ON THE BEHAVIOR OF INTEGRABLE FUNCTIONS AT INFINITY

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Abstract. We investigate the behavior of sequences $(f(c_nx))$ for Lebesgue integrable functions $f : \mathbb{R}^d \to \mathbb{R}$. In particular, we give a description of classes of multipliers (c_n) and (d_n) such that $f(c_nx) \to 0$ or $\sum_{n=1}^{\infty} |f(d_nx)| < \infty$ for λ almost every $x \in \mathbb{R}^d$.

It is well known that if a series $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$. It may seem surprising that a similar result does not hold for integrals. Namely, if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, then it is not necessary that $\lim_{x\to\infty} f(x) = 0$. Various authors investigated the behavior of integrable functions at infinity, see e.g. [2, 3, 4, 5, 6].

E. Lesigne showed in [2] that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, then for λ almost every $x \in \mathbb{R}$ one has $f(nx) \to 0$. In this paper we generalize Lesigne's investigations in several directions. One way is to replace the domain of f by the space $\mathbb{R}^d$ equipped with d-dimensional Lebesgue measure λ. On the other hand, we want to describe a possibly large class of multipliers c_n which may be substituted for n in Lesigne's result. As the first result going in this direction we present the following theorem:

Theorem 1. *Let $d \in \mathbb{N}$ and let (c_n) be a sequence of positive numbers such that for some permutation (c'_n) of (c_n) the sequence $(\sqrt[d]{n}/c'_n)$ is bounded. Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue integrable ($\int |f(x)|dx < \infty$). Then for λ almost every $x \in \mathbb{R}^d$ one has $\sum_{n=1}^{\infty} |f(c_nx)| < \infty$ (hence $f(c_nx) \to 0$).*

A little comment is necessary to explain the assumption on the sequence (c_n). Theorem 1 would be valid if we just assumed that $(\sqrt[d]{n}/c_n)$ is bounded. However, the conclusion of the theorem is permutation invariant, i.e., if it holds for a sequence (c_n), then it also holds for any permutation of (c_n). If any form of the reversal of Theorem 1 should hold true, then its assumptions have to be permutation invariant as well. Unfortunately, the condition "$(\sqrt[d]{n}/c_n)$ is bounded" is not permutation invariant. For this reason an additional sequence (c'_n) (being a permutation of (c_n)) has to be explicitly introduced.

We note that in Theorem 1 we obtain more than we intended. Namely, we get $\sum_{n=1}^{\infty} |f(c_nx)| < \infty$ instead of $f(c_nx) \to 0$. If one wishes to conclude that $f(c_nx) \to 0$, then weaker assumptions on the function f are needed:

Theorem 2. *Let $d \in \mathbb{N}$ and let (c_n) be a sequence of positive numbers such that for some permutation (c'_n) of (c_n) the sequence $(\sqrt[d]{n}/c'_n)$ is bounded. Moreover, let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable and such that for every $\varepsilon > 0$ one has $\lambda(\{x \in \mathbb{R}^d : |f(x)| \geq \varepsilon\}) < \infty$. Then for λ almost every $x \in \mathbb{R}^d$ one has $f(c_nx) \to 0$.*

In the conclusion of the above theorem we cannot keep the stronger statement $\sum_{n=1}^{\infty} |f(c_nx)| < \infty$ from Theorem 1. Indeed, if $f(x) = 1/(1 + \|x\|)$, then $\sum_{n=1}^{\infty} |f(nx)| = \infty$ for every x.

2010 *Mathematics Subject Classification.* Primary 26B15; Secondary 40A05, 26A42, 28A25.

Key words and phrases. Lebesgue integral, behavior of a function at infinity, convergence of series, convergence of sequences.

The next theorem shows that the assumption on the sequence (c_n) in Theorem 1 cannot be weakened.

Theorem 3. *Let $d \in \mathbb{N}$ and let (c_n) be a sequence of positive numbers such that for every permutation (c_n') of (c_n) the sequence $(\sqrt[d]{n}/c_n')$ is unbounded. Then there exists a continuous, nonnegative function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\int |f(x)|dx < \infty$ and $\sum_{n=1}^{\infty} |f(c_n x)| = \infty$ for every $x \in \mathbb{R}^d$.*

The above theorem may be seen as the inverse of Theorem 1. The situation is much more delicate when we try to inverse Theorem 2. Consider the following example: Let (c_n) satisfy the assumption of Theorem 2, for simplicity set $c_n = n$. Then for any integrable f we have $f(nx) \to 0$ for λ almost every $x \in \mathbb{R}^d$. Now, we define a sequence (d_n) such that it tends to infinity arbitrarily slowly, yet $f(d_n x) \to 0$ for λ almost every $x \in \mathbb{R}^d$. It suffices to take (d_n) which is formed by repeating each term of the sequence $(c_n = n)$ finitely many times. Indeed, the convergence of $f(d_n x)$ to zero follows from $f(nx) \to 0$. On the other hand, (d_n) may tend to infinity slowly enough to ensure that $(\sqrt[d]{n}/d_n')$ is unbounded for every permutation (d_n') of (d_n). All this shows that Theorem 2 cannot be fully inversed. Instead, we show the following theorem:

Theorem 4. *Let $d \in \mathbb{N}$ and let (c_n) be a sequence of positive numbers such that for every permutation (c_n') of (c_n) the sequence $(\sqrt[d]{n}/c_n')$ is unbounded. Then there exist a sequence (b_n) of positive numbers and a continuous, nonnegative, integrable function $f : \mathbb{R}^d \to \mathbb{R}$ such that $b_n/c_n \to 1$ and $f(b_n x) \not\to 0$ for every $x \in \mathbb{R}^d$.*

In fact we prove a bit more: If $c_n \to \infty$, then additionally $\limsup_{n\to\infty} f(b_n x) = \infty$ for every $x \neq 0$.

In Theorem 4 we claim that if a sequence (c_n) does not satisfy the assumption of Theorem 2, then even if it is not "bad" itself, it can be slightly modified to a "bad" sequence. On the other hand, each sequence (c_n) with $c_n \to \infty$ can be improved in the following sense:

Theorem 5. *Let $d \in \mathbb{N}$ and let (c_n) be a sequence of positive numbers tending to infinity. There exists a sequence (b_n) of positive numbers with $b_n/c_n \to 1$ such that: For any measurable $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\forall_{\varepsilon>0}\ \lambda(\{x \in \mathbb{R}^d : |f(x)| \geq \varepsilon\}) < \infty$ one has $f(b_n x) \to 0$ for λ almost every $x \in \mathbb{R}^d$.*

In [2] Lesigne also investigated the rate of convergence of $(f(nx))$ to zero. In particular, he showed that for any sequence (a_n) with $0 \leq a_n \to \infty$ there exists a continuous, integrable function $f : \mathbb{R} \to \mathbb{R}$ such that $\limsup_{n\to\infty} a_n f(nx) = \infty$ for λ almost every $x \in \mathbb{R}$. Moreover, if we drop the continuity requirement (we only require integrability of f), then we may obtain $\limsup_{n\to\infty} a_n f(nx) = \infty$ for every $x \in \mathbb{R}$. Lesigne asked if we may have both: continuity of f and $\limsup_{n\to\infty} a_n f(nx) = \infty$ for every x. This question has been positively answered by G. Batten in [1]. The original Batten's paper is accessible through arXiv, but (to best our knowledge) has never been published. Here we present a much shorter proof of Batten's result in $\mathbb{R}^d$, based on completely different ideas.

Theorem 6. *Let $d \in \mathbb{N}$ and let a sequence (a_n) satisfy $0 \leq a_n \to \infty$. There exists a continuous, nonnegative, integrable function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\limsup_{n\to\infty} a_n f(\sqrt[d]{n}x) = \infty$ for every $x \in \mathbb{R}^d$.*

Proofs

The following lemma plays a very important role in the proofs of (almost) all theorems in this paper:

Lemma 7. *Let $d>0$ and $a>1$ be real numbers and let (c_n) be a sequence of positive numbers. The following conditions are equivalent:*

(i) *There exists a permutation (c'_n) of (c_n), such that the sequence $(\sqrt[d]{n}/c'_n)$ is bounded.*

(i') *There exists a (unique) nondecreasing sequence (c'_n) being a permutation of (c_n) and for this permutation the sequence $(\sqrt[d]{n}/c'_n)$ is bounded.*

(ii) *There exists $M>0$, such that $\forall_{t>0}\ \sum_{\{n:\ t\le c_n<at\}} \frac{1}{c_n^d} \le M$.*

(iii) *There exists $M'>0$, such that $\forall_{k\in\mathbb{Z}}\ \frac{|\{n: a^k\le c_n<a^{k+1}\}|}{a^{kd}} \le M'$*

Proof. Clearly (i') implies (i). We will show (i)⇒(iii)⇒(ii)⇒(i')

(i)⇒(iii). Let $L>0$ satisfy $\sqrt[d]{n}/c'_n \le L$ for every n. If we put $M'=(La)^d$, then for every $k\in\mathbb{Z}$ we have

$$\frac{|\{n: a^k\le c_n<a^{k+1}\}|}{a^{kd}} \le \frac{|\{n: c_n<a^{k+1}\}|}{a^{kd}} = \frac{|\{n: c'_n<a^{k+1}\}|}{a^{kd}} \le \frac{|\{n: \sqrt[d]{n}<La^{k+1}\}|}{a^{kd}} =$$
$$\frac{|\{n: n<(La^{k+1})^d\}|}{a^{kd}} < \frac{(La^{k+1})^d}{a^{kd}} = (La)^d = M'.$$

(iii)⇒(ii). We put $M=2M'$. Let $t>0$. We have $a^{k-1}\le t<a^k$ for some $k\in\mathbb{Z}$ and then

$$\sum_{\{n:\ t\le c_n<at\}} \frac{1}{c_n^d} \le \sum_{\{n:\ a^{k-1}\le c_n<a^k\}} \frac{1}{c_n^d} + \sum_{\{n:\ a^k\le c_n<a^{k+1}\}} \frac{1}{c_n^d} \le$$
$$\frac{|\{n:\ a^{k-1}\le c_n<a^k\}|}{a^{(k-1)d}} + \frac{|\{n:\ a^k\le c_n<a^{k+1}\}|}{a^{kd}} \le M'+M'=M.$$

(ii)⇒(i'). For any $t>0$ we have

$$|\{n: c_n<t\}| = \sum_{k=1}^{\infty} |\{n: ta^{-k}\le c_n<ata^{-k}\}| \le \sum_{k=1}^{\infty} \sum_{\{n:\ ta^{-k}\le c_n<ta^{1-k}\}} \frac{(ta^{1-k})^d}{c_n^d} \le$$
$$\sum_{k=1}^{\infty}(ta^{1-k})^d\cdot M = t^d\cdot\frac{M}{1-1/a^d}.$$

In particular, for every $t>0$ the set $\{n: c_n<t\}$ is finite, hence there exists a nondecreasing permutation (c'_n) of (c_n). For this permutation we have $|\{n: c'_n<t\}| = |\{n: c_n<t\}| \le t^d\cdot\frac{M}{1-1/a^d}$. Since (c'_n) is nondecreasing, for every $m\in\mathbb{N}$ we have:

$$m \le \inf_{t>c'_m} |\{n: c'_n<t\}| \le \inf_{t>c'_m}\left(t^d\cdot\frac{M}{1-1/a^d}\right) = c'^d_m\cdot\frac{M}{1-1/a^d},$$

hence $\sqrt[d]{m}/c'_m \le \sqrt[d]{\frac{M}{1-1/a^d}}$. □

Proof of Theorem 1. In the first part of the proof we show that for λ almost every $x\in\mathbb{R}^d$ satisfying $\frac{1}{2}<\|x\|\le 1$ we have $\sum_{n=1}^{\infty}|f(c_nx)|<\infty$. We define $f_n:\mathbb{R}^d\to\mathbb{R}$ by the formula

$$f_n(x) = \frac{1}{c_n^d}\cdot|f(x)|\cdot\mathbf{1}_{c_n/2<\|x\|\le c_n}.$$

Functions f_n are nonnegative and $\sum_{n=1}^{\infty} f_n(0) = 0$. For every $x \neq 0$ we use Lemma 7 ((i)⇒(ii) with $a = 2$ and $t = \|x\|$) to obtain:

$$\sum_{n=1}^{\infty} f_n(x) = |f(x)| \cdot \sum_{\{n:\ \|x\| \leq c_n < 2\|x\|\}} \frac{1}{c_n^d} \leq |f(x)| \cdot M.$$

It follows that the function series $\sum_{n=1}^{\infty} f_n(x)$ is convergent and $\int \sum_{n=1}^{\infty} f_n(x)dx \leq M \cdot \int |f(x)|dx < \infty$. Hence

$$\sum_{n=1}^{\infty} \int_{\{x: \frac{1}{2} < \|x\| \leq 1\}} |f(c_n x)|dx = \sum_{n=1}^{\infty} \int |f(c_n x)| \cdot \mathbf{1}_{\frac{1}{2} < \|x\| \leq 1} dx = \sum_{n=1}^{\infty} \int |f(x)| \cdot \mathbf{1}_{c_n/2 < \|x\| \leq c_n} \cdot \frac{1}{c_n^d} dx =$$
$$\sum_{n=1}^{\infty} \int f_n(x)dx = \int \sum_{n=1}^{\infty} f_n(x)dx < \infty.$$

Thus, the function series $\sum_{n=1}^{\infty} |f(c_n x)|$ is convergent λ almost everywhere on $\{x \in \mathbb{R}^d : \frac{1}{2} < \|x\| \leq 1\}$ and the first part of the proof is completed.

Now, for $k \in \mathbb{Z}$ we consider the function $g_k(x) = f(2^k x)$. Clearly g_k is integrable, hence, by the first part of the proof, for λ almost every y satisfying $\frac{1}{2} < \|y\| \leq 1$ the series $\sum_{n=1}^{\infty} |f(c_n 2^k y)| = \sum_{n=1}^{\infty} |g_k(c_n y)|$ converges. Denoting $x = 2^k y$ we obtain that for λ almost every x satisfying $2^{k-1} < \|x\| \leq 2^k$ we have $\sum_{n=1}^{\infty} |f(c_n x)| < \infty$. This observation completes the proof, because $\mathbb{R}^d = \{0\} \cup \bigcup_{k \in \mathbb{Z}} \{x \in \mathbb{R}^d : 2^{k-1} < \|x\| \leq 2^k\}$. □

Proof of Theorem 2. For $k = 1, 2, \ldots$ we apply Theorem 1 for an integrable function $f_k(x) = \mathbf{1}_{|f(x)| \geq 1/k}$. As a result, we obtain a set $A_k \subset \mathbb{R}^d$, such that $\lambda(A_k) = 0$ and for every $x \in \mathbb{R}^d \setminus A_k$ we have $f_k(c_n x) \to 0$ when $n \to \infty$. Clearly, $\lambda(\bigcup_{k=1}^{\infty} A_k) = 0$. The convergence $f_k(c_n x) \to 0$ implies that the set $\{n : |f(c_n x)| \geq 1/k\}$ is finite. It follows that if $x \in \mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} A_k$, then $\forall_{k \in \mathbb{N}}\ |\{n : |f(c_n x)| \geq 1/k\}| < \infty$, which means $f(c_n x) \to 0$. □

Proof of Theorem 3. If $c_n \not\to \infty$, then there exists $c \geq 0$ and a subsequence (c_{n_i}) such that $c_{n_i} \to c$. In this case we can take any f which is strictly positive, integrable and continuous, e.g. $f(x) = 1/(1 + \|x\|^{d+1})$. Indeed, if $x \in \mathbb{R}^d$, then $f(c_{n_i} x) \to f(cx) > 0$, hence $\sum_{n=1}^{\infty} |f(c_n x)| \geq \sum_{i=1}^{\infty} |f(c_{n_i} x)| = \infty$. In the remaining part of the proof we assume $c_n \to \infty$.

For $k \in \mathbb{Z}$ let $A_k = \{n : 2^k \leq c_n < 2^{k+1}\}$ and $l_k = \sum_{n \in A_k} \frac{1}{c_n^d}$. The assumption $c_n \to \infty$ implies that the sets A_k are finite. Moreover, the sets A_k are pairwise disjoint and $\mathbb{N} = \bigcup_{k \in \mathbb{Z}} A_k$. It follows, that for every $n \in \mathbb{N}$ there exists the unique $k(n) \in \mathbb{Z}$ such that $n \in A_{k(n)}$. By Lemma 7 (¬(i)⇒ ¬(iii) with $a = 2$) and by the inequality $l_k \geq \frac{|A_k|}{2^{(k+1)d}}$ we obtain that the set $\{l_k : k \in \mathbb{Z}\}$ is unbounded. We take a sequence (k_i) such that k_i's are pairwise different and $l_{k_i} \geq i$ for every i. We define nonnegative numbers $(r_k)_{k \in \mathbb{Z}}$ by the formula

$$r_k = \begin{cases} \frac{1}{i^2 |A_{k_i}|} & \text{if } k = k_i, \\ 0 & \text{if } k \neq k_i \text{ for every } i. \end{cases}$$

(note that $l_{k_i} > 0$ implies $A_{k_i} \neq \emptyset$). Then

$$\sum_{m=1}^{\infty} r_{k(m)} = \sum_{k \in \mathbb{Z}} \sum_{m \in A_k} r_k = \sum_{k \in \mathbb{Z}} r_k |A_k| = \sum_{i=1}^{\infty} r_{k_i} |A_{k_i}| = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

and $\sum_{k \in \mathbb{Z}} r_k |A_k| l_k = \sum_{i=1}^{\infty} r_{k_i} |A_{k_i}| l_{k_i} \geq \sum_{i=1}^{\infty} \frac{1}{i} = \infty$.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be any bounded, strictly positive, integrable and continuous function, such that $g(x)$ is a nonincreasing function of $\|x\|$ (e.g., $g(x) = 1/(1 + \|x\|^{d+1})$). We define

$$f(x) = \sum_{m=1}^{\infty} \frac{r_{k(m)}}{c_m^d} \cdot g\left(\frac{x}{c_m}\right).$$

Note that the above function series converges uniformly, because g is bounded, $c_m \to \infty$ and $\sum_{m=1}^{\infty} r_{k(m)} < \infty$. In particular f is continuous. Clearly f is positive. Moreover,

$$\sum_{m=1}^{\infty} \int \frac{r_{k(m)}}{c_m^d} \cdot g\left(\frac{x}{c_m}\right) dx = \sum_{m=1}^{\infty} \int r_{k(m)} \cdot g(x)dx = \int g(x)dx \cdot \sum_{m=1}^{\infty} r_{k(m)} < \infty,$$

hence f is integrable.

If $x = 0$, then $\sum_{n=1}^{\infty} |f(c_n x)| = \sum_{n=1}^{\infty} |f(0)| = \infty$, because $f(0) > 0$. For $x \neq 0$ we have

$$\sum_{n=1}^{\infty} |f(c_n x)| = \sum_{k\in\mathbb{Z}} \sum_{n\in A_k} f(c_n x) \geq \sum_{k\in\mathbb{Z}} \sum_{n\in A_k} \sum_{m\in A_k} \frac{r_{k(m)}}{c_m^d} \cdot g\left(\frac{c_n}{c_m} \cdot x\right) \geq$$

$$\sum_{k\in\mathbb{Z}} r_k \sum_{n\in A_k} \sum_{m\in A_k} \frac{1}{c_m^d} \cdot g(2x) = g(2x) \cdot \sum_{k\in\mathbb{Z}} r_k |A_k| l_k = \infty$$

(we used the following observation: if $m, n \in A_k$, then $\frac{c_n}{c_m} < 2$). □

The proof of Theorem 4 is presented at the end of the paper. It is the hardest proof and it uses some ideas presented in the proof of Theorem 6. For this reasons leaving it for the end is a good idea.

Proof of Theorem 5. Let $(b_n) = (\lceil c_n \rceil)$. Then all the terms of (b_n) are in $\mathbb{N}$. The assumption $c_n \to \infty$ assures that for every $k \in \mathbb{N}$ the set $\{n : b_n = k\}$ is finite. By Theorem 2 we have $f(kx) \to 0$ for λ almost every $x \in \mathbb{R}^d$. Thus $f(b_n x) \to 0$ for λ almost every $x \in \mathbb{R}^d$. Moreover, $c_n \to \infty$ implies $\frac{b_n}{c_n} = \frac{\lceil c_n \rceil}{c_n} \to 1$. □

Proof of Theorem 6. It is enough to construct a continuous, nonnegative, integrable function $\widetilde{f} : [0, \infty) \to \mathbb{R}$, such that $\limsup_{n\to\infty} a_n \widetilde{f}(nx) = \infty$ for every $x \in [0, \infty)$. Then we define $f : \mathbb{R}^d \to \mathbb{R}$ by $f(x) = \widetilde{f}(\|x\|^d)$. Clearly, f is continuous, nonnegative and $\limsup_{n\to\infty} a_n f(\sqrt[d]{n}x) = \limsup_{n\to\infty} a_n \widetilde{f}(n\|x\|^d) = \infty$ for every $x \in \mathbb{R}^d$. Moreover,

$$\int f(x)dx = \int \widetilde{f}(\|x\|^d)dx = S_d \cdot \int_{r=0}^{\infty} \widetilde{f}(r^d) r^{d-1} dr = \frac{S_d}{d} \int_{y=0}^{\infty} \widetilde{f}(y)dy < \infty$$

(here S_d is $d-1$-dimensional measure of the unit sphere in $\mathbb{R}^d$).

For $k \in \mathbb{N}$ let $t_k > 0$ be such that $n \geq t_k \Rightarrow a_n \geq k^4$ for every $n \in \mathbb{N}$. Let $h : \mathbb{R} \to \mathbb{R}$ be any continuous, bounded, nonnegative, integrable function satisfying $h|_{[0,1]} \geq 1$. We define $\widetilde{f} : [0, \infty) \to \mathbb{R}$ as follows:

$$\widetilde{f}(x) = h(x) + \sum_{l=1}^{\infty} \frac{h(\frac{x}{l} - t_l)}{l^3}.$$

Function $\widetilde{f}$ is nonnegative and continuous (the series converges uniformly). It is also integrable:

$$\int_0^{\infty} \widetilde{f}(x)dx = \int_0^{\infty} h(x)dx + \sum_{l=1}^{\infty} \int_0^{\infty} \frac{h(\frac{x}{l} - t_l)}{l^3} dx = \int_0^{\infty} h(x)dx + \sum_{l=1}^{\infty} \int_0^{\infty} \frac{h(x - t_l)}{l^2} dx \leq$$

$$\int h(x)dx + \sum_{l=1}^{\infty} \int \frac{h(x)}{l^2} dx = \int h(x)dx \cdot \left(1 + \sum_{l=1}^{\infty} \frac{1}{l^2}\right) < \infty.$$

If $x = 0$, then $\limsup_{n\to\infty} a_n \widetilde{f}(nx) \geq \limsup_{n\to\infty} a_n h(nx) = \limsup_{n\to\infty} a_n h(0) \geq \limsup_{n\to\infty} a_n = \infty$. Let $x > 0$. Then for every $k \in \mathbb{N}$ satisfying $k > x$ we have $0 < \frac{x}{k} < 1$ and there exists $n_k \in \mathbb{N}$ such that $n_k \cdot \frac{x}{k} \in [t_k, t_k + 1]$, i.e., $\frac{n_k x}{k} - t_k \in [0,1]$. In particular, $n_k \geq t_k \cdot \frac{k}{x} > t_k$, hence $a_{n_k} \geq k^4$. It follows that

$$a_{n_k} \widetilde{f}(n_k x) \geq a_{n_k} \cdot \frac{1}{k^3} \cdot h\left(\frac{n_k x}{k} - t_k\right) \geq k^4 \cdot \frac{1}{k^3} \cdot 1 = k,$$

thus $\limsup_{n\to\infty} a_n \widetilde{f}(nx) \geq \limsup_{k\to\infty} a_{n_k} \widetilde{f}(n_k x) \geq \limsup_{k\to\infty} k = \infty$. □

The following technical lemma is helpful to perform an inductive construction in the proof of Theorem 4.

Lemma 8. *Let (c_n) be a sequence of positive numbers such that $c_n \to \infty$ and for every permutation (c'_n) of (c_n) the sequence (n/c'_n) is unbounded. Then for every $a > 1$, $\varepsilon > 0$, $S > 0$, $l \in \mathbb{Z}$ and $M \in \mathbb{N} \cup \{0\}$ there exist $T > S$, $\mathbb{N} \ni N > M$, $b_{M+1}, b_{M+2}, \ldots, b_N > 0$ and a continuous, integrable, nonnegative function $g : [0,\infty) \to \mathbb{R}$ satisfying $\frac{1}{a} \leq \frac{b_n}{c_n} \leq a$ for $n = M+1, M+2, \ldots, N$, $\int_0^\infty g(x)dx < \varepsilon$, $g|_{[0,\infty)\setminus[S,T]} = 0$ and $\forall_{x\in[a^{l-1},a^l]} \max_{M<n\leq N} g(b_n x) \geq 1$.*

Proof. For $k \in \mathbb{Z}$ let $A_k = \{n : a^k \leq c_n < a^{k+1}\}$. According to Lemma 7 ($\neg$(i)$\Rightarrow$ $\neg$(iii)) there exists a sequence (k_i) such that $\frac{|A_{k_i}|}{a^{k_i}} \to \infty$. We can assume that $A_{k_i} \neq \emptyset$ and $k_i > 1 - l + \log_a S$ and $k_i > \max\{\log_a c_n : n \leq M\}$ for every i. The last inequality ensures that for every n if $n \in A_{k_i}$, then $n > M$. We consider a term $a^{k_i+l}(1 - a^{-1/|A_{k_i}|})$ and its limit when $i \to \infty$:

$$\lim_{i\to\infty} a^{k_i+l}(1 - a^{-1/|A_{k_i}|}) = \lim_{i\to\infty} a^l \cdot \frac{a^{k_i}}{|A_{k_i}|} \cdot \frac{1 - a^{-1/|A_{k_i}|}}{0 - (-1/|A_{k_i}|)} = a^l \cdot 0 \cdot \log_e a = 0.$$

It follows that we can choose $K \in \{k_i : i \in \mathbb{N}\}$ satisfying $a^{K+l}(1 - a^{-1/|A_K|}) < \varepsilon$. We put $N = \max A_K$. Then $A_K \subset \{M+1, M+2, \ldots, N\}$.

We define $b_{M+1}, b_{M+2}, \ldots, b_N$: If $n \in \{M+1, \ldots, N\} \setminus A_K$, then we put $b_n = c_n$. The remaining b_n's (with $n \in A_K$) are chosen in any way satisfying $\{b_n : n \in A_K\} = \{a^{K+\frac{j}{|A_K|}} : j = 0, 1, \ldots, |A_K| - 1\}$. If $n \in \{M+1, \ldots, N\} \setminus A_K$, then $\frac{1}{a} \leq 1 = \frac{b_n}{c_n} \leq a$. If $n \in A_K$, then both b_n and c_n are in $[a^K, a^{K+1})$, hence $\frac{1}{a} \leq \frac{b_n}{c_n} \leq a$.

We choose any $T > a^{K+l}$. The inequality $K > 1 - l + \log_a S$ implies $a^{K+l-\frac{1}{|A_K|}} \geq a^{K+l-1} > S$. Hence $[a^{K+l-\frac{1}{|A_K|}}, a^{K+l}] \subset (S,T)$. We also have $\int_0^\infty \mathbf{1}_{[a^{K+l-\frac{1}{|A_K|}}, a^{K+l}]} dx = \lambda([a^{K+l-\frac{1}{|A_K|}}, a^{K+l}]) = a^{K+l}(1 - a^{-1/|A_K|}) < \varepsilon$. All these observations show that there exists a nonnegative, continuous function $g : [0,\infty) \to \mathbb{R}$ such that g equals 0 outside $[S,T]$, g equals 1 on $[a^{K+l-\frac{1}{|A_K|}}, a^{K+l}]$ and $\int_0^\infty g(x)dx < \varepsilon$.

It remains to check that $\forall_{x\in[a^{l-1},a^l]} \max_{M<n\leq N} g(b_n x) \geq 1$. We have

$$[a^{l-1}, a^l] = \bigcup_{j=0}^{|A_K|-1} \left[a^{l-\frac{j+1}{|A_K|}}, a^{l-\frac{j}{|A_K|}}\right] = \bigcup_{n\in A_K} \left[\frac{a^{K+l-\frac{1}{|A_K|}}}{b_n}, \frac{a^{K+l}}{b_n}\right].$$

It follows, that if $x \in [a^{l-1}, a^l]$, then $b_{n_0} x \in [a^{K+l-\frac{1}{|A_K|}}, a^{K+l}]$ for some $n_0 \in A_K$. Consequently, $\max_{M<n\leq N} g(b_n x) \geq g(b_{n_0} x) = 1$. □

Proof of Theorem 4. If $c_n \not\to \infty$, then there exists $c \geq 0$ and a subsequence (c_{n_i}) such that $c_{n_i} \to c$. In this case we can take any f which is strictly positive, integrable and continuous, e.g. $f(x) = 1/(1 + \|x\|^{d+1})$

and $(b_n) = (c_n)$. Indeed, if $x \in \mathbb{R}^d$, then $f(c_{n_i}x) \to f(cx) > 0$, hence $f(b_n x) = f(c_n x) \not\to 0$. In the remaining part of the proof we assume $c_n \to \infty$.

Let $(\widetilde{c}_n) = (c_n^d)$. Then $\widetilde{c}_n \to \infty$ and for every permutation $(\widetilde{c}'_n)$ of $(\widetilde{c}_n)$ the sequence $(n/\widetilde{c}'_n)$ is unbounded. To finish the proof it is enough to construct a continuous, nonnegative, integrable function $\widetilde{f} : [0,\infty) \to \mathbb{R}$ and a sequence $(\widetilde{b}_n)$ such that $\frac{\widetilde{b}_n}{\widetilde{c}_n} \to 1$ and $\widetilde{f}(\widetilde{b}_n x) \not\to 0$ for every $x \in [0,\infty)$. Then the function $f : \mathbb{R}^d \to \mathbb{R}$ defined by $f(x) = \widetilde{f}(\|x\|^d)$ is continuous, nonnegative and integrable (see the beginning of the proof of Theorem 6). For $(b_n) = (\sqrt[d]{\widetilde{b}_n})$ we have $\frac{b_n}{c_n} = \sqrt[d]{\frac{\widetilde{b}_n}{\widetilde{c}_n}} \to 1$ and $f(b_n x) = \widetilde{f}(\widetilde{b}_n \|x\|^d) \not\to 0$ for every $x \in \mathbb{R}^d$.

We fix two sequences: (a_i) and (l_i) such that $a_i > 1$ and $l_i \in \mathbb{Z}$ for every $i \in \mathbb{N}$, $a_i \to 1$ and every $x > 0$ is an element of infinitely many of the intervals $[a_i^{l_i-1}, a_i^{l_i}]$. One may put for example

$$\begin{aligned} a_i &= 1 + \frac{1}{k} \\ l_i &= i - \frac{(k+1)^3 + k^3 - 1}{2} \end{aligned} \qquad \text{for } k^3 \le i < (k+1)^3, \quad k \in \mathbb{N}$$

(it is easy to compute that for such (a_i) and (l_i) one has $\bigcup_{i=k^3}^{(k+1)^3-1}[a_i^{l_i-1}, a_i^{l_i}] \supset [2^{-k}, 2^k]$).

We construct the function $\widetilde{f}$ and the sequence $(\widetilde{b}_n)$ piecewise, by induction. In each step we apply Lemma 8 to obtain the next part of the function $\widetilde{f}$ and the next part of the sequence $(\widetilde{b}_n)$. More precisely, in the i-th step of the induction we define $\widetilde{f}$ on an interval $[S_i, T_i]$ and $\widetilde{b}_n$'s with $n = M_i + 1, \ldots, N_i$. At the beginning no $\widetilde{b}_n$'s are defined, so we put $M_1 = 0$. We choose S_1 arbitrarily, e.g. $S_1 = 1$. Then we apply Lemma 8 with $a = a_1$, $l = l_1$, $M = M_1$, $S = S_1$ and $\varepsilon = 1/4$. As a result we obtain $N_1 = N$, $T_1 = T$, function $g_1 = g : [0,\infty) \to \mathbb{R}$ such that g_1 is zero outside $[S_1, T_1]$ and $\widetilde{b}_n$'s for $n = M_1 + 1, \ldots, N_1$. We repeat this procedure infinitely many times. In the i-th step we apply Lemma 8 with $a = a_i$, $l = l_i$, $M = M_i = N_{i-1}$, $S = S_i = T_{i-1} + 1$ and $\varepsilon = 1/4^i$. As a result we obtain $N_i = N$, $T_i = T$, function $g_i = g : [0,\infty) \to \mathbb{R}$ such that g_i is zero outside $[S_i, T_i]$ and $\widetilde{b}_n$'s for $n = M_i + 1, \ldots, N_i$.

The whole sequence $(\widetilde{b}_n)$ satisfies $\frac{1}{a_i} \le \frac{\widetilde{b}_n}{\widetilde{c}_n} \le a_i$ for $M_i < n \le N_i$, which (together with $a_i \to 1$) implies $\frac{\widetilde{b}_n}{\widetilde{c}_n} \to 1$. Let

$$\widetilde{f}(x) = h(x) + \sum_{i=1}^{\infty} 2^i g_i(x),$$

where $h : [0,\infty) \to \mathbb{R}$ is an arbitrary continuous, positive and integrable function. Function $\widetilde{f}$ is nonnegative, continuous (the series converges almost uniformly) and integrable ($\int_0^\infty \widetilde{f}(x)dx < \int_0^\infty h(x)dx + \sum_{i=1}^\infty 2^i/4^i < \infty$).

Finally, let $x \in [0,\infty)$. If $x = 0$, then $\widetilde{f}(\widetilde{b}_n x) \ge h(0) > 0$, hence $\widetilde{f}(\widetilde{b}_n x) \not\to 0$. If $x > 0$, then there exists an increasing sequence (i_j) satisfying $x \in [a_{i_j}^{l_{i_j}-1}, a_{i_j}^{l_{i_j}}]$ and we have

$$\limsup_{n\to\infty} \widetilde{f}(\widetilde{b}_n x) = \limsup_{i\to\infty} \max_{M_i < n \le N_i} \widetilde{f}(\widetilde{b}_n x) \ge \limsup_{j\to\infty} \max_{M_{i_j} < n \le N_{i_j}} 2^{i_j} g_{i_j}(\widetilde{b}_n x) \ge \limsup_{j\to\infty} 2^{i_j} = \infty.$$

□

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