans a spectrum of homogeneity of the cognitive blend (represented as the overlap in Fig. 1.) Researchers focused on students use of relatively sophisticated mathematics [3, 4, 42] commonly model the mathematical world and physical world as a more heterogeneous thinking space, as students grapple with multistep mathematical operations within physics contexts. Considering the use of algebra at the introductory level, Tuminaro also describes heterogeneity, "students invoke ideas from mathematics—such as equations, graphs, etc.— to help them understand the physics."[43] By contrast, from their case study research Hull *et al*. hypothesize a more homogeneous blend of these thinking spaces in the context of introductory physics.[20] From the mathematics research perspective, Czocher [7] conducted a microscopic study with engineering students enrolled in a differential equations course and observed them solving a variety of physics problems over the course of the semester. She reports that successful students functioned most of the time in a "mathematically structured real-world" in which they moved back and forth fluidly between physics ideas and mathematical concepts. Czocher describes this thinking space as being between the "real world" and the "math world",

We suggest that student learning in introductory physics is best supported through a completely homogeneous blend (as observed by Czocher), such that there is no distinction between the physics and the arithmetic worlds. We propose a thinking space we refer to as the *mathematization of introductory physics*, in which physical sensemaking is essential for and integrated with mathematical reasoning. In the context of arithmetic thinking we claim that the optimum thinking space is a heterogeneous blend representing a continuous interdependence between the physical world and conceptual understanding of arithmetic operations and representations. We analyze our results using this framework and draw conclusions that can inform both instruction and curriculum development.

III. Research methods

In order to uncover specific challenges that students encounter, we administered multiple choice questions at the beginning and end of introductory physics and chemistry courses taught at a large, public research university. The assessment items investigate facets of first-year engineering students' arithmetic reasoning. In this section, we present the research tasks, and describe the student population and methods of data collection and analysis.

A. Research tasks

The research tasks were drawn from a large set of items used by the authors in a previous investigation of the proportional reasoning of introductory physics students. The development and validation of the items is described in detail elsewhere;[30] here we summarize the process. Rather than procedural or computational skill, the questions focus on sensemaking and conceptualization of ratio quantities. The initial versions of the items asked students to explain their reasoning and show their work. Question validity was established through in-depth, think-aloud interviews with more than twenty individual students. We used the written responses and interview transcripts to create multiple-choice versions of the questions, with distractors based on the difficulties identified through analysis of students' verbal explanations. Because the current study identifies trends in large populations of students, we focus on quantitative results from the multiple choice versions.

Throughout the development of the items, we observed variations in student reasoning associated with physical context, with the level of abstraction of the ratio or product quantity, and with the numerical complexity of the quantities involved. These findings, consistent with previous studies of student reasoning about ratio,[23, 44, 45] led us to develop parallel versions of several of the assessment questions. We hoped to isolate triggers for variations in student reasoning by changing only a single surface feature of each question.

For the current study, we selected items that involve the mathematical cognitive domains described earlier: generalized structural reasoning, quantification, and symbolizing. These items are presented in Tables I-III. Due to the interrelated nature of these cognitive domains, it is not possible to design items that target one domain at the exclusion of the others. We thus present items that *highlight* the cognitive domain of interest, acknowledging that the other domains may also be represented in student responses.

Table I presents items used to investigate our first research question, associated with generalized structural reasoning. These items require students to either apply a given ratio (items I and II), or identify ratio as an appropriate measure (items III and IV). Students who have internalized ratio as a general mathematical structure will have a powerful resource for determining how to use the relevant ratio appropriately, and for checking the validity of their answer. In contrast, poor performance would suggest lack of an internalized ratio structure.

Table I: Items used to assess generalized structural reasoning.

<table>
<tr><th>Item name</th><th>Item text</th></tr>
<tr><td>I. Olive Oil[1]</td><td>You go to the farmer's market to buy olive oil. When you arrive you realize that you have only one dollar in your pocket. The clerk sells you 0.26 pints of olive oil for one dollar. You plan next week to buy 3 pints of olive oil. Which of the following expressions helps figure out how much this will cost (in dollars)?
a. 3/0.26 b. 0.26/3 c. 3•0.26 d. (3+1)•0.26 e. none of these</td></tr>
<tr><td>II. Traxolene[1]</td><td>You are part of a team that has invented a new, high-tech material called "traxolene." One gram of traxolene has a volume of 0.41 cm^3. For a laboratory experiment, you are working with a piece of traxolene that has a volume of 3 cm^3. Which of the following expressions helps figure out the mass of this piece of traxolene (in grams)?
a. 3/0.41 b.0.41/3 c.3•0.41 d. (3+1)•0.41 e. none of these</td></tr>
<tr><td>III. Square Buildings[2]</td><td>You are riding in an airplane. Below you see three rectangular buildings with the rooftop dimensions shown at right. You are interested in how close the shapes of the rooftops of the buildings are to being square. You decide to rank them by "squareness," from most square to least square. Which of the following choices is the best ranking?
a. A, B, C b. B, A, C c. C, A, B d. C, B, A e. B, C, A
Building A: 77 ft by 93 ft
Building B: 51 ft by 64 ft
Building C: 96 ft by 150 ft</td></tr>
<tr><td>IV. Force Vectors[1]</td><td>Each of three different objects (A, B, C) experience two forces, one in the +x direction and one in the +y direction. Rank each object according to how close the direction of the net force is to a 45° angle between the x-direction and the y-direction, from closest to 45° to farthest from 45°.
a. A, B, C b. B, A, C c. C, A, B d. C, B, A e. B, C, A
<table><tr><th></th><th>Force in x-direct</th><th>Force in y-direct</th></tr><tr><td>A</td><td>77 N</td><td>93 N</td></tr><tr><td>B</td><td>51 N</td><td>64 N</td></tr><tr><td>C</td><td>96 N</td><td>150 N</td></tr></table></td></tr>
</table>

1: Administered as online posttest in chemistry course.
2: Administered as in-class posttest in physics course.

While each item I-IV stands on its own as an assessment of generalized structural reasoning, the items together consist of pairs of questions that can be compared to probe the context dependence of student reasoning. Pairs (I & II and III & IV) are isomorphic in that experts recognize them as involving identical mathematical reasoning. Only the surface context varies, with one item in each pair involving an everyday context and the other a physics context. For example, item I involves purchasing olive oil at the market while item II involves a high-tech material called traxolene; both can be completed, however, by using a characteristic ratio to find an unknown amount associated with a specific case.

Table II presents items used to investigate student difficulties with quantification. Items V and VI require students to interpret an unfamiliar quantity by attaching a specific meaning to, and understanding of, the units that result from dividing one physical quantity by another. On items VII a and b, students must construct a ratio of quantities when provided with a description of the meaning of the yet-to-be-determined ratio. Students who recognize that a physical quantity is instantiated by a numerical value (regardless of the complexity of the numerical structure) linked to an associated unit will be more likely to succeed in these tasks.

Table II: Items used to assess quantification.

Item name	Item text
V. Paint[1]	Catherine is hired to paint the ceiling of her aunt's living room. She covers the ceiling with a uniform coat of paint. The ceiling has a surface area of 580 square feet. After finishing, Catherine notes that she used 2.4 gallons of paint. Catherine divides 580 by 2.4 and gets 241.7. Which of the following statements about the number 241.7 is true? a. 241.7 is the total number of gallons of paint used b. 241.7 is the total number of square feet of surface area covered by the paint c. 241.7 is the number of gallons of paint that covers one square foot d. 241.7 is the number of square feet that one gallon of paint covers e. none of the above
VI. Door Knob[1]	Catherine shuffles her feet across her living room carpet and then she touches a doorknob, which has a surface area of 580 square centimeters. When she touches the doorknob she transfers 2.4 microcoulombs of electric charge that spreads out uniformly over the doorknob's surface. Catherine divides 580 by 2.4 and gets 241.7. Which of the following statements about the number 241.7 is true? a. 241.7 is the total number of microcoulombs of charge transferred b. 241.7 is the total number of square centimeters of surface area covered by the charge c. 241.7 is the number of microcoulombs of charge that covers one square centimeter d. 241.7 is the number of square centimeters that one microcoulomb of charge covers e. none of the above
VIIa. Rice[2] – Whole	Bartholomew is making rice pudding using his grandmother's recipe. For three servings of pudding the ingredients include 4 pints of milk and 2 cups of rice. Bartholomew looks in his refrigerator and sees he has one pint of milk. Given that he wants to use all of the milk, which of the following expressions will help Bartholomew figure out how many cups of rice he should use? a. 4/2 b. 2/4 c. 2•4 d. (2+1)•4 e. none of these
VIIb. Rice[2] – Decimal	Same as VIIa except decimal quantities are used "… include 0.75 pints of milk and 0.5 cups of rice…" a. 0.5/0.75 b. 0.75/0.5 c.0.5•0.75 d. (0.5+1)•0.75 e. none of these

1: Administered as online posttest in chemistry course. 2: Administered in-class in physics course.

Items V and VI constitute a matched pair, identical in the underlying mathematical reasoning, but different in surface context. Items VII a and b are identical save for differences in the way the quantities are represented: version a) involves whole numbers, while version b) involves decimal numbers. This allows the impact of numerical complexity on student quantification to be examined. Together, items V-VIIb are used to investigate our second research question.

Table III: Items used to assess symbolizing, administered inclass in physics course.

Item name	Item text
VIIc. Rice – Fraction	Same as Table 2, VIIa except a fractional and a decimal quantity used "… include 0.75 pints of milk and 5/8 cups of rice…" a. (5/8) /0.75 b. 0.75/(5/8) c. (5/8) • 0.75 d. ((5/8) +1) • 0.75 e. none of these
VIId. Rice – Variable	Same as Table 2, VIIa except a fractional and a variable quantity is used "… include *N* pints of milk and 5/8 cups of rice…" a. (5/8)/*N* b. *N*/(5/8)] c. *N* x (5/8) d. ((5/8)+1) x *N* e. none of these
VIII. Woozles	Consider the following statement about Winnie the Pooh's dream: *"There are three times as many heffalumps as woozles*." Some students were asked to write an equation to represent this statement, using *h* for the number of heffalumps and *w* for the number of woozles. Which of the following is correct? a. *3h*/*w* b. *3h = w* c. *3h + w* d. *h = 3w* e. both a and b

Table III presents items used to uncover symbolizing challenges using a variable and equals sign. Item VIII requires students to effectively represent a proportional relationship with an algebraic statement. Items VII c and d require analogous reasoning to VII b and a, but in the face of a single variable. This trio of questions is used to measure an effect of symbolizing on student reasoning.

B. Study population

This study's population is freshman non-honors engineering students at Rutgers University taking traditionally taught calculus-based physics and chemistry. These students tend to be well prepared mathematically, with a mean mathematics SAT (2011/2012 test version) score of 680. The data was collected as part of routine course pre and post testing, which includes concept inventories in addition to a suite of questions associated with ratio reasoning. Throughout the semester, the lecturer in the physics course modeled proportional reasoning (and other mathematical methods) in the context of the physics content being taught.

C. Data collection

We administered the research tasks under exam conditions as an ungraded in-class pretest during the first week of the introductory, calculus-based physics course and the general chemistry course in Fall 2013. In the physics course, the same tasks were administered again as a posttest, 10 days before the end of the semester, also under exam conditions. In the chemistry course, however, the post-test was administered online outside of class, and there was a substantial drop in the number of students participating. Table IV summarizes the administration of pre- and post-test questions.

Table IV. Administration of written assessment items reported on in this paper.

Class	**Subject**	**Items reported on in this paper**	**Pretest**	**Posttest**	**No. of versions**
Freshmen	Mechanics	III, VII a-d, VIII	Supervised In class	Supervised In class	8
Freshman	Gen Chem	I, II, IV, V VI	Supervised In class	Unsupervised Online	6

In all, 14 multiple-choice items, probing different features of proportional reasoning, were administered on the pretest and again on the posttest. Seven items were administered in the mechanics course (n_{pre}=770 and n_{post}=737), and seven in the chemistry course (n_{pre}=628 and n_{post}=332). A subset of the students took both the mechanics and the chemistry tests (479 on the pretests and 287 on the posttest). In both courses, the items were bundled with a standardized concept inventory (the Force Concept Inventory [(46)] in the physics course and the Chemical Concepts Inventory [(47)] in the chemistry course). In a single sitting, students first completed the proportional reasoning items, and then immediately completed the concept inventory. The students were not constrained by time and were awarded credit for participation. Note that these considerations apply to both the pretest and posttest, which were administered under identical conditions (except that in chemistry, the posttest was given online, see Table IV).

As mentioned in section IIIA, we probed the effect of surface features on student reasoning by administering matched versions of items on different versions of the tests (see Table IV). Tests were administered in the recitation section of the course, and within a given recitation different test versions were assigned randomly. Thus, for a given isomorphic question pair, half of the students in a given course received one version of the question and half received the other. Each student in the study received the same version of the question suite on the pretest and the posttest.

D. Data analysis

We compare the fraction of correct responses in a given sample under three possible conditions: 1) a single question using a matched set of students who were sampled at two different times (*i.e.,* pre- and post-instruction), 2) two different questions using a matched set of students who were sampled simultaneously, or 3) two different samples taken from the same population who were tested simultaneously. In cases 1 and 2, we use the McNemar test of significance, and in the case of 3 we use the Mann-Whitney test. In case 3, we established baseline equivalence between samples using FCI pretest and SAT Math scores, assuring that the effect size associated with the difference in the standard error was less than 0.5 on both measures.

IV. Results

In this section, we summarize responses to the multiple choice items as evidence of the level of student facility with generalized structural reasoning, quantification, and symbolizing.

A. Generalized structural reasoning

Facility with generalized structural reasoning (GSR) includes the ability to reason quantitatively about a given mathematical form (e.g., a ratio) in order to make sense of a specific physical context. Items I-IV (shown in Table I) assess student ability to apply GSR in ratio contexts. Results are summarized in Table V.

Table V. Results on items that assess generalized structural reasoning.

Item:	*I. Olive Oil[1]*	*II. Traxo-lene[1]*	*III. Square Buildings[2]*	*IV. Force Vectors[1]*
N:	155	177	275	275
Correct:	55%	66%	17%	23%
Comparison:	I to II		III to IV	
p-value:	.06		.05	
Effect size:	N/A		1.9	

1: Administered as online posttest in chemistry course.
2: Administered as in-class posttest in physics course.

The mathematical reasoning necessary for items I and II is at the level of middle school story problems. The reasoning involves only a single step and can easily be checked using dimensional analysis. Even after completing a one-semester introductory calculus-based mechanics course, however, over one-third of the engineering freshman in our study answered these questions incorrectly. The most common incorrect response on item I was c) 0.26•3, given by 24% of the students. This response corresponds to multiplication of a number of pints of olive oil and the olive oil cost in pints per dollar. Even without applying a detailed argument about ratio and proportion, a student could use dimensional analysis to recognize that multiplication yields a quantity measured in the physically meaningless units of square pints per dollar, whereas dividing the total number of pints by the number of pints per dollar would yield the desired outcome of total cost in dollars. It seems that many students failed to apply ratio reasoning or even to use dimensional analysis for sensemaking in this case. A similar interpretation applies to results on the Traxolene question (item II).

Items III and IV each require a student to identify ratio as the appropriate measure for a specific physical context in order to make a judgment. In comparing a rectangular shape to a square, the difference in lengths of the two sides of the rectangle is meaningful only in comparison to the absolute length of those sides. A dimensionless ratio of the lengths thus serves as an appropriate comparison that is both independent of the units that are used and of the overall size of the rectangle. However, only 17% of the freshman engineering students selected the ratio-based answer, while more than 70% selected the difference-based answer.

For item IV, the force vectors question, a student can recognize that the closer the ratio of the x- and y-components of a vector is to unity, the smaller the deviation of the direction of the vector is from 45°. Fewer than one-quarter of the students answered correctly. One-half of the students gave the response consistent with use of the difference in the vector components.

Results on items I-IV provide evidence that for many freshmen engineering students, functional understanding of ratio is often not applied in even very basic contexts. Given that ratio is one of the most basic generalized mathematical forms, and that engineering students have likely had many years of exposure to ratios, we may speculate that many students will lack facility with other forms as well (e.g., an inverse square relationship or an exponential).

The lower rows of Table V compare responses on two pairs of isomorphic questions (I & II, and III & IV). In each pair, the two items involve reasoning that to an expert seems identical. The first item in each pair uses an everyday context while the second uses a "physics" context. On each pair of items, student performance was slightly stronger on the physics context item in comparison to the item using an everyday context (e.g., students were more successful applying a ratio in the olive oil context than in the traxolene context). The differences, however, were only marginally significant, with p-values ~ 0.05.

The stronger performance on the physics context was unexpected; we hypothesized that an everyday context would cue reasoning resources that a physics context would not. In hindsight, we speculate that any effect of this type was overshadowed by other contextual differences. In the first isomorphic pair, item II involves a likely well-memorized relationship between density, mass, and volume. In interviews, students readily recalled the formula $d = m/v$ and used it in this context. No such formula exists for the olive oil context of item I. While rote use of formulas can often be problematic, it is possible that in this case, use of formulas, triggered in the physics context but not the everyday context, led to a slightly stronger performance. Regarding the second pair of items, on free-response written versions of item IV, students commonly performed extensive trigonometry calculations with their calculators, while on item III, we only rarely observed students using trigonometry. As in the case of the previous pair, increased use of the mathematical formalism, even if it was not connected to physical sense making, may have led to the slightly higher correct response rate.

B. Quantification

Physics experts conceptualize *quantity* as a numerical value tightly linked with an associated unit. Our investigation of student understanding of quantity has involved questions in which students must either verbally interpret a given ratio (items V and VI), or construct an appropriate ratio from measured values (items VIIa and b). Student performance on these questions is summarized in Table VI.

Item VI involves interpreting the ratio of the surface area of a doorknob to the net electric charge distributed on that doorknob. Though the students are enrolled in mechanics, charge units were part of their chemistry curriculum. The number of square centimeters of area required for each microcoulomb of charge is a non-standard quantity; it is the inverse of the more common surface charge density ratio, making it difficult for students to complete this item successfully through a memorized definition. Instead, students must apply ratio reasoning. More than two-fifths of students answered item VI incorrectly. We expect facility with quantification would lead a student to use the units (cm^2/microcoulomb) associated with the given numerical value as a guide for interpreting the quantity.

Table VI. Results on isomorphic pairs of items that assess quantification.

Item:	*V. Paint*[1]	*VI. Door knob*[1]	*VIIa. Rice-Whole*[2]	*VIIb. Rice-Decimal*[2]
N	280	291	171	177
Correct	88%	59%	*Pre:* 78% *Post:* 75%	*Pre:* 60% *Post:* 66%
Comparison	V to VI		VIIa[3] to VII b[3]	
p-value	< 10^{-4}		*Pre:* < 10^{-4} *Post:* .02	
Effect size	11.6		*Pre:* 5.3 *Post:* N/A	

1: Administered as online posttest in chemistry course.
2: Administered in-class in physics course.
3: p-value > .20 for pre-to-post comparison of single item.

Item V requires the same reasoning as item VI, and even involves identical numerical values. The context for item V, however, is a less abstract, more familiar situation: an amount of paint applied to a wall, rather than electric charge distributed over the surface of a doorknob. As shown in Table VI, performance on item V was significantly stronger than that on item VI. It seems that electric charge as a quantity in item VI may interfere with student ability to apply the necessary quantitative reasoning, which is consistent with prior work.[48] The difference in performance, together with the relatively high absolute success rate on item V (88% correct), suggests that while most freshman engineering students may, in a sense, "possess" the reasoning resources needed to interpret a ratio quantity in context, many lack the facility needed to do so reliably.

Items VIIa and VIIb are nearly identical. Both involve a recipe context in which students must construct a ratio to find the number of cups of rice for each pint of milk, given the total numbers of cups of rice and pints of milk. The items differ only in numerical complexity; version a involves whole number quantities (2 cups of rice and 4 pints of milk) while version b involves decimal quantities (0.5 cups of rice and 0.75 pints of milk). As shown in Table VI, the correct response rate on item VIIa was significantly higher than that on VIIb. It seems that for some of the engineering students, reasoning arithmetically with decimal numbers presents an obstacle not present when reasoning with whole numbers. This surprising result provides additional evidence that for students enrolled in calculus-based introductory physics, facility with quantity is not yet robust.

C. Symbolizing

Symbolizing involves representing quantities and quantitative concepts with symbols. We focus here on the use of letters to represent quantities, and the equals sign to represent balance. Item VIIc represents both relevant quantities with numbers (5/8 cups of rice and 0.75 pints of milk), while version d replaces the decimal quantity with a general variable "*N*" (5/8 cups of rice and *N* pints of milk). Prior research has shown that students struggle with purely multi-variable expressions,[41] here we investigate whether student reasoning is affected by including only a single variable. Item VIII is taken from Cohen and Kanim;[40] we include it to probe symbolizing of a verbal statement. Table VII summarizes results on items VIIc, VIId, and VIII.

Table VII. Results on items that assess student ability to work with symbols in context.

Item:	*VIIc. Rice-Fraction*[2]		*VIId. Rice-Variable*[2]		*VIII. Woozles*[2]	
N:	179		142		685	
Correct:	*Pre:*	58%	*Pre:*	42%	*Pre:*	48%
	Post:	60%	*Post:*	30%	*Post:*	49%
Comparison:	VIIc[3] to VIId[4]				Pre/Post	
p-value:	*Pre:*	$< 10^{-3}$			0.85	
	Post:	$< 10^{-9}$				
Effect size:	*Pre:*	4.7			0.53	
	Post:	8.8				

Notes 2: administered in-class as part of physics course
3: p-value>.20 for pre-to-post comparison of single item
4: p-value=.04 for pre-to-post comparison of VIId with an effect size of 3.0

We see a significant and large difference between correct response rates on items VIIc and VIId, providing strong evidence that the presence of just one symbolic variable quantity presents a significant obstacle to students' algebraic reasoning – even among mathematically well-prepared students. Furthermore, at the p-value < .05 significance level it appears that this obstacle is in fact increased by a one semester physics course, with the difference in performance on the two versions of the question increasing from 16% on the pretest to 30% on the posttest.

On item VIII, which involves expressing a natural language statement as a mathematical equation, results are consistent with the findings of Cohen and Kanim.[(40)] Student performance shows no improvement after one semester of mechanics, even though the problem solving focus of the course involved frequent modeling and practice of this skill. As a group, results on items VIIc, VIId, and VIII suggest that even engineering students struggle with the use of symbols to represent quantities in physical contexts.

V. Discussion

A weighted average of the assessment items presented in the previous section yields an average correct response rate of just over 50%, suggesting that an introductory calculus-based mechanics course has only limited ability to help even well prepared students learn to reason consistently about ratio and proportion. This finding is disturbing, given that proportional reasoning is fundamental to all of physics, and that instructors commonly model it when teaching a physics course. The adage "practice makes perfect" would seem not to apply for all students in the case of learning to reason algebraically in introductory physics.

From the cognitive blending perspective, algebraic reasoning is inextricably bound to the context in which it is being used. We argue that this cognitive space is entirely homogeneous, in the sense that in a physics course, "doing algebra" is not separate from "doing physics." It is not uncommon for experts to make errors similar to those of students when they initially answer items I, II, VII (all versions) and VIII. Experts, however, expect an answer to make sense in context, and employ a variety of tactics to evaluate answers before they consider the problem to be completed. It seems likely that this tight binding of algebraic reasoning and physical sense making is not part of what all students are learning from experts when they take a physics course.

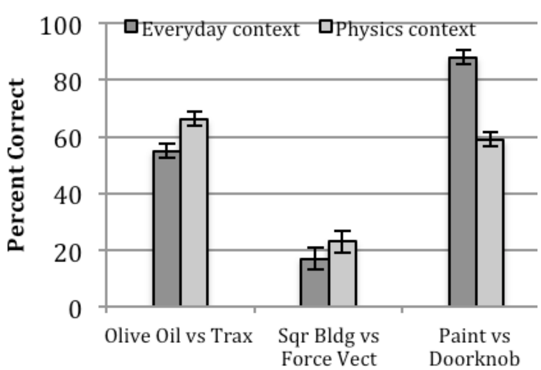
Below we revisit our research questions, which focused on the mathematical cognitive domains of general structural reasoning, quantification, and symbolizing, in light of the results presented in the previous section.

A. General Structural Reasoning

The first research question asked *To what extent are engineering students able to reason successfully with ratio and proportion after a one semester calculus-based physics course?* Our results reveal that many students continue to struggle with proportional reasoning in spite of having completed not only a physics course, but also calculus and calculus-based chemistry courses. Analysis of the free-response versions of the items presented in this paper indicates that many students believed they were being asked to recall a prefabricated formula. In interviews, such students spent time trying to remember formulas as they responded to these items. Similar to the case study described in Von Korff, Elby, Hu, and Rebello,[16] many students rely on the authority of formulas rather than engaging in the reasoning needed to understand what the formulas represent.

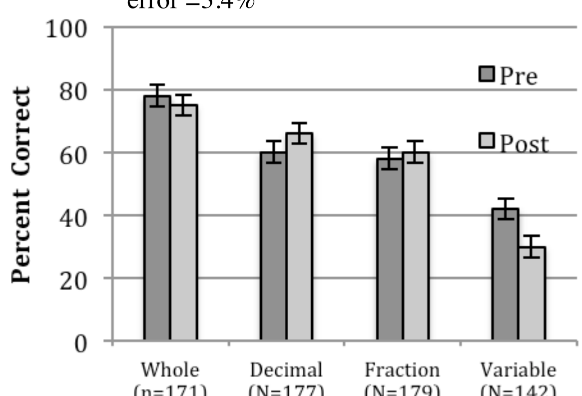
Item II (Traxolene) involves mass and volume, quantities that introductory science courses commonly deal with in the context of density. While the unit rate given is not itself the density, students could rearrange the typically memorized density formula in order to obtain the correct response. Item I (Olive oil), on the other hand, does not lend itself to a ready formula. While the market context is likely familiar to students, success requires flexibility in the use of the unit rate structure. We regard performance on both of these items, summarized in Fig. 2, as weak, given the basic nature of the items and the mathematics preparation of the student population. These results indicate that forming a unit rate from measured values is a challenge, even for engineering students. This in-context, generative use of mathematics, while central to physics, may not be a reliable cognitive resource for many students. In addition, many physics instructors might assume (understandably) that, because of the mathematics prerequisites, such mathematical reasoning is not necessary to cover as part of the physics course.

Figure 2: Context comparisons, pooled standard error show for each comparison



Item# **I II** **III IV** **V VI**

Figure 3: Rice questions, pooled standard error =3.4%



Item# **VIIa** **VIIb** **VIIc** **VIId**

B. Quantification

Results from questions designed to assess student reasoning about quantity demonstrate the extent to which this reasoning is sensitive to surface features of the quantities involved.

Tasks that involve more abstract quantities (such as electric charge covering a doorknob in Item VI) seem to inhibit the reasoning students are successful with in less abstract contexts (such as the paint context of item V). While a physicist would likely regard the paint and doorknob questions as similar, Fig. 2 shows that student performance on the doorknob question was substantially weaker, suggesting that many students have difficulty generalizing the relevant reasoning about quantity across contexts with different surface features.

Similarly, on Items VII a-c performance varies with the complexity of the numeric values. Although the necessary reasoning is identical, Fig. 3 shows that performance on the decimal and fraction versions of the Rice question was significantly weaker than that on the whole numbers version. Students apparently are distracted from or are less likely to cue the appropriate ratio reasoning when presented with decimal and fractional values.

Our results reveal that it is not always the case that physics contexts are more challenging for students than everyday contexts. We observe that many contexts can be difficult, especially when they involve unfamiliar or abstract quantities – a result similar to what has been observed on the FCI by researchers using modified question contexts.[49, 50] While the issues discussed in our study may be generalizable to contexts outside of physics, we consider them to be specifically relevant in introductory physics because of the important role that algebraic reasoning plays in its discourse, and the immediate and constant introduction of new and abstract quantities.

Even after having taken a physics course, students appear to lack the flexibility in ratio reasoning necessary for success across varied contexts and types of quantities. Quantification is cognitively challenging; unfamiliar units and complexities in the representations of value can derail students as they struggle to reason about the many new quantities they encounter in a physics course. Physicists use units and dimension to guide their reasoning about new physical quantities. This generative use of mathematics may be foreign to introductory physics students, and any nascent abilities may be overwhelmed in situations in which the complexity of the numbers or the level of abstraction of the quantities is high.

C. Symbolizing

Results from questions designed to assess student reasoning using symbolic representations demonstrate that students struggle with variables, both as general unspecified quantities and in functional relationships.

Comparison of the fraction and variable versions of item VII (see Fig. 3) reveals that substituting even just one generalized variable, represented by a letter, for a numeric value inhibits the appropriate reasoning at least as much as replacement of whole numbers with decimals numbers. Our results here are consistent with those of Torigoe and Gladding[41] in which they compared algebraic computations with numeric ones. Our work, however, extends the effort to disentangle the source of students' difficulty. While most instructors would probably agree that a purely symbolic equation is more challenging for students than one that has just one symbol, it is perhaps surprising that only a single generalized variable can hinder student reasoning to the extent we have measured here. For nearly half of the population tested, the presence of a single variable quantity disrupts reasoning the students were successful with in the context of whole number quantities.

We speculate that it is not that students cannot do algebra, or even manipulate algebraic statements necessarily, but that even after a semester of calculus-based physics they are already lost at the problem statement – there are many students who are derailed once they see the first variable quantity.

Also consistent with published results,[38, 40] we find that students struggle to interpret and manipulate symbols appropriately to generate a mathematically statement from a verbal one, and have trouble determining whether or not their symbolic statement makes physical sense. Fewer than one-half of the engineering students select the equation that matches the given verbal statement on item VIII. The most common incorrect selection corresponds to the previously identified reversal error. These results help quantify how students in introductory physics struggle to make sense with variables in the varied ways physicists use them.

VI. Conclusion

We have used written assessment questions to investigate the ability of engineering students in an introductory calculus-based physics course at a selective, public research university to apply proportional reasoning in simple contexts. We characterize our findings using a cognitive blending framework,[5, 6] which treats the necessary mathematical reasoning and the associated physics content as a single thinking space. We identify student reasoning difficulties in three mathematical cognitive domains: generalized structural reasoning, quantification, and symbolizing. These domains are foundational for the quantitative analysis that characterizes physics.

We find that roughly half of the students struggle with generalized structural reasoning, quantification, and symbolizing, and that students make little progress with these basic modes of mathematical thinking over the course of their first semester of calculus-based physics. Through high achievement on the SAT and other tests, the engineering students in our study have demonstrated competence in problem solving. Our findings reveal, however, that these students are not necessarily thinking mathematically in physics contexts, even after one semester of mechanics.

Instructors may be unintentionally creating barriers to success for many students in introductory physics by using “pre-fabricated” reasoning resources when modeling mathematical sensemaking in physics, without attending to the fabrication process explicitly. In a physics course students must develop a sense of what the new and abstract quantities actually represent, while simultaneously learning to reason with them algebraically. Much instructional effort and research has been done in the service of the former; we here argue for the need for additional research on the latter.

We conclude that successfully completing mathematics prerequisites primarily prepares students for the *procedural* aspects of the mathematics necessary for productive work in physics, but not for the mathematical reasoning needed for physical sensemaking. Physicists, because of their deep knowledge of the contextualized use of mathematics within the discipline, are best positioned to help students develop this reasoning. We suggest that for the majority of students in a physics course to understand and adopt the mathematical reasoning of physics, physics instructors and education researchers must develop instructional approaches effective in supporting student mathematization.

We view recent increased interest in physics students’ mathematical reasoning and increased dialogue with mathematics education researchers as important trends in physics education research. We believe these trends can support further elucidation of how experts use mathematics for sensemaking in physics, increased understanding of how student thinking aligns with and diverges from expert mathematization, and development of instructional approaches that can help bridge the gaps. The work described in this paper is intended to contribute to these goals.

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