A "QUANTUM" RAMSEY THEOREM FOR OPERATOR SYSTEMS

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ABSTRACT. Let $\mathcal{V}$ be a linear subspace of $M_n(\mathbb{C})$ which contains the identity matrix and is stable under the formation of Hermitian adjoints. We prove that if n is sufficiently large then there exists a rank k orthogonal projection P such that $\dim(P\mathcal{V}P) = 1$ or k^2.

1. Background

An operator system in finite dimensions is a linear subspace $\mathcal{V}$ of $M_n(\mathbb{C})$ with the properties

- $I_n \in \mathcal{V}$
- $A \in \mathcal{V} \Rightarrow A^* \in \mathcal{V}$

where I_n is the $n \times n$ identity matrix and A^* is the Hermitian adjoint of A. In this paper the scalar field will be complex and we will write $M_n = M_n(\mathbb{C})$.

Operator systems play a role in the theory of quantum error correction. In classical information theory, the "confusability graph" is a bookkeeping device which keeps track of possible ambiguity that can result when a message is transmitted through a noisy channel. It is defined by taking as vertices all possible source messages, and placing an edge between two messages if they are sufficiently similar that data corruption could lead to them being indistinguishable on reception. Once the confusability graph is known, one is able to overcome the problem of information loss by using an independent subset of the confusability graph, which is known as a "code". If it is agreed that only code messages will be sent, then we can be sure that the intended message is recoverable.

When information is stored in quantum mechanical systems, the problem of error correction changes radically. The basic theory of quantum error correction was laid down in [3]. In [2] it was suggested that in this setting the role of the confusability graph is played by an operator system, and it was shown that for every operator system a "quantum Lovász number" could be defined, in analogy to the classical Lovász number of a graph. This is an important parameter in classical information theory. See also [5] for much more along these lines.

The interpretation of operator systems as "quantum graphs" was also proposed in [8], based on the more general idea of regarding linear subspaces of M_n as "quantum relations", and taking the conditions $I_n \in \mathcal{V}$ and $A \in \mathcal{V} \Rightarrow A^* \in \mathcal{V}$ to respectively express reflexivity and symmetry conditions. The idea is that the edge structure of a classical graph can be encoded in an obvious way as a reflexive, symmetric relation on a set. This point of view was explicitly connected to the quantum error correction literature in [9].

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Ramsey's theorem states that for any k there exists n such that every graph with at least n vertices contains either a k-clique or a k-anticlique, i.e., a set of k vertices among which either all edges are present or no edges are present. Simone Severini asked the author whether there is a "quantum" version of this theorem for operator systems. The natural notion of a *quantum k-clique* for an operator system $\mathcal{V}$ is an orthogonal projection $P \in M_n$ (i.e., a matrix satisfying $P = P^2 = P^*$) whose rank is k, such that $P\mathcal{V}P = \{PAP : A \in \mathcal{V}\}$ is maximal; that is, such that $P\mathcal{V}P = PM_nP \cong M_k$, or equivalently, $\dim(P\mathcal{V}P) = k^2$. The natural notion of a *quantum k-anticlique* is a rank k projection P such that $P\mathcal{V}P = \mathbb{C} \cdot P \cong M_1$, or equivalently, $\dim(P\mathcal{V}P) = 1$. This proposal is supported by the fact that in quantum error correction a code is taken to be the range of a projection satisfying just this condition, $P\mathcal{V}P = \mathbb{C}\cdot P$ [3]. As mentioned earlier, classical codes are taken to be independent sets, which is to say, anticliques.

The main result of this paper is a quantum Ramsey theorem which states that for every k there exists n such that every operator system in M_n has either a quantum k-clique or a quantum k-anticlique. This answers Severini's question positively.

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2. Examples

If $G = (V, E)$ is any finite simple graph, without loss of generality suppose $V = \{1, \ldots, n\}$ and define $\mathcal{V}_G$ to be the operator system

$$\mathcal{V}_G = \mathrm{span}\{E_{ij} : i = j \text{ or } \{i, j\} \in E\} \subseteq M_n.$$

Here we use the notation E_{ij} for the $n \times n$ matrix with a 1 in the (i, j) entry and 0's elsewhere. Also, let (e_i) be the standard basis of $\mathbb{C}^n$, so that $E_{ij} = e_i e_j^*$.

The inclusion of the diagonal E_{ii} matrices corresponds to including a loop at each vertex. (In the error correction setting this is natural: we place an edge between any two messages that might be indistinguishable on reception, and this is certainly true of any message and itself.) Once we adopt the convention that every graph has a loop at each vertex, an anticlique should no longer be a subset $S \subseteq V$ which contains no edges, it should be a subset which contains no edges except loops. Such a set corresponds to the projection P_S onto $\mathrm{span}\{e_i : i \in S\}$ with the property that $P_S\mathcal{V}_G P_S = \mathrm{span}\{E_{ii} : i \in S\}$. Or course this is very different from a quantum anticlique where $P\mathcal{V}P$ is one-dimensional.

To illustrate the dissimilarity between classical and quantum cliques and anticliques, consider the *diagonal operator system* $D_n \subseteq M_n$ consisting of the diagonal $n \times n$ complex matrices. In the notation used above, this is just the operator system $\mathcal{V}_G$ corresponding to the empty graph on n vertices. It might at first appear to falsify the desired quantum Ramsey theorem, because of the following fact.

Proposition 2.1. *D_n has no quantum k-anticlique for $k \geq 2$.*

Proof. Let $P \in M_n$ be a projection of rank $k \geq 2$. Since $\mathrm{rank}(E_{ii}) = 1$ for all i, it follows that $\mathrm{rank}(PE_{ii}P) = 0$ or 1 for each i. If $PE_{ii}P = 0$ for all i then $P = \sum_{i=1}^n PE_{ii}P = 0$, contradiction. Thus we must have $\mathrm{rank}(PE_{ii}P) = 1$ for some i, but then $PE_{ii}P$ cannot belong to $\mathbb{C} \cdot P = \{aP : a \in \mathbb{C}\}$, since every matrix in this set has rank 0 or k. So $PD_nP \neq \mathbb{C} \cdot P$. ☐

Since every operator system of the form $\mathcal{V}_G$ contains the diagonal matrices, none of these operator systems has nontrivial quantum anticliques. The surprising thing is that for n sufficiently large, they all have quantum k-cliques. This follows from the next result.

Proposition 2.2. *If $n \geq k^2 + k - 1$ then there is a rank k projection $P \in M_n$ such that $\dim(PD_nP) = k^2$.*

Proof. Without loss of generality let $n = k^2 + k - 1$. Start by considering M_k acting on $\mathbb{C}^k$. Find k^2 vectors $v_1, \ldots, v_{k^2}$ in $\mathbb{C}^k$ such that the rank 1 matrices $v_i v_i^*$ are linearly independent. (For example, we could take the k standard basis vectors e_i plus the $\frac{k^2-k}{2}$ vectors $e_i + e_j$ for $i \neq j$ plus the $\frac{k^2-k}{2}$ vectors $e_i + ie_j$ for $i \neq j$. The corresponding rank 1 matrices span M_k and thus they must be independent since $\dim(M_k) = k^2$.) Regarding $\mathbb{C}^k$ as a subspace of $\mathbb{C}^n$, we can extend the v_i to orthogonal vectors $w_i \in \mathbb{C}^n$ as follows: take $w_1 = v_1 \oplus (1, 0, \ldots, 0)$, $w_2 = v_2 \oplus (a_1, 1, 0, \ldots, 0)$, $w_3 = v_3 \oplus (b_1, b_2, 1, 0, \ldots, 0)$, etc., with $a_1, b_1, b_2, \ldots$ successively chosen so that $\langle w_i, w_j \rangle = 0$ for $i \neq j$. We need $k^2 - 1$ extra dimensions to accomplish this. Now let P be the rank k projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$ and let D_n be the diagonal operator system relative to any orthonormal basis of $\mathbb{C}^n$ that contains the vectors $\frac{w_i}{\|w_i\|}$ for $1 \leq i \leq k^2$. Then PD_nP contains $Pw_i w_i^* P = v_i v_i^*$ for all i, so $\dim(PD_nP) = k^2$. $\square$

A stronger version of this result will be proven in Lemma 4.3. The value $n = k^2 + k - 1$ may not be optimal, but note that in order for D_n to have a quantum k-clique n must be at least k^2, since $\dim(D_n) = n$ and we need $\dim(PD_nP) = k^2$.

Next, we show that operator systems of arbitrarily large dimension may lack quantum 3-cliques.

Proposition 2.3. *Let $\mathcal{V}_n = \mathrm{span}\{I_n, E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, \ldots, E_{n1}\} \subseteq M_n$. Then $\mathcal{V}_n$ has no quantum 3-cliques.*

Proof. Let $P \in M_n$ be any projection. If $Pe_1 = 0$ then $P\mathcal{V}_nP = \mathbf{C} \cdot P$, so P is a quantum anticlique. Otherwise let $k = \mathrm{rank}(P)$ and let $f_1, \ldots, f_k$ be an orthonormal basis of $\mathrm{ran}(P)$ with $f_1 = \frac{Pe_1}{\|Pe_1\|}$. Then $PE_{1i}P = Pe_1e_i^*P = f_1 v_i^*$ where $v_i = \|Pe_1\| Pe_i$. The span of these matrices is precisely $\mathrm{span}\{f_1 f_i^*\}$, since the projections of the e_i span $\mathrm{ran}(P)$. So $P\mathcal{V}_nP$ is just $\mathcal{V}_k \subseteq M_k \cong PM_nP$, relative to the (f_i) basis. If $k \geq 3$ then $\dim(\mathcal{V}_k) = 2k < k^2$, so P cannot be a quantum clique. $\square$

3. Quantum 2-cliques

In contrast to Proposition 2.3, we will show in this section that any operator system whose dimension is at least four must have a quantum 2-clique. This result is clearly sharp. It is somewhat analogous to the trivial classical fact that any graph that contains at least one edge must have a 2-clique.

Define the Hilbert-Schmidt inner product of $A, B \in M_n$ to be $\mathrm{Tr}(AB^*)$. Denote the set of Hermitian $n \times n$ matrices by M_n^h. Observe that any operator system is spanned by its Hermitian part since any matrix A satisfies $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$ where $\mathrm{Re}(A) = \frac{1}{2}(A + A^*)$ and $\mathrm{Im}(A) = \frac{1}{2i}(A - A^*)$.

Lemma 3.1. *Let $\mathcal{V} \subseteq M_n$ be an operator system and suppose $\dim(\mathcal{V}) \leq 3$. Then its Hilbert-Schmidt orthocomplement is spanned by rank 2 Hermitian matrices.*

Proof. Work in M_n^h. Let $\mathcal{V}_0 = \mathcal{V} \cap M_n^h$ and let $\mathcal{W}_0$ be the real span of the Hermitian matrices in $\mathcal{V}_0^\perp$ whose rank is 2. We will show that $\mathcal{W}_0 = \mathcal{V}_0^\perp$; taking complex spans then yields the desired result.

Suppose to the contrary that there exists a nonzero Hermitian matrix $B \in \mathcal{V}_0^\perp$ which is orthogonal to $\mathcal{W}_0$. Say $\mathcal{V}_0 = \mathrm{span}\{I_n, A_1, A_2\}$, where the A_i are not necessarily distinct from I_n. Since $B \in \mathcal{V}_0^\perp$, we have $\mathrm{Tr}(I_n B) = \mathrm{Tr}(A_1 B) = \mathrm{Tr}(A_2 B) = 0$, but $\mathrm{Tr}(B^2) \neq 0$. We will show that there is a rank 2 Hermitian matrix C whose inner products against I_n, A_1, A_2, and B are the same as their inner products against B. This will be a matrix in $\mathcal{W}_0$ which is not orthogonal to B, a contradiction.

Since B is Hermitian, we can choose an orthonormal basis (f_i) of $\mathbb{C}^n$ with respect to which it is diagonal, say $B = \mathrm{diag}(b_1, \ldots, b_n)$. We may assume $b_1, \ldots, b_j \geq 0$ and $b_{j+1}, \ldots, b_n < 0$. Let $B^+ = \mathrm{diag}(b_1, \ldots, b_j, 0, \ldots, 0)$ and $B^- = \mathrm{diag}(0, \ldots, 0, -b_{j+1}, \ldots, -b_n)$ be the positive and negative parts of B, so that $B = B^+ - B^-$. Let $\alpha = \mathrm{Tr}(B^+) = \mathrm{Tr}(B^-)$ (they are equal since $\mathrm{Tr}(B) = \mathrm{Tr}(I_n B) = 0$). Then $\frac{1}{\alpha}B^+$ is a convex combination of the rank 1 matrices $f_1 f_1^*, \ldots, f_j f_j^*$; that is, the linear functional $A \mapsto \frac{1}{\alpha}\mathrm{Tr}(AB^+)$ is a convex combination of the linear functionals $A \mapsto \langle A f_i, f_i\rangle$ for $1 \leq i \leq j$. By the convexity of the joint numerical range of three Hermitian matrices [1], there exists a unit vector $v \in \mathbb{C}$ such that $\frac{1}{\alpha}\mathrm{Tr}(AB^+) = \langle Av, v\rangle$ for $A = A_1$, A_2, and B. Similarly, there exists a unit vector w such that $\frac{1}{\alpha}\mathrm{Tr}(AB^-) = \langle Aw, w\rangle$ for $A = A_1$, A_2, and B. Then $C = \alpha(vv^* - ww^*)$ is a rank 2 matrix whose inner products against I_n, A_1, A_2, and B are the same as their inner products against B. So C has the desired properties. □

Lemma 3.2. *Let $\mathcal{V} \subseteq M_3$ be an operator system and suppose* $\dim(V) = 4$. *Then $\mathcal{V}$ has a quantum 2-clique.*

Proof. Let $\mathcal{V} = \mathrm{span}\{I_3, A_1, A_2, A_3\}$ where the A_i are Hermitian. Denote the set of unit vectors in $\mathbb{C}^3$ by $[\mathbb{C}^3]_1$. If $v \in [\mathbb{C}^3]_1$ then v and $A_1 v$ are linearly independent unless v is an eigenvector of A_1. Since A_1 is not a scalar multiple of I_3, the set of unit vectors v for which v and $A_1 v$ are linearly independent constitutes an open, dense subset of $[\mathbb{C}^3]_1$. For each such v, the vector $\tilde{v} = \overline{\frac{v \times A_1 v}{\|v \times A_1 v\|}} \in [\mathbb{C}^3]_1$ (the complex conjugate of the normalized cross product) is orthogonal to both v and $A_1 v$.

Claim 1: the set of $v \in [\mathbb{C}^3]_1$ such that $\langle A_1 v, v\rangle \neq \langle A_1 \tilde{v}, \tilde{v}\rangle$ is open and dense. Claim 2: the set of $v \in [\mathbb{C}^3]_1$ such that $\langle A_2 v, \tilde{v}\rangle \neq 0$ is open and dense. Claim 3: the set of $v \in [\mathbb{C}^3]_1$ such that $\langle A_2 v, \tilde{v}\rangle\overline{\langle A_3 v, \tilde{v}\rangle} \notin \mathbb{R}$ is open and dense. Then for any v in the intersection of these sets, the projection onto $\mathrm{span}\{v, \tilde{v}\}$ is a quantum 2-clique for $\mathcal{V}$.

[The proofs of the claims are computational; omitted until I find something better.] □

Theorem 3.3. *Let $\mathcal{V} \subseteq M_n$ be an operator system and suppose* $\dim(\mathcal{V}) \geq 4$. *Then $\mathcal{V}$ has a quantum 2-clique.*

Proof. Without loss of generality we can suppose that $\dim(\mathcal{V}) = 4$. Say $\mathcal{V} = \mathrm{span}\{I_n, A_1, A_2, A_3\}$ where the A_i are Hermitian.

We first claim that there is a projection P of rank at most 3 such that PI_nP, PA_1P, and PA_2P are linearly independent. If A_1 and A_2 are jointly diagonalizable (but, together with I_n, linearly independent) then we can find three common eigenvectors v_1, v_2, and v_3 such that the vectors $(1,1,1),(\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3) \in \mathbb{C}^3$ are linearly independent, where λ_i and μ_i are the eigenvalues belonging to v_i for A_1 and A_2, respectively. Then the projection onto $\mathrm{span}\{v_1,v_2,v_3\}$ verifies the claim. If A_1 and A_2 are not jointly diagonalizable, then we can find two eigenvectors v_1 and v_2 of A_1 such that $\langle A_2v_1, v_2\rangle \neq 0$. Letting v_3 be a third eigenvector of A_1 with the property that the eigenvalues of A_1 belonging to v_1, v_2, and v_3 are not all equal, we can again use the projection onto $\mathrm{span}\{v_1,v_2,v_3\}$. This establishes the claim.

Now let P be as in the claim and find $B \in M_n$ such that PI_nP, PA_1P, PA_2P, and PBP are linearly independent. By Lemma 3.2 we can then find a rank 2 projection $Q \leq P$ such that QI_nQ, QA_1Q, QA_2Q, and QBQ are linearly independent.

If QI_nQ, QA_1Q, QA_2Q, and QA_3Q are linearly independent then we are done. Otherwise, let α, β, and γ be the unique scalars such that $QA_3Q = \alpha QI_nQ + \beta QA_1Q + \gamma QA_2Q$. By Lemma 3.1 we can find a rank 2 Hermitian matrix C such that $\mathrm{Tr}(I_nC) = \mathrm{Tr}(A_1C) = \mathrm{Tr}(A_2C) = 0$ but $\mathrm{Tr}(A_3C) \neq 0$. Then $C = vv^* - ww^*$ for some orthogonal vectors v and w. Thus, $\langle Av, v\rangle = \langle Aw, w\rangle$ for $A = I_n$, A_1, and A_2, but not for $A = A_3$. It follows that the two conditions

$$\langle A_3v, v\rangle = \alpha\langle I_nv, v\rangle + \beta\langle A_1v, v\rangle + \gamma\langle A_2v, v\rangle$$

and

$$\langle A_3w, w\rangle = \alpha\langle I_nw, w\rangle + \beta\langle A_1w, w\rangle + \gamma\langle A_2w, w\rangle$$

cannot both hold. Without loss of generality suppose the first fails. Then letting Q' be the projection onto $\mathrm{span}(\mathrm{ran}(Q) \cup \{v\})$, we cannot have $Q'A_3Q' = \alpha Q'I_nQ' + \beta Q'A_1Q' + \gamma Q'A_2Q'$. Thus $\mathrm{rank}(Q') = 3$ and $\dim(Q'\mathcal{V}Q') = 4$. The theorem now follows by applying Lemma 3.2 to $Q'\mathcal{V}Q'$. ☐

4. The main theorem

The proof of our main theorem proceeds through a series of lemmas.

Lemma 4.1. *Suppose the operator system $\mathcal{V}$ is contained in D_n. If $\dim(\mathcal{V}) \geq k^2 + k - 1$ then $\mathcal{V}$ has a quantum k-clique. If $\dim(\mathcal{V}) \leq \frac{n-k}{k-1}$ then $\mathcal{V}$ has a quantum k-anticlique. If $n \geq k^3 - k + 1$ then $\mathcal{V}$ has either a quantum k-clique or a quantum k-anticlique.*

Proof. If $\dim(\mathcal{V}) \geq k^2 + k - 1 = m$ then we can find a set of indices $S \subseteq \{1, \ldots, n\}$ of cardinality m such that $\dim(P\mathcal{V}P) = m$ where P is the orthogonal projection onto $\mathrm{span}\{e_i : i \in S\}$. Then $P\mathcal{V}P \cong D_m \subseteq M_m \cong PM_nP$ and Proposition 2.2 yields that $P\mathcal{V}P$, and hence also $\mathcal{V}$, has a quantum k-clique. If $\dim(\mathcal{V}) \leq \frac{n-k}{k-1}$ then a result of Tverberg [6, 7] can be used to extract a quantum k-anticlique; this is essentially Theorem 4 of [4]. Thus if $k^2 + k - 1 \leq \frac{n-k}{k-1}$ then one of the two cases must obtain, i.e., $\mathcal{V}$ must have either a quantum k-clique or a quantum k-anticlique. A little algebra shows that this inequality is equivalent to $n \geq k^3 - k + 1$. ☐

Lemma 4.2. *Let $v_1, \ldots, v_r$ be vectors in $\mathbb{C}^s$. Then there are vectors $w_1, \ldots, w_r \in \mathbb{C}^{r-1}$ such that the vectors $v_i \oplus w_i \in \mathbb{C}^{s+r-1}$ are pairwise orthogonal and all have the same norm.*

Proof. Let G be the Gramian matrix of the vectors v_i and let $\|G\|$ be its operator norm. Then $\mathrm{rank}(\|G\|I_r - G) \leq r - 1$, so we can find vectors $w_i \in \mathbb{C}^{r-1}$ whose

Grammian matrix is $\|G\|I_r - G$. The Grammian matrix of the vectors $v_i \oplus w_i$ is then $\|G\|I_r$, as desired. □

Then next lemma improves Proposition 2.2.

Lemma 4.3. *Let $n = k^2 + k - 1$ and suppose $A_1, \ldots, A_{k^2}$ are Hermitian matrices in M_n such that for each i we have $\langle A_i e_i, e_i\rangle = 1$, and also $\langle A_i e_r, e_s\rangle = 0$ whenever $\max\{r, s\} > i$. Then $\mathcal{V} = \mathrm{span}\{I, A_1, \ldots, A_{k^2}\}$ has a quantum k-clique.*

Proof. Let A_i have matrix entries (a^i_{rs}). The goal is to find vectors $v_1, \ldots, v_{k^2} \in \mathbb{C}^k$ such that the matrices

$$A'_i = \sum_{1\leq r,s\leq k} a^i_{rs} v_r v_s^*$$

are linearly independent. Once we have done this, find vectors $w_i \in \mathbf{C}^{k^2-1}$ as in Lemma 4.2 and let $f_i = \frac{1}{N}(v_i \oplus w_i) \in \mathbb{C}^n$ where N is the common norm of the $v_i \oplus w_i$. Then the f_i form an orthonormal set in $\mathbb{C}^n$, so they can be extended to an orthonormal basis, and the operators whose matrices for this basis are the A_i compress to the matrices $\frac{1}{N^2}A'_i$ on the initial $\mathbb{C}^k$, which are linearly independent. So $P\mathcal{V}P$ contains k^2 linearly independent matrices, where P is the orthogonal projection onto $\mathbb{C}^k$, showing that $\mathcal{V}$ has a quantum k-clique.

The vectors v_i are constructed inductively. Once $v_1, \ldots, v_i$ are chosen so that $A'_1, \ldots, A'_i$ are independent, future choices of the v's cannot change this since $A_1, \ldots, A_i$ all live on the initial $i \times i$ block. We can let v_1 be any nonzero vector in $\mathbb{C}^k$, since $A_1 = e_1 e_1^*$, so that $A'_1 = v_1 v_1^*$ and this only has to be nonzero. Now suppose $v_1, \ldots, v_{i-1}$ have been chosen and we need to select v_i so that A'_i is independent of $A'_1, \ldots, A'_{i-1}$. After choosing v_i we will have $A'_i = \sum_{1\leq r,s\leq i} a^i_{rs} v_r v_s^*$. Let B be this sum restricted to $1 \leq r, s \leq i - 1$. That part is already determined since v_i does not appear. Also let

$$u = a^i_{1i} v_1 + \cdots + a^i_{(i-1)i} v_{i-1};$$

then we will have

$$A'_i = B + uv_i^* + v_i u^* + v_i v_i^*$$

(using the assumption that $a^i_{ii} = 1$). That is,

$$A'_i = (B - uu^*) + (u + v_i)(u + v_i)^* = B' + \tilde{u}\tilde{u}^*$$

where $\tilde{u} = u + v_i$ is arbitrary, and the question is whether $\tilde{u}$ can be chosen to make this matrix independent of $A'_1, \ldots, A'_{i-1}$. But the possible choices of A'_i clearly span M_k — there is no matrix which is Hilbert-Schmidt orthogonal to $B' + \tilde{u}\tilde{u}^*$ for all $\tilde{u}$ — so there must be a choice of $\tilde{u}$ which makes A'_i independent of $A'_1, \ldots, A'_{i-1}$, as desired. □

Next we prove a technical variation on Lemma 4.3.

Lemma 4.4. *Let $n = k^4 + k^3 + k - 1$ and let $\mathcal{V}$ be an operator system contained in M_n. Suppose $\mathcal{V}$ contains matrices $A_1, \ldots, A_{k^4+k^3}$ such that for each i we have $\langle A_i e_i, e_{i+1}\rangle \neq 0$, and also $\langle A_i e_r, e_s\rangle = 0$ whenever $\max\{r, s\} > i + 1$ and $r \neq s$. Then $\mathcal{V}$ has a quantum k-clique.*

Proof. Let A_i have matrix entries (a^i_{rs}). Observe that for each i the compression of A_i to $\mathrm{span}\{e_{i+2}, \ldots, e_n\}$ is diagonal. For each $r > i + 1$ let the *r-tail* of A_i be the vector $(a^i_{rr}, \ldots, a^i_{nn})$. Suppose there exist indices $i_1, \ldots, i_{k^2+k-1}$ such that the r-tails of the A_{i_j}, $1 \leq j \leq k^2 + k - 1$, are linearly independent, where $r = \max_j\{i_j + 2\}$. Then the compression of $\mathcal{V}$ to $\mathrm{span}\{e_r, \ldots, e_n\}$ contains $k^2 + k - 1$ linearly independent diagonal matrices, so it has a quantum k-clique by the first assertion of Lemma 4.1. Thus, we may assume that for any $k^2 + k - 1$ distinct indices i_j the matrices A_{i_j} have linearly dependent r-tails.

We construct an orthonormal sequence of vectors v_i and a sequence of Hermitian matrices $B_i \in \mathcal{V}$, $1 \leq i \leq k^2$, such that the compressions of the B_i to $\mathrm{span}\{v_1, \ldots, v_{k^2}, e_{k^4+k^3+1}, \ldots, e_{k^4+k^3+k-1}\}$ satisfy the hypotheses of Lemma 4.3. This will ensure the existence of a quantum k-clique.

The first k^2+k-1 matrices $A_1, \ldots, A_{k^2+k-1}$ have linearly dependent r-tails for $r = k^2 + k + 1$. Thus there is a nontrivial linear combination $B_1' = \sum_{i=1}^{k^2+k-1} \alpha_i A_i$ whose r-tail is the zero vector. Letting j be the largest index such that α_j is nonzero, we have $\langle B_1' e_j, e_{j+1}\rangle \neq 0$ because $\langle A_j e_j, e_{j+1}\rangle \neq 0$ but $\langle A_i e_j, e_{j+1}\rangle = 0$ for $i < j$. Thus the compression of B_1' to $\mathrm{span}\{e_1, \ldots, e_{k^2+k}\}$ is nonzero, so there exists a unit vector v_1 in this span such that $\langle B_1' v_1, v_1\rangle \neq 0$. Then let B_1 be a scalar multiple of either the real or imaginary part of B_1' which satisfies $\langle B_1 v_1, v_1\rangle = 1$. Note that $\langle B_1 e_r, e_s\rangle = 0$ for any r, s with $\max\{r, s\} > k^2 + k$. Apply the same reasoning to the next block of $k^2 + k - 1$ matrices $A_{k^2+k+1}, \ldots, A_{2k^2+2k-1}$ to find v_2 and B_2, and proceed inductively. After k^2 steps, $k^2(k^2 + k) = k^4 + k^3$ indices will have been used up and $k - 1$ (namely, $e_{k^4+k^3+1}, \ldots, e_{k^4+k^3+k-1}$) will remain, as needed. □

Theorem 4.5. *For any k there exists n such that any operator system in M_n has either a quantum k-clique or a quantum k-anticlique.*

Proof. Take $n = 8k^{11}$ and let $\mathcal{V}$ be an operator system in M_n. Find a unit vector $v_1 \in \mathbb{C}^n$, if one exists, such that the dimension of $\mathcal{V}v_1 = \{Av_1 : A \in \mathcal{V}\}$ is less than $8k^8$. Then find a unit vector $v_2 \in (\mathcal{V}v_1)^\perp$, if one exists, such that the dimension of $(\mathcal{V}v_1)^\perp \cap (\mathcal{V}v_2)$ is less than $8k^8$. Proceed in this fashion, at the rth step trying to find a unit vector v_r in

$$(\mathcal{V}v_1)^\perp \cap \cdots \cap (\mathcal{V}v_{r-1})^\perp$$

such that the dimension of

$$(\mathcal{V}v_1)^\perp \cap \cdots \cap (\mathcal{V}v_{r-1})^\perp \cap (\mathcal{V}v_r)$$

is less than $8k^8$. If this construction lasts for k^3 steps then the compression of $\mathcal{V}$ to $\mathrm{span}\{v_1, \ldots, v_{k^3}\} \cong M_m$ is contained in D_{k^3}, so this compression, and hence also $\mathcal{V}$, has either a quantum k-clique or a quantum k-anticlique by Lemma 4.1.

Otherwise, the construction fails at some stage d. This means that the compression $\mathcal{V}'$ of $\mathcal{V}$ to $E = (\mathcal{V}v_1)^\perp \cap \cdots \cap (\mathcal{V}v_d)^\perp$ has the property that the dimension of $\mathcal{V}'v$ is at least $8k^8$, for every unit vector $v \in E$.

Work in E. Choose any nonzero vector $w_1 \in E$ and find $A_1 \in \mathcal{V}'$ such that $w_2 = A_1 w_1$ is nonzero and orthogonal to w_1. Then find $A_2 \in \mathcal{V}'$ such that $w_3 = A_2 w_2$ is nonzero and orthogonal to $\mathrm{span}\{w_i, A_1 w_i, A_1^* w_i : i = 1, 2\}$. Continue in this way, at the rth step finding $A_r \in \mathcal{V}'$ such that $w_{r+1} = A_r w_r$ is nonzero and orthogonal to $\mathrm{span}\{w_j, A_i w_j, A_i^* w_j : i < r \text{ and } j \leq r\}$. The dimension of this span is at most $2r^2 - r$, so as long as $r \leq 2k^4$ its dimension is less than $8k^8$ and a vector w_{r+1} can be found. Compressing to the span of the w_i then puts us in the situation of Lemma 4.4 with $n = 2k^4$, which is more than enough. So there exists a quantum k-clique by that lemma. □

The constants in the proof could be improved, but only marginally. Very likely the problem of determining bounds on quantum Ramsey numbers is open-ended, just as in the classical case.

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