# Necessary and sufficient conditions for existence of Blow-up solutions for elliptic problems in Orlicz-Sobolev spaces

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**Abstract**

This paper is principally devoted to revisit the remarkable works of Keller and Osserman and generalize some previous results related to the those for the class of quasilinear elliptic problem

$$\begin{cases} \operatorname{div}(\phi(|\nabla u|)\nabla u) = a(x)f(u) & \text{in } \Omega, \\ u \geq 0 \ \text{ in } \Omega, \quad u = \infty \text{ on } \partial\Omega, \end{cases}$$

where either $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a smooth bounded domain or $\Omega = \mathbb{R}^N$. The function $\phi$ includes special cases appearing in mathematical models in nonlinear elasticity, plasticity, generalized Newtonian fluids, and in quantum physics. The proofs are based on comparison principle, variational methods and topological arguments on the Orlicz-Sobolev spaces.

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## 1 Introduction

In this paper, let us consider the problems

$$\begin{cases} \Delta_\phi u = a(x)f(u) \ \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \ u = \infty \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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and

$$\begin{cases} \Delta_\phi u = a(x)f(u) \ \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \ u(x) \stackrel{|x|\to\infty}{\longrightarrow} \infty, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a smooth bounded domain, $f$ is a continuous function that satisfies $f(0) = 0$, $f(s) > 0$ for $s > 0$, the assumptions on $a(x)$ will be fixed later on, and $\Delta_\phi u = div(\phi(|\nabla u|)\nabla u)$ is called the $\phi-$*Laplacian* operator, where the $C^1$-function $\phi : (0, +\infty) \to (0, +\infty)$ satisfies:

**$(\phi)_1$:** $(\phi(t)t)' > 0$ for all $t > 0$;

**$(\phi)_2$:** there exist $l, m > 1$ such that

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m \text{ for all } t > 0,$$

where $\Phi(t) = \int_0^{|t|} \phi(s)sds$, $t \in \mathbb{R}$;

**$(\phi)_3$:** there exist $l_1,\ m_1 > 0$ such that

$$l_1 \leq \frac{\Phi''(t)t}{\Phi'(t)} \leq m_1 \text{ for all } t > 0.$$

Under the $(\phi)_1$-$(\phi)_3$ hypotheses, we have an wide class of $\phi-$*Laplacian* operators, for instance:

(1) $\phi(t) = 2$, $t > 0$. So, $\Delta_\phi u = \Delta u$ is the *Laplacian* operator,

(2) $\phi(t) = p|t|^{p-2}$, $t > 0$ and $p > 1$. In this case, $\Delta_\phi u = \Delta_p u$ is called the $p-$*Laplacian* operator,

(3) $\phi(t) = p|t|^{p-2} + q|t|^{q-2}$, $t > 0$ and $1 < p < q$. The $\Delta_\phi u = \Delta_p u + \Delta_q u$ is called as $(p\&q)-$*Laplacian* operator and it appears in quantum physics [7],

(4) $\phi(t) = 2\gamma(1 + t^2)^{\gamma-1}$, $t > 0$ and $\gamma > 1$. With this $\phi$, the $\Delta_\phi$ operator models problems in nonlinear elasticity problems [12],

(5) $\phi(t) = \gamma\dfrac{(\sqrt{(1+t^2)} - 1)^{\gamma-1}}{\sqrt{1+t^2}}$, $t > 0$ and $\gamma \geq 1$. It appears in models of nonlinear elasticity. See, for instance, [9] for $\gamma = 1$ and [11] for $\gamma > 1$,

(6) $\phi(t) = \dfrac{pt^{p-2}(1+t)\ln(1+t) + t^{p-1}}{1+t}$, $t > 0$ and $(-1 + \sqrt{1+4N})/2 > 1$ appears in plasticity problems [12].

The problems (1.1) (blow-up on the boundary) and (1.2) also model problems that appear in the theory of automorphic functions, Riemann surfaces, population dynamics, subsonic motion of a gas, non-Newtonian fluids, non-Newtonian filtration as well as in the theory of the electric potential in a glowing hollow metal body.

Researches related to Problem (1.1) was initiated with the case $\phi(t) = 2$, $a = 1$, and $f(u) = exp(u)$ by Bieberbach [8] (if $N = 2$) and Rademacher [27] (if $N = 3$). Problems of this type arise in Riemannian geometry, namely if a Riemannian metric of the form $|ds|^2 = exp(2u(x))|dx|^2$ has constant Gaussian curvature $-c^2$, then $\Delta u = c^2 exp(2u)$. Lazer and McKenna [19] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniform external sphere condition and for exponential-type nonlinearities.

Still for $\phi(t) \equiv 2$, a remarkable development in the study of problem (1.1) is due to Keller [15] and Osserman [24] that in 1957 established necessary and sufficient conditions for existence of solutions for the problems

$$(I): \quad \begin{cases} \Delta u = f(u) \ \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \ u = \infty \text{ on } \partial\Omega, \end{cases} \qquad (II): \quad \begin{cases} \Delta u = f(u) \ \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \Omega, \ u(x) \overset{|x|\to\infty}{\longrightarrow} \infty, \end{cases}$$

where $f$ is a non-decreasing continuous function. Keller established that

$$\int_1^{+\infty} \frac{dt}{\sqrt{F(t)}} < +\infty, \text{ where } F(t) = \int_0^t f(s)ds, \ t > 0 \tag{1.3}$$

is a sufficient condition for the problem $(I)$ to have a solution and $(II)$ to have no solution. Besides this, Keller showed that $(II)$ has radially symmetric solutions if, and only if, $f$ does not satisfies (1.3). In this same year, Osserman proved this same result for sub solutions in $(II)$. After these works, (1.3) has become well-known as the $Keller - Osserman$ condition for $f$.

The attention of researchers has turned in considering non-autonomous potentials in $(I)$ and $(II)$, that is, particular operators of (1.1). One goal has been to enlarge the class of terms $a(x)$ that still assures existence or nonexistence of solutions for particular cases of (1.1). Another branch of attention of researchers has been in the direction to extend the class of operators in the problems $(I)$ or $(II)$. For instance, Mohammed in [23] and Radulescu at all in [1] have considered the problem (1.1) with $\phi(t) = p|t|^{p-2}$, $t > 0$ with $p > 1$ and $\Omega$ a bounded domain. Recently, Zhang in [30] considered the $p(x)-Laplacian$ operator in the problem (1.1). About $\Omega = \mathbb{R}^N$, the problem (1.1) was considered in [6] and references therein with $\phi(t) = p|t|^{p-2}$, $t > 0$ and $2 \leq p \leq N$.

Before doing an overview about these classes of problems, we set that a solution of (1.1) (or (1.2)) is a non-negative function $u \in C^1(\Omega)$ (or a positive function $u \in C^1(\mathbb{R}^N)$) such that $u = \infty$ on $\partial\Omega$, that is, $u(x) \to \infty$ as $d(x) = \inf\{\|x - y\| \ / \ y \in \partial\Omega\} \to 0$ (or $u \to \infty$ as $|x| \to \infty$) and

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla \psi dx + \int_\Omega a(x) f(u)\psi dx = 0,$$

holds for all $\psi \in C_0^\infty(\Omega)$ (or $\psi \in C_0^\infty(\mathbb{R}^N)$).

*Overview about* (1.1). For $\phi(t) \equiv 2$, the question of existence of solutions to (1.1) was investigated in [4, 3] with $0 < a \in C(\overline{\Omega})$ and $f$ a Keller-Osserman function. In [16], Lair showed that (1.3) is a necessary and sufficient condition for (1.1) to have a solution under the more general hypothesis on $a$ than previous paper, namely, $a$ satisfying

**(a):** $a : \overline{\Omega} \to [0, +\infty)$ is a $c_\Omega - positive$ continuous function, that is, if $a(x_0) = 0$ for some $x_0 \in \Omega$, then there exists an open set $O_{x_0} \subset \Omega$ such that $x_0 \in O_{x_0}$ and $a(x) > 0$ for all $x \in \partial O_{x_0}$.

Now for $\phi(t) = |t|^{p-2}$, $t > 0$, we believe that the issue of existence of solutions for (1.1) was first studied in [10]. There the term $a$ was considered equal 1. After this work, a number of important papers have been considering issues as existence, uniqueness and asymptotic behavior for different kinds of weight $a$ and nonlinearities $f$. See, for instance, [21], [23], [13], [1], and references therein.

Our objective related to the problem (1.1) is two-fold: first, we generalise previous results to the $\phi-$*Laplacian* operator whose appropriate setting is the Orlicz-Sobolev space. The lack of homogeneity of the operator becomes our estimatives very delicate. Another purpose of this work is to establish necessary and sufficient conditions on the term $f$ for existence of solutions for (1.1) for $a$ in a class of potentials given satisfying $(\mathbf{a})$. This result is new even for the context of $p$-Laplacian operator.

To do this, let us consider

$(\underline{f})$: $\liminf_{s\to+\infty}(\inf\{f(t) \ / \ t \geq s\}/f(s)) > 0,$ $\qquad$ $(\overline{f})$: $\limsup_{s\to+\infty}(\sup\{f(t) \ / \ 0 \leq t \leq s\}/f(s)) < \infty$

and note that these assumptions are necessary because we require no kind of monotonicity on $f$, that is, if for instance we assume that $f$ is non-decreasing, as is made in the most of the prior works, the $\underline{f}$ and $\overline{f}$ hypotheses immediately are true.

Besides this, let us consider

$$\mathbf{(F)}: \int_1^{+\infty} \frac{dt}{\Phi^{-1}(F(t))} < +\infty$$

and refer to it *as $f$ satisfying the $\phi$–Keller–Osserman condition* in reason from the $Keller-Osserman$ condition as it is known for the particular case $\phi(t) = 2$.

Ou first result is.

**Theorem 1.1** *Assume that $(\phi)_1 - (\phi)_3$ hold and $a$ satisfies $(\mathbf{a})$. Then:*

*$(i)$ if $f$ satisfies $(\underline{f})$ and $(\mathbf{F})$, then Problem (1.1) admits at least one solution,*

*$(ii)$ if $f$ satisfies $(\overline{f})$ and does not satisfy $(\mathbf{F})$, then Problem (1.1) have no solution.*

To highlight the last theorem and emphasize the importance of hypothesis $(F)$, we state.

**Corollary 1.1** *Assume that $(\phi)_1 - (\phi)_3$ hold, $a$ satisfies $(\mathbf{a})$, and $f$ is a non-increasing function. Then Hypothesis $(F)$ is a necessary and sufficient condition for the problem (1.1) has a solution.*

We note that the above Corollary requires that $f$ goes to infinity at infinity in a strong way when $a$ is bounded in $\Omega$. Yet, if instead of this we permit that $a(x)$ goes to infinity at boundary of $\Omega$ in a strong way, we still can have solutions to (1.1) for $f$ going to infinity at infinity in a slowly way. These was showed in [2] or in [22] for $\phi(t) \equiv 2$, $\Omega = B_1(0)$ and $a(x)$ a symmetric radially function and $f$ does not satisfying the Keller-Osserman condition. So, we can infer that $a(x)$ at boundary of $\Omega$ and $f$ at infinity must behave in a inverse way sense to assure existence of solution. The exactly way is an interesting open issue.

*Overview about* (1.2). In $\mathbb{R}^N$, the last researches have showed that the existence of solutions for (1.2) depends on how $a(x)$ goes to 0 at infinity and $f$ goes to infinity at infinity. In general, it is known that for either "fast velocities for both" or "slow velocities for both" are sufficient conditions for (1.2) has solutions. Interesting open issues are "how should behave $a(x)$ and $f$ at infinity in a inverse way to ensure existence of solutions yet?".

Besides this, the existence of solutions for (1.2) is sensible to the another measure related to $a(x)$, more exactly, "how radial is $a(x)$ at infinity?". To understand this and state our results, let us introduce

$$a_{osc}(r) = \overline{a}(r) - \underline{a}(r), \ r \geq 0,$$

where

$$\underline{a}(r) = \min\{a(x) \ / \ |x| = r\} \ \text{ and } \ \overline{a}(r) = \max\{a(x) \ / \ |x| = r\}, \ r \geq 0,$$

and note that $a_{osc}(r) = 0$, $r \geq r_0$ if, and only if, $a$ is symmetric radially in $|x| \geq r_0$, for some $r_0 \geq 0$.

For $\phi(t) \equiv 2$, Lair and Wood in [18] considered $a$ being a symmetric radially continuous function (that is, $a_{osc}(r) = 0$ for all $r \geq 0$), $f(u) = u^\gamma$, $u \geq 0$ with $0 < \gamma \leq 1$ (that is, $f$ does not satisfies ($\mathbf{F}$)) and showed that problem (1.1) has a solution if, and only if,

$$\int_1^\infty ra(r)dr = \infty. \tag{1.4}$$

Still in this context, in 2003, Lair [17] enlarged the class of potentials $a(x)$ by permitting $a_{osc}$ to assume not identically null values, but not too big ones. More exactly, he assumed

$$\int_0^\infty ra_{osc}(r)exp(\underline{A}(r))dr < \infty, \ \text{ where } \ \underline{A}(r) = \int_0^r s\underline{a}(s)ds, \ r \geq 0 \tag{1.5}$$

and proved that (1.2) with suitable $f$ that includes $u^\gamma$, $0 < \gamma \leq 1$, has a solution if, and only if, (1.4) holds with $\underline{a}$ in the place of $a$.

Keeping us in this context, Mabroux and Hansen in [20] improved the above results by consider

$$\int_0^\infty ra_{osc}(r)(1 + \underline{A}(r))^{\gamma/(1-\gamma)}dr < \infty$$

in the place of (1.5). In the line of the previous results, recently, Rhouma and Drissi [6] considered $\phi(t) = |t|^{p-2}, \ t \in \mathbb{R}$ with $2 \leq p \leq N$, $f$ a differentiable function that includes $u^\gamma$ with $0 < \gamma \leq 1$, and they established necessary and sufficient conditions for existence of solutions for (1.2) around the term $a$.

Our objective for the problem (1.2) is a little bit different of the above ones, because we are principally concerned in establishing necessary and sufficient conditions for existence of solutions of (1.2) around the function $f$, once fixed a potential $a(x)$ in a bigger class than above results. Besides this, we generalise the prior existence results to the context of $\phi - Laplacian$ operator.

To do this, first we note that $(\phi)_1$ and $(\phi)_2$ permit us to consider $h^{-1} : (0,\infty) \to (0,\infty)$ being the inverse of $h(t) = \phi(t)t, \ t > 0$. So, let us assume

$(\mathbf{A}_\rho)$: $\displaystyle\int_1^\infty h^{-1}(\mathcal{A}_\rho(s))dr = \infty, \text{ where } \mathcal{A}_\rho(s) = s^{1-N}\int_0^s t^{N-1}\rho(t)dt, \ s > 0$

for suitable continuous function $\rho : [0,\infty) \to [0,\infty)$ given, and $\mathcal{F} : (0,\infty) \to (0,\infty)$, defined by $\mathcal{F}(t) = \frac{t}{2}f(t)^{-1/l_1}$, be a non-increasing and bijective function such that

**($\mathcal{F}$):** $\mathcal{F}$ is invertible and

$$0 \leq \overline{H} := \int_0^\infty \eta_4(\mathcal{A}_{a_{osc}}(t))h^{-1}\Big(f\Big(\mathcal{F}^{-1}\Big(\int_0^s h^{-1}(\mathcal{A}_{\overline{a}}(t))dt\Big)\Big)\Big)ds < \infty.$$

After this, we state our existence result.

**Theorem 1.2** *Assume that $(\phi)_1$–$(\phi)_3$ hold. Suppose that $a(x)$ is a non-negative function satisfying $(\mathbf{A}_{\underline{\mathbf{a}}})$, $f$ is a non-decreasing function that does not satisfies $(\mathbf{F})$ and such that $(\mathcal{F})$ holds. If*

$$h^{-1}(s+t) \leq h^{-1}(s) + h^{-1}(t), \text{ for all } s,t \geq 0, \tag{1.6}$$

*then there exists a solution $u \in C^1(\mathbb{R}^N)$ of the problem (1.2) satisfying $\alpha \leq u(0) \leq (\alpha+\varepsilon) + \overline{H}$, for each $\alpha, \varepsilon > 0$ given.*

**Remark 1.1** *About the above hypotheses, we have:*

*i) the hypothesis (1.6) is satisfied for $\Delta_p$-laplacian operator with $2 \leq p < \infty$,*

*ii) the hypothesis $(\mathcal{F})$ is trivially satisfied for a radial functions $a(x)$ at infinity. In particular, if a symmetric radially, then $\overline{H} = 0$,*

*iii) as showed in [20] for the particular case $\phi(t) = 2$, we can not ensure that the problem (1.2) has solution, if we remove the hypothesis $(\mathcal{F})$,*

*iii) If $f(s) = s^\gamma$, $s > 0$ with $0 < \gamma < l_1$, then $\mathcal{F}(s) = s^{(l_1-\gamma)/l_1}/2$, $s > 0$.*

Our next goal is to establish necessary conditions, around on $f$, for existence of solutions for (1.2), once fixed a potential $a$ satisfying $(A_{\overline{a}})$. To do this, let us consider the problem

$$\begin{cases} \Delta_\phi u = \overline{a}(|x|)f(u) \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N, \ u(x) \stackrel{|x|\to\infty}{\longrightarrow} \infty, \end{cases} \tag{1.7}$$

and denote by $A = \sup \mathbb{A}$, where

$$\mathbb{A} = \{\alpha > 0 \ / \ (1.7) \text{ has a radial solution with } u(0) = \alpha\}.$$

So, we have.

**Theorem 1.3** *Assume that $(\phi)_1 - (\phi)_3$ hold, $a(x)$ is a non-negative continuous function (not necessarily radial) satisfying $(\mathbf{A}_{\overline{\mathbf{a}}})$, and $f$ is a non-decreasing function. If (1.2) admits a positive solution, then $A > 0$ and $(0, A) \subset \mathbb{A}$. Moreover, $A = \infty$ if, and only if, $f$ does not satisfies $(\mathbf{F})$.*

To become clearest our two last Theorems, let us restate them as below.

**Corollary 1.2** *Assume that $(\phi)_1 - (\phi)_3$ hold. Suppose that $a(x)$ is a non-negative symmetric radially function satisfying $(\mathbf{A}_{\mathbf{a}})$ and $f$ is a non-decreasing function. Then problem (1.2) admits a sequence of symmetric radial solutions $u_k(|x|) \in C^1(\mathbb{R}^N)$ with $u_k(0) \to \infty$ as $k \to \infty$ if, and only if, $f$ does not satisfies $(\mathbf{F})$. Besides this, $u_k' \geq 0$ in $[0,\infty)$.*

**Remark 1.2** *We emphasize that the equivalence stated in above Corollary is sharp in the sense that we can have solutions for (1.2) with $f$ satisfying $(\mathbf{F})$ and $a(x)$ as in $(\mathbf{A}_{\mathbf{a}})$. In particular, this shows that $A < \infty$ may occur as well. We quote Gladkov and Slepchenkov [14], where an example was build for the case $\phi(t) = 2$, $t > 0$.*

This paper is organized in the following way. In the section 2, we present the Orlicz and Orlicz-Sobolev spaces where we work variationally some approximated problems. In the section 3, we present some results of some auxiliary problems. In the section 4, we completed the proof of Theorems 1.1 and Corollary 1.1 and in section 5 we prove Theorem 1.2 and 1.3.

## 2 The Orlicz and Orlicz-Sobolev setting

In this section, we present an overview about Orlicz and Orlicz-Sobolev spaces, that will be the appropriate settings to deal approximated problems of (1.1) in variational way, and also we give some technical results which will be need later. For more details about Orlicz and Orlicz-Sobolev spaces, see for instance [28].

A function $M : \mathbb{R} \to [0, +\infty)$ is called an $N$-function if it is convex, even, $M(t) = 0$ if, and only if, $t = 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to +\infty$ as $t \to +\infty$. It is well known that an $N$-function $M$ can be rewritten as

$$M(t) = \int_0^{|t|} m(s)ds, \ t \in \mathbb{R}, \tag{1.8}$$

where $m : [0, \infty) \to [0, \infty)$ is a right derivative of $M$, non-decreasing, right continuous function, $m(0) = 0$, $m(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} m(s) = \infty$. Reciprocally, if $m$ satisfies the former properties, then $M$ defined in (1.8) is an $N$-function.

For an $N$-function $M$ and an open set $\Omega \subset \mathbb{R}^N$, the Orlicz class is the set of function defined by

$$K_M(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ / \ \ u \text{ is measurable and } \int_\Omega M(u(x))\, dx < \infty \right\}$$

and the vector space $L^M(\Omega)$ generated by $K_M(\Omega)$ is called Orlicz space. When $M$ satisfies the $\Delta_2$-condition, namely, there exists a constant $k > 0$ such that

$$M(2t) \leq kM(t), \ \text{for all } t \geq 0,$$

the Orlicz class $K_M(\Omega)$ is a vector space, and hence equal to $L^M(\Omega)$.

Defining the following norm (Luxemburg norm) on $L^M(\Omega)$ by

$$|u|_M = \inf \left\{ \lambda > 0 \ / \ \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\},$$

we have that the space $\left(L^M(\Omega), |\cdot|_M\right)$ is a Banach space. The complement function of $M$ is defined by

$$\tilde{M}(t) = \sup_{s>0} \{ts - M(s)\} \ \text{ (note that } \tilde{\tilde{M}} = M).$$

In the spaces $L^M(\Omega)$ and $L^{\tilde{M}}(\Omega)$ an extension of Hölder's inequality holds:

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leq 2|u|_M |v|_{\tilde{M}}, \ \text{for all } u \in L^M(\Omega) \text{ and } v \in L^{\tilde{M}}(\Omega).$$

As a consequence, to every $\tilde{u} \in L^{\tilde{M}}(\Omega)$ there exists a corresponds $f_{\tilde{u}} \in (L^{\tilde{M}}(\Omega))^*$ such that

$$f_{\tilde{u}}(v) = \int_\Omega \tilde{u}(x)v(x)\, dx, \ \ v \in L^M(\Omega).$$

Thus, we can define the Orlicz norm on the space $L^{\tilde{M}}(\Omega)$ by

$$\|\tilde{u}\|_{\tilde{M}} = \sup_{|v|_M \leq 1} \int_\Omega \tilde{u}(x)v(x)\, dx,$$

and, in a similar way, we can define the Orlicz norm $\|\cdot\|_M$ on $L^M(\Omega)$. The norms $|\cdot|_M$ and $\|\cdot\|_M$ are equivalent and satisfy

$$|u|_M \leq \|u\|_M \leq 2|u|_M.$$

It is important to detach that the $L^M(\Omega)$ is reflexive if and only if $M$ and $\tilde{M}$ satisfy the $\Delta_2$-condition and that

$$\left(L^M(\Omega), |\cdot|_M\right)^* = \left(L^{\tilde{M}}(\Omega), \|\cdot\|_{\tilde{M}}\right) \quad \text{and} \quad \left(L^{\tilde{M}}(\Omega), |\cdot|_{\tilde{M}}\right)^* = \left(L^M(\Omega), \|\cdot\|_M\right)$$

are true.

Now, setting

$$W^{1,M}(\Omega) = \Big\{u \in L^M \,/\, \exists\, v_i \in L^M(\Omega)\,; \int u\frac{\partial \varphi}{\partial x_i}dx = -\int v_i\varphi dx, \text{ for } i = 1, \cdots, N \text{ and } \forall\, \varphi \in C_0^\infty(\Omega)\Big\},$$

we have that $W^{1,M}(\Omega) = (W^{1,M}(\Omega), |\cdot|_{1,M})$ is a Banach space, where

$$|u|_{1,M} = |u|_M + |\nabla u|_M. \tag{1.9}$$

Besides this, it is well known that $W_0^{1,M}(\Omega)$, denoting the completion of $C_0^\infty(\Omega)$ in the norm (1.9), is a Banach space as well. It is reflexive if and only if $M$ and $\tilde{M}$ satisfy the $\Delta_2$-condition. Still in this case, we have

$$\int_\Omega M(u) \leq \int_\Omega M(2d|\nabla u|) \text{ and } |u|_M \leq 2d|\nabla u|_M \text{ for all } u \in W_0^{1,M}(\Omega),$$

where $0 < d < \infty$ is the diameter of $\Omega$. This inequality is well known as Poincaré's inequality. As a consequence of this, we have that $\|u\| := |\nabla u|_M$ is an equivalent norm to the norm $|u|_{1,M}$ on $W_0^{1,M}(\Omega)$. From now on, we consider $\|\cdot\|$ as the norm on $W_0^{1,M}(\Omega)$.

Hereafter, let us assume that $(\phi)_1$ and $(\phi)_2$ hold, that is, $\Phi$ is an $N$-function. We state three lemmas. Some items of two first ones are due to Fugakai et all [12, Lemma 2.1].

**Lemma 2.1** *Suppose $\phi$ satisfies $(\phi)_1$ and $(\phi)_2$. Considerer*

$$\xi_1(t) = \min\{t^l, t^m\},\ \xi_2(t) = \max\{t^l, t^m\},\ \eta_1(t) = \min\{t^{1/l}, t^{1/m}\},\ \eta_2(t) = \max\{t^{1/l}, t^{1/m}\},\ t \geq 0.$$

*Then,*

*(i)* $\xi_1(\rho)\Phi(t) \leq \Phi(\rho t) \leq \xi_2(\rho)\Phi(t), \quad$ *for* $\rho, t \geq 0,$

*(ii)* $\xi_1(|u|_\Phi) \leq \int_\Omega \Phi(|u|)\, dx \leq \xi_2(|u|_\Phi), \quad$ *for* $u \in L^\Phi(\Omega),$

*(iii)* $\eta_1(\rho)\Phi^{-1}(t) \leq \Phi^{-1}(\rho t) \leq \eta_2(\rho)\Phi^{-1}(t), \quad$ *for* $\rho, t \geq 0,$

*(iv)* $L^\Phi(\Omega)$, $W^{1,\Phi}(\Omega)$ *and* $W_0^{1,\Phi}(\Omega)$ *are reflexives and separable.*

Now, remembering that $h$ denotes $h(t) := \phi(t)t$, $t > 0$, we can show that.

**Lemma 2.2** *Suppose $\phi$ satisfies $(\phi)_1 - (\phi)_3$. If*

$$\xi_3(t) = \min\{t^{l_1}, t^{m_1}\},\ \xi_4(t) = \max\{t^{l_1}, t^{m_1}\},\quad \eta_3(t) = \min\{t^{1/l_1}, t^{1/m_1}\}$$
$$\textit{and}\ \ \eta_4(t) = \max\{t^{1/l_1}, t^{1/m_1}\},\ t \geq 0,$$

*then*

(*i*) $\xi_3(\rho)h(t) \leq h(\rho t) \leq \xi_4(\rho)h(t),\ \textit{for } \rho, t \geq 0,$

(*ii*) $\eta_3(\rho)h^{-1}(t) \leq h^{-1}(\rho t) \leq \eta_4(\rho)h^{-1}(t),\ \textit{for } \rho, t \geq 0.$

The next Lemma is very important in our approach to control solutions of approximated problems.

**Lemma 2.3** *Suppose $\phi$ satisfies $(\phi)_1$, $(\phi)_2$ and $B = B(x,t) \in L^\infty_{loc}(\Omega \times \mathbb{R})$ is non-decreasing in $t \in \mathbb{R}$. Considerer $u, v \in C^1(\Omega)$ satisfying*

$$\begin{cases} \Delta_\phi u \geq B(x,u) & \textit{in } \Omega, \\ \Delta_\phi v \leq B(x,v) & \textit{in } \Omega. \end{cases}$$

*If $u \leq v$ on $\partial\Omega$ (that is, for each $\delta > 0$ given there exists a neighborhood of $\partial\Omega$ in what $u < v + \delta$), then $u \leq v$ in $\Omega$.*

*Proof* Since $(\phi)_1$ and $(\phi)_2$ hold, we able to show that

$$\langle \phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v, \nabla u - \nabla v\rangle > 0 \text{ for all } u, v \in C^1(\Omega) \text{ with } u \neq v \tag{1.10}$$

holds. So, by Theorem 2.4.1 in [26], it follows the claim.

■

## 3 Auxiliary results

Our principal strategy to show existence of solutions for (1.1) and (1.2) is to solve boundary value problems with finite data, to control these solutions and then to get the solution for (1.1) or (1.2) by a limit process. To do this, first we prove a variational sub and super solution theorem for the problem

$$\begin{cases} \Delta_\phi u = g(x,u) \text{ in } \Omega, \\ u = k \text{ on } \partial\Omega, \end{cases} \tag{1.11}$$

where $k \geq 0$ is an appropriate real number and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a $C(\bar{\Omega} \times \mathbb{R})$ function.

We define a sub solution of (1.11) as being a function $\underline{u} \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ such that $\underline{u} \leq k$ on $\partial\Omega$ and

$$\int_\Omega \phi(|\nabla \underline{u}|)\nabla \underline{u} \nabla \psi dx + \int_\Omega g(x, \underline{u})\psi dx \leq 0$$

holds for all $\psi \in W^{1,\Phi}_0(\Omega)$ with $\psi \geq 0$ in $\Omega$. A super solution is a function $\overline{u} \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ satisfying the converse above inequalities. So, a solution is a function $u \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ that is simultaneously a sub and a super solution for (1.11).

**Proposition 3.1** *Suppose that $(\phi)_1 - (\phi)_3$ hold and $\underline{u}, \bar{u} \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ with $\underline{u} \leq \bar{u}$ are sub e super solutions of* (1.11)*, respectively. Then there exists a $u \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ with $\underline{u} \leq u \leq \bar{u}$ solution of* (1.11)*. Besides this, if $g \in L^\infty(\Omega \times \mathbb{R})$, then $u \in C^{1,\alpha}(\bar{\Omega})$, for some $0 < \alpha < 1$.*

*Proof* Defining

$$\mathfrak{h}(x,t)=\begin{cases} g(x,\underline{u}(x)), & \text{if } t\le \underline{u}(x)-k,\\ g(x,t+k), & \text{if } \underline{u}(x)-k\le t\le \bar{u}(x)-k,\\ g(x,\bar{u}(x)), & \text{if } t\ge \bar{u}(x)-k,\end{cases}$$

we have that $\mathfrak{h}\in C(\bar{\Omega}\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$. Since, $(\phi)_1$ and $(\phi)_2$ hold, we have that

$$I(u)=\int_\Omega \Phi(|\nabla u|)dx+\int_\Omega H(x,u)dx,\ \ u\in W_0^{1,\Phi}(\Omega)$$

is well-defined and $I\in C^1(W_0^{1,\Phi}(\Omega),\mathbb{R})$, where

$$H(x,t)=\int_0^t \mathfrak{h}(x,s)ds,\ \ t\in\mathbb{R}.$$

Besides this, it follows from Poincaré's inequality and Lemma 2.1 that $I$ is coercive, because

$$\begin{aligned} I(u) &\ge \int_\Omega \Phi(|\nabla u|)dx - C\int_\Omega |u|dx \ge \xi_1(\|u\|)-C|u|_{L^1}\\ &\ge \xi_1(\|u\|)-\bar{C}\|u\|\to\infty \text{ as } \|u\|\to\infty,\end{aligned}$$

where $C,\bar{C}>0$ are real constants. Finally, as $\mathfrak{h}\in L^\infty(\Omega\times\mathbb{R})$, we have that $I$ is weak s.c.i. Since $W_0^{1,\Phi}(\Omega)$ is a reflexive space, there exists a

$$u_0\in W_0^{1,\Phi}(\Omega)\ \text{ such that }\ I'(u_0)=0 \text{ and } I(u_0)=\min_{v\in W_0^{1,\Phi}(\Omega)} I(v),$$

that is, $u_0$ is a weak solution of

$$\begin{cases} \Delta_\phi v=\mathfrak{h}(x,v) \text{ in } \Omega,\\ v=0 \text{ on } \partial\Omega.\end{cases} \tag{1.12}$$

Now, we show that $\underline{u}-k\le u_0\le \bar{u}-k$ a.e. in $\Omega$. In fact, by taking $(u_0-(\bar{u}-k))^+\in W_0^{1,\Phi}(\Omega)$ as a test function and using that $(\bar{u}-k)$ is a super solution of (1.12), we obtain

$$\begin{aligned}\int_\Omega \phi(|\nabla u_0|)\nabla u_0\nabla\left(u_0-(\bar{u}-k)\right)^+dx &= -\int_\Omega \mathfrak{h}(x,u_0)\left(u_0-(\bar{u}-k)\right)^+dx\\ &= -\int_\Omega g(x,\bar{u})\left(u_0-(\bar{u}-k)\right)^+dx\\ &\le \int_\Omega \phi(|\nabla\bar{u}|)\nabla\bar{u}\nabla\left(u_0-(\bar{u}-k)\right)^+dx,\end{aligned}$$

that is,

$$\int_{\{u_0>\bar{u}-k\}}\langle \phi(|\nabla u_0|)\nabla u_0-\phi(|\nabla\bar{u}|)\nabla\bar{u},\nabla(u_0-\bar{u})\rangle dx\le 0.$$

So, by using (1.10) we get the claim. In analogue way, we obtain $\underline{u}-k\le u_0$ in $\Omega$. Setting $u=u_0+k$, we have that it is a solution of (1.11) with $\underline{u}\le u\le\bar{u}$ a.e. in $\Omega$. Finally, the regularity follows from the Lemma 3.3 in [12]. This ends the proof.

■

Now, setting the reflexive Banach space

$$W^{1,\Phi}_{rad}(B_R(0)) = \{u \in W^{1,\Phi}(B_R(0)) \ / \ u \text{ is symmetric radially}\},$$

where $B_R(x_0)$ stands for the ball in $\mathbb{R}^N$ centred at $x_0$ with radius $R > 0$. So, we have the below result.

**Corollary 3.1** *Suppose that* $(\phi)_1 - (\phi)_3$ *hold and* $\underline{u}, \bar{u} \in W^{1,\Phi}_{rad}(B_R(0)) \cap L^\infty(B_R(0))$*, with* $\underline{u} \le \bar{u}$*, are sub e super solutions for* (1.11)*, respectively. Then there exists a* $u \in W^{1,\Phi}_{rad}(B_R(0)) \cap L^\infty(B_R(0))$ *with* $\underline{u} \le u \le \bar{u}$ *solution of* (1.11).

*Proof* Following the same arguments as in the above proof, we obtain an $u_0 \in W^{1,\Phi}_{0,rad}(B_R(0))$, with $\underline{u} - k \le u_0 \le \bar{u} - k$, and

$$I(u_0) = \min\{I(v) \ / \ v \in W^{1,\Phi}_{0,rad}(B_R(0))\},$$

where

$$W^{1,\Phi}_{0,rad}(B_R(0)) = \{u \in W^{1,\Phi}_0(B_R(0)) \ / \ u \text{ is symmetric radially}\}.$$

So, by the principle of symmetric criticality [25], we obtain that

$$I'(u_0) = 0 \text{ and } I(u_0) = \min_{u \in W^{1,\Phi}_0(B_R(0))} I(u).$$

In analogue way, we show that $u = u_0 + k$ is a symmetric radially solution for (1.11), with $\underline{u} \le u \le \bar{u}$. ■

Now, let us emphasize the solution of problem

$$\begin{cases} \Delta_\phi v = cg(v) \text{ in } B_L(0), \\ v \ge 0 \text{ in } B_L(0), \quad v = k \text{ on } \partial B_L(0), \end{cases} \tag{1.13}$$

where $c, k, L > 0$ are real constants given.

**Corollary 3.2** *Suppose that* $(\phi)_1$-$(\phi)_3$ *hold. If* $g$ *is a non-decreasing and continuous function on* $[0, \infty)$ *such that* $g(t) > 0$ *for* $t > 0$*, then there exists a* $v(|x|) = v_{k,L}(|x|) \in W^{1,\Phi}_{rad}(B_L(0)) \cap C^{1,\alpha}(\bar{B}_L(0))$*, for some* $0 < \alpha < 1$*, satisfying:*

*(i)* $0 \le v(0) \le v(|x|) \le k$*,* $v'(0) = 0$*, and* $v' \ge 0$ *on* $[0, L]$*,*

*(ii)* $v_{k,L} \le v_{k+1,L}$ *and* $v_{k,L} \ge v_{k,L+1}$ *on* $[0, L]$*,*

*(iii) the inequalities*

$$\eta_2^{-1}\left(\frac{c}{l_1}\right)\int_{v(0)}^{r} \frac{d\tau}{\Phi^{-1}(G(v(0),\tau))} \le r \le \eta_1^{-1}\left(\frac{c}{m_1 N}\right)\int_{v(0)}^{r} \frac{d\tau}{\Phi^{-1}(G(v(0),\tau))} \tag{1.14}$$

*hold for all* $0 \le r \le L$*, where* $\eta_1$ *and* $\eta_2$ *were defined at Lemma* 2.1 *and*

$$G(x, y) := \int_x^y g(t)dt \text{ for } x, y \in \mathbb{R} \text{ with } 0 \le x < y.$$

*Proof* By applying the Proposition 3.1 and Corollary 3.1 with $\underline{u} = 0$ and $\overline{u} = k$ as sub solution and super solution, respectively, we obtain a $v(|x|) = v_{k,L}(|x|) \in W^{1,\Phi}_{rad}(B_L(0)) \cap C^{1,\alpha}(\bar{B}_L(0))$ solution of (1.13). Besides this, by using Lemma 2.3, we prove $(ii)$. To show $(i)$ and $(iii)$, let us set

$$v_{r,\epsilon}(t) = \begin{cases} 1 & \text{if } 0 \le t \le r, \\ \text{linear} & \text{if } r \le t \le r+\epsilon, \\ 0 & \text{if } r+\epsilon \le t \le L, \end{cases}$$

for each $\varepsilon > 0$ given such that $0 \le r < r+\epsilon < L$.

Now, by taking $\psi(x) = v_{r,\epsilon}(|x|)$ as a test function, we have

$$-\int_{B_L(0)} \phi(|\nabla v|)\nabla v \nabla \psi dx = c\int_{B_L(0)} g(v(x))\psi dx,$$

holds. So,

$$\frac{1}{\epsilon}\int_{A_{r,r+\epsilon}} \phi(|v'|)v' dx = c\int_{B_r(0)} g(v)dx + c\int_{A_{r,r+\epsilon}} g(v)v_{r,\epsilon}(|x|)dx,$$

where $A_{r,r+\epsilon} = \{x \in \mathbb{R}^N, r \le |x| \le r+\epsilon\}$. That is,

$$\frac{1}{\epsilon}\int_r^{r+\epsilon} t^{N-1}\phi(|v'|)v' dt = c\int_0^r t^{N-1}g(v)dt + c\int_r^{r+\epsilon} t^{N-1}g(v)v_{r,\epsilon}dt$$

and by taking $\epsilon \to 0$, we obtain

$$r^{N-1}\phi(|v'|)v' = c\int_0^r t^{N-1}g(v)dt \ge 0, \; 0 < r < L, \tag{1.15}$$

because $g$ is non-negative, that is, $v' \ge 0$ on $[0, L]$.

Besides this, it follows from $(\phi)_1$ and $(\phi)_2$ that $h(s) = \phi(s)s$, $s > 0$, is a $C^1$-increasing homeomorphism on $[0, \infty)$ such that $h(0) = 0$. As $g$ is continuous, it follows from (1.15) that

$$v' \in C^1([0, R)) \text{ and } v'(r) = h^{-1}\Big(cr^{1-N}\int_0^r t^{N-1}g(v)dt\Big), \; 0 < r < L, \tag{1.16}$$

and in particular, we have $v'(0) = 0$. This ends the proof of $(i)$.

To prove $(iii)$, first we note that (1.15) implies that

$$(\phi(v')v')' + \frac{N-1}{r}\phi(v')v' = cg(v)$$

is holds true for all $0 < r < L$. As a consequence of these, we have that

$$[\Phi'(v'(r))]' \le cg(v(r)), \text{ for all } 0 < r < L.$$

Besides this, by using (1.15), $g$ , and $v$ increasing, we obtain

$$\begin{aligned} [\Phi'(v'(r))]' &= cg(v) - \frac{N-1}{r}\phi(v')v' \\ &\ge cg(v) - \frac{N-1}{r}\Big(\frac{cr}{N}g(v)\Big) \\ &= \frac{c}{N}g(v(r)), \text{ for all } 0 < r < L, \end{aligned}$$

that is,

$$\frac{1}{N}cg(v(r)) \leq [\Phi'(v'(r))]' \leq cg(v(r)),\ 0<r<L.$$

Now, by using of $v'(0)=0$ and $(\phi)_3$, we have

$$\begin{aligned}\frac{c}{N}\int_{v(0)}^{v(r)} g(t)dt &\leq \int_0^r [\Phi'(v'(t))]'v'(t)dt = \int_0^r \Phi''(v'(t))v'(t)v''(t)dt \\ &\leq m_1\int_0^r [\Phi(v'(t))]'dt = m_1\Phi(v'(r)),\ 0<r<L\end{aligned}$$

and, in similar way,

$$c\int_{v(0)}^{v(r)} g(t)dt \geq l_1\Phi(v'(r)),\ 0<r<L,$$

that is,

$$\frac{c}{m_1 N}\int_{v(0)}^{v(r)} g(t)dt \leq \Phi(v'(r)) \leq \frac{c}{l_1}\int_{v(0)}^{v(r)} g(t)dt,\ 0<r<L. \tag{1.17}$$

Now by using the definition of $G(x,y)$, we can rewrite (1.17) as

$$\Phi^{-1}\Big(\frac{c}{m_1 N}G(v(0),v(r))\Big) \leq v'(r) \leq \Phi^{-1}\Big(\frac{c}{l_1}G(v(0),v(r))\Big),\ 0<r\leq L,$$

to obtain

$$\int_{v(0)}^{v(r)} \frac{d\tau}{\Phi^{-1}(\frac{c}{l_1}G(v(0),\tau))} \leq r \leq \int_{v(0)}^{v(r)} \frac{d\tau}{\Phi^{-1}(\frac{c}{m_1 N}G(v(0),\tau))},\quad 0<r\leq L.$$

So, by using the Lemma 2.1, we obtain (1.14). This completes the proof of the Corollary. ■

## 4 On bounded domain

Before proving Theorems 1.1, we need of the next result to help us to control a sequence of solutions of an approximate problem.

**Lemma 4.1** *Assume that $(\phi)_1-(\phi)_3$ hold. If $\Omega = B_R(0)$, $a(x)$ is a positive symmetric radially function and $f$ satisfies $(\underline{f})$ and $(\mathbf{F})$, then Problem (1.1) admits at least one symmetric radially solution $u(|x|) \in C^1(\overline{B_R(0)})$ such that $u'(0)=0$ and $u' \geq 0$ for $0<r<R$.*

*Proof* By applying Corollary 3.1, with $\underline{u}=0$ and $\overline{u}=k-1$, we obtain an $u_k \in W^{1,\Phi}_{rad}(B_R(0)) \cap C^{1,\alpha}(\bar{B}_R(0))$, for some $\alpha \in (0,1)$, satisfying

$$\begin{cases} \Delta_\phi u_k = a(|x|)f(u_k) \text{ in } B_R(0), \\ u_k \geq 0 \text{ in } B_R(0),\quad u_k = k-1 \text{ on } \partial B_R(0), \end{cases} \tag{1.18}$$

for each $k \in \mathbb{N}$ with $k \geq 2$ given. Besides this, it follows from Lemma 2.3, that

$$0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq \cdots \text{ in } \overline{B}_R(0), \tag{1.19}$$

and a consequence of this, it there exists

$$0 \leq u(x) := \lim_{k\to\infty} u_k(x) \leq \infty, \text{ for each } x \in B_R(0).$$

In the following, let us show that $0 \leq u(x) < \infty$, for each $x \in B_R(0)$, and it is a solution of (1.1). To do this, given an $x_0 \in B_R(0)$, let us denote by $a_\infty = \min\{a(t) \ / \ 0 \leq t = |x - x_0| \leq L\} > 0$ and apply Corollary 3.2 with $L = R - |x_0| > 0$, to obtain a $v_k(|x - x_0|) \in W^{1,\Phi}_{rad}(B_L(0)) \cap C^{1,\alpha}(\bar{B}_L(0))$ that satisfies

$$\begin{cases} \Delta_\phi v_k = a_\infty f(v_k) \text{ in } B_L(0), \\ v_k \geq 0 \text{ in } B_L(0), \quad v_k = k \text{ on } \partial B_L(0), \end{cases} \tag{1.20}$$

for each $k \in \mathbb{N}$ given. As another consequence of Corollary 3.2 and Lemma 2.3, we have

$$0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq v_k \leq v_{k+1} \leq \cdots \text{ in } \overline{B}_R(0), \tag{1.21}$$

that is, there exists

$$0 \leq v_{x_0,L} := \lim_{k\to\infty} v_k(x_0) \leq \infty.$$

We claim that $v_{x_0,L} < \infty$. In fact, assume that there exists a $k_0$ such that $v_{k_0}(x_0) > 0$ (on the contrary, we have nothing to do). Let us assume $x_0 = 0$ to simplify our reasoning. So, given $M > 1$ and denoting by $F(x, y) = F(y) - F(x)$ for $x, y \in \mathbb{R}$, with $F$ defined at (1.3), it follows from Lemma 2.1 and $\Phi^{-1}$ non-decreasing, that

$$\begin{aligned} \int_{v_k(0)}^{\infty} \frac{d\tau}{\Phi^{-1}(F(v_k(0),\tau))} &\leq \int_0^\infty \frac{d\tau}{\Phi^{-1}(F(v_k(0), v_k(0)+\tau))} \\ &\leq \int_0^1 \frac{d\tau}{\Phi^{-1}(F(v_k(0), v_k(0)+\tau))} + \int_1^M \frac{d\tau}{\Phi^{-1}(F(v_k(0), v_k(0)+\tau))} \\ &+ \int_M^\infty \frac{d\tau}{\Phi^{-1}(F(v_k(0), v_k(0)+\tau))} \\ &\leq \frac{1}{\Phi^{-1}(f(v_k(0)))} \int_0^1 \frac{d\tau}{\tau^{1/l}} + \frac{1}{\Phi^{-1}(f(v_k(0)))} \int_1^M \frac{d\tau}{\tau^{1/m}} \\ &+ \int_M^\infty \frac{d\tau}{\Phi^{-1}(F(0,\tau))} \\ &= \frac{1}{\Phi^{-1}(f(v_k(0)))} \Big[\frac{l}{l-1} - \frac{m}{m-1} + \frac{m}{m-1} M^{\frac{m-1}{m}}\Big] \\ &+ \int_M^\infty \frac{d\tau}{\Phi^{-1}(F(\tau))}, \end{aligned} \tag{1.22}$$

where we used in the fourth inequality that

$$F(v_k(0), v_k(0)+\tau) \geq F(0,\tau) = F(\tau) \text{ and } F(v_k(0), v_k(0)+\tau) \geq f(v_k(0))\tau \text{ for all } \tau \geq 0,$$

because $f$ is non-decreasing.

Now, supposing by contradiction that $v_{x_0,L} = \infty$, it follows from (1.14) and (1.22), that

$$\begin{aligned} 0 < L &\leq \eta_1^{-1}\left(\frac{a_\infty}{m_1 N}\right) \lim_{k\to\infty} \int_{v_k(0)}^{k} \frac{d\tau}{\Phi^{-1}(F(v_k(0),\tau))} \\ &\leq \eta_1^{-1}\left(\frac{a_\infty}{m_1 N}\right) \lim_{k\to\infty} \int_{v_k(0)}^{\infty} \frac{d\tau}{\Phi^{-1}(F(v_k(0),\tau))} \leq \eta_1^{-1}\left(\frac{a_\infty}{m_1 N}\right) \int_M^\infty \frac{d\tau}{\Phi^{-1}(F(\tau))}. \end{aligned}$$

for all $M > 1$ given. So, doing $M \to \infty$, we obtain a contradiction by using the hypothesis $(\mathbf{F})$.

So, we showed that $u(x) = \lim_{k\to\infty} u_k(x)$ for each $x \in B_R(0)$ is well defined and by using standard arguments we are able to prove that $u \in C^1(B_R(0))$ is a symmetric radially solution for (1.1). Besides this, by using similar arguments as those used to show that $v' \geq 0$ in Corollary 3.2, we can show that $u' \geq 0$ as well. These complete the proof of Theorem 4.1. ■

Now, we prove the Theorem 1.1, by using the Lemma 4.1.

*Proof of (i)*: By applying the Proposition 3.1, we get a $u_k \in C^{1,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$ that satisfies $u_k \leq u_{k+1}$ and

$$\begin{cases} \Delta_\phi u_k = a(x) f(u_k) \text{ in } \Omega, \\ u_k \geq 0 \text{ in } \Omega, \ u_k = k \text{ on } \partial\Omega, \end{cases} \tag{1.23}$$

for all $k \in \mathbb{N}$ given. Now, given a $x_0 \in \Omega$, let us consider two cases.

First, assume $a(x_0) > 0$. Then, there exists a neighborhood $V \subset \Omega$ of $x_0$ such that $a(x) > 0$ for all $x \in \bar{V}$, that is, $\mathfrak{m} = \min_{\bar{V}} a > 0$. Besides this, it follows from the hypothesis $(\underline{f})$ that the non-increasing function

$$\underline{f}(t) = \inf\{f(s),\ s \geq t\},\ t \geq 0 \tag{1.24}$$

is well defined and fulfils: $0 \leq \underline{f} \leq f$, $\underline{f}(s) = 0$ if, and only if $s = 0$, satisfies the *ϕ-Keller-Osserman* condition. In fact, let $\tau = \liminf_{t\to+\infty}\{\underline{f}(t)/f(t)\}$. It follows from $(\underline{f})$, there exists a $M > 0$ such that $\underline{f}(t) \geq \tau f(t)$ for all $t > M$. So,

$$\underline{F}(t) = \int_0^t \underline{f}(s)ds = \int_0^M \underline{f}(s)ds + \int_M^t \underline{f}(s)ds \geq \frac{\tau}{2}F(t),\ t \geq M,$$

that is, by Lemma 2.1, we have

$$\Phi^{-1}(\underline{F}(t)) \geq \Phi^{-1}(\frac{\tau}{2}F(t)) \geq \eta_1(\tau/2)\Phi^{-1}(F(t)),\ t \geq M.$$

These show our claim.

Now, let us consider the problem

$$\begin{cases} \Delta_\phi v = \mathfrak{m}\underline{f}(v) \text{ in } V, \\ v \geq 0 \text{ in } V,\ v = \infty \text{ on } \partial V, \end{cases} \tag{1.25}$$

where $\underline{f}$ is defined at (1.24). So, it follows from Lemmas 4.1 and 2.3 that, there exists a $v \in C^1(\Omega)$ solution of (1.25) that satisfies $0 \leq u_k \leq u_{k+1} \leq v$ in $V$.

Now, if $x_0 \in \Omega$ is such that $a(x_0) = 0$, then it follows from $c_\Omega-$*positive* hypothesis under $a$ that there exists a neighborhood $V \subset \Omega$ of $x_0$ such that $a(x) > 0$ for all $x \in \partial V$. By the compactness of $\partial V$, there exist open sets $V_i,\ i = 1, \ldots, n$ such that

$$\partial V \subset \cup_{i=1}^n V_i \text{ and } a(x) > 0, \text{ for all } x \in V_i.$$

So by above arguments, we obtain that $u_k \leq v_i$ in $V_i$, where $v_i \in C^1(V_i)$ is a solution of

$$\begin{cases} \Delta_\phi v = \mathfrak{m}_i\underline{f}(v) \text{ in } V_i, \\ v \geq 0 \text{ in } V_i,\ v = \infty \text{ on } \partial V_i, \end{cases} \tag{1.26}$$

and $\mathfrak{m}_i = \min_{\bar{V}_i} a > 0$. Hence, there exists a real constant $C = C_V > 0$ such that $0 \leq u_k \leq C$ in $\partial V$. Again, by the Lemma 2.3, we obtain $u_k \leq C$ in $V$. That is, in both cases, $u_k$ is bounded locally on $\Omega$. So, by standard arguments, we can show that $u_k \to u \in C^1(\Omega)$ and $u$ is a solution of (1.1).

*Proof of (ii)*: Denoting by

$$\mathfrak{M} = \max_{\overline{\Omega}} a > 0 \text{ and } \overline{f}(t) = \sup\{f(s),\ 0 \leq s \leq t\},\ t \geq 0,$$

it follows from the hypotheses $(a)$ and $(\overline{f})$ that: $M > 0$, $\overline{f}(0) = 0$, $\overline{f}(t) > 0$ for $t > 0$, $\overline{f}$ is non-decreasing function, and $\overline{f}$ does not satisfy $(\mathbf{F})$.

Now, assume by contradiction, that problem (1.1) admits a solution $u \in C^1(\Omega)$. So, it follows from above informations that $u$ satisfies

$$\begin{cases} \Delta_\phi u = a(x) f(u) \leq \mathfrak{M}\overline{f}(u) \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega,\ u = \infty \text{ on } \partial\Omega. \end{cases} \tag{1.27}$$

On the other hand, it follows from Proposition 3.1, that there exists a $u_k \in C^1(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta_\phi u_k = \mathfrak{M}\overline{f}(u_k) \text{ in } \Omega, \\ u_k \geq 0 \text{ in } \Omega,\ u_k = k \text{ on } \partial\Omega, \end{cases} \tag{1.28}$$

and

$$0 \leq u_1 \leq u_2 \leq \ldots \leq u_k \leq \ldots \leq u,$$

as a consequence of Lemma 2.3. So, there exists an $\omega \in C^1(\Omega)$ such that $u_k \to \omega$ in $C^1(\Omega)$, $0 \leq \omega \leq u$, and

$$\begin{cases} \Delta_\phi \omega = \mathfrak{M}\overline{f}(\omega) \text{ in } \Omega, \\ \omega \geq 0 \text{ in } \Omega,\ \omega = \infty \text{ on } \partial\Omega. \end{cases} \tag{1.29}$$

Finally, considering $R > 0$ such that $\overline{\Omega} \subset\subset B_R(0)$, it follows from Corollary 3.1 and (1.14) that there exists a $w_k \in C^1(\overline{B}_R(0))$ with $w_k' \geq 0$ satisfying

$$\begin{cases} \Delta_\phi w_k = \mathfrak{M}\overline{f}(w_k) \text{ in } B_R(0), \\ w_k \geq 0 \text{ in } B_R(0),\ w_k = k \text{ on } \partial B_R(0), \end{cases} \tag{1.30}$$

and

$$\eta_2^{-1}\left(\frac{\mathfrak{M}}{l_1}\right)\int_{w_k(0)}^{k} \frac{d\tau}{\Phi^{-1}(\overline{F}(\tau))} \leq \eta_2^{-1}\left(\frac{\mathfrak{M}}{l_1}\right)\int_{w_k(0)}^{k} \frac{d\tau}{\Phi^{-1}(\overline{F}(\tau) - \overline{F}(w_k(0)))} \leq R,$$

for each $k > 0$ given, where $\overline{F}(s) = \int_0^s \overline{f}(t)dt,\ s \geq 0$.

Since, $\overline{f}$ does not satisfies $(\mathbf{F})$, there exists a $k_0 > 0$ such that $w_{k_0}(0) > \min\{\omega(x)\ /\ x \in \Omega\}$. This is impossible, because $w_{k_0} \leq \omega$ in $\Omega$, by using the Lemma 2.3. This completes the proof of Theorem 1.1. ■

# 5 On whole $\mathbb{R}^N$

We begin this section with the below Lemma which is a radial version of Theorem 1.2. We emphasize that in it we do not require the hypotheses $(\mathcal{F})$ and (1.6). So, let us consider

$$\begin{cases} \Delta_\phi u = \rho(|x|) f(u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N,\ u(x) \overset{|x|\to\infty}{\longrightarrow} \infty, \end{cases} \tag{1.31}$$

and state our below lemma.

**Lemma 5.1** *Assume that $(\phi)_1 - (\phi)_3$ hold, $\rho$ is a non-negative continuous function satisfying $(\mathbf{A}_\rho)$, and $f$ is a non-decreasing function such that $(\mathbf{F})$ is not satisfied. Then $\mathbb{A}_\rho = (0, \infty)$, where $\mathbb{A}_\rho = \{\alpha > 0 \; / \; (1.31)$ has a radial solution with $u(0) = \alpha\}$.*

*Proof* Given $\alpha > 0$, consider the problem

$$\begin{cases} (r^{N-1}\phi(|u'|u'))' = r^{N-1}\rho(r)f(u(r)), \quad r > 0 \\ u'(0) = 0, \quad u(0) = \alpha. \end{cases} \tag{1.32}$$

Since $\rho$ and $f$ are continuous, we can follow the arguments in [29], to conclude that there exist a $\Gamma(\alpha) > 0$ (maximal extreme to the right for the existence interval of solutions for (1.32) and a $u_\alpha \in C^2(0, \Gamma(\alpha)) \cap C^1([0, \Gamma(\alpha)))$ solution of (1.32) on $(0, \Gamma(\alpha))$. If we had $\Gamma(\alpha) < \infty$ for some $\alpha > 0$, then we would have, by ordinary differential equations theory, that $u_\alpha(r) \to \infty$ as $r \to \Gamma(\alpha)^-$.

So, $u_\alpha(|x|)$ would be a symmetric radially solution of the problem

$$\begin{cases} \Delta_\phi u = \rho(|x|)f(u) \text{ in } B_{\Gamma(\alpha)}(0), \\ u \geq 0 \text{ in } B_{\Gamma(\alpha)}(0), \; u = \infty \text{ on } \partial B_{\Gamma(\alpha)}(0), \end{cases}$$

but this is impossible by Theorem 1.1, because $f$ does not satisfy $(\mathbf{F})$.

Besides this, it follows from $f$ non-decreasing, Lemma 2.2 and $(\mathbf{A}_\rho)$, that

$$\begin{aligned} u_\alpha(r) &\geq \alpha + \int_0^r h^{-1}\Big(f(\alpha)s^{1-N}\int_0^s t^{N-1}\rho(t)dt\Big)ds \\ &\geq \alpha + \eta_3(f(\alpha))\int_0^r h^{-1}\Big(s^{1-N}\int_0^s t^{N-1}\rho(t)dt\Big)ds \to \infty, \text{ as } r \to \infty, \end{aligned}$$

that is, $u_\alpha(|x|)$, $x \in \mathbb{R}^N$ radial solution of

$$\begin{cases} \Delta_\phi u = \rho(|x|)f(u) \text{ in } \mathbb{R}^N, \\ u \geq \alpha \text{ in } \mathbb{R}^N, \; u(x) \overset{|x|\to\infty}{\longrightarrow} \infty, \end{cases}$$

with $u_\alpha(0) = \alpha$. That is, $\alpha \in \mathbb{A}_\rho$. This ends our proof. ■

*Proof of Theorem* 1.2. Given $\beta > \alpha > 0$, it follows from the Lemma 5.1 that there exist positive radial solutions $u_\alpha$ and $u_\beta$ of the problems

$$\begin{cases} \Delta_\phi u = \overline{a}(|x|)f(u) \; \mathbb{R}^N, \\ u_\alpha(0) = \alpha, \; u(x) \overset{|x|\to\infty}{\longrightarrow} \infty, \end{cases} \quad \text{and} \quad \begin{cases} \Delta_\phi u = \underline{a}(|x|)f(u) \; \mathbb{R}^N, \\ u_\beta(0) = \beta, \; u(x) \overset{|x|\to\infty}{\longrightarrow} \infty, \end{cases}$$

respectively.

Besides this, it follows from $u_\alpha$, $f$ non-decreasing, Lemma 2.2, and $(\mathbf{A}_{\underline{\mathbf{a}}})$ that

$$\begin{aligned} u_\alpha(r) &\leq \alpha + \int_0^r h^{-1}(\mathcal{A}_{\overline{a}}(t))f(u_\alpha(t))dt \leq 2\eta_4(f(u_\alpha(r)))\int_0^r h^{-1}(\mathcal{A}_{\overline{a}}(t))dt \\ &\leq 2(f(u_\alpha(r)))^{1/l_1}\int_0^r h^{-1}(\mathcal{A}_{\overline{a}}(t))dt, \end{aligned}$$

for all $r > 0$ sufficiently large. That is,

$$u_\alpha(r) \leq \mathcal{F}^{-1}\Big(\int_0^r h^{-1}(\mathcal{A}_{\overline{a}}(t))dt\Big), \text{ for all } r >> 0. \tag{1.33}$$

Now, setting

$$0 < S(\beta) = sup\{r > 0 \ / \ u_\alpha(r) < u_\beta(r)\} \leq \infty,$$

for $\alpha > 0$ given, we claim that $S(\beta) = \infty$ for all $\beta > \alpha + \overline{H}$. In fact, by assuming this is not true, then there exists a $\beta_0 > \alpha + \overline{H}$ such that $u_\alpha(S(\beta_0)) = u_\beta(S(\beta_0))$. So, by using that $f$ is non-increasing and $u_\alpha \leq u_\beta$ on $[0, S(\beta_0)]$, we obtain that

$$\beta_0 \leq \alpha + \int_0^{S(\beta_0)} \Big[h^{-1}\Big(s^{1-N}\int_0^s t^{N-1}\overline{a}(t)f(u_\alpha(t))dt\Big) - h^{-1}\Big(s^{1-N}\int_0^s t^{N-1}\underline{a}(t)f(u_\alpha(t))dt\Big)\Big]ds \quad (1.34)$$

holds.

On the other hands, it follows from $f$, $u_\alpha$ non-decreasing, (1.6), and Lemma 2.2, that

$$\begin{aligned}
0 \ &\leq \ \Big[h^{-1}\Big(s^{1-N}\textstyle\int_0^s t^{N-1}\overline{a}(t)f(u_\alpha(t))dt\Big) - h^{-1}\Big(s^{1-N}\int_0^s t^{N-1}\underline{a}(t)f(u_\alpha(t))dt\Big)\Big]\chi_{[0,S(\beta)]}(s) \\
&= \ \Big[h^{-1}\Big(s^{1-N}\textstyle\int_0^s t^{N-1}[\overline{a}(t) - \underline{a}(t)]f(u_\alpha(t))dt + h^{-1}\Big(s^{1-N}\int_0^s t^{N-1}\underline{a}(t)f(u_\alpha(t))dt\Big)\Big) \\
&- \ h^{-1}\Big(s^{1-N}\textstyle\int_0^s t^{N-1}\underline{a}(t)f(u_\alpha(t))dt\Big)\Big]\chi_{[0,S(\beta)]}(s) \leq h^{-1}\Big(s^{1-N}\int_0^s t^{N-1}[\overline{a}(t) - \underline{a}(t)]f(u_\alpha(t))dt\Big) \\
&\leq \ h^{-1}(\mathcal{A}_{a_{osc}}(s)f(u_\alpha(s)) \leq \eta_4(\mathcal{A}_{a_{osc}}(s))h^{-1}(f(u_\alpha(s))) \\
&\leq \ \eta_4(\mathcal{A}_{a_{osc}}(s))h^{-1}\Big(f\Big(\mathcal{F}^{-1}\Big(\textstyle\int_0^r h^{-1}(\mathcal{A}_{\overline{a}}(t))dt\Big)\Big)\Big) := \mathcal{H}(s), \ \ s \geq 0,
\end{aligned}$$

where $\chi_{[0,S(\beta)]}$ stands for the characteristic function of $[0, S(\beta)]$.

So, it follows from the hypothesis $(\mathcal{F})$ and (1.34), that

$$\beta_0 \leq \alpha + \int_0^\infty \mathcal{H}(s)ds \leq \alpha + \overline{H},$$

but this is impossible.

Now, by setting $\beta = (\alpha + \epsilon) + \overline{H}$, for each $\alpha, \epsilon > 0$ given, and considering the problem

$$\begin{cases} \Delta_\phi u = a(x)f(u) \text{ in } B_n(0), \\ u \geq 0 \text{ in } B_n(0), \quad u = u_\alpha \text{ on } \partial B_n(0), \end{cases} \quad (1.35)$$

we can infer from Proposition 3.1 that there exists a $w_n = w_{n,\alpha} \in C^{1,\nu}(\overline{B}_n)$, for some $0 < \nu < 1$, solution of (1.35) satisfying $0 < \alpha \leq u_\alpha \leq w_n \leq u_\beta$ in $B_n$ for all $n \in \mathbb{N}$. So, by compactness, there exists a $w \in C^1(\mathbb{R}^N)$ such that $w(x) = \lim_{n\to\infty} w_n(x)$ is a solution of (1.1). ■

By adjusting the above arguments, we are able to prove the below remark, which generalises the main result in [5].

**Remark 5.1** *If we assume the stronger hypothesis*

$$\tilde{H} := \int_0^\infty [\eta_4(a^*(s)) - \eta_3(a_*(s))]h^{-1}\Big(sf\Big(\mathcal{F}^{-1}\Big(\int_0^s h^{-1}(\mathcal{A}_{a^*}(t))dt\Big)\Big)\Big)ds < \infty,$$

*instead of* $(\mathcal{F})$*, we obtain the same results of Theorem* 1.2 *with* $\tilde{H}$ *in the place of* $\overline{H}$*, without assuming* (1.6)*, where*

$$a_*(r) = \min\{a(x); \ |x| \leq r\}, \quad a^*(r) = \max\{a(x); \ |x| \leq r\}, \ r \geq 0.$$

*Proof of Theorem* 1.3. Assume that $w \in C^1(\mathbb{R}^N)$ is a positive solution of (1.1). So, given $0 < \alpha < w(0)$, it follows by the arguments in the proof of Lemma 5.1 that there exists a radial solution $u_\alpha \in C^1(B_{\Gamma(\alpha)}(0))$ of the problem

$$\begin{cases} \Delta_\phi u = \overline{a}(|x|)f(u) \text{ in } B_{\Gamma(\alpha)}(0), \\ u \geq 0 \text{ in } B_{\Gamma(\alpha)}(0),\ u = \infty \text{ on } \partial B_{\Gamma(\alpha)}(0), \end{cases} \tag{1.36}$$

if $\Gamma(\alpha) < \infty$. Yet, this is impossible by Lemma 2.3, that is, $\Gamma(\alpha) = \infty$. Besides this, it follows from $(\mathbf{A}_{\underline{\mathbf{a}}})$ that $u_\alpha$ is a solution of the problem (1.7), that is, $\alpha \in \mathbb{A}$. In particular, it follows from the above arguments that $(0, A) \subset \mathbb{A}$.

Finally, if we assume that $f$ satisfies the $\phi-$Keller-Osserman condition, it follows from Theorem 1.1 that the problem

$$\begin{cases} \Delta_\phi u = \overline{a}(|x|)f(u) \text{ in } B_1(0), \\ u \geq 0 \text{ in } B_1(0),\ u = \infty \text{ on } \partial B_1(0), \end{cases}$$

admits a solution. So, it follows from Lemma 2.3 that $A < \infty$. Reciprocally, if $A < \infty$, then $u_{A+1}$ is a radial solution of

$$\begin{cases} \Delta_\phi u = \overline{a}(|x|)f(u) \text{ in } B_{\Gamma(A+1)}(0), \\ u \geq 0 \text{ in } B_{\Gamma(A+1)}(0),\ u = \infty \text{ on } \partial B_{\Gamma(A+1)}(0), \end{cases}$$

where $0 < \Gamma(A+1) < \infty$. So, by Theorem 1.1, we have that $f$ satisfies $(\mathbf{F})$. These ends the proof. ■

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