The number of dominating k-sets of paths, cycles and wheels

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Abstract

We give a shorter proof of the recurrence relation for the domination polynomial $\gamma(P_n, t)$ and for the number $\gamma_k(P_n)$ of dominating k-sets of the path with n vertices. For every positive integers n and k, numbers $\gamma_k(P_n)$ are determined solving a problem posed by S. Alikhani in CID 2015. Moreover, the numbers of dominating k-sets $\gamma_k(C_n)$ of cycles and $\gamma_k(W_n)$ of wheels with n vertices are computed.

Keywords: dominating set, domination polynomial, path, cycle, wheel

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1 Introduction and preliminaries

Let $G = (V, E)$ be a simple graph and a subset of vertices $D \subseteq V$. A set D is a *dominating set* if for every $y \in V - D$ there exists $x \in D$ such that $\{x, y\} \in E$. For every $x \in V$, let $N(x)$ denote the neighborhood of x and $N[x] = N(x) \cup \{x\}$ is the closed neighborhood of x. Using this notation, $D \subseteq V$ is dominating set if $N[y] \cap D \neq \emptyset$ for every $y \in V - D$. The minimum cardinality of a dominating set in G is the *domination number* of G and it is usually denoted by $\gamma(G)$.

Let $\mathcal{D}_G$ be the set of every dominating set of G. Observe that $\emptyset \in \mathcal{D}_G$ and if $D \in \mathcal{D}_G$ and $D \subset D'$, then $D' \in \mathcal{D}_G$. In [5], the so called *domination polynomial* of a graph G was first introduced as follows.

For every graph G, we denote by $\gamma_k(G)$ the *number of dominating sets of cardinality* k (briefly, the *dominating k-sets*) of G. The domination polynomial is define to be

$$\gamma(G,t)=\sum_{k=1}^{n}\gamma_k(G)\,t^{n-k},$$

where n is the number of vertices of G.

By definition, we set $\gamma_0(G)=0$. If Φ denotes the empty graph, then $\gamma(\Phi,t)=0$ since $\mathcal{D}_\Phi=\{\emptyset\}$. In the aforementioned paper, we determined the following domination polynomials:

$$\gamma(K_n,t)=\sum_{k=1}^{n}\binom{n}{k}t^{n-k}=(1+t)^n-t^n, \tag{1}$$

where K_n is the complete graph with n vertices,

$$\gamma(\cup_{i=1}^{n}G_i,t)=\gamma(G_1,t)\gamma(G_2,t)\cdots\gamma(G_n,t), \tag{2}$$

where $\cup_{i=1}^{n}G_i$ is the disjoint union of the graphs G_i for $1\le i\le n$ ([5], Theorem 3.1 and Corollary 3.2) and

$$\begin{aligned}\gamma(G+H,t)=&\,\gamma(K_{n+m},t)-t^m\left[\gamma(K_n,t)-\gamma(G,t)\right]\\&-t^n\left[\gamma(K_m,t)-\gamma(H,t)\right],\end{aligned} \tag{3}$$

where G and H are graphs with n and m vertices, respectively, and $G+H$ denotes the sum of G and H ([5], Theorem 3.3).

Equivalently (see [2], [11] and [14]), the domination polynomial can be defined as

$$D(G,t)=\sum_{k=0}^{n}\gamma_k(G)\,t^k$$

and therefore, $\gamma(G,t)=t^nD(G,\frac{1}{t})$.

In recent years, the domination polynomial of graphs has received a lot of attention. In [11], the polynomial is related to network reliability measure for some special service networks. This new measure was defined by the authors as the domination and the corresponding domination reliability polynomial is associated. Some interesting theoretical problems are studied. It is also proved that computing domination reliability is NP-hard. The book [7] by Beichelt and Tittmann broadens the study of network reliability. The connection of reliability with graph theory and combinatorial analysis is widely displayed. The book is an excellent example of applying nontrivial graph theoretical tools to solve key problems in reliability analysis.

The roots of the domination polynomial have recently been studied, see for example [1], [8] and [15]. Interesting problems are still open, particularly, those relating the existence of some special roots of the domination polynomial to specific properties of the graph.

Let us define the following binary relation between graphs G and H. We say that $G\sim H$ if and only if $\gamma(G,t)=\gamma(H,t)$. Clearly, this is an equivalence relation. The determination of the equivalence classes of the quotient set is a hard important problem. A graph G is *uniquely determined by its domination polynomial* if $[G] = \{G\}$. In [1], it is proved that $C_n = \{C_n\}$ for $n \equiv 0, 2 \pmod 3$ and $[P_n]$ contains two graphs for $n \equiv 0 \pmod 3$ (P_n denotes the path with n vertices). It is also conjectured that $C_n = \{C_n\}$ for $n \equiv 1 \pmod 3$.

Problem 1. *Give a characterization of $[P_n]$ for $n \equiv 1, 2 \pmod 3$.*

In [4], the authors characterize those complete r-partite graphs that are uniquely determined by their domination polynomials (see Theorem 2).

The computation of the equivalence classes of almost all other graphs remains an interesting open question.

The total domination polynomial of a graph is studied in [9]. It is the generating function for the number of total dominating $k-$sets in a graph G. A generalization of the total domination polynomial, called the trivariate total domination polynomial is investigated, in particular, this kind of polynomial is studied for some graph products. There are other variations and generalizations of the domination polynomial as the independent domination polynomial or the bipartition polynomial. The already mentioned paper by Dod and the recent work [10] by Dod, Kotek, Preen and Tittmann are suitable references for these topics.

For a simple graph $G = (V, E)$, $v \in V$ and $e = \{u, v\} \in E$, we define the following operations: $G - v$ is the usual *vertex deletion*; $G - N[v]$ is the *vertex extraction* defined to be the graph $\bigcup_{u \in N[v]} (G - u)$; G/v is the *vertex contraction* defined as the graph obtained from G by removing v and adding the edges between any pair of non-adjacent neighbors of v; $G/N(v)$ is the *neighborhood contraction* defined as the graph defined from G by removing every vertex of $N(v)$ (but not itself v) and adding the edges $\{v, w\}$ for every $w \in N(N(v))$, where $N(N(v))$ is the *second neighborhood* of v; $G - e$ is the usual *edge deletion*; G/e is the usual *edge contraction* and $G \dagger e$ is the *edge extraction* defined to be the graph $G - \{u, v\} = (G - u) \cup (G - v)$.

One of the main results of [14] is the following

Theorem 2 (([14], Theorems 2.4 and 2.5))**.** *There do not exist rational functions $f_i, g_j \in \mathbb{R}(t)$ with $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$ such that for every graph G, $v \in V$ and $e \in E$, it holds that*

$$\begin{aligned} D(G,t) &= f_1 D(G - v, t) + f_2 D(G/v, t) + f_3 D(G - N[v], t) \\ &\quad + f_4 D(G/N(v), t) \text{ and} \\ D(G,t) &= g_1 D(G - e, t) + g_2 D(G/e, t) + g_3 D(G \dagger e, t). \end{aligned}$$

Despite the theorem above states that the domination polynomial satisfies no linear recurrence relation with the defined operations, it is possible to defined a very useful recurrence relation using an special class of oriented graphs obtained from the undirected ones. This approach was introduced and applied in [5]. We summarize the main definitions and results that will be used later in this paper.

Consider the oriented graph $\Gamma = (U, A)$. We denote by $N^-(x)$ and $N^+(x)$ the in- and out-neighborhood of a vertex $x \in U$, respectively, and $N^-[x] = N^-(x) \cup \{x\}$ and $N^+[x] = N^+(x) \cup \{x\}$ the respective closed in- and out-neighborhood of x. We say that $D \subseteq U$ is a *dominating set* of Γ if for every $v \in U - D$ there exists $u \in D$ such that $(u, v) \in A$, that is $N^-[v] \cap D \neq \emptyset$. From this definition, if D is a dominating set of

Γ, then there exits $u \in D$ such that $N^{+}(u) \neq \emptyset$. Similarly, the domination polynomial of an oriented graph Γ is defined as

$$\gamma(\Gamma, t) = \sum_{k=1}^{n} \gamma_k(\Gamma)\, t^{n-k},$$

where $\gamma_k(\Gamma)$ denotes the number of dominating k-sets of Γ.

Let $\Gamma = (U_1, U_2, A)$ a bipartite one-way oriented graph, that is, a bipartite oriented graph with partite sets U_1 and U_2 such that every arc is oriented from a vertex of U_1 to a vertex of U_2. Observe that if D is a dominating set of a bipartite one-way oriented graph Γ, then $D \subseteq U_1$ by definition.

Remark 3. *If $U_1 = \emptyset$, then $\gamma(\Gamma, t) = 0$ (there is no dominating set) and if $U_2 = \emptyset$, then by convention we define $\gamma(\Gamma, t) = (1+t)^{|U_1|}$.*

We recall that if Γ_1 and Γ_2 are bipartite components (not necessarily connected) then the domination polynomial is multiplicative respect to the components, that is $\gamma(\Gamma, t) = \gamma(\Gamma_1, t)\ \gamma(\Gamma_2, t)$ (compare with the domination polynomial (2) for the disjoint union of graphs). The oriented graph operations $\Gamma - i$ and $\Gamma - N^{+}[i]$ are analogously defined as the corresponding vertex deletion and vertex extraction for graphs.

Let $G = (V, E)$ a simple graph. We construct a bipartite one-way oriented graph $\Gamma_G = (U_1, U_2, A)$ from G such that U_1 and U_2 are disjoint copies of V and

$$A = \{(i,i) : i \in V\} \cup \{(i,j), (j,i) : \{i,j\} \in E\}.$$

Theorem 4 (([5], Lemma 3.5, Theorem 3.6))**.** *Let G be a simple graph. Then*

$$\begin{aligned} \gamma(G, t) &= \gamma(\Gamma_G, t) \text{ and} \\ \gamma(\Gamma, t) &= t\gamma(\Gamma - i, t) + \gamma\left(\Gamma - N^{+}[i], t\right) \end{aligned} \tag{4}$$

for every bipartite one-way oriented graph $\Gamma = (U_1, U_2, A)$ and $i \in U_1$.

Corollary 5. *$\gamma(G, t) = \gamma(\Gamma_G, t) = t\gamma(\Gamma_G - i, t) + \gamma(\Gamma_G - N^{+}[i], t)$ for every simple graph G.*

In this paper, we solve the following open problem posed at the Problem Session of CID 2015 by S. Alikhani:

Problem 6. *Let P_n be the path with n vertices. Find an explicit formula for $\gamma_k(P_n)$, where k and n are positive integers.*

For this purpose, in Section 2 we give a much shorter proof for the recurrence relation involving the domination polynomials of paths proved in [2]. Using similar tools, we can show the recurrence relation for the polynomial of cycles. As a consequence, we give the domination polynomial of wheels. There are two corollaries following the respective theorems for the recurrence relation involving the numbers $\gamma_k(P_n)$ and $\gamma_k(C_n)$. The number of dominating k-sets $\gamma_k(W_n)$ for the wheel with n vertices is a consequence of its domination polynomial depending on the domination polynomial of a cycle with $n-1$ vertices. In Section 3 we give the explicit formulas for the number of the dominating k-sets of paths, cycles and wheels using ordinary generating functions of two variables. In particular, Theorem 16 is a solution to Problem 6.

For the terminology on graph an digraphs used in what follows, see [6].

2 The domination polynomial of paths, cycles and wheels

Let n be a positive integer, $[n] = \{1, ..., n\}$ and P_n the path with vertex set $[n]$. In [2], the following theorem is proved.

Theorem 7 (([2], Theorem 3.1)). *Let n and k be positive integers. Then for every $n \geq 4$ and $k \geq 2$*

(i) $\gamma_k(P_n) = \gamma_{k-1}(P_{n-1}) + \gamma_{k-1}(P_{n-2}) + \gamma_{k-1}(P_{n-3})$ *and*

(ii) $D(P_n, t) = t\ [D(P_{n-1}, t) + D(P_{n-2}, t) + D(P_{n-3}, t)]$ *with initial conditions* $D(P_1, t) = t$, $D(P_2, t) = t^2 + 2t$ *and* $D(P_3, t) = t^3 + 3t^2 + t$.

The long proof of this theorem is based on six lemmas and Theorem 2.7. We give a much shorter proof of this theorem. For this aim, we define bipartite one-way oriented graphs $I_{m+1,m}$ and $J_{m,m+1}$ for every positive integer m. First, observe that Γ_{P_n} is given by $U_1(\Gamma_{P_n}) = U_2(\Gamma_{P_n}) = [n]$ and

$$A(\Gamma_{P_n}) = \{(i, i) : i \in [n]\} \cup \{(i, i+1), (i+1, i) : i \in [n-1]\}.$$

Let $U_1(I_{m+1,m}) = [m+1]$, $U_2(I_{m+1,m}) = [m]$ and

$$A(I_{m+1,m}) = A(\Gamma_{P_m}) \cup \{(m+1, m)\}.$$

Similarly, $U_1(J_{m,m+1}) = [m]$, $U_2(J_{m,m+1}) = [m+1]$ and

$$A(J_{m,m+1}) = A(\Gamma_{P_m}) \cup \{(m, m+1)\}.$$

Theorem 8. *Let n be a positive integer. Then*

$$\gamma(P_n, t) = \gamma(P_{n-1}, t) + t\ \gamma(P_{n-2}, t) + t^2 \gamma(P_{n-3}, t)$$

for every $n \geq 4$ with initial conditions $\gamma(P_1, t) = 1$, $\gamma(P_2, t) = 1 + 2t$ and $\gamma(P_3, t) = 1 + 3t + t^2$.

Proof. Let $n \geq 4$. We apply recurrence relation (4) of Theorem 4 to the domination polynomial $\gamma(\Gamma_{P_n}, t)$ and $\gamma(I_{n,n-1}, t)$ using vertex n of $U_1(\Gamma_{P_n})$ and $U_1(I_{n,n-1})$, respectively. We obtain that

$$\begin{aligned} \gamma(\Gamma_{P_n}, t) &= t\ \gamma(J_{n-1,n}, t) + \gamma(I_{n-1,n-2}, t) \text{ and} \\ \gamma(I_{n,n-1}, t) &= t\ \gamma(\Gamma_{P_{n-1}}, t) + \gamma(I_{n-1,n-2}, t). \end{aligned}$$

Applying again (4) of Theorem 4 to $\gamma(J_{n-1,n}, t)$ using vertex $n-1$ of $U_1(J_{n-1,n})$, it follows that

$$\gamma(J_{n-1,n}, t) = t\ \gamma(J_{n-2,n-1} \cup J_{0,1}, t) + \gamma(I_{n-2,n-3}, t).$$

Since

$$\gamma(J_{n-2,n-1} \cup J_{0,1}, t) = \gamma(J_{n-2,n-1}, t)\ \gamma(J_{0,1}, t)$$

and by Remark 3, $\gamma(J_{0,1}, t) = 0$ ($U_1(J_{0,1}) = \emptyset$), we have that $\gamma(J_{n-2,n-1} \cup J_{0,1}, t) = 0$ and hence $\gamma(J_{n-1,n}, t) = \gamma(I_{n-2,n-3}, t)$. Then

$$\begin{aligned} \gamma(\Gamma_{P_n}, t) &= \gamma(I_{n-1,n-2}, t) + t\ \gamma(I_{n-2,n-3}, t) \text{ and} \\ \gamma(I_{n,n-1}, t) &= t\ [\gamma(I_{n-2,n-3}, t) + t\ \gamma(I_{n-3,n-4}, t)] + \gamma(I_{n-1,n-2}, t) \\ &= \gamma(I_{n-1,n-2}, t) + t\ \gamma(I_{n-2,n-3}, t) + t^2 \gamma(I_{n-3,n-4}, t). \end{aligned}$$

Since $n \geq 4$, the last recurrence relation for $\gamma(I_{n,n-1},t)$ is true with initial conditions $\gamma(I_{1,0},t) = 1+t$ (by Remark 3), $\gamma(I_{2,1},t) = 1+2t$ and $\gamma(I_{3,2},t) = 1+3t+2t^2$. Finally,

$$\begin{aligned}
\gamma(P_n,t) &= \gamma(\Gamma_{P_n},t) = \gamma(I_{n-1,n-2},t) + t\ \gamma(I_{n-2,n-3},t) \\
&= \gamma(I_{n-2,n-3},t) + t\ \gamma(I_{n-3,n-4},t) + t^2\gamma(I_{n-4,n-5},t) \\
&\quad +t\left[\gamma(I_{n-3,n-4},t) + t\ \gamma(I_{n-4,n-5},t) + t^2\gamma(I_{n-5,n-6},t)\right] \\
&= \gamma(I_{n-2,n-3},t) + t\ \gamma(I_{n-3,n-4},t) + t\left[\gamma(I_{n-3,n-4},t) + t\ \gamma(I_{n-4,n-5},t)\right] \\
&\quad +t^2\left[\gamma(I_{n-4,n-5},t) + t\ \gamma(I_{n-5,n-6},t)\right] \\
&= \gamma(\Gamma_{P_{n-1}},t) + t\ \gamma(\Gamma_{P_{n-2}},t) + t^2\gamma(\Gamma_{P_{n-3}},t) \\
&= \gamma(P_{n-1},t) + t\ \gamma(P_{n-2},t) + t^2\gamma(P_{n-3},t),
\end{aligned}$$

which proves the theorem. □

Recalling that $\gamma\left(P_n,t\right) = t^n D(P_n,\frac{1}{t})$, we obtain the recurrence relation (ii) of Theorem 7.

From this theorem and using the definition of the domination polynomial we have that

$$\begin{aligned}
\sum_{k=1}^{n}\gamma_k\left(P_n\right)t^{n-k} &= \sum_{k=1}^{n-1}\gamma_k\left(P_{n-1}\right)t^{n-1-k} + t\sum_{k=1}^{n-2}\gamma_k\left(P_{n-2}\right)t^{n-2-k} \\
&\quad +t^2\sum_{k=1}^{n-3}\gamma_k\left(P_{n-3}\right)t^{n-3-k}
\end{aligned}$$

$$= \sum_{k=1}^{n-1}\gamma_k\left(P_{n-1}\right)t^{n-1-k} + \sum_{k=1}^{n-2}\gamma_k\left(P_{n-2}\right)t^{n-1-k} + \sum_{k=1}^{n-3}\gamma_k\left(P_{n-3}\right)t^{n-1-k}$$

$$\begin{aligned}
&= \sum_{k=2}^{n-2}\left[\gamma_{k-1}(P_{n-1}) + \gamma_{k-1}(P_{n-2}) + \gamma_{k-1}(P_{n-3})\right]t^{n-k} \\
&\quad +\gamma_{n-2}(P_{n-1})\ t + \gamma_{n-1}(P_{n-1}) + \gamma_{n-2}(P_{n-2})t
\end{aligned}$$

$$= \sum_{k=2}^{n-2}\left[\gamma_{k-1}(P_{n-1}) + \gamma_{k-1}(P_{n-2}) + \gamma_{k-1}(P_{n-3})\right]t^{n-k} + n\ t + 1$$

and therefore, we have the recurrence relation (i) of Theorem 7.

Corollary 9. *$\gamma_k(P_n) = \gamma_{k-1}(P_{n-1}) + \gamma_{k-1}(P_{n-2}) + \gamma_{k-1}(P_{n-3})$ for every $n \geq 4$ and $2 \leq k \leq n-2$ with initial conditions $\gamma_1(P_1) = 1$, $\gamma_1(P_2) = 2$, $\gamma_1(P_3) = 1$, $\gamma_2(P_2) = 1$ and $\gamma_2(P_3) = 3$.*

Notice that $\gamma_{n-1}(P_n) = n$ and $\gamma_n(P_n) = 1$.

Let C_n be the cycle with n vertices. We define $C_1 = K_1$, the cycle with one vertex, and C_2 is the so called *dicycle* (two vertices joined by two parallel edges). The domination polynomials are $\gamma(C_1,t) = 1$ and $\gamma(C_2,t) = 1+2t$, respectively. If we appropriately apply recurrence relation (4) of Theorem 4 to the domination polynomial $\gamma(\Gamma_{C_n},t)$ and define suitable bipartite one-way oriented graphs, then it can be analogously proved the following result for the domination polynomial of cycles (see Theorem 4.5 of [3])

Theorem 10. *Let n be a positive integer. Then*

$$\gamma(C_n, t) = \gamma(C_{n-1}, t) + t\gamma(C_{n-2}, t) + t^2\gamma(C_{n-3}, t)$$

for every $n \geq 4$ with initial conditions $\gamma(C_1, t) = 1$, $\gamma(C_2, t) = 1 + 2t$ and $\gamma(C_3, t) = 1 + 3t + 3t^2$.

Similarly as done before for paths, we have the following (Theorem 4.4 of [3])

Corollary 11. $\gamma_k(C_n) = \gamma_{k-1}(C_{n-1}) + \gamma_{k-1}(C_{n-2}) + \gamma_{k-1}(C_{n-3})$ *for every $n \geq 4$ and $2 \leq k \leq n-2$ with initial conditions $\gamma_1(C_1) = 1$, $\gamma_1(C_2) = 2$, $\gamma_1(C_3) = 3$, $\gamma_2(C_2) = 1$ and $\gamma_2(C_3) = 3$.*

Notice that $\gamma_{n-1}(C_n) = n$ and $\gamma_n(C_n) = 1$.

The recurrence relations for $\gamma_k(P_n)$ and $\gamma_k(C_n)$ of Corollaries 9 and 11 are generalizations to the well-known Tribonacci numbers defined by the recurrence relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for every $n \geq 3$ with initial conditions $T_0 = T_1 = 1$ and $T_2 = 2$. Originally, the Tribonacci sequence was first discussed by Feinberg in [12]. In [13], these numbers are expressed as sums of numbers along diagonal planes of the the so-called Pascal's pyramid, a natural generalization of the Pascal's triangle.

Later, a closed formula for the Tribonacci numbers was proved by Shannon in [16].

Theorem 12 ([16])**.** *For every nonnegative integer n,*

$$T_n = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-m-2r}{m+r} \binom{m+r}{r}.$$

The identity of this theorem generates sequence A000073 of [17]. This formula is in some sense "analogous" to the identities of Theorems 16 and 19 for the number of dominating k-sets of paths and cycles, respectively.

Let W_n be the wheel with n vertices. Recall that $W_n = K_1 + C_{n-1}$. Using the identities (1) and (3) for domination polynomials of complete graphs and the sum of two graphs, respectively, we have the following results.

Theorem 13. *Let n be a positive integer. Then $\gamma(W_n, t) = (1 + t)^{n-1} + t\gamma(C_{n-1}, t)$ for every $n \geq 4$.*

Corollary 14. $\gamma_k(W_n) = \binom{n-1}{n-k} + \gamma_k(C_{n-1})$ *for every $n \geq 4$ and $1 \leq k \leq n$.*

3 The number of dominating k-sets of paths, cycles and wheels with n vertices

In this section, we explicitly compute the numbers $\gamma_k(P_n)$ for every positive integers n and k and so, we give a solution to Problem 6. By Corollary 9, we know that $\gamma_k(P_n) = \gamma_{k-1}(P_{n-1}) + \gamma_{k-1}(P_{n-2}) + \gamma_{k-1}(P_{n-3})$ for every $n \geq 4$ and $2 \leq k \leq n-2$ with initial conditions $\gamma_1(P_1) = 1$, $\gamma_1(P_2) = 2$ and $\gamma_1(P_3) = 1$. Let us denote $\gamma_{n,k} = \gamma_k(P_n)$. We define the ordinary generating function for the numbers $\gamma_{n,k} = \gamma_k(P_n)$ as

$$G(x, y) = \sum_{n \geq 1} \sum_{k \geq 1} \gamma_{n,k}\ x^n\ y^k.$$

Theorem 15.

$$G(x,y) = \frac{x(1+x)^2 y}{1-(x+x^2+x^3)y}.$$

Proof. Consider the following identity:

$$\begin{aligned}
& \left[1-(x+x^2+x^3)y\right] G(x,y) \\
= \;& G(x,y) - x\ y\ G(x,y) - x^2 y\ G(x,y) - x^3 y\ G(x,y) \\
= \;& \sum_{n\ge 1}\sum_{k\ge 1} \gamma_{n,k}\ x^n\ y^k - \sum_{n\ge 1}\sum_{k\ge 1} \gamma_{n,k}\ x^{n+1}\ y^{k+1} \\
& - \sum_{n\ge 1}\sum_{k\ge 1} \gamma_{n,k}\ x^{n+2}\ y^{k+1} - \sum_{n\ge 1}\sum_{k\ge 1} \gamma_{n,k}\ x^{n+3}\ y^{k+1} \\
= \;& \sum_{n\ge 1}\sum_{k\ge 1} \gamma_{n,k}\ x^n\ y^k - \sum_{n\ge 2}\sum_{k\ge 2} \gamma_{n-1,k-1}\ x^n\ y^k \\
& - \sum_{n\ge 3}\sum_{k\ge 2} \gamma_{n-2,k-1}\ x^n\ y^k - \sum_{n\ge 4}\sum_{k\ge 2} \gamma_{n-3,k-1}\ x^n\ y^k
\end{aligned}$$

$$\begin{aligned}
= \;& \gamma_{1,1}\ x\ y + \gamma_{2,1}\ x^2\ y + \gamma_{3,1}\ x^3\ y + \gamma_{1,2}\ x\ y^2 + \gamma_{2,2}\ x^2\ y^2 + \gamma_{3,2}\ x^3\ y^2 \\
& + \sum_{n\ge 4}\sum_{k\ge 2} \gamma_{n,k}\ x^n\ y^k - \left[\gamma_{1,1}\ x^2 y^2 + \gamma_{2,1}\ x^3\ y^2\right] - \sum_{n\ge 4}\sum_{k\ge 2} \gamma_{n-1,k-1}\ x^n\ y^k \\
& - \gamma_{1,1}\ x^3\ y^2 - \sum_{n\ge 4}\sum_{k\ge 2} \gamma_{n-2,k-1}\ x^n\ y^k - \sum_{n\ge 4}\sum_{k\ge 2} \gamma_{n-3,k-1}\ x^n\ y^k
\end{aligned}$$

$$\begin{aligned}
= \;& \gamma_{1,1}\left[x\ y - x^2\ y^2 - x^3\ y^2\right] + \gamma_{2,1}\left[\ x^2\ y - x^3\ y^2\right] + \gamma_{3,1}\ x^3\ y \\
& + \gamma_{1,2}\ x\ y^2 + \gamma_{2,2}\ x^2\ y^2 + \gamma_{3,2}\ x^3\ y^2 \\
& + \sum_{n\ge 4}\sum_{k\ge 2} \left[\gamma_{n,k} - \gamma_{n-1,k-1} - \gamma_{n-2,k-1} - \gamma_{n-3,k-1}\right]\ x^n\ y^k
\end{aligned}$$

$$\begin{aligned}
= \;& x\ y - x^2\ y^2 - x^3\ y^2 + 2\ x^2\ y - 2\ x^3\ y^2 +\ x^3\ y + x^2\ y^2 + 3\ x^3\ y^2 \\
= \;& xy + 2x^2 y + x^3 y = x(1+x)^2 y,
\end{aligned}$$

where $\gamma_{n,k} = \gamma_k(P_n) = 0$ if $n < k$ and using Corollary 9. $\square$

If we determine the formal power series of $G(x,y)$ expanded in powers of y, we have that

$$\begin{aligned}
G(x,y) &= \frac{x(1+x)^2 y}{1-(x+x^2+x^3)y} \\
&= x(1+x)^2 y \sum_{k\ge 0} (x+x^2+x^3)^k y^k \\
&= \sum_{k\ge 1} x(1+x)^2 (x+x^2+x^3)^{k-1} y^k \\
&= \sum_{k\ge 1} \left(\sum_{n\ge 1} \gamma_k(P_n)\ x^n \right) y^k
\end{aligned}$$

and then for every $k \geq 1$

$$\begin{aligned}\sum_{n\geq 1} \gamma_k(P_n)\ x^n &= x(1+x)^2(x+x^2+x^3)^{k-1} \\ &= x^k(1+x)^2(1+x+x^2)^{k-1} \qquad (5)\end{aligned}$$

is a monic polynomial denoted by $g_k(x)$ of degree $3k$ such that

$$g_k(x) = \sum_{n=k}^{3k} \gamma_k(P_n)\ x^n = \sum_{t=0}^{2k} \gamma_k(P_{k+t})\ x^{k+t}$$

since $\gamma_k(P_n) = 0$ if $n < k$ and the well-known fact that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ for every $n \geq 1$. Observe that $g_k(x)$ has $2k+1$ terms, $g_k(x) = x^{2k} g_k(\frac{1}{x})$ and so, $\gamma_k(P_n) = \gamma_k(P_{4k-n})$ which means that the polynomial is symmetric with respect to the $(k+1)$-th term.

Theorem 16. *For every $k \geq 1$ and $t \geq 0$, the number of dominating k-sets of the path P_{k+t} is*

$$\gamma_k(P_{k+t}) = \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \binom{t-m+2}{m}.$$

Proof. From identity (5), we have that

$$(1+x+x^2)^{k-1} = \sum_{l=0}^{k-1} \binom{k-1}{l} (x+x^2)^l = \sum_{l=0}^{k-1} \binom{k-1}{l} \sum_{m=0}^{l} \binom{l}{m} x^{l-m} x^{2m}$$

and hence

$$(1+x+x^2)^{k-1} = \sum_{l=0}^{k-1} \sum_{m=0}^{l} \binom{k-1}{l} \binom{l}{m} x^{l+m}. \qquad (6)$$

Therefore,

$$g_k(x) = (x^k + 2x^{k+1} + x^{k+2})(1+x+x^2)^{k-1}$$

$$\begin{aligned} =\ & x^k \sum_{l=0}^{k-1} \binom{k-1}{l} \left(\sum_{m=0}^{l} \binom{l}{m} x^m \right) x^l + 2x^{k+1} \sum_{l=0}^{k-1} \binom{k-1}{l} \left(\sum_{m=0}^{l} \binom{l}{m} x^m \right) x^l \\ & + x^{k+2} \sum_{l=0}^{k-1} \binom{k-1}{l} \left(\sum_{m=0}^{l} \binom{l}{m} x^m \right) x^l \end{aligned}$$

$$\begin{aligned} =\ & \sum_{l=0}^{k-1} \sum_{m=0}^{l} \binom{k-1}{l} \binom{l}{m} x^{k+l+m} + \sum_{l=0}^{k-1} \sum_{m=0}^{l} 2 \binom{k-1}{l} \binom{l}{m} x^{k+l+m+1} \\ & + \sum_{l=0}^{k-1} \sum_{m=0}^{l} \binom{k-1}{l} \binom{l}{m} x^{k+l+m+2} \end{aligned}$$

$$= \sum_{l=0}^{k-1} \binom{k-1}{l} x^k \sum_{m=0}^{l} \left[\binom{l}{m} x^{l+m} + 2 \binom{l}{m} x^{l+m+1} + \binom{l}{m} x^{l+m+2} \right]$$

$$= \sum_{l=0}^{k-1} \binom{k-1}{l} x^k \left[\sum_{m=0}^{l+2} \left(\binom{l}{m} + 2\binom{l}{m-1} + \binom{l}{m-2} \right) x^{m+l} \right].$$

Using Pascal's formula,

$$\begin{aligned} g_k(x) &= x^k \sum_{l=0}^{k-1} \binom{k-1}{l} \left[\sum_{m=0}^{l+2} \binom{l+2}{m} x^m \right] x^l \\ &= x^k \sum_{l=0}^{k-1} \sum_{m=0}^{l+2} \binom{k-1}{l} \binom{l+2}{m} x^{l+m} \\ &= x^k \sum_{t=0}^{2k} \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \binom{t-m+2}{m} x^t \\ &= \sum_{t=0}^{2k} \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \binom{t-m+2}{m} x^{k+t}. \end{aligned}$$

Hence,

$$\gamma_k(P_{k+t}) = \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \binom{t-m+2}{m}$$

which is what had to be proven. □

The result of this theorem is an explicit formula for sequence A212633 of [17].

Corollary 17. *Let k be a positive integer. The number of the dominating sets of minimum cardinality for paths is*

$$\begin{aligned} \gamma_k(P_{3k}) &= 1 && \text{if } n = 3k, \\ \gamma_{k+1}(P_{3k+1}) &= \tbinom{k+2}{2} + k && \text{if } n = 3k+1 \text{ and} \\ \gamma_{k+1}(P_{3k+2}) &= k+2 && \text{if } n = 3k+2. \end{aligned}$$

By Corollary 11, we have that $\gamma_k(C_n) = \gamma_{k-1}(C_{n-1}) + \gamma_{k-1}(C_{n-2}) + \gamma_{k-1}(C_{n-3})$ for every $n \geq 4$ and $2 \leq k \leq n-2$ with initial conditions $\gamma_1(C_1) = 1$, $\gamma_1(C_2) = 2$, $\gamma_1(C_3) = 3$, $\gamma_2(C_2) = 1$ and $\gamma_2(C_3) = 3$. Let us define the ordinary generating function for the numbers $\gamma_k(C_n)$ as

$$H(x, y) = \sum_{n \geq 1} \sum_{k \geq 1} \gamma_k(C_n) x^n \ y^k.$$

Analogously as we proved Theorem 15, we have the following generating function for $\gamma_k(C_n)$.

Theorem 18.

$$H(x, y) = \frac{x(1 + 2x + 3x^2)y}{1 - (x + x^2 + x^3)y}.$$

We determine the formal power series of $H(x,y)$ expanded in powers of y to obtain

$$\begin{aligned}
H(x,y) &= \frac{x(1+2x+3x^2)y}{1-(x+x^2+x^3)y} \\
&= x(1+2x+3x^2)y\sum_{k\geq 0}(x+x^2+x^3)^k y^k \\
&= \sum_{k\geq 1} x(1+2x+3x^2)(x+x^2+x^3)^{k-1}y^k \\
&= \sum_{k\geq 1}\left(\sum_{n\geq 1}\gamma_k(C_n)\ x^n\right) y^k.
\end{aligned}$$

Therefore, for every $k \geq 1$

$$\begin{aligned}
\sum_{n\geq 1}\gamma_k(C_n)\ x^n &= x(1+2x+3x^2)(x+x^2+x^3)^{k-1} \\
&= x^k(1+2x+3x^2)(1+x+x^2)^{k-1} \qquad (7)
\end{aligned}$$

is a monic polynomial denoted by $h_k(x)$ of degree $3k$ such that

$$h_k(x) = \sum_{n=k}^{3k}\gamma_k(C_n)\ x^n = \sum_{t=0}^{2k}\gamma_k(C_{k+t})\ x^{k+t}$$

since $\gamma_k(C_n) = 0$ if $n < k$ and the well-known fact that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ for every $n \geq 1$. Observe that $h_k(x)$ has $2k+1$ terms as $g_k(x)$ does. In this case, polynomial $h_k(x)$ has no symmetry.

Theorem 19. *For every $k \geq 1$ and $t \geq 0$, the number of dominating k-sets of the cycle C_{k+t} is*

$$\gamma_k(C_{k+t}) = \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1}\binom{k-1}{t-m}\left(\binom{t-m+2}{m+2}\binom{t-m}{m-2}\right).$$

Proof. Using equalities (6) and (7), we have that

$$h_k(x) = (x^k + 2x^{k+1} + 3x^{k+2})(1+x+x^2)^{k-1}$$

$$\begin{aligned}
&= x^k\sum_{l=0}^{k-1}\binom{k-1}{l}\left(\sum_{m=0}^{l}\binom{l}{m}x^m\right)x^l + 2x^{k+1}\sum_{l=0}^{k-1}\binom{k-1}{l}\left(\sum_{m=0}^{l}\binom{l}{m}x^m\right)x^l \\
&\quad +3x^{k+2}\sum_{l=0}^{k-1}\binom{k-1}{l}\left(\sum_{m=0}^{l}\binom{l}{m}x^m\right)x^l
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{k-1}\sum_{m=0}^{l}\binom{k-1}{l}\binom{l}{m}x^{k+l+m} + \sum_{l=0}^{k-1}\sum_{m=0}^{l}2\binom{k-1}{l}\binom{l}{m}x^{k+l+m+1} \\
&\quad + \sum_{l=0}^{k-1}\sum_{m=0}^{l}3\binom{k-1}{l}\binom{l}{m}x^{k+l+m+2}
\end{aligned}$$

$$= \sum_{l=0}^{k-1} \binom{k-1}{l} x^k \sum_{m=0}^{l} \left[\binom{l}{m} x^{l+m} + 2\binom{l}{m} x^{l+m+1} + 3\binom{l}{m} x^{l+m+2} \right]$$

$$= \sum_{l=0}^{k-1} \binom{k-1}{l} x^k \left[\sum_{m=0}^{l+2} \left(\binom{l}{m} + 2\binom{l}{m-1} + 3\binom{l}{m-2} \right) x^{m+l} \right].$$

Using Pascal's formula,

$$\begin{aligned} h_k(x) &= x^k \sum_{l=0}^{k-1} \binom{k-1}{l} \left[\sum_{m=0}^{l+2} \left(\binom{l+2}{m} + 2\binom{l}{m-2} \right) x^m \right] x^l \\ &= x^k \sum_{l=0}^{k-1} \sum_{m=0}^{l+2} \binom{k-1}{l} \left(\binom{l+2}{m} + 2\binom{l}{m-2} \right) x^{l+m} \end{aligned}$$

$$\begin{aligned} &= x^k \sum_{t=0}^{2k} \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \left(\binom{t-m+2}{m} + 2\binom{t-m}{m-2} \right) x^t \\ &= \sum_{t=0}^{2k} \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \left(\binom{t-m+2}{m} + 2\binom{t-m}{m-2} \right) x^{k+t}. \end{aligned}$$

Hence,

$$\gamma_k(C_{k+t}) = \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-1}{t-m} \left(\binom{t-m+2}{m} + 2\binom{t-m}{m-2} \right),$$

the equality of the theorem. □

The explicit formula of the theorem generates sequence A212634 of [17].

Corollary 20. *Let k be a positive integer. The number of the dominating sets of minimum cardinality for cycles is*

$$\begin{array}{ll} \gamma_k(C_{3k}) = 3 & \text{if } n = 3k, \\ \gamma_{k+1}(C_{3k+1}) = \binom{k+2}{2} + 2\binom{k}{2} + 3k & \text{if } n = 3k+1 \text{ and} \\ \gamma_{k+1}(C_{3k+2}) = 3k+2 & \text{if } n = 3k+2. \end{array}$$

By Theorem 19 and Corollary 14, we conclude that

$$\begin{aligned} \gamma_k(W_{k+t}) &= \binom{k+t-1}{t} + \gamma_k(C_{k-1+t}) \\ &= \binom{k+t-1}{t} + \sum_{m=0}^{\lfloor \frac{t}{2} \rfloor + 1} \binom{k-2}{t-m} \left(\binom{t-m+2}{m} + 2\binom{t-m}{m-2} \right), \end{aligned}$$

for every $k \geq 1$ and $t \geq 0$ such that $k + t \geq 4$. This identity generates the terms of sequence A212635 of [17].

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