Structure Formation in the Early Universe

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Abstract

The evolution of the perturbations in the energy density and the particle number density in a flat Friedmann-Lemaître-Robertson-Walker universe in the radiation-dominated era and in the epoch after decoupling of matter and radiation is studied. For large-scale perturbations the outcome is in accordance with treatments in the literature. For small-scale perturbations the differences are conspicuous. Firstly, in the radiation-dominated era small-scale perturbations grew proportional to the square root of time. Secondly, perturbations in the Cold Dark Matter particle number density were, due to gravitation, coupled to perturbations in the total energy density. This implies that structure formation has commenced successfully only after decoupling of matter and radiation. Finally, after decoupling density perturbations evolved diabatically, i.e., they exchanged heat with their environment. This heat exchange may have enhanced the growth rate of its mass sufficiently to explain structure formation in the early universe, a phenomenon which cannot be understood from adiabatic density perturbations.

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Contents

1 Introduction

The global properties of our universe are very well described by a ΛCDM model with a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric within the context of the General Theory of Relativity. To explain structure formation after decoupling of matter and radiation in this model, one has to assume that before decoupling Cold Dark Matter (CDM) has already contracted to form seeds into which the baryons (i.e., ordinary matter) could fall after decoupling. In this article it will be shown that CDM did not contract faster than baryons before decoupling and that structure formation started off successfully only after decoupling.

The perturbation equations for FLRW universes derived in a companion article [1] will be applied to a flat FLRW universe in its three main phases, namely the radiation-dominated era, the plasma era, and the epoch after decoupling of matter and radiation. In the derivation of these equations, an equation of state for the pressure of the form $p = p(n, \varepsilon)$ has been taken into account, as is required by thermodynamics. As a consequence, in addition to a usual second-order evolution equation (3a) for density perturbations, a first-order evolution equation (3b) for entropy perturbations follows also from the perturbed Einstein equations. This entropy evolution equation is absent in former treatments of the subject. Therefore, the system (3) leads to further reaching conclusions than is possible from treatments in the literature.

Analytic expressions for the fluctuations in the energy density δ_ε and the particle number density δ_n in the radiation-dominated era and the epoch after decoupling will be determined. It is shown that the evolution equations (3) corroborate the standard perturbation theory in both eras in the limiting case of *infinite* scale perturbations. For *finite* scales, however, the differences are conspicuous. Therefore, only finite scale perturbations are considered in detail.

A first result is that in the radiation-dominated era oscillating density perturbations with an *increasing* amplitude proportional to $t^{1/2}$ are found, whereas the standard perturbation equation yields oscillating density perturbations with a *constant* amplitude. This difference is due to the fact that in the new perturbation equations (3) the divergence $\vartheta_{(1)}$ of the spatial part of the fluid four-velocity is taken into account, whereas $\vartheta_{(1)} = 0$ in the standard equation (61). In fact, $\vartheta_{(1)} = 0$ is one of the conditions for the non-relativistic limit, as has been shown in Section 4*[1]. In Section A.1 it is made clear why $\vartheta_{(1)}$ may not be omitted.

In the radiation-dominated era and the plasma era baryons were tightly coupled to radiation via Thomson and Coulomb scattering until decoupling. A second result is that Cold Dark Matter (CDM) was also tightly coupled to radiation, not through Thomson and Coulomb scattering, but through *gravitation*. This implies that before decoupling perturbations in CDM have contracted as fast as perturbations in the baryon density. As a consequence, CDM could not have triggered structure formation after decoupling. This result follows from the entropy evolution equation (3b) since $p_n \leq 0$, (5), throughout the history of the universe as will be shown in Section 3.

From observations [2] of the Cosmic Microwave Background (CMB) it follows that perturbations were adiabatic at the moment of decoupling, and density fluctuations δ_ε and δ_n were of the order of 10^{-5} or less. Since the growth rate of *adiabatic* perturbations in the era after decoupling was too small to explain structure in the universe, there must have been, in addition to gravitation, some other mechanism which has enhanced the growth rate sufficiently to form the first stars from small density perturbations. The result of the present study is that after decoupling such a mechanism did indeed exist in the early universe, as will now be explained.

At the moment of decoupling of matter and radiation, photons could not ionize matter any more and the two constituents fell out of thermal equilibrium. As a consequence, the pressure dropped from a very high radiation pressure just before decoupling to a very low gas pressure after decoupling. This fast and chaotic transition from a high pressure epoch to a very low pressure era may have resulted in large relative *diabatic* pressure perturbations due to very small fluctuations in the kinetic energy density. Consequently, the pressure perturbations did not vanish in the *perturbed* universe just after decoupling. It is found that the growth of a density perturbation has not only been governed by gravitation, but also by heat exchange of a perturbation with its environment. The growth rate depended strongly on the scale of a perturbation. For perturbations with a scale of $6.5\,\mathrm{pc} \approx 21\,\mathrm{ly}$ (see the peak value in Figure 1) gravity and heat exchange worked perfectly together, resulting in a fast growth rate. Perturbations larger than this scale reached, despite their stronger gravitational field, their non-linear phase at a later time since heat exchange was slower due to their larger scales. On the other hand, for perturbations with scales smaller than 6.5 pc gravity was weak and heat exchange was not sufficient to let perturbations grow. Therefore, density perturbations with scales smaller than 6.5 pc did not reach the non-linear regime within 13.81 Gyr. Since there was a sharp decline in

[1] Section and equation numbers with a $*$ refer to sections and equations in the companion article [1].

growth rate below a scale of 6.5 pc, this scale will be called the *relativistic Jeans scale.*

The new evolution equations (3) have solutions which are the relativistic counterparts of the Newtonian energy density perturbation and particle number density perturbation. Moreover, their solutions are free of spurious gauge modes. Therefore, these equations describe unambiguously, Section 2.6*, the evolution of cosmological density perturbations. Consequently, the ΛCDM model of the universe and its evolution equations for density perturbations (3) explain the first structures in the universe, several hundreds of million years after the Big Bang [3, 4].

2 Einstein Equations for a Flat FLRW Universe

In this section the equations needed for the study of the evolution of density perturbations in the early universe are written down for an equation of state for the pressure, $p = p(n, \varepsilon)$.

2.1 Background Equations

The set of zeroth-order Einstein equations and conservation laws for a flat, i.e., $R_{(0)} = 0$, FLRW universe filled with a perfect fluid with energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p g^{\mu\nu}, \quad p = p(n, \varepsilon), \tag{1}$$

is given by

$$3H^2 = \kappa\varepsilon_{(0)}, \qquad \kappa = 8\pi G_{\mathrm{N}}/c^4, \tag{2a}$$

$$\dot{\varepsilon}_{(0)} = -3H\varepsilon_{(0)}(1 + w), \quad w \coloneqq p_{(0)}/\varepsilon_{(0)}, \tag{2b}$$

$$\dot{n}_{(0)} = -3Hn_{(0)}. \tag{2c}$$

The evolution of density perturbations has been taken place in the early universe, when $\Lambda \ll \kappa\varepsilon_{(0)}$. Therefore, the cosmological constant Λ has been neglected.

2.2 Evolution Equations for Density Perturbations

The complete set of perturbation equations for the two independent density contrast functions δ_n and δ_ε is given by [1]

$$\ddot{\delta}_\varepsilon + b_1\dot{\delta}_\varepsilon + b_2\delta_\varepsilon = b_3\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right], \tag{3a}$$

$$\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right] = \frac{3Hn_{(0)}p_n}{\varepsilon_{(0)}(1+w)}\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right], \tag{3b}$$

where the coefficients b_1, b_2 and b_3 are, for a flat FLRW universe, filled with a perfect fluid described by an equation of state $p = p(n, \varepsilon)$ given by

$$b_1 = H(1 - 3w - 3\beta^2) - 2\frac{\dot{\beta}}{\beta}, \tag{4a}$$

$$b_2 = \kappa\varepsilon_{(0)}\Big[2\beta^2(2 + 3w) - \tfrac{1}{6}(1 + 18w + 9w^2)\Big] + 2H\frac{\dot{\beta}}{\beta}(1 + 3w) - \beta^2\frac{\nabla^2}{a^2}, \tag{4b}$$

$$b_3 = \left\{\frac{-2}{1+w}\left[\varepsilon_{(0)}p_{\varepsilon n}(1+w) + \frac{2p_n}{3H}\frac{\dot{\beta}}{\beta} + p_n(p_\varepsilon - \beta^2) + n_{(0)}p_{nn}\right] + p_n\right\}\frac{n_{(0)}}{\varepsilon_{(0)}}\frac{\nabla^2}{a^2}, \tag{4c}$$

where $p_n(n,\varepsilon)$ and $p_\varepsilon(n,\varepsilon)$ are the partial derivatives of the equation of state $p(n,\varepsilon)$:

$$p_n := \left(\frac{\partial p}{\partial n}\right)_\varepsilon, \quad p_\varepsilon := \left(\frac{\partial p}{\partial \varepsilon}\right)_n. \tag{5}$$

The symbol ∇^2 denotes the Laplace operator. The quantity $\beta(t)$ is defined by $\beta^2 := \dot{p}_{(0)}/\dot{\varepsilon}_{(0)}$. Using that $\dot{p}_{(0)} = p_n\dot{n}_{(0)} + p_\varepsilon\dot{\varepsilon}_{(0)}$ and the conservation laws (2b) and (2c) one gets

$$\beta^2 = p_\varepsilon + \frac{n_{(0)}p_n}{\varepsilon_{(0)}(1+w)}. \tag{6}$$

Using the definitions $w := p_{(0)}/\varepsilon_{(0)}$ and $\beta^2 := \dot{p}_{(0)}/\dot{\varepsilon}_{(0)}$ and the energy conservation law (2b), one finds for the time-derivative of w

$$\dot{w} = 3H(1+w)(w-\beta^2). \tag{7}$$

This expression is *independent* of the equation of state.

The pressure perturbation is given by [1]

$$p^{\rm gi}_{(1)} = \beta^2\varepsilon_{(0)}\delta_\varepsilon + n_{(0)}p_n\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right], \tag{8}$$

where the first term is the adiabatic part and the second term the diabatic part of the pressure perturbation.

The combined First and Second Law of Thermodynamics reads [1]

$$T_{(0)}s^{\rm gi}_{(1)} = -\frac{\varepsilon_{(0)}(1+w)}{n_{(0)}}\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right]. \tag{9}$$

Density perturbations evolve adiabatically if and only if the source term of the evolution equation (3a) vanishes, so that this equation is homogeneous and describes, therefore, a closed system that does not exchange heat with its environment. This can only be achieved for $p_n \approx 0$, or, equivalently, $p \approx p(\varepsilon)$, i.e., if the particle number density does not contribute to the pressure. In this case, the coefficient b_3, (4c), vanishes.

3 Analytic Solutions

In this section analytic solutions of equations (3) are derived for a flat FLRW universe with a vanishing cosmological constant in its radiation-dominated phase and in the era after decoupling of matter and radiation. It is shown that $p_n \leq 0$ throughout the history of the universe. In this case, the entropy evolution equation (3b) implies that fluctuations in the particle number density, δ_n, are coupled to fluctuations in the total energy density, δ_ε, through gravitation, independent of the nature of the particles. In particular, this holds true for perturbations in CDM. Consequently, CDM fluctuations have evolved in the same way as perturbations in ordinary matter. This may rule out CDM as a means to facilitate the formation of structure in the universe after decoupling. The same conclusion has also been reached by Nieuwenhuizen *et al.* [5], on different grounds. Consequently, in the radiation-dominated universe CDM did not contract faster than baryons, so that structure formation could commence only after decoupling.

3.1 Radiation-dominated Era

At very high temperatures, radiation and ordinary matter are in thermal equilibrium, coupled via Thomson scattering with the photons dominating over the nucleons ($n_\gamma/n_\mathrm{p} \approx 10^9$). Therefore the primordial fluid can be treated as radiation-dominated with equations of state

$$\varepsilon = a_\mathrm{B} T_\gamma^4, \quad p = \tfrac{1}{3} a_\mathrm{B} T_\gamma^4, \tag{10}$$

where a_B is the black body constant and T_γ the radiation temperature. The equations of state (10) imply the equation of state for the pressure $p = \frac{1}{3}\varepsilon$, so that, with (5),

$$p_n = 0, \quad p_\varepsilon = \tfrac{1}{3}. \tag{11}$$

Therefore, one has from (6),

$$\beta^2 = w = \tfrac{1}{3}. \tag{12}$$

Using (11) and (12), the perturbation equations (3) reduce to

$$\ddot{\delta}_\varepsilon - H\dot{\delta}_\varepsilon - \left[\frac{1}{3}\frac{\nabla^2}{a^2} - \tfrac{2}{3}\kappa\varepsilon_{(0)}\right]\delta_\varepsilon = 0, \tag{13a}$$

$$\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_n - \tfrac{3}{4}\delta_\varepsilon\right) = 0. \tag{13b}$$

Since $p_n = 0$ the right-hand side of (13a) vanishes, implying that density perturbations evolved adiabatically: they did not exchange heat with their environment. Moreover, baryons were tightly coupled to radiation through Thomson and Coulomb scattering, i.e., baryons obey $\delta_{n,\,\mathrm{baryon}} = \frac{3}{4}\delta_\varepsilon$. Thus, for baryons (13b) is identically satisfied. In contrast to baryons, CDM is *not* coupled to radiation through Thomson and Coulomb scattering. However, equation (13b) follows from the General Theory of Relativity, as has been shown in Section 2.7*, equation (44b*). As a consequence, equation (13b) should be obeyed by *all* kinds of particles that interact through gravitation. In other words, equation (13b) holds true for baryons as well as CDM. Since CDM interacts only via gravity with baryons and radiation, the fluctuations in CDM are coupled through gravitation to fluctuations in the energy density, so that fluctuations in CDM also satisfy equation (13b).

In order to solve equation (13a) it will first be rewritten in a form using dimensionless quantities. The solutions of the background equations (2) are given by

$$H \propto t^{-1}, \quad \varepsilon_{(0)} \propto t^{-2}, \quad n_{(0)} \propto t^{-3/2}, \quad a \propto t^{1/2}, \tag{14}$$

implying that $T_{(0)\gamma} \propto a^{-1}$. The dimensionless time τ is defined by $\tau := t/t_0$. Since $H := \dot{a}/a$, one finds that

$$\frac{\mathrm{d}^k}{c^k \mathrm{d}t^k} = \left[\frac{1}{ct_0}\right]^k \frac{\mathrm{d}^k}{\mathrm{d}\tau^k} = [2H(t_0)]^k \frac{\mathrm{d}^k}{\mathrm{d}\tau^k}, \quad k = 1, 2. \tag{15}$$

Substituting $\delta_\varepsilon(t, \boldsymbol{x}) = \delta_\varepsilon(t, \boldsymbol{q}) \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})$ into equation (13a) and using (15) yields

$$\delta_\varepsilon'' - \frac{1}{2\tau}\delta_\varepsilon' + \left[\frac{\mu_\mathrm{r}^2}{4\tau} + \frac{1}{2\tau^2}\right]\delta_\varepsilon = 0, \quad \tau \geq 1, \tag{16}$$

where a prime denotes differentiation with respect to τ. The parameter μ_{r} is given by

$$\mu_{\mathrm{r}} := \frac{2\pi}{\lambda_0}\frac{1}{H(t_0)}\frac{1}{\sqrt{3}}, \quad \lambda_0 := \lambda a(t_0), \tag{17}$$

with λ_0 the physical scale of a perturbation at time t_0 ($\tau = 1$), and $|\boldsymbol{q}| = 2\pi/\lambda$. To solve equation (16), replace τ by $x := \mu_{\mathrm{r}}\sqrt{\tau}$. After transforming back to τ, one finds

$$\delta_\varepsilon(\tau, \boldsymbol{q}) = \Big[A_1(\boldsymbol{q})\sin\big(\mu_{\mathrm{r}}\sqrt{\tau}\big) + A_2(\boldsymbol{q})\cos\big(\mu_{\mathrm{r}}\sqrt{\tau}\big)\Big]\sqrt{\tau}, \tag{18}$$

where the 'constants' of integration $A_1(\boldsymbol{q})$ and $A_2(\boldsymbol{q})$ are given by

$$A_{\frac{1}{2}}(\boldsymbol{q}) = \delta_\varepsilon(t_0, \boldsymbol{q})\frac{\sin\mu_{\mathrm{r}}}{\cos\mu_{\mathrm{r}}} \mp \frac{1}{\mu_{\mathrm{r}}}\frac{\cos\mu_{\mathrm{r}}}{\sin\mu_{\mathrm{r}}}\left[\delta_\varepsilon(t_0, \boldsymbol{q}) - \frac{\dot{\delta}_\varepsilon(t_0, \boldsymbol{q})}{H(t_0)}\right]. \tag{19}$$

For large-scale perturbations ($\lambda \to \infty$), it follows from (18) and (19) that

$$\delta_\varepsilon(t) = -\left[\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)}\right]\frac{t}{t_0} + \left[2\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)}\right]\left(\frac{t}{t_0}\right)^{\frac{1}{2}}. \tag{20}$$

The energy density contrast has two contributions to the growth rate, one proportional to t and one proportional to $t^{1/2}$. These two solutions have been found, with the exception of the precise factors of proportionality, by a large number of authors [6–11]. Consequently, the evolution equations (13) corroborates for large-scale perturbations the results of the literature.

Small-scale perturbations ($\lambda \to 0$) oscillate with an *increasing* amplitude according to

$$\delta_\varepsilon(t, \boldsymbol{q}) \approx \delta_\varepsilon(t_0, \boldsymbol{q})\left(\frac{t}{t_0}\right)^{\frac{1}{2}}\cos\left[\mu_{\mathrm{r}} - \mu_{\mathrm{r}}\left(\frac{t}{t_0}\right)^{\frac{1}{2}}\right], \tag{21}$$

as follows from (18) and (19). Thus, the evolution equations (13) yield oscillating density perturbations with an *increasing* amplitude, since in these equations $\vartheta_{(1)} \neq 0$, as follows from their derivation, see Section 2.7*. In contrast, the standard equation (61), which has $\vartheta_{(1)} = 0$, yields oscillating density perturbations with a *constant* amplitude. Note that $\vartheta_{(1)} \to 0$ is one of the requirements of the non-relativistic limit, see Section 4*.

Finally, the plasma era has begun at time t_{eq}, where the energy density of ordinary matter was equal to the energy density of radiation, (58), and ends at time t_{dec}, the time of decoupling of matter and radiation. In the plasma era the matter-radiation mixture can be characterized by the equations of state (Kodama and Sasaki [12], Chapter V)

$$\varepsilon(n, T) = nmc^2 + a_{\mathrm{B}}T_\gamma^4, \quad p(n, T) = \tfrac{1}{3}a_{\mathrm{B}}T_\gamma^4, \tag{22}$$

where the contributions to the pressure of ordinary matter and CDM have not been taken into account, since these contributions are negligible with respect to the radiation energy density. Eliminating T_γ from (22), one finds for the equation of state for the pressure

$$p(n, \varepsilon) = \tfrac{1}{3}(\varepsilon - nmc^2), \tag{23}$$

so that with (5) one gets

$$p_n = -\tfrac{1}{3}mc^2, \quad p_\varepsilon = \tfrac{1}{3}. \tag{24}$$

Since $p_n < 0$, equation (3b) implies that fluctuations in the particle number density, δ_n, were coupled to fluctuations in the total energy density, δ_ε, through gravitation, independent of the nature of the particles.

3.2 Era after Decoupling of Matter and Radiation

Once protons and electrons combined to yield hydrogen, the radiation pressure was negligible, and the equations of state have become those of a non-relativistic monatomic perfect gas with three degrees of freedom

$$\varepsilon(n,T) = nmc^2 + \tfrac{3}{2}nk_{\mathrm{B}}T, \tag{25a}$$

$$p(n,T) = nk_{\mathrm{B}}T, \tag{25b}$$

where k_{B} is Boltzmann's constant, m the mean particle mass, and T the temperature of the matter. For the calculations in this subsection it is only needed that the CDM particle mass is such that for the mean particle mass m one has $mc^2 \gg k_{\mathrm{B}}T$, so that $w := p_{(0)}/\varepsilon_{(0)} \ll 1$. Therefore, as follows from the background equations (2a) and (2b), one may neglect the pressure $nk_{\mathrm{B}}T$ and the kinetic energy density $\frac{3}{2}nk_{\mathrm{B}}T$ with respect to the rest mass energy density nmc^2 in the *un*perturbed universe. However, neglecting the pressure in the *perturbed* universe yields non-evolving density perturbations with a static gravitational field, as has been demonstrated in Section 4*. Consequently, it is important to take the pressure perturbations into account.

Eliminating T from (25) yields, see Section 2.1*, the equation of state for the pressure

$$p(n,\varepsilon) = \tfrac{2}{3}(\varepsilon - nmc^2), \tag{26}$$

so that with (5) one has

$$p_n = -\tfrac{2}{3}mc^2, \quad p_\varepsilon = \tfrac{2}{3}. \tag{27}$$

Substituting p_n, p_ε and (25a) into (6) on finds, using that $mc^2 \gg k_{\mathrm{B}}T$,

$$\beta \approx \frac{v_{\mathrm{s}}}{c} = \sqrt{\frac{5}{3}\frac{k_{\mathrm{B}}T_{(0)}}{mc^2}}, \tag{28}$$

with v_{s} the adiabatic speed of sound and $T_{(0)}$ the matter temperature. Using that $\beta^2 \approx \frac{5}{3}w$ and $w \ll 1$, expression (7) reduces to $\dot{w} \approx -2Hw$, so that with $H := \dot{a}/a$ one has $w \propto a^{-2}$. This implies that the matter temperature decays as

$$T_{(0)} \propto a^{-2}. \tag{29}$$

This, in turn, implies with (28) that $\dot{\beta}/\beta = -H$. The system (3) can now be rewritten as

$$\ddot{\delta}_\varepsilon + 3H\dot{\delta}_\varepsilon - \left[\beta^2\frac{\nabla^2}{a^2} + \tfrac{5}{6}\kappa\varepsilon_{(0)}\right]\delta_\varepsilon = -\frac{2}{3}\frac{\nabla^2}{a^2}\left(\delta_n - \delta_\varepsilon\right), \tag{30a}$$

$$\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_n - \delta_\varepsilon\right) = -2H\left(\delta_n - \delta_\varepsilon\right), \tag{30b}$$

where $w \ll 1$ and $\beta^2 \ll 1$ have been neglected with respect to constants of order unity. From equation (30b) it follows with $H := \dot{a}/a$ that

$$\delta_n - \delta_\varepsilon \propto a^{-2}. \tag{31}$$

Since the system (30) is derived from the General Theory of Relativity, it should be obeyed by all kinds of particles which interact through gravity, in particular baryons and CDM.

It will now be shown that the right-hand side of equation (30a) is proportional to the mean kinetic energy density fluctuation of the particles of a density perturbation. To that end, an expression for $\varepsilon^{\text{gi}}_{(1)}$ will be derived from (25a). Multiplying $\dot{\varepsilon}_{(0)}$ by $\theta_{(1)}/\dot{\theta}_{(0)}$ and subtracting the result from $\varepsilon_{(1)}$, one finds

$$\varepsilon^{\text{gi}}_{(1)} = n^{\text{gi}}_{(1)}mc^2 + \tfrac{3}{2}n^{\text{gi}}_{(1)}k_{\text{B}}T_{(0)} + \tfrac{3}{2}n_{(0)}k_{\text{B}}T^{\text{gi}}_{(1)}, \tag{32}$$

where also the definitions (40a*) and (52*) have been used. Dividing the result by $\varepsilon_{(0)}$ and using that $k_{\text{B}}T_{(0)} \ll mc^2$, one finds

$$\delta_\varepsilon \approx \delta_n + \frac{3}{2}\frac{k_{\text{B}}T_{(0)}}{mc^2}\delta_T, \tag{33}$$

to a very good approximation. In this expression δ_ε is the relative perturbation in the *total* energy density. Since $mc^2 \gg \frac{3}{2}k_{\text{B}}T_{(0)}$, it follows from the derivation of (33) that δ_n can be considered as the relative perturbation in the *rest* energy density. Consequently, the second term is the fluctuation in the *kinetic* energy density, i.e., $\delta_{\text{kin}} \approx \delta_\varepsilon - \delta_n$. The relative kinetic energy density perturbation occurs in the source term of the evolution equation (30a) and is of the same order of magnitude as the term with β^2. That is why the pressure may not be neglected in the perturbed universe: for $p = 0$, one has $\delta_\varepsilon = \delta_n$.

Combining (29) and (31) one finds from (33) that δ_T is constant

$$\delta_T(t, \boldsymbol{x}) \approx \delta_T(t_0, \boldsymbol{x}), \tag{34}$$

to a very good approximation, so that the kinetic energy density fluctuation is given by

$$\delta_{\text{kin}}(t, \boldsymbol{x}) \approx \delta_\varepsilon(t, \boldsymbol{x}) - \delta_n(t, \boldsymbol{x}) \approx \frac{3}{2}\frac{k_{\text{B}}T_{(0)}(t)}{mc^2}\delta_T(t_0, \boldsymbol{x}). \tag{35}$$

In Section 4 it will be shown that the kinetic energy density fluctuation has played, in addition to gravitation, a role in the evolution of density perturbations. In fact, if a density perturbation was somewhat cooler than its environment, i.e., $\delta_T < 0$, its growth rate was, depending on its scale, enhanced.

Using (27) and (33), one finds from (8)

$$\delta_p \approx \tfrac{5}{3}\delta_\varepsilon + \delta_T, \tag{36}$$

where δ_p is the relative pressure perturbation defined by $\delta_p := p^{\text{gi}}_{(1)}/p_{(0)}$, with $p_{(0)}$ given by (25b). The term $\frac{5}{3}\delta_\varepsilon$ is the adiabatic part and δ_T is the diabatic part of the relative pressure perturbation. The factor $\frac{5}{3}$ is the so-called adiabatic index for a monatomic ideal gas with three degrees of freedom. Thus, relative kinetic energy density perturbations give rise to diabatic pressure fluctuations.

Finally, the perturbed entropy per particle follows from (9) and (33), i.e.,

$$s^{\text{gi}}_{(1)} \approx \tfrac{3}{2}k_{\text{B}}\delta_T. \tag{37}$$

The background entropy per particle is independent of time, i.e., $\dot{s}_{(0)} = 0$. In a *linear* perturbation theory the perturbed entropy per particle is approximately constant, i.e., $\dot{s}^{\text{gi}}_{(1)} \approx 0$. Therefore, heat exchange of a perturbation with its environment decays proportional to the temperature, i.e., $T_{(0)}s^{\text{gi}}_{(1)} \propto a^{-2}$, as follows from (29).

In order to solve equation (30a) it will first be rewritten in a form using dimensionless quantities. The solutions of the background equations (2) are given by

$$H \propto t^{-1}, \quad \varepsilon_{(0)} \propto t^{-2}, \quad n_{(0)} \propto t^{-2}, \quad a \propto t^{2/3}, \tag{38}$$

where the kinetic energy density and pressure have been neglected with respect to the rest mass energy density. The dimensionless time τ is defined by $\tau := t/t_0$. Using that $H := \dot{a}/a$, one gets

$$\frac{\mathrm{d}^k}{c^k \mathrm{d}t^k} = \left[\frac{1}{ct_0}\right]^k \frac{\mathrm{d}^k}{\mathrm{d}\tau^k} = \left[\tfrac{3}{2}H(t_0)\right]^k \frac{\mathrm{d}^k}{\mathrm{d}\tau^k}, \quad k = 1, 2. \tag{39}$$

Substituting $\delta_\varepsilon(t,\boldsymbol{x}) = \delta_\varepsilon(t,\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})$, $\delta_n(t,\boldsymbol{x}) = \delta_n(t,\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})$, (28) and (35) into equations (30) and using (29) and (39) one finds that equations (30) can be combined into one equation

$$\delta_\varepsilon'' + \frac{2}{\tau}\delta_\varepsilon' + \left[\frac{4}{9}\frac{\mu_{\mathrm{m}}^2}{\tau^{8/3}} - \frac{10}{9\tau^2}\right]\delta_\varepsilon = -\frac{4}{15}\frac{\mu_{\mathrm{m}}^2}{\tau^{8/3}}\delta_T(t_0,\boldsymbol{q}), \tag{40}$$

where a prime denotes differentiation with respect to τ. The parameter μ_{m} is given by

$$\mu_{\mathrm{m}} := \frac{2\pi}{\lambda_0}\frac{1}{H(t_0)}\frac{v_{\mathrm{s}}(t_0)}{c}, \quad \lambda_0 := \lambda a(t_0), \tag{41}$$

with λ_0 the physical scale of a perturbation at time t_0 ($\tau = 1$), and $|\boldsymbol{q}| = 2\pi/\lambda$. To solve equation (40) replace τ by $x := 2\mu_{\mathrm{m}}\tau^{-1/3}$. After transforming back to τ, one finds for the general solution of the evolution equation (40)

$$\delta_\varepsilon(\tau,\boldsymbol{q}) = \Big[B_1(\boldsymbol{q})J_{+\frac{7}{2}}\big(2\mu_{\mathrm{m}}\tau^{-1/3}\big) + B_2(\boldsymbol{q})J_{-\frac{7}{2}}\big(2\mu_{\mathrm{m}}\tau^{-1/3}\big)\Big]\tau^{-1/2} - \frac{3}{5}\left[1 + \frac{5\tau^{2/3}}{2\mu_{\mathrm{m}}^2}\right]\delta_T(t_0,\boldsymbol{q}), \tag{42}$$

where $J_{\pm 7/2}(x)$ are Bessel functions of the first kind and $B_1(\boldsymbol{q})$ and $B_2(\boldsymbol{q})$ are the 'constants' of integration, calculated with the help of Maxima [13]:

$$\begin{aligned} B_{\frac{1}{2}}(\boldsymbol{q}) = {} & \frac{3\sqrt{\pi}}{20\mu_{\mathrm{m}}^{3/2}}\left[\left(4\mu_{\mathrm{m}}^2 - 5\right)\begin{matrix}\cos 2\mu_{\mathrm{m}} \\ \sin 2\mu_{\mathrm{m}}\end{matrix} \mp 10\mu_{\mathrm{m}}\begin{matrix}\sin 2\mu_{\mathrm{m}} \\ \cos 2\mu_{\mathrm{m}}\end{matrix}\right]\delta_T(t_0,\boldsymbol{q}) + {} \\ & \frac{\sqrt{\pi}}{8\mu_{\mathrm{m}}^{7/2}}\left[\left(8\mu_{\mathrm{m}}^4 - 30\mu_{\mathrm{m}}^2 + 15\right)\begin{matrix}\cos 2\mu_{\mathrm{m}} \\ \sin 2\mu_{\mathrm{m}}\end{matrix} \mp \left(20\mu_{\mathrm{m}}^3 - 30\mu_{\mathrm{m}}\right)\begin{matrix}\sin 2\mu_{\mathrm{m}} \\ \cos 2\mu_{\mathrm{m}}\end{matrix}\right]\delta_\varepsilon(t_0,\boldsymbol{q}) + {} \\ & \frac{\sqrt{\pi}}{8\mu_{\mathrm{m}}^{7/2}}\left[\left(24\mu_{\mathrm{m}}^2 - 15\right)\begin{matrix}\cos 2\mu_{\mathrm{m}} \\ \sin 2\mu_{\mathrm{m}}\end{matrix} \pm \left(8\mu_{\mathrm{m}}^3 - 30\mu_{\mathrm{m}}\right)\begin{matrix}\sin 2\mu_{\mathrm{m}} \\ \cos 2\mu_{\mathrm{m}}\end{matrix}\right]\frac{\dot{\delta}_\varepsilon(t_0,\boldsymbol{q})}{H(t_0)}. \end{aligned} \tag{43}$$

The particle number density contrast $\delta_n(t,\boldsymbol{q})$ follows from equation (33), (34) and (42). In (42) the first term (i.e., the solution of the homogeneous equation) is the adiabatic part of a density perturbation, whereas the second term (i.e., the particular solution) is the diabatic part.

In the large-scale limit $\lambda \to \infty$ terms with ∇^2 vanish. Therefore, the general solution of equation (40) becomes

$$\delta_\varepsilon(t) = \frac{1}{7}\left[5\delta_\varepsilon(t_0) + \frac{2\dot{\delta}_\varepsilon(t_0)}{H(t_0)}\right]\left(\frac{t}{t_0}\right)^{\frac{2}{3}} + \frac{2}{7}\left[\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)}\right]\left(\frac{t}{t_0}\right)^{-\frac{5}{3}}. \tag{44}$$

Thus, for large-scale perturbations the diabatic pressure fluctuation $\delta_T(t_0,\boldsymbol{q})$ did not play a role during the evolution: large-scale perturbations were adiabatic and evolved only under the influence of gravity. These perturbations were so large that heat exchange did not play a role during their evolution in the linear phase. For perturbations much larger than the Jeans scale (i.e., the peak value in Figure 1), gravity alone was insufficient to explain structure formation within 13.81 Gyr, since they grow as $\delta_\varepsilon \propto t^{2/3}$.

The solution proportional to $t^{2/3}$ is a standard result [6–11]. Since δ_ε is gauge-invariant, the standard non-physical gauge mode proportional to t^{-1} is absent from the solution set of the evolution equations (30). Instead, a physical mode proportional to $t^{-5/3}$ is found. This mode follows also from the standard perturbation equations if one does *not* neglect the divergence $\vartheta_{(1)}$, as is shown in the appendix. Consequently, only the growing mode of (44) is in agreement with results given in the literature.

In the small-scale limit $\lambda \to 0$, one finds from (42) and (43)

$$\delta_\varepsilon(t,\boldsymbol{q}) \approx -\tfrac{3}{5}\delta_T(t_0,\boldsymbol{q}) + \left(\frac{t}{t_0}\right)^{-\frac{1}{3}}\left[\delta_\varepsilon(t_0,\boldsymbol{q}) + \tfrac{3}{5}\delta_T(t_0,\boldsymbol{q})\right]\cos\left[2\mu_{\mathrm{m}} - 2\mu_{\mathrm{m}}\left(\frac{t}{t_0}\right)^{-\frac{1}{3}}\right], \tag{45a}$$

$$\delta_p(t,\boldsymbol{q}) \approx \left(\frac{t}{t_0}\right)^{-\frac{1}{3}}\left[\tfrac{5}{3}\delta_\varepsilon(t_0,\boldsymbol{q}) + \delta_T(t_0,\boldsymbol{q})\right]\cos\left[2\mu_{\mathrm{m}} - 2\mu_{\mathrm{m}}\left(\frac{t}{t_0}\right)^{-\frac{1}{3}}\right], \tag{45b}$$

where (36) has been used to calculate the fluctuation δ_p in the pressure. Thus, density perturbations with scales smaller than the Jeans scale oscillated with a decaying amplitude which was smaller than unity: these perturbations were so small that gravity was insufficient to let perturbations grow. Heat exchange alone was not enough for the growth of density perturbations. Consequently, perturbations with scales smaller than the Jeans scale did never reach the non-linear regime.

In the next section it is shown that for density perturbations with scales of the order of the Jeans scale, the action of both gravity and heat exchange together may result in massive structures several hundred million years after decoupling of matter and radiation.

4 Structure Formation after Decoupling

In this section it is demonstrated that the relativistic evolution equations, which include a realistic equation of state for the pressure $p = p(n,\varepsilon)$ yields that in the era after decoupling of matter and radiation density perturbations may have grown fast.

Up till now it is only assumed that $mc^2 \gg k_{\mathrm{B}}T$ for baryons and CDM, without specifying the mass of the baryon and CDM particles. From now on it is convenient to assume that the mass of a CDM particle is of the order of magnitude of the proton mass.

4.1 Introducing Observable Quantities

The parameter μ_{m} (41) will be expressed in observable quantities, namely the present values of the background radiation temperature, $T_{(0)\gamma}(t_{\mathrm{p}})$, the Hubble parameter, $\mathcal{H}(t_{\mathrm{p}}) = cH(t_{\mathrm{p}})$, and the redshift at decoupling, $z(t_{\mathrm{dec}})$. From now on the initial time is taken to be the time at decoupling of matter and radiation: $t_0 = t_{\mathrm{dec}}$, so that $\tau := t/t_{\mathrm{dec}}$.

The redshift $z(t)$ as a function of the scale factor $a(t)$ is given by

$$z(t) = \frac{a(t_\mathrm{p})}{a(t)} - 1, \tag{46}$$

where $a(t_\mathrm{p})$ is the present value of the scale factor and $z(t_\mathrm{p}) = 0$. For a flat FLRW universe one may take $a(t_\mathrm{p}) = 1$. Using the background solutions (38), one finds from (46)

$$H(t) = H(t_\mathrm{p})\big[z(t) + 1\big]^{3/2}, \tag{47a}$$

$$t = t_\mathrm{p}\big[z(t) + 1\big]^{-3/2}, \tag{47b}$$

$$T_{(0)\gamma}(t) = T_{(0)\gamma}(t_\mathrm{p})\big[z(t) + 1\big], \tag{47c}$$

where it is used that $T_{(0)\gamma} \propto a^{-1}$ after decoupling, as follows from (10) and (14).

The dimensionless time $\tau := t/t_\mathrm{dec}$ can be expressed in the redshift

$$\tau = \left[\frac{z(t_\mathrm{dec}) + 1)}{z(t) + 1}\right]^{3/2}, \tag{48}$$

by using that $\tau = (t/t_\mathrm{p})(t_\mathrm{p}/t_\mathrm{dec})$ and (47b).

Substituting (28) into (41), one gets

$$\mu_\mathrm{m} = \frac{2\pi}{\lambda_\mathrm{dec}}\frac{1}{H(t_\mathrm{dec})}\sqrt{\frac{5}{3}\frac{k_\mathrm{B}T_{(0)}(t_\mathrm{dec})}{mc^2}}, \quad \lambda_\mathrm{dec} := \lambda a(t_\mathrm{dec}), \tag{49}$$

where t_dec is the time when a perturbation starts to contract and λ_dec the physical scale of a perturbation at time t_dec. From (47) one finds

$$\mu_\mathrm{m} = \frac{2\pi}{\lambda_\mathrm{dec}}\frac{1}{\mathcal{H}(t_\mathrm{p})\big[z(t_\mathrm{dec}) + 1\big]}\sqrt{\frac{5}{3}\frac{k_\mathrm{B}T_{(0)\gamma}(t_\mathrm{p})}{m}}, \tag{50}$$

where it is used that $T_{(0)}(t_\mathrm{dec}) = T_{(0)\gamma}(t_\mathrm{dec})$. With (50) the parameter μ_m is expressed in observable quantities.

4.2 Initial Values from the Planck Satellite

The physical quantities measured by Planck [14] and needed in the parameter μ_m (50) of the evolution equation (40) are the redshift at decoupling, the present values of the Hubble function and the background radiation temperature, the age of the universe and the fluctuations in the background radiation temperature. The numerical values of these quantities are

$$z(t_\mathrm{dec}) = 1090, \tag{51a}$$

$$cH(t_\mathrm{p}) = \mathcal{H}(t_\mathrm{p}) = 67.31\,\mathrm{km/sec/Mpc} = 2.181 \times 10^{-18}\,\mathrm{sec}^{-1}, \tag{51b}$$

$$T_{(0)\gamma}(t_\mathrm{p}) = 2.725\,\mathrm{K}, \tag{51c}$$

$$t_\mathrm{p} = 13.81\,\mathrm{Gyr}, \tag{51d}$$

$$\delta_{T_\gamma}(t_\mathrm{dec}) \lesssim 10^{-5}. \tag{51e}$$

Substituting the observed values (51a)–(51c) into (50), one finds

$$\mu_\mathrm{m} = \frac{16.57}{\lambda_\mathrm{dec}}, \quad \lambda_\mathrm{dec} \text{ in pc}, \tag{52}$$

where it is used that the proton mass is $m = m_{\mathrm{H}} = 1.6726 \times 10^{-27}\,\mathrm{kg}$, $1\,\mathrm{pc} = 3.0857 \times 10^{16}\,\mathrm{m} = 3.2616\,\mathrm{ly}$, the speed of light $c = 2.9979 \times 10^{8}\,\mathrm{m/s}$ and Boltzmann's constant $k_{\mathrm{B}} = 1.3806 \times 10^{-23}\,\mathrm{J\,K^{-1}}$.

The Planck observations of the fluctuations $\delta_{T_\gamma}(t_{\mathrm{dec}})$, (51e), in the background radiation temperature yield for the initial value of the fluctuations in the energy density

$$|\delta_\varepsilon(t_{\mathrm{dec}}, \boldsymbol{q})| \lesssim 10^{-5}. \tag{53}$$

In addition, it is assumed that

$$\dot{\delta}_\varepsilon(t_{\mathrm{dec}}, \boldsymbol{q}) \approx 0, \tag{54}$$

i.e., during the transition from the radiation-dominated era to the era after decoupling, perturbations in the energy density were approximately constant with respect to time.

During the linear phase of the evolution, $\delta_n(t, \boldsymbol{q})$ follows from (33) so that the initial values $\delta_n(t_{\mathrm{dec}}, \boldsymbol{q})$ and $\dot{\delta}_n(t_{\mathrm{dec}}, \boldsymbol{q})$ need not be specified.

4.3 Diabatic Pressure Perturbations

At the moment of decoupling of matter and radiation, photons could not ionize matter any more and the two constituents fell out of thermal equilibrium. As a consequence, the high radiation pressure $p = \frac{1}{3}a_{\mathrm{B}}T_\gamma^4$ just before decoupling did go over into the low gas pressure $p = nk_{\mathrm{B}}T$ after decoupling. In fact, from (47c) and (59) it follows that at decoupling one has

$$\frac{n_{(0)}(t_{\mathrm{dec}})k_{\mathrm{B}}T_{(0)}(t_{\mathrm{dec}})}{\frac{1}{3}a_{\mathrm{B}}T^4_{(0)\gamma}(t_{\mathrm{dec}})} = \frac{3k_{\mathrm{B}}T_{(0)\gamma}(t_{\mathrm{p}})}{mc^2}\big[z(t_{\mathrm{eq}}) + 1\big] \approx 2.5 \times 10^{-9}, \tag{55}$$

where it is used that at the moment of decoupling the matter temperature was equal to the radiation temperature. Moreover, it is used that $k_{\mathrm{B}} = 1.3806 \times 10^{-23}\,\mathrm{J\,K^{-1}}$, and the redshift at matter-radiation equality $z(t_{\mathrm{eq}}) = 3393$, Planck [14]. The fast and chaotic transition from a high pressure epoch to a very low pressure era may have resulted in large relative *diabatic* pressure perturbations δ_T, (36), due to very small fluctuations δ_{kin}, (35), in the kinetic energy density. It will be shown in Section 4.4 that density perturbations which were cooler than their environments may have collapsed fast, depending on their scales. In fact, perturbations for which

$$\delta_T(t_{\mathrm{dec}}, \boldsymbol{q}) \lesssim -0.005, \tag{56}$$

may have resulted in primordial stars, the so-called (hypothetical) Population III stars, and larger structures, several hundred million years after the Big Bang.

4.4 Structure Formation in the Early Universe

In this section the evolution equation (40) is solved numerically [15, 16] and the results are summarized in Figure 1, which is constructed as follows. For each choice of $\delta_T(t_{\mathrm{dec}}, \boldsymbol{q})$ in the range -0.005, -0.01, -0.02, …, -0.1 equation (40) is integrated for a large number of values for the initial perturbation scale λ_{dec} using the initial values (53) and (54). The integration starts at $\tau = 1$, i.e., at $z(t_{\mathrm{dec}}) = 1090$ and will be halted if either $z = 0$, i.e., $\tau = [z(t_{\mathrm{dec}}) + 1]^{3/2}$, see (48), or $\delta_\varepsilon(t, \boldsymbol{q}) = 1$ for $z > 0$ has been reached. One integration run yields one point on the curve for a particular choice of the scale λ_{dec} if $\delta_\varepsilon(t, \boldsymbol{q}) = 1$ has been reached for $z > 0$. If

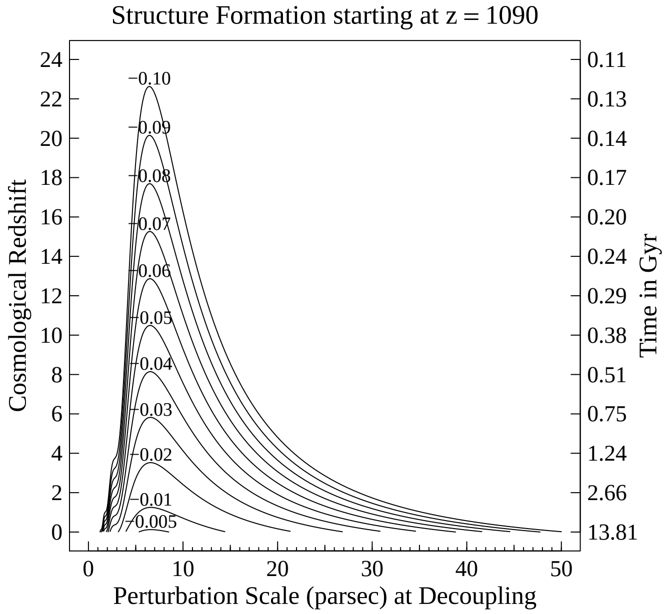

Figure 1: The curves give the redshift and time, as a function of $\lambda_{\rm dec}$, when a linear perturbation in the energy density with initial values $\delta_\varepsilon(t_{\rm dec},\boldsymbol{q}) \lesssim 10^{-5}$ and $\dot\delta_\varepsilon(t_{\rm dec},\boldsymbol{q}) \approx 0$ starting to grow at an initial redshift of $z(t_{\rm dec}) = 1090$ has become non-linear, i.e., $\delta_\varepsilon(t,\boldsymbol{q}) = 1$. The curves are labeled with the initial values of the relative perturbations $\delta_T(t_{\rm dec},\boldsymbol{q})$ in the diabatic part of the pressure. For each curve, the Jeans scale, i.e., the peak value, is at 6.5 pc.

the integration halts at $z = 0$ and still $\delta_\varepsilon(t_{\rm p},\boldsymbol{q}) < 1$, then the perturbation pertaining to that particular scale $\lambda_{\rm dec}$ has not yet reached its non-linear phase today, i.e., at $t_{\rm p} = 13.81\,$Gyr. On the other hand, if the integration is stopped at $\delta_\varepsilon(t,\boldsymbol{q}) = 1$ and $z > 0$, then the perturbation has become non-linear within 13.81 Gyr. Each curve denotes the time and scale for which $\delta_\varepsilon(t,\boldsymbol{q}) = 1$ for a particular value of $\delta_T(t_{\rm dec},\boldsymbol{q})$.

The growth of a perturbation was governed by both gravity as well as heat exchange. From Figure 1 one may infer that the optimal scale for growth was around $6.5\,{\rm pc} \approx 21\,{\rm ly}$. At this scale, which is independent of the initial value of the diabatic pressure perturbation $\delta_T(t_{\rm dec},\boldsymbol{q})$, see (8) and (36), heat exchange and gravity worked together perfectly, resulting in a fast growth. Perturbations with scales smaller than 6.5 pc reached their non-linear phase at a much later time, because their internal gravity was weaker than for large-scale perturbations and heat exchange was insufficient to enhance the growth. On the other hand, perturbations with scales larger than 6.5 pc exchanged heat at a slower rate due to their large scales, resulting also in a smaller growth rate. Perturbations larger than 50 pc grew proportional to $t^{2/3}$, (44), a well-known result. Since the growth rate decreased rapidly for perturbations with scales below 6.5 pc, this scale will be considered as the *relativistic* counterpart of the classical *Jeans scale.* The relativistic Jeans scale 6.5 pc was much smaller than the horizon size at decoupling, given by $d_{\rm H}(t_{\rm dec}) = 3ct_{\rm dec} \approx 3.5 \times 10^5\,{\rm pc} \approx 1.1 \times 10^6\,{\rm ly}$.

4.5 Relativistic Jeans Mass

The Jeans mass at decoupling, $M_{\mathrm{J}}(t_{\mathrm{dec}})$, can be estimated by assuming that a density perturbation has a spherical symmetry with diameter the relativistic Jeans scale $\lambda_{\mathrm{J,dec}} := \lambda_{\mathrm{J}} a(t_{\mathrm{dec}})$. The relativistic Jeans mass at decoupling is then given by

$$M_{\mathrm{J}}(t_{\mathrm{dec}}) = \frac{4\pi}{3}\left[\tfrac{1}{2}\lambda_{\mathrm{J,dec}}\right]^3 n_{(0)}(t_{\mathrm{dec}})m. \tag{57}$$

The particle number density $n_{(0)}(t_{\mathrm{dec}})$ can be calculated from its value $n_{(0)}(t_{\mathrm{eq}})$ at the end of the radiation-dominated era. By definition, at the end of the radiation-domination era the matter energy density $n_{(0)}mc^2$ was equal to the energy density of the radiation:

$$n_{(0)}(t_{\mathrm{eq}})mc^2 = a_{\mathrm{B}} T^4_{(0)\gamma}(t_{\mathrm{eq}}). \tag{58}$$

Since $n_{(0)} \propto a^{-3}$ and $T_{(0)\gamma} \propto a^{-1}$, one finds, using (46), (47c) and (58), for the particle number density at the time of decoupling t_{dec}

$$n_{(0)}(t_{\mathrm{dec}}) = \frac{a_{\mathrm{B}} T^4_{(0)\gamma}(t_{\mathrm{p}})}{mc^2}\left[z(t_{\mathrm{eq}})+1\right]\left[z(t_{\mathrm{dec}})+1\right]^3. \tag{59}$$

Using (51a), the black body constant $a_{\mathrm{B}} = 7.5657\times 10^{-16}\,\mathrm{J/m^3/K^4}$, the redshift at matter-radiation equality, $z(t_{\mathrm{eq}}) = 3393$, the redshift at decoupling (51a) Planck [14], and the speed of light $c = 2.9979\times 10^8\,\mathrm{m/s}$, one finds for the Jeans mass (57) at decoupling

$$M_{\mathrm{J}}(t_{\mathrm{dec}}) \approx 4.4\times 10^3\,\mathrm{M_\odot}, \tag{60}$$

where it is used that one solar mass $1\,\mathrm{M_\odot} = 1.9889\times 10^{30}\,\mathrm{kg}$ and the relativistic Jeans scale $\lambda_{\mathrm{J,dec}} = 6.5\,\mathrm{pc}$, the peak value in Figure 1.

A Standard Evolution Equation derived from the General Theory of Relativity

The standard evolution equation for relative density perturbations $\delta(t,\boldsymbol{x})$ in a flat, $R_{(0)} = 0$, FLRW universe with vanishing cosmological constant, $\Lambda = 0$, reads

$$\ddot{\delta} + 2H\dot{\delta} - \left[\beta^2\frac{\nabla^2}{a^2} + \tfrac{1}{2}\kappa\varepsilon_{(0)}(1+w)(1+3w)\right]\delta = 0. \tag{61}$$

In the radiation-dominated universe one has $\beta^2 = w = \frac{1}{3}$. In the epoch after decoupling of matter and radiation β^2 is given by (28), so that $w \approx \frac{3}{5}\beta^2 \ll 1$.

In this appendix it will be shown that the standard equation is inadequate to study the evolution of density perturbations in the universe. To that end an exact derivation of this equation will be presented, using the General Theory of Relativity, for a flat FLRW universe filled with a perfect fluid which is described by a barotropic equation of state $p = p(\varepsilon)$. This implies that $p_n = 0$, so that $\dot{p}_{(0)} = p_\varepsilon\dot{\varepsilon}_{(0)}$ and $p_{(1)} = p_\varepsilon\varepsilon_{(1)}$. Therefore, the evolution equations for the background particle number density $n_{(0)}$, (2c), and its first-order perturbation $n_{(1)}$, (41b*), need not be considered. From (6) one finds that $p_\varepsilon = \beta^2$ so that $p_{(1)} = \beta^2 \varepsilon_{(1)}$. Using that $\delta := \varepsilon_{(1)}/\varepsilon_{(0)}$ equations (41*) for scalar perturbations can be written in the form

$$\dot{\delta} + 3H\delta\Big[\beta^2 + \tfrac{1}{2}(1-w)\Big] + (1+w)\left[\vartheta_{(1)} + \frac{R_{(1)}}{4H}\right] = 0, \tag{62a}$$

$$\dot{\vartheta}_{(1)} + H(2-3\beta^2)\vartheta_{(1)} + \frac{\beta^2}{1+w}\frac{\nabla^2\delta}{a^2} = 0, \tag{62b}$$

$$\dot{R}_{(1)} + 2HR_{(1)} - 2\kappa\varepsilon_{(0)}(1+w)\vartheta_{(1)} = 0, \tag{62c}$$

where $\kappa\varepsilon_{(0)} = 3H^2$, (2a), has been used. Differentiating (62a) with respect to time and eliminating the time-derivatives of H, $\vartheta_{(1)}$ and $R_{(1)}$ with the help of the background equations (2) and perturbation equations (62b) and (62c), respectively, and, subsequently, eliminating $R_{(1)}$ with the help of (62a), one finds, using Maxima [13], that the set of equations (62) reduces to the system

$$\begin{aligned}\ddot{\delta} + 2H\dot{\delta}\Big[1 + 3\beta^2 - 3w\Big] - \Big[\beta^2\frac{\nabla^2}{a^2} + \tfrac{1}{2}\kappa\varepsilon_{(0)}\Big((1+w)(1+3w)\\ + 4w - 6w^2 + 12\beta^2 w - 4\beta^2 - 6\beta^4\Big) - 6\beta\dot{\beta}H\Big]\delta = -3H\beta^2(1+w)\vartheta_{(1)},\end{aligned} \tag{63a}$$

$$\dot{\vartheta}_{(1)} + H(2-3\beta^2)\vartheta_{(1)} + \frac{\beta^2}{1+w}\frac{\nabla^2\delta}{a^2} = 0, \tag{63b}$$

where $\dot{w}$ has been eliminated using (7). The system (63) consists of two *relativistic* equations for two unknown quantities, namely the density fluctuation δ and the divergence $\vartheta_{(1)}$ of the spatial part of the fluid four-velocity. Thus, the relativistic perturbation equations (41*) for open, flat or closed FLRW universes and a general equation of state for the pressure $p = p(n, \varepsilon)$ reduce for a flat universe and a barotropic equation of state $p = p(\varepsilon)$ to the relativistic system (63).

The gauge modes (39a*)

$$\hat{\delta}(t, \boldsymbol{x}) = \frac{\psi(\boldsymbol{x})\dot{\varepsilon}_{(0)}(t)}{\varepsilon_{(0)}(t)} = -3H(t)\psi(\boldsymbol{x})\big[1 + w(t)\big], \quad \hat{\vartheta}_{(1)}(t, \boldsymbol{x}) = -\frac{\nabla^2\psi(\boldsymbol{x})}{a^2(t)}, \tag{64}$$

are, for all scales, solutions of equations (63), with $\dot{w}$ given by (7).

The relativistic equations (63) are exact for first-order perturbations. This fact has consequences for the standard evolution equation (61), which will be discussed in detail in the next two subsections.

A.1 Radiation-dominated Era

In this era, the pressure is given by a linear barotropic equation of state $p = w\varepsilon$, so that $p_n = 0$ and $p_\varepsilon = w$. Since $p_\varepsilon = \beta^2$, (6), one finds from (7) that $\beta^2 = w$ is *constant.* In the case of a radiation-dominated universe this constant is $w = \beta^2 = \frac{1}{3}$. For a linear barotropic equation of state $p = w\varepsilon$ equations (63) reduce to

$$\ddot{\delta} + 2H\dot{\delta} - \left[w\frac{\nabla^2}{a^2} + \tfrac{1}{2}\kappa\varepsilon_{(0)}(1+w)(1+3w)\right]\delta = -3Hw(1+w)\vartheta_{(1)}, \tag{65a}$$

$$\dot{\vartheta}_{(1)} + H(2-3w)\vartheta_{(1)} + \frac{w}{1+w}\frac{\nabla^2\delta}{a^2} = 0. \tag{65b}$$

The gauge modes (64) are solutions of the system (65) for $\dot{w} = 0$.

For large-scale perturbations, $\nabla^2\delta \to 0$, δ does not influence the evolution of $\vartheta_{(1)}$. Using that $w = \frac{1}{3}$ the solutions (14) of the background equations imply that (65b) yields $\vartheta_{(1)} \propto t^{-1/2}$, so that with (14) one has $H\vartheta_{(1)} \propto t^{-3/2}$. Therefore, the particular solution of (65a) is $\delta \propto t^{1/2}$. The solutions of the homogeneous part of (65a) are $\delta \propto t$ and the gauge mode $\delta \propto t^{-1}$. This explains the physical modes $\delta \propto t^{1/2}$ and $\delta \propto t$ in (20). The standard equation (61) has only one physical mode $\delta \propto t$ as solution. The physical mode $\delta \propto t^{1/2}$ cannot be found since the standard equation has $\vartheta_{(1)} = 0$. As a consequence, for large-scale perturbations the outcome of the evolution equations (13) corroborates the outcome of the standard perturbation equation (61) with the exception of the physical mode $\delta \propto t^{1/2}$.

For small-scale perturbations, however, the case is entirely different. The standard equation (61) implies that $\vartheta_{(1)} = 0$, so that (65b) yields $\nabla^2\delta = 0$. Since $\nabla^2\delta$ can be large for small-scale perturbations, the standard equation (61) is inadequate to study small-scale density perturbations in the radiation-dominated era. Consequently, $\vartheta_{(1)}$ is important for the evolution of density perturbations.

The evolution equations (3) take $\vartheta_{(1)}$ into account, so that the system (13) yields oscillating density perturbations with an *increasing* amplitude, given by (18). In contrast, the standard equation (61) for which $\vartheta_{(1)} = 0$ yields oscillating perturbations with a *constant* amplitude.

The conclusion must be that the standard equation (61) cannot be used to study the evolution of density perturbations in the radiation-dominated era of the universe.

A.2 Era after Decoupling of Matter and Radiation

In this era one has $w \ll 1$, and $\beta^2 \ll 1$. Since β^2 is given by (28) it follows that $\dot{\beta}/\beta = -H$. Using that $3H^2 = \kappa\varepsilon_{(0)}$, (2a), one gets $6\beta\dot{\beta}H = -2\kappa\varepsilon_{(0)}\beta^2$. Neglecting w and β^2 with respect to constants of order unity, the system (63) reduces to

$$\ddot{\delta} + 2H\dot{\delta} - \left[\beta^2\frac{\nabla^2}{a^2} + \tfrac{1}{2}\kappa\varepsilon_{(0)}\right]\delta = -3H\beta^2\vartheta_{(1)}, \tag{66a}$$

$$\dot{\vartheta}_{(1)} + 2H\vartheta_{(1)} + \beta^2\frac{\nabla^2\delta}{a^2} = 0. \tag{66b}$$

The gauge modes (64) are solutions of the system (66) for $w \ll 1$ and $\nabla^2\psi = 0$. Consequently, for the system (66) ψ is an arbitrary infinitesimal constant C so that $\vartheta_{(1)}$ is a purely physical quantity, since its gauge mode (64) vanishes identically. However, δ is still gauge-dependent with gauge mode $\hat{\delta} = -3H(t)C \propto t^{-1}$, (38) and (64), implying that one cannot impose *physical* initial conditions $\delta(t_0, \boldsymbol{x})$ and $\dot{\delta}(t_0, \boldsymbol{x})$. These facts are in accordance with the residual gauge transformation (64*)

$$x^0 \mapsto x^0 - C, \quad x^i \mapsto x^i - \chi^i(\boldsymbol{x}), \tag{67}$$

in the non-relativistic limit, since a cosmological fluid for which $w \ll 1$ and $\beta^2 \ll 1$ can be described by a non-relativistic equation of state. Thus, the standard equation (61) yields for *all* scales gauge-dependent solutions.

Using the background solutions (38) one finds that for large-scale perturbations, $\nabla^2\delta \to 0$, equation (66b) yields $\vartheta_{(1)} \propto t^{-4/3}$, so that with $\beta \propto a^{-1}$ one finds that $H\beta^2\vartheta_{(1)} \propto t^{-11/3}$. Therefore, the particular solution of (66a) is $\delta \propto t^{-5/3}$. The solutions of the homogeneous part of equation (66a) are $\delta \propto t^{2/3}$ and the gauge mode $\delta \propto t^{-1}$. This explains the two physical modes in (44).

Just as in the radiation-dominated era, the standard equation (61) implies that $\vartheta_{(1)} = 0$ and $\nabla^2\delta = 0$. However, since $\beta^2 \ll 1$, the source term of (66a) is small, so that the influence of $\vartheta_{(1)}$ on the evolution of a density perturbation is, although non-zero, small. This explains the fact that both the standard equation (61) as well as the homogeneous part of equation (30a) yield oscillating solutions with a *decreasing* amplitude, as can be inferred from (42) with $\delta_T = 0$.

The main disadvantage of the standard equation (61) is that it is only adapted to a barotropic equation of state $p = p(\varepsilon)$. Therefore, the phenomenon of heat exchange of a density perturbation with its environment is not taken into account by equation (61). As a consequence, this equation does not explain structure formation in the early universe. In contrast to (61), the evolution equations (3) are adapted to the more realistic equation of state $p = p(n, \varepsilon)$, so that heat exchange is taken into account. As a consequence, the evolution equations (30) may explain the existence of the so-called (hypothetical) Population III stars and larger structures, as has been demonstrated in Section 4.

It has to be concluded that the standard equation (61) is inadequate to study the evolution of density perturbations in the universe in the era after decoupling of matter and radiation.

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```r
1   #                  Structure Formation in the Early Universe
2
3   #                                  P.G.Miedema
4
5   #                      Program to calculate Figure 1 in the main text
6   #                    The R file will be send to the reader upon request:
7   #                              pieter.miedema@gmail.com
8
9   #           The R Project for Statistical Computing:  http://www.r-project.org
10
11  ############################################################################################
12
13  library(deSolve)  # load package "deSolve" to use the solver "lsodar" at line 109
14
15  m <- 1.6726e-27  # proton mass in kg
16  c <- 2.9979e8    # speed of light in m/s
17  parsec <- 3.0857e16  # 1 parsec (pc) in m
18  k_B <- 1.3806e-23  # Boltzmann's constant in J/K
19  T_gamma <- 2.725 # present value of the background radiation temperature in K
20  H_p <- 67.31  #  present value of the Hubble parameter in km/s/Mpc
21  H_sec <- H_p * 1000 / (parsec * 1e6) # present value of the Hubble parameter in 1/s
22  H_m <- H_sec / c  # present value of the Hubble parameter in 1/m
23  H_parsec <- H_m * parsec # present value of the Hubble parameter in 1/pc
24  t_p <- 13.81 # years after Big Bang in Gyr
25  delta_e <- 1.0e-5  # (53)
26  dot.delta_e <- 0.0 # (54)
27  z_dec <- 1090 # redshift at decoupling
28  tau_dec <- 1.0 # value of dimensionless time tau at decoupling, start of integration
29  tau_p <- (z_dec+1)^(3/2) # dimensionless time tau at 13.81 Gyr, end of integration (48)
30  t_dec <- t_p / tau_p # time of decoupling in Gyr
31  factor <- 2*pi/(z_dec+1) / H_parsec * sqrt(5/3*k_B*T_gamma/(m*c^2)) # factor in (50) and (52)
32
33  ############################################################################################
34
35  equation.40 <- function (tau, y, parms)
36    {
37        ydot <- vector(len=2)
38        aux <- mu_m^2/tau^(8/3)
39        ydot[1] <- y[2]
40        ydot[2] <- (-2/tau)*y[2] - ((4/9) * aux - (10/9)/tau^2) * y[1] - (4/15) * aux * delta_T
41        return(list(ydot))
42    }
43
44  stop.conditions <- function (tau, y, parms)
45    {
46        stop <- vector(len=2)
47        stop[1] <- 1.0 - y[1]    # delta=1
48        stop[2] <- tau_p - tau   # z=0
49        return(stop)
50    }
51
52  ############################################################################################
53
54  #pdf(file="fig1.pdf", family="Times") # open a plotfile in pdf-format
55
56  par(mar=c(3,3,2,4), cex=1.2, cex.axis=1.2, pty="s")
57  plot.new()
58  plot.window(xlim=c(0, 50), ylim=c(0,24))
59  title(main=expression(paste("Structure Formation starting at ", z==1090)),
60        cex.main = 1.4, font.main=1, col.main="black", line=1.0)
61
62  pc <- seq(0,50,by=10)
63  axis(1, las=1, at=pc, tick=TRUE, label=pc, tcl=0.4, mgp=c(2, 0.3, 0))
64  tussen <- seq(5,45,by=10)
65  axis(1, las=1, at=tussen, tick=TRUE, label=FALSE, tcl=0.25, mgp=c(2, 0.3, 0))
66  eenheden <- seq(1,50,by=1)
67  axis(1, las=1, at=eenheden, tick=TRUE, label=FALSE, tcl=0.15, mgp=c(2, 0.3, 0))
68  mtext("Perturbation Scale (parsec) at Decoupling", cex=1.6, side=1, line=1.5)
69
70  zt <- seq(0, 24, by=2);
71  axis(2, at=zt, labels=TRUE, las=1, tcl=0.4, mgp=c(2, 0.3, 0))
72  mtext("Cosmological Redshift",cex=1.6, side=2, line=1.7)
73
74  axis(4, at=zt, labels=sprintf("%.2f", t_dec * ((z_dec+1)/(zt+1))^(3/2)),
75       las=2, tcl=0.4, mgp=c(2, 0.3, 0))  # (48)
76  mtext("Time in Gyr", cex=1.6, side=4, line=2.5)
77
78  box()
79
80  ############################################################################################
81
82  # perturbations with scales outside the interval [0.5, 60] parsec do not become
83  # non-linear within 13.81 Gyr:
84  scale_min <- 0.5;  scale_max <- 60; increment <- 0.01
85  # initially the increment should be small, since the line is steep:
86  range.lambda_dec <- 10^(seq(log10(scale_min), log10(scale_max), increment))
87  Jeans.scale <- vector()
88  for (k in 1:11)
89    {
90      if (k==1)  delta_T <- -0.005
91      if (k==2)  delta_T <- -0.01
92      if (k==3)  delta_T <- -0.02
93      if (k==4)  delta_T <- -0.03
94      if (k==5)  delta_T <- -0.04
95      if (k==6)  delta_T <- -0.05
96      if (k==7)  delta_T <- -0.06
97      if (k==8)  delta_T <- -0.07
98      if (k==9)  delta_T <- -0.08
99      if (k==10) delta_T <- -0.09
100     if (k==11) delta_T <- -0.10
101
102     z <- vector(); lambda.nonlin <- vector()
103     i <- 0
104     for (lambda_dec in range.lambda_dec)
105       {
106         mu_m <- factor/lambda_dec  # see (50) and (52)
107         y <- c(delta_e, dot.delta_e)  # initial values at tau_dec (start of integration)
108         tau.start.end <- c(tau_dec, 1.1*tau_p) # 10% overshoot at the end time
109         result <- lsodar(y, tau.start.end, fun=equation.40, rootfun=stop.conditions, parms)
110
111 ############################################################################################
112
113 # Only the end values, i.e., result[2,..], are needed:
114         tau.end <- result[2,1]; delta <- result[2,2]
115         if (round(delta, 6)==1.0)
116           {
117             i <- i+1
118             lambda.nonlin[i] <- lambda_dec
119             z[i] <- (z_dec+1) / tau.end^(2/3)-1.0  # (48)
120           }
121       }
122
123     z_max <- max(z)
124     lambda.nonlin_max <- lambda.nonlin[z==z_max]; Jeans.scale[k] <- lambda.nonlin_max
125
126     if (k==1)  text(lambda.nonlin_max, z_max, "-0.005", adj=c(0.5,-0.15))
127     if (k==2)  text(lambda.nonlin_max, z_max, "-0.01", adj=c(0.5,-0.15))
128     if (k==3)  text(lambda.nonlin_max, z_max, "-0.02", adj=c(0.5,-0.15))
129     if (k==4)  text(lambda.nonlin_max, z_max, "-0.03", adj=c(0.5,-0.15))
130     if (k==5)  text(lambda.nonlin_max, z_max, "-0.04", adj=c(0.5,-0.15))
131     if (k==6)  text(lambda.nonlin_max, z_max, "-0.05", adj=c(0.5,-0.15))
132     if (k==7)  text(lambda.nonlin_max, z_max, "-0.06", adj=c(0.5,-0.15))
133     if (k==8)  text(lambda.nonlin_max, z_max, "-0.07", adj=c(0.5,-0.15))
134     if (k==9)  text(lambda.nonlin_max, z_max, "-0.08", adj=c(0.5,-0.15))
135     if (k==10) text(lambda.nonlin_max, z_max, "-0.09", adj=c(0.5,-0.15))
136     if (k==11) text(lambda.nonlin_max, z_max, "-0.10", adj=c(0.5,-0.15))
137
138     points(z ~ lambda.nonlin, type="l")
139   }
140 #dev.off() # close the plotfile
141
142 # Calculation of the Jeans mass expressed in sun's mass:
143 z_eq <- 3393 # redshift at matter-radiation equality
144 a_B <- 7.5657e-16 # black-body constant in J/m^3/K^4
145 m_sun <- 1.9889e30 # sun's mass in kg
146 Js <- mean(Jeans.scale) # Jeans scale in pc
147 n_dec <- a_B*T_gamma^4/(m*c^2)*(z_eq+1)*(z_dec+1)^3 # (59)
148 M_J <- (4/3)*pi*((1/2)*Js*parsec)^3*n_dec*m / m_sun  # (57)
```