NEF LINE BUNDLES ON CALABI-YAU THREEFOLDS, I

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ABSTRACT. We prove that a nef line bundle $\mathcal{L}$ with $c_1(\mathcal{L})^2 \neq 0$ on a Calabi-Yau threefold X with Picard number 2 and with $c_3(X) \neq 0$ is semiample, i.e. some multiple of $\mathcal{L}$ is generated by global sections.

CONTENTS

1. INTRODUCTION

The following is a standard conjecture in the theory of simply connected Ricci-flat compact Kähler manifolds.

Conjecture 1.1. *Let X be a simply connected compact Kähler manifold with $c_1(X) = 0$. Let $\mathcal{L}$ be a nef line bundle on X. Then $\mathcal{L}$ is semiample, i.e. there exists a positive integer m such that $\mathcal{L}^{\otimes m}$ is generated by global sections.*

Recall that a line bundle $\mathcal{L}$ on a projective manifold X of dimension n is nef if $c_1(\mathcal{L}) \cdot C \geq 0$ for every irreducible curve C on X; in the Kähler setting, $\mathcal{L}$ being nef means that $c_1(\mathcal{L})$ is in the closure of the Kähler cone. In this paper we consider the case of dimension three, in which setting X is automatically projective. Conjecture 1.1, which should be seen as a stronger

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form of log abundance on Calabi-Yau manifolds, is rather mysterious unless $c_1(\mathcal{L})^{\dim X} > 0$, and almost nothing is known when $\dim X > 2$.

Our result towards Conjecture 1.1 is the following.

Theorem 1.2. *Let X be a three-dimensional simply connected projective manifold with $c_1(X) = 0$ and with Picard number 2. Suppose further that $c_3(X) \neq 0$. Then any nef line bundle $\mathcal{L}$ on X with $c_1(\mathcal{L})^2 \neq 0$ is semiample.*

Theorem 1.2 has been claimed in [Wil94], even in a more general setting, but we were unable to follow the proof. However, the general ideas of [Wil94] are important.

The assumption that $\rho(X) = 2$ in Theorem 1.2 is essential in the proof, but we hope that some of the methods can be useful also in the general case. We also emphasize that there is a wealth of interesting families of Calabi-Yau threefolds of Picard number $\rho(X) = 2$; we refer to [LP13, Ogu14] and the references given therein. Notice also that if $\mathcal{L}$ is a semiample line bundle on a Calabi-Yau threefold with $c_1(\mathcal{L})^2 \neq 0$ and $c_1(\mathcal{L}) \cdot c_2(X) = 0$, then $\rho(X) \geq 3$ by [Ogu93]. So the potentially missing case in Theorem 1.2 should not exist.

The semiampleness of a nef line bundle $\mathcal{L}$ on a simply connected projective threefold X with $c_1(X) = 0$ is obvious if $c_1(\mathcal{L}) = 0$, and when $\mathcal{L}$ is big, it is a consequence of the basepoint free theorem [Sho85, Kaw85]. However, if there is no bigness assumption, the existence of sections is notoriously difficult.

Remarkably, as a consequence of log abundance for threefolds, in order to show semiampleness of $\mathcal{L}$, by [Ogu93, KMM94, Kaw92], it suffices to show that

$$H^0(X, \mathcal{L}^{\otimes m}) \neq 0 \quad \text{for some positive } m.$$

Our results are also related to the Cone conjecture of Morrison and Kawamata. As we discuss in Section 2, a consequence of Kawamata's formulation of the Cone conjecture is the following structural prediction.

Conjecture 1.3. *Let X be a $\mathbb{Q}$-factorial projective variety with klt singularities and with numerically trivial canonical class K_X. Then $\mathrm{Nef}(X)^+ = \mathrm{Nef}(X)^e$.*

Here, $\mathrm{Nef}(X)^+$ and $\mathrm{Nef}(X)^e$ are the parts of the nef cone of X spanned by rational, respectively effective, classes. It seems to have been unknown thus far whether one of these cones is a subset of the other, unless X is an abelian variety or a hyperkähler manifold [Bou04]; see also [CO15] for results on special Calabi-Yau manifolds in any dimension.

Theorem 2.12 below gives a short proof that in the most general setting we have $\mathrm{Nef}(X)^e \subseteq \mathrm{Nef}(X)^+$. Thus, our main result Theorem 1.2, together with Theorem 2.12 can be restated as follows.

Corollary 1.4. *Let X be a three-dimensional simply connected projective manifold with $c_1(X) = 0$ and with Picard number 2. Assume that $c_3(X) \neq 0$ and that X does not carry a non-trivial line bundle $\mathcal{L}$ with $c_1(\mathcal{L})^2 = 0$ and $c_1(\mathcal{L}) \cdot c_2(X) = 0$. Then Conjecture 1.3 holds.*

Another application concerns the existence of rational curves on Calabi-Yau threefolds. It is known that rational curves exist on Calabi-Yau threefolds with Picard number at least 14, cf. [HBW92, Theorem], but almost nothing is known for smaller Picard number apart from [DF14]. Using [Pet91, Ogu93], in Section 6 we deduce:

Corollary 1.5. *Let X be a three-dimensional simply connected projective manifold with $c_1(X) = 0$, $c_3(X) \neq 0$ and with Picard number 2. Assume that not both boundary rays of the cone* $\mathrm{Nef}(X)$ *are irrational. Then X has a rational curve.*

We spend a few words about the method of the proof of Theorem 1.2, given in Section 5. Suppose that $\mathcal{L}$ is a nef line bundle on a Calabi-Yau threefold X with Picard number 2 such that $c_1(\mathcal{L})^2 \neq 0$ which is not semiample. Possibly replacing $\mathcal{L}$ by a multiple, we first observe that

$$H^0(X, \Omega^1_X \otimes \mathcal{L}^{\otimes m}) = 0 \quad \text{for all } m,$$

cf. [Wil94, 3.1]. Our method is then to study the cohomology of logarithmic differentials with poles along a carefully chosen smooth very ample divisor D and with values in $\mathcal{L}^{\otimes m}$ and its behaviour under deformation of D. As the final outcome, we show the crucial vanishing

$$H^2(X, \Omega^1_X \otimes \mathcal{L}^{\otimes m}) = 0 \quad \text{for large } m.$$

Then it is not difficult to see that actually

$$H^q(X, \Omega^p_X \otimes \mathcal{L}^{\otimes m}) = 0 \quad \text{for all } p \text{ and } q, \text{ and all } m \text{ with } |m| \gg 0.$$

Calculating $\chi(X, \Omega^1_X \otimes \mathcal{L}^{\otimes m})$ by the Hirzebruch-Riemann-Roch, we deduce that $c_3(X) = 0$.

2. Preliminaries

Throughout the paper we work over the field of complex numbers $\mathbb{C}$. For a variety X, $\mathrm{Pic}(X)$ is the Picard group of X and $N^1(X)$ is the Néron-Severi group of X. The Picard number of X is $\rho(X) = \mathrm{rk}\, N^1(X)$. If $\mathcal{L}$ is a nef Cartier divisor on X, the *numerical dimension of* $\mathcal{L}$ is

$$\nu(X, \mathcal{L}) = \max\{k \in \mathbb{N} \mid \mathcal{L}^k \not\equiv 0\}.$$

If $D = \sum_{i=1}^k D_i$ is a reduced divisor with simple normal crossings on a smooth variety X, recall that we have the locally free sheaf of logarithmic differentials $\Omega^1_X(\log D)$ together with the exact *residue sequences*

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \bigoplus_{i=1}^k \mathcal{O}_{D_i} \to 0, \tag{1}$$

$$0 \to \Omega^1_X(\log(D - D_k)) \to \Omega^1_X(\log D) \to \mathcal{O}_{D_k} \to 0, \tag{2}$$

and

$$0 \to \Omega^1_X(\log D)(-D_k) \to \Omega^1_X(\log(D-D_k)) \to \Omega^1_{D_k}(\log(D-D_k)|_{D_k}) \to 0, \tag{3}$$

cf. [EV92, §2].

2.1. **Calabi-Yau threefolds.** A Calabi-Yau manifold is by definition a compact Kähler manifold X which is simply connected and has trivial canonical bundle $\omega_X \simeq \mathcal{O}_X$. A three-dimensional Calabi-Yau manifold is simply called *Calabi-Yau threefold*, and thus a Calabi-Yau threefold in this paper is always smooth and projective. We notice that $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$ and hence $\operatorname{Pic}(X) \simeq N^1(X)$.

If $\mathcal{E}$ is a vector bundle on a Calabi-Yau threefold X, then by the Hirzebruch-Riemann-Roch theorem [Har77, p. 432] we have

$$\chi(X, \mathcal{E}) = \frac{1}{12}c_1(\mathcal{E})c_2(X) + \frac{1}{6}\big(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})\big). \tag{4}$$

In particular, for a Cartier divisor L on X this gives

$$\chi(X, L) = \frac{1}{6}L^3 + \frac{1}{12}L \cdot c_2(X). \tag{5}$$

For a vector bundle $\mathcal{E}$ of rank 3 and a line bundle $\mathcal{L}$ on X, [Ful98, Example 3.2.2] gives

$$\begin{aligned} c_1(\mathcal{E} \otimes \mathcal{L}) &= 3c_1(\mathcal{L}) + c_1(\mathcal{E}),\\ c_2(\mathcal{E} \otimes \mathcal{L}) &= 3c_1(\mathcal{L})^2 + 2c_1(\mathcal{E})c_1(\mathcal{L}) + c_2(\mathcal{E}),\\ c_3(\mathcal{E} \otimes \mathcal{L}) &= c_1(\mathcal{L})^3 + c_1(\mathcal{E})c_1(\mathcal{L})^2 + c_2(\mathcal{E})c_1(\mathcal{L}) + c_3(\mathcal{E}). \end{aligned} \tag{6}$$

Given a locally free sheaf $\mathcal{E}$, we denote its k-th Segre class by $s_k(\mathcal{E})$. The sign is determined in such a way that if $\mathcal{E}$ has rank r and $\dim X = n$, then

$$s_r(\mathcal{E}) = c_1\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\big)^{n+r-1}.$$

Here $\mathbb{P}(\mathcal{E})$ is defined by taking hyperplanes; in particular, $s_k(\mathcal{E})$ differs in sign from the definition in [Ful98, Chapter 3] in the case when k is odd.

Proposition 2.1. *If X is a Calabi-Yau threefold, then $c_2(X) \in \overline{\mathrm{NE}}(X)\setminus\{0\}$.*

Proof. We have $c_2(X) \in \overline{\mathrm{NE}}(X)$ by [Miy87] and $c_2(X) \neq 0$ by Yau's theorem, see e.g. [Kob87, IV.4.15]. □

Proposition 2.2. *Let X be a Calabi-Yau threefold, let L be a nef Cartier divisor on X which is not semiample, and let D be a smooth divisor on X. Denote*

$$\mathcal{E}_m = \Omega^1_X(\log D) \otimes \mathcal{O}_X(mL).$$

Then

(i) $\kappa(X, L) = -\infty$,
(ii) $L^3 = L \cdot c_2(X) = 0$,
(iii) $\chi\big(X, \mathcal{O}_X(mL)\big) = 0$ *for every m,*

(iv) $\chi\big(X,\Omega^1_X\otimes\mathcal{O}_X(mL)\big)=-\frac{1}{2}c_3(X)$ *for every* m,

(v) *for every* m *we have*

$$\chi(X,\mathcal{E}_m)=\frac{1}{2}m^2L^2\cdot D-\frac{1}{2}mL\cdot D^2+\frac{1}{6}D^3+\frac{1}{12}D\cdot c_2(X)-\frac{1}{2}c_3(X),$$

(vi) *for every* m *we have*

$$s_3(\mathcal{E}_m)=10m^2L^2\cdot D-D\cdot c_2(X)-c_3(X).$$

Proof. The statement (i) is [Ogu93], or it can be viewed as a consequence of log abundance for threefolds [KMM94]. Statement (ii) is the basepoint free theorem together with [Ogu93, 2.7]. From (ii) and (5) we obtain (iii).

Set $\mathcal{F}_m=\Omega^1_X\otimes\mathcal{O}_X(mL)$. Using (6) and (ii), we calculate:

$$c_1(\mathcal{F}_m)=3mL,\quad c_2(\mathcal{F}_m)=3m^2L^2+c_2(X),\quad c_3(\mathcal{F}_m)=-c_3(X),$$

which gives (iv) by (4) and (ii).

Since $c_j(\mathcal{O}_D)=D^j$ for $1\leq j\leq 3$, from (1) we obtain

$$c_1\big(\Omega^1_X(\log D)\big)=D,\quad c_2\big(\Omega^1_X(\log D)\big)=c_2(X)+D^2,$$
$$c_3\big(\Omega^1_X(\log D)\big)=c_2(X)\cdot D-c_3(X)+D^3.$$

Using (6) and (ii), we calculate:

$$\begin{aligned}(7)\qquad & c_1(\mathcal{E}_m)=3mL+D,\quad c_2(\mathcal{E}_m)=3m^2L^2+2mL\cdot D+c_2(X)+D^2,\\ & c_3(\mathcal{E}_m)=m^2L^2\cdot D+mL\cdot D^2+c_2(X)\cdot D-c_3(X)+D^3,\end{aligned}$$

which gives (v) by (4) and (ii).

Finally, (vi) follows from the formula

$$s_3(\mathcal{E}_m)=c_1(\mathcal{E}_m)^3-2c_1(\mathcal{E}_m)\cdot c_2(\mathcal{E}_m)+c_3(\mathcal{E}_m)$$

and from (7) and (ii). □

Lemma 2.3. *Let* X *be a Calabi-Yau threefold with* $\rho(X)=2$*, and assume that there exists a nef Cartier divisor* L *on* X *which is not semiample. If* $S\subseteq X$ *is an irreducible surface, then* $S\cdot c_2(X)>0$.

Proof. Fix an ample divisor H on X. Since $\rho(X)=2$, we can write

$$S=\alpha L+\beta H\quad\text{with }\alpha,\beta\in\mathbb{Q}.$$

Notice that $\beta>0$, since otherwise $\kappa(X,L)\geq 0$, which contradicts Proposition 2.2(i). Hence

$$S\cdot c_2(X)=\beta H\cdot c_2(X)>0,$$

by Proposition 2.2(ii) and Proposition 2.1. □

Lemma 2.4. *Let* X *be a Calabi-Yau threefold with* $\rho(X)=2$*, and assume that there exists a nef Cartier divisor* L *on* X *with* $\nu(X,L)=2$ *which is not semiample. Then for every surface* S *on* X *we have* $L^2\cdot S>0$.

Proof. Assume that S is a surface on X with $L^2 \cdot S = 0$. Since L is not semiample, S is not proportional to L by Proposition 2.2(i). Therefore, as $\rho(X) = 2$, the divisors L and S form a basis of $N^1(X)_{\mathbb{R}}$. Since also $L^3 = 0$, the 1-cycle L^2 is orthogonal to L and S, hence $L^2 = 0$, a contradiction. □

Lemma 2.5. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$, and assume that there exists a nef Cartier divisor L on X with $\nu(X, L) = 2$ which is not semiample.*

(a) *There exist only countably many curves C_i on X such that $L \cdot C_i = 0$.*
(b) *If A is a basepoint free ample divisor and if D is a very general member of the linear system $|A|$, then $C_i \not\subseteq D$ for all i, and $L|_D$ is ample.*

Proof. Assume (a) does not hold. Then, since the Hilbert scheme of X (for any fixed polarization) has only countably many components, there exists a one-dimensional family $(C_t)_{t \in T}$ of generically irreducible curves such that

$$L \cdot C_t = 0 \quad \text{for all } t \in T. \tag{8}$$

Let S be the irreducible surface covered by the curves C_t. Then $L|_S$ is a nef divisor which is not big: otherwise, by Kodaira's trick we could write $L|_S \sim_{\mathbb{Q}} H + E$ for ample, respectively effective, $\mathbb{Q}$-divisors H and E on S. But then (8) implies $E \cdot C_t < 0$ for all $t \in T$, hence $C_t \subseteq E$, a contradiction. Therefore $(L|_S)^2 = 0$, a contradiction with Lemma 2.4. This shows (a).

For (b), if $D \in |A|$ is general, then clearly $C_i \not\subseteq D$ by (a), and $L|_D$ is ample by the Nakai-Moishezon criterion due to $L^2 \cdot D > 0$ by Lemma 2.4. □

2.2. **Positivity of locally free sheaves.** Recall that a locally free sheaf $\mathcal{E}$ on a variety X is nef if the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef. The following properties are well known.

Lemma 2.6. *Let X be projective manifold.*

(a) *If $\mathcal{E}$ is a nef locally free sheaf on X, then every locally free quotient sheaf of $\mathcal{E}$ is nef.*
(b) *Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of locally free sheaves. If $\mathcal{E}$ and $\mathcal{G}$ are nef, then $\mathcal{F}$ is nef.*
(c) *Assume that X is a curve. A locally free sheaf $\mathcal{E}$ on X is nef if and only if all locally free sheaves have non-negative degree, and is ample if and only if all locally free quotients have positive degree. A semistable locally free sheaf $\mathcal{E}$ on X is nef if and only if it has a non-negative degree, and is ample if and only if it has a positive degree.*
(d) *Assume that X is a curve, and let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \otimes \mathcal{T} \to 0$ be an exact sequence, where $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, $\mathcal{E}$ and $\mathcal{G}$ are nef and $\mathcal{T}$ is a torsion sheaf. Then $\mathcal{F}$ is nef.*

Proof. For (a), (b) and (c), see for instance [Laz04, Proposition 6.1.2, Theorem 6.2.12, and the proof of Theorem 6.4.15]. For (d), let $\mathcal{E}'$ be the saturation of $\mathcal{E}$ in $\mathcal{F}$, so that we have the exact sequence

$$0 \to \mathcal{E}' \to \mathcal{F} \to \mathcal{G} \to 0.$$

By (b), it suffices to show that $\mathcal{E}'$ is a nef. Assuming otherwise, by (c) there exists a quotient $\mathcal{E}' \to \mathcal{Q}' \to 0$ with $\deg \mathcal{Q}' < 0$, and it induces a quotient $\mathcal{E} \to \mathcal{Q} \to 0$ with $\mathcal{Q} \subseteq \mathcal{Q}'$. In particular, $\deg \mathcal{Q} \leq \deg \mathcal{Q}'$, a contradiction since $\mathcal{E}$ is nef. □

The following result, see [CP11, Theorem 0.1] and [CP15, Theorem 1.2], will be crucial in the proof of our main theorem.

Theorem 2.7. *Let X be a projective manifold, and let $(\Omega^1_X)^{\otimes m} \to \mathcal{Q}$ be a torsion free coherent quotient for some $m \geq 1$. If K_X is pseudoeffective, then $c_1(\mathcal{Q})$ is pseudoeffective.*

The following lemma is well known.

Lemma 2.8. *Let Y be a projective manifold, let $\mathcal{M}$ be a nef line bundle on Y, and let $\mathcal{E}$ be a nef locally free sheaf on Y. If $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is big, then*

$$H^q(Y, \omega_Y \otimes \mathcal{E} \otimes \det \mathcal{E} \otimes \mathcal{M}) = 0 \quad \text{for } q \geq 1.$$

Proof. Let $\pi \colon \mathbb{P}(\mathcal{E}) \to Y$ be the projection morphism, and let r be the rank of $\mathcal{E}$. Note that

$$\omega_{\mathbb{P}(\mathcal{E})} = \pi^*(\omega_Y \otimes \det \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-r) \quad \text{and} \quad \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E},$$

hence for every $q \geq 1$,

$$H^q(Y, \omega_Y \otimes \mathcal{E} \otimes \det \mathcal{E} \otimes \mathcal{M}) \simeq H^q\big(\mathbb{P}(\mathcal{E}), \omega_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r+1) \otimes \pi^*\mathcal{M}\big) = 0$$

by the Kawamata-Viehweg vanishing theorem. □

We often below use the following result [Fuj83, 6.2], see also [Laz04, Theorem 1.4.40].

Theorem 2.9. *Let X be a projective variety of dimension n and let D be a nef divisor on X. Then for any coherent sheaf $\mathcal{F}$ on X and for every i we have*

$$h^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) = O(m^{n-i}).$$

2.3. **The Cone conjecture.** Let V be a real vector space equipped with a rational structure, let $\mathcal{C}$ be a cone in V, and let Γ be a subgroup of $\mathrm{GL}(V)$ which preserves $\mathcal{C}$. A rational polyhedral cone $\Pi \subseteq \mathcal{C}$ is a *fundamental domain* for the action of Γ on $\mathcal{C}$ if $\mathcal{C} = \bigcup_{g\in\Gamma} g\Pi$ and $\operatorname{int}\Pi \cap \operatorname{int} g\Pi = \emptyset$ if $g \neq \mathrm{id}$.

The Cone conjecture (for the nef cone) deals with the action of the automorphism group of X on $\mathrm{Nef}(X)$, where X is a $\mathbb{Q}$-factorial projective klt variety with $K_X \equiv 0$. According to the original version by Morrison [Mor93], inspired by Mirror Symmetry, there exists a fundamental domain of the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)^+$, which is the convex hull of the cone spanned by all *rational* divisors in $\mathrm{Nef}(X)$.

The version of the conjecture by Kawamata [Kaw97] postulates that there exists a fundamental domain of the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)^e$, which is the cone spanned by all *effective* divisors in $\mathrm{Nef}(X)$. This version is more natural from the point of view of birational geometry, especially since it implies, in its most general form, that the number of minimal models of terminal varieties is finite [CL14, Theorem 2.14].

On the other hand, we have the following consequence of [Loo14, Theorem 4.1, Application 4.14], which is a result which belongs completely to the realm of convex geometry.

Lemma 2.10. *Let X be a $\mathbb{Q}$-factorial projective klt variety with $K_X \equiv 0$. Assume that there exists a polyhedral cone $\Pi \subseteq \mathrm{Nef}(X)^+$ such that $\mathrm{Aut}(X)\cdot\Pi$ contains the ample cone of X. Then $\mathrm{Aut}(X) \cdot \Pi = \mathrm{Nef}(X)^+$, and there exists a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)^+$.*

In particular, if we assume that Kawamata's version of the Cone conjecture holds, then necessarily Conjecture 1.3 holds. In Theorem 2.12 we show that at least one part of Conjecture 1.3 is true, that

$$\mathrm{Nef}(X)^e \subseteq \mathrm{Nef}(X)^+.$$

We need the following result of Shokurov and Birkar, [Bir11, Proposition 3.2]; note that it is a careful application of the boundedness of extremal rays, which is a consequence of Mori's bend-and-break.

Theorem 2.11. *Let X be a $\mathbb{Q}$-factorial projective variety, let $D_1, \dots, D_r$ be prime divisors on X and denote $V = \bigoplus_{i=1}^r \mathbb{R}D_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Then the set*

$$\mathcal{N}(V) = \{\Delta \in V \mid (X, \Delta) \text{ is log canonical and } K_X + \Delta \text{ is nef}\}$$

is a rational polytope.

Theorem 2.12. *Let X be a $\mathbb{Q}$-factorial projective klt variety with $K_X \equiv 0$. Then*

$$\mathrm{Nef}(X)^e \subseteq \mathrm{Nef}(X)^+.$$

Proof. Let D be an $\mathbb{R}$-divisor whose class is in $\mathrm{Nef}(X)^e$, and let $V \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the vector space spanned by all the components $D_1, \dots, D_r$ of D. Replacing D by εD for $0 < \varepsilon \ll 1$, we may assume that (X, D) is a klt pair, and in particular, with notation from Theorem 2.11, $D \in \mathcal{N}(V)$. On the other hand, clearly $D \in \sum_{i=1}^r \mathbb{R}_+ D_i \subseteq V$. By Theorem 2.11, the set

$$\mathcal{N}(V) \cap \sum_{i=1}^r \mathbb{R}_+ D_i$$

is a rational polytope, hence D is spanned by nef $\mathbb{Q}$-divisors. □

3. Differentials with coefficients in a line bundle

In this section, we prove several properties of nef line bundles on a Calabi-Yau threefold. For future applications in Part II to this paper, we mostly do not restrict ourselves to varieties with Picard number 2 or line bundles with numerical dimension 2.

Lemma 3.1. *Let X be a smooth projective surface and let L and M be divisors on X such that L is nef and $L^2 = M^2 = L \cdot M = 0$. If L and M are not numerically trivial, then L and M are numerically proportional.*

Proof. Let H be an ample divisor on X. By the Hodge index theorem we have $\lambda = L \cdot H \neq 0$ and $\mu = M \cdot H \neq 0$, and set $D = \lambda M - \mu L$. Then $D^2 = D \cdot H = 0$, hence $D \equiv 0$ again by the Hodge index theorem. □

Lemma 3.2. *Let X be a smooth projective threefold with $H^1(X, \mathcal{O}_X) = 0$ and let L be a nef divisor on X with $\nu(X, L) = 1$. Assume that $\kappa(X, L) = -\infty$ and let D be a non-zero effective divisor on X. Then the divisor $L - D$ is not pseudoeffective.*

Proof. Denote $G = L - D$, and assume that G is pseudoeffective. Denote $P = P_\sigma(G)$ and $N = N_\sigma(G)$, see [Nak04, Chapter III]. Let H be a general very ample divisor on X. Then $P|_H$ is nef by [Nak04, paragraph after Corollary V.1.5], and in particular

$$(P|_H)^2 \geq 0. \tag{9}$$

On the other hand, we have

$$0 = (L|_H)^2 = L|_H \cdot P|_H + L|_H \cdot N|_H + L|_H \cdot D|_H,$$

hence

$$L|_H \cdot P|_H = L|_H \cdot N|_H = L|_H \cdot D|_H = 0.$$

Now the Hodge index theorem implies $(P|_H)^2 \leq 0$, and hence $(P|_H)^2 = 0$ by (9). Then Lemma 3.1 yields $P|_H \equiv \lambda L|_H$ for some real number $\lambda \geq 0$, and hence $P \equiv \lambda L$ by the Lefschetz hyperplane section theorem. Note that $\lambda < 1$ since D is non-zero. Therefore, setting $E = \frac{1}{1-\lambda}(N + D)$, we obtain

$$L \equiv E,$$

and the Weil $\mathbb{R}$-divisor E is effective. Let $E_1, \dots, E_r$ be components of E and let $\pi\colon \mathrm{Div}_\mathbb{R}(X) \to N^1(X)_\mathbb{R}$ be the standard projection. Then $\pi^{-1}(\pi(L)) \cap \sum \mathbb{R}_+ E_i$ is a rational affine subspace of $\sum \mathbb{R} E_i$ which contains E, hence there exists a rational point

$$E' \in \pi^{-1}(\pi(L)) \cap \sum \mathbb{R}_+ E_i.$$

Therefore $L \equiv E'$, and consequently $L \sim_\mathbb{Q} E'$, which is a contradiction with $\kappa(X, L) = -\infty$. □

Remark 3.3. The assertion of Lemma 3.2 is obviously also true when $\nu(X, L) = 2$, provided that $\rho(X) = 2$.

The following result has been claimed in [Wil94, 3.1] when $\nu(X,L)=2$. The method of proof below was already used in [HPR13, Theorem 5.1].

Proposition 3.4. *Let X be a Calabi-Yau threefold and let L be a nef divisor on X such that $\kappa(X,L)=-\infty$. If $\nu(X,L)=2$, assume also that $\rho(X)=2$. Then there is a positive integer k such that*

$$H^0\big(X,\Omega_X^q\otimes\mathcal{O}_X(mkL)\big)=0\quad\textit{for all } m \textit{ and all } q.$$

Proof. Assume to the contrary that there exists q such that

$$H^0\big(X,\Omega_X^q\otimes\mathcal{O}_X(mL)\big)\neq 0$$

for infinitely many m. Every nontrivial section of $H^0\big(X,\Omega_X^q\otimes\mathcal{O}_X(mL)\big)$ gives an inclusion $\mathcal{O}_X(-mL)\to\Omega_X^q$, and consider the smallest subsheaf $\mathcal{F}\subseteq\Omega_X^q$ containing the images of all these inclusions. Let r be the generic rank of $\mathcal{F}$. Then, without loss of generality, we may find infinitely many r-tuples $(m_1,\dots,m_r)$ such that the image of the map

$$\mathcal{O}_X(-m_1L)\oplus\cdots\oplus\mathcal{O}_X(-m_rL)\to\mathcal{F}$$

has rank r. Taking determinants, we obtain infinitely many inclusions $\mathcal{O}_X(-m'L)\to\det\mathcal{F}$. Let F be a Cartier divisor such that $\mathcal{O}_X(-F)$ is the saturation of $\det\mathcal{F}$ in $\bigwedge^r\Omega_X^q$. Therefore

$$H^0\big(X,\mathcal{O}_X(-F)\otimes\mathcal{O}_X(m'L)\big)\neq 0\quad\text{for infinitely many } m'. \tag{10}$$

Consider the exact sequence

$$0\to\mathcal{O}_X(-F)\to\bigwedge^r\Omega_X^q\to\mathcal{Q}\to 0.$$

Since $\mathcal{O}_X(-F)$ is saturated, it follows that $\mathcal{Q}$ is torsion free, and hence $c_1(\mathcal{Q})$ is pseudoeffective by Theorem 2.7. As $\omega_X\simeq\mathcal{O}_X$, we deduce from the exact sequence above that $F=c_1(\mathcal{Q})$, hence the divisor F is pseudoeffective.

From (10), for every such m' we obtain an effective divisor $N_{m'}$ such that

$$N_{m'}+F\sim m'L.$$

Now Lemma 3.2 and Remark 3.3 yield $N_{m'}=0$ for all m', hence some multiple of L is linearly equivalent to 0, a contradiction with $\kappa(X,L)=-\infty$. □

Corollary 3.5. *Let X be a Calabi-Yau threefold with $\rho(X)=2$, and let L be a nef Cartier divisor with $\nu(X,L)=2$ which is not semiample. Then there exists a positive integer k such that*

$$h^j\big(X,\Omega_X^q\otimes\mathcal{O}_X(kmL)\big)=O(m)\quad\textit{for all } q \textit{ and } j.$$

Proof. By Proposition 2.2 and by Serre duality, for all m we have

$$\chi\big(X,\mathcal{O}_X(mL)\big)=0,$$
$$\chi\big(X,\Omega^1_X\otimes\mathcal{O}_X(mL)\big)=-\frac{1}{2}c_3(X),$$
$$\chi\big(X,\Omega^2_X\otimes\mathcal{O}_X(mL)\big)=\frac{1}{2}c_3(X).$$

By Proposition 3.4 and by Serre duality,

$$h^j\big(X,\Omega^q_X\otimes\mathcal{O}_X(mL)\big)=0\quad\text{for } m\gg 0,\ j\in\{0,3\},\ \text{and all } q.$$

Since

$$h^2\big(X,\Omega^q_X\otimes\mathcal{O}_X(mL)\big)=O(m)$$

by Theorem 2.9, we obtain $h^1\big(X,\Omega^q_X\otimes\mathcal{O}_X(mL)\big)=O(m)$. □

The following lemma is well known to the experts.

Lemma 3.6. *Let X be a projective manifold and let L be a pseudoeffective Cartier divisor on X. Let h be a singular hermitian metric on $\mathcal{O}_X(L)$ with semipositive curvature current and multiplier ideal sheaf $\mathcal{I}(h)$. Let D be an effective Cartier divisor such that $\mathcal{I}(h)\subseteq\mathcal{O}_X(-D)$. Then $L-D$ is pseudoeffective.*

Proof. By [DEL00, Theorem 1.10] there exists an ample line bundle G on X such that $\mathcal{O}_X(G+mL)\otimes\mathcal{I}(h^{\otimes m})$ is globally generated for all $m\geq 1$. Since

$$\mathcal{I}(h^{\otimes m})\subseteq\mathcal{I}(h)^m\subseteq\mathcal{O}_X(-mD),$$

where the first inclusion follows from [DEL00, Theorem 2.6], for all $m\geq 1$ we have

$$H^0\big(X,G+m(L-D)\big)\neq 0.$$

Hence $L-D=\lim\limits_{m\to\infty}\frac{1}{m}\big(m(L-D)+G\big)$ is pseudoeffective. □

Corollary 3.7. *Let X be a Calabi-Yau threefold and let L be a nef Cartier divisor on X with $\kappa(X,L)=-\infty$. Then there exists a positive integer m_0 such that*

$$H^q(X,mL)=0\quad\textit{for all } q \textit{ and } m\geq m_0.$$

Proof. If $\nu(X,L)=2$, then $H^q(X,mL)=0$ for $q\geq 2$ and all $m\geq 1$ by the Kawamata-Viehweg vanishing [Kaw82, Corollary]. Since $\chi(X,mL)=0$ for all m by Proposition 2.2(iii), and since $H^0(X,mL)=0$ by assumption, we also have $H^1(X,mL)=0$ for $m\geq 1$.

So we may assume that $\nu(X,L)=1$. Let h be a singular metric on $\mathcal{O}_X(L)$ with semipositive curvature current. Let $\mathcal{I}(h^{\otimes m})$ be the multiplier ideal of the associated metric $h^{\otimes m}$ on $\mathcal{O}_X(mL)$ and denote by $V_m\subseteq X$ the subspace defined by $\mathcal{I}(h^{\otimes m})$.

The subspace V_m cannot contain an effective divisor D: otherwise, by Lemma 3.6, $mL-D$ would be pseudoeffective, which would contradict

Lemma 3.2. Thus $\dim V_m \leq 1$. The Hard Lefschetz theorem [DPS01, Theorem 0.1] gives the surjection

$$H^0\big(X, \Omega_X^1 \otimes \mathcal{O}_X(mL) \otimes \mathcal{I}(h^{\otimes m})\big) \to H^2\big(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^{\otimes m})\big)$$

hence $H^2\big(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^{\otimes m})\big) = 0$ by Proposition 3.4. From the long cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_X(mL) \otimes \mathcal{I}(h^{\otimes m}) \to \mathcal{O}_X(mL) \to \mathcal{O}_{V_m}(mL) \to 0$$

we obtain

$$H^2(X, mL) = 0 \quad \text{for } m \geq m_0.$$

Since $H^3(X, mL) = 0$ for $m \geq 1$ by Serre duality, we conclude as above. □

In this context, we note

Theorem 3.8. *Let X be a Calabi-Yau threefold and let L be a nef divisor on X with $\nu(X, L) = 1$. Assume that there is a singular metric h on $\mathcal{O}_X(L)$ with semipositive curvature current such that the multiplier ideal sheaf $\mathcal{I}(h)$ is different from $\mathcal{O}_X$. Then L is semiample.*

Proof. Let $V \subseteq X$ be the subspace defined by $\mathcal{I}(h)$, and let x be a closed point in V with ideal sheaf $\mathcal{I}_x$ in X. Let $\pi\colon \hat{X} \to X$ be the blowup of X at x and let $E = \pi^{-1}(x)$ be the exceptional divisor. Let $\hat{h}$ be the induced metric on $\pi^*\mathcal{O}_X(L)$. By [Dem01, Proposition 14.3], we have

$$\mathcal{I}(\hat{h}) \subseteq \pi^{-1}\mathcal{I}(h) \cdot \mathcal{O}_{\hat{X}} \subseteq \pi^{-1}\mathcal{I}_x \cdot \mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(-E).$$

By Lemma 3.6, the divisor $\pi^*L - E$ is pseudoeffective, hence π^*L is semiample by Lemma 3.2. □

4. Log differentials

The following result is crucial in the proof of Theorem 1.2.

Lemma 4.1. *Let X be a projective manifold and let $C \subseteq X$ be an irreducible curve on X such that $K_X \cdot C \geq 0$. Let $\mathcal{L}$ be an ample line bundle on X. Then there exists a positive integer m_0 such that for all $m \geq m_0$ and for a general element $D \in |\mathcal{L}^{\otimes m}|$, the sheaf $\Omega_X^1(\log D)|_C$ is nef.*

Proof. Let $\nu\colon \tilde{C} \to C$ be the normalization. If $\Omega_X^1|_C$ is nef, the assertion follows from the residue sequence and from Lemma 2.6(d). So we may assume that $\Omega_X^1|_C$ is not nef, or equivalently, that $\nu^*(\Omega_X^1|_C)$ is not nef. Since $\deg \nu^*(\Omega_X^1|_C) \geq 0$ by assumption, the bundle $\nu^*(\Omega_X^1|_C)$ is not semistable by Lemma 2.6(c), and let $\mathcal{F}$ be its maximal destabilising subsheaf. Then $\mathcal{F}$ has positive slope, hence is ample by Lemma 2.6(c).

We obtain an exact sequence

$$0 \to \mathcal{F} \to \nu^*(\Omega_X^1|_C) \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is a locally free sheaf on $\tilde{C}$. Choose a positive integer N and a very ample line bundle $\mathcal{A}$ on C with $\deg \mathcal{A} = N$, such that $\mathcal{G} \otimes \mathcal{A}$ is globally generated. Then it follows that $\mathcal{G} \otimes \mathcal{A}'$ is nef for all ample divisors $\mathcal{A}'$ on C of degree $\geq N$.

Fix smooth points $x_1, \dots x_N \in C$ and let $\mathcal{I}$ be the corresponding ideal sheaf; since ν is an isomorphism around x_j, we consider x_j also as points on $\tilde{C}$. For $j = 1, \dots, N$, fix

$$v_j \in \nu^*(\Omega^1_X|_C) \otimes \mathbb{C}(x_j) \setminus \mathcal{F} \otimes \mathbb{C}(x_j),$$

which we also view as elements of $\Omega^1_X|_C \otimes \mathbb{C}(x_j)$. Choose a positive integer $m_0 \geq N$ such that $\mathcal{L}^{\otimes m}$ is very ample for $m \geq m_0$ and $H^1(X, \mathcal{I}^2 \otimes \mathcal{L}^{\otimes m}) = 0$. This implies that the restriction map

$$H^0(X, \mathcal{L}^{\otimes m}) \to H^0(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}^2)$$

is surjective. Since locally $\mathcal{O}_X/\mathcal{I}^2 \simeq \mathcal{O}_X/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2$, there exists a section $s \in H^0(X, \mathcal{L}^{\otimes m})$ such that

$$s(x_j) = 0 \quad \text{and} \quad ds(x_j) = v_j \quad \text{for every } j. \tag{11}$$

Now, let

$$M \subseteq H^0(X, \mathcal{L}^{\otimes m})$$

be the subspace of all sections $s \in H^0(X, \mathcal{L}^{\otimes m})$ for which there exists points $y_1, \dots, y_N \in \tilde{C}$ such that for all j we have

$$s(y_j) = 0 \quad \text{and} \quad ds(y_j) \notin \mathcal{F} \otimes \mathbb{C}(y_j).$$

Then $M \neq \emptyset$ by (11), and the set M is clearly open. Therefore, by Bertini, there exists a smooth element $D \in |\mathcal{L}^{\otimes m}|$ smooth, meeting C transversally at points $z_1, \dots, z_\ell$ in the smooth locus of C, and such that D is in M. By relabelling, we may assume that $A = \{z_1, \dots, z_N\} \subseteq C$ is the set of points such that

$$d\varphi(z_i) \notin \mathcal{F} \otimes \mathbb{C}(z_i), \tag{12}$$

where φ is the local equation of D.

We claim that the sheaf $\mathcal{F}$ is saturated in $\nu^*(\Omega^1_X(\log D)|_C)$ at the points of a subset $A' \subseteq \{z_1, \dots, z_\ell\}$, where $A \subseteq A'$. Granting the claim for the moment, let us see how it implies the lemma. We obtain the exact sequence

$$0 \to \mathcal{F} \to \nu^*(\Omega^1_X(\log D)|_C) \to \mathcal{G} \otimes \mathcal{O}_{\tilde{C}}\big(\textstyle\sum_{z_i \in A'} z_i\big) \otimes \mathcal{T} \to 0,$$

where $\mathcal{T}$ is a torsion sheaf whose support is contained in the set $(D \cap C) \setminus A'$. By our choice of N, the vector bundle $\mathcal{G} \otimes \mathcal{O}_{\tilde{C}}\big(\sum_{z_i \in A'} z_i\big)$ is nef, and since $\mathcal{F}$ is ample, the bundle $\nu^*(\Omega^1_X(\log D)|_C)$ is nef by Lemma 2.6(d).

Finally, we prove the claim. It is enough to show that for any $z_i \in A$, the linear map

$$\mathcal{F} \otimes \mathbb{C}(z_i) \to \nu^*(\Omega^1_X(\log D)|_C) \otimes \mathbb{C}(z_i)$$

has rank $\operatorname{rk}\mathcal{F}$. Consider the diagram:

$$\begin{array}{ccc} \mathcal{F}\otimes\mathbb{C}(z_i) & \longrightarrow & \nu^*(\Omega^1_X|_C)\otimes\mathbb{C}(z_i) \\ & \searrow & \big\downarrow \alpha \\ & & \nu^*(\Omega^1_X(\log D)|_C)\otimes\mathbb{C}(z_i) \end{array}$$

and note that $\ker\alpha = \mathbb{C}(z_i)d\varphi(z_i)$. Therefore, $\big(\mathcal{F}\otimes\mathbb{C}(z_i)\big)\cap\ker\alpha = 0$ by (12), and the claim follows. □

5. Proof of the Main Theorem

In this section we prove

Theorem 5.1. *Let X be a Calabi-Yau threefold with $\rho(X)=2$ and let L be a nef Cartier divisor on X. Suppose that $\nu(X,L)=2$ and that $c_3(X)\neq 0$. Then L is semiample.*

In the proof we argue by contradiction. Hence for the remainder of this section, we assume the following:

Notation 5.2. Let D be a smooth very ample divisor on X.

Denote by B a non-empty Zariski open affine subset of the linear system $\mathbb{P}\big(H^0(X,\mathcal{O}_X(D))\big)$ contained in the locus of smooth elements of that linear system. Set $\mathcal{X}=X\times B$, let $\pi\colon\mathcal{X}\to B$ be the projection map and let $\mathcal{L}$ be the pullback of L by the projection $\mathcal{X}\to X$. Let $\mathcal{D}\subseteq\mathcal{X}$ be the universal family of divisors parametrised by B. Note that $\mathcal{D}$ is a smooth divisor in $\mathcal{X}$. We consider the relative logarithmic cotangent sheaf

$$\Omega^1_{\mathcal{X}/B}(\log\mathcal{D})$$

with log poles along $\mathcal{D}$, which is a locally free sheaf of rank 3 on $\mathcal{X}$. Denote by

$$T_{\mathcal{X}/B}(-\log\mathcal{D})$$

its dual. For every point $b\in B$, denote $X_b=\pi^{-1}(b)$, $D_b=\mathcal{D}\cap X_b$ and $L_b=\mathcal{L}|_{X_b}$. Note that $X_b=X$, $L_b=L$ and

$$\Omega^1_{\mathcal{X}/B}(\log\mathcal{D})|_{X_b}=\Omega^1_{X_b}(\log D_b)\quad\text{and}\quad T_{\mathcal{X}/B}(-\log\mathcal{D})|_{X_b}=T_{X_b}(-\log D_b).$$

For each positive integer m, denote

$$\mathcal{E}_m=\Omega^1_{\mathcal{X}/B}(\log\mathcal{D})\otimes\mathcal{O}_{\mathcal{X}}(m\mathcal{L})\quad\text{and}\quad\mathcal{E}_{m,b}=\mathcal{E}_m|_{X_b}.$$

In the remainder of the section, we freely shrink B if necessary.

Lemma 5.3. *Let X be a Calabi-Yau threefold with $\rho(X)=2$ and let L be a nef Cartier divisor on X with $\nu(X,L)=2$ which is not semiample. Assuming Notation 5.2, the following holds:*

(a) *there exist a positive integer m_0 and a positive constant C, such that for every $b\in B$ and for all $m\geq m_0$, we have*

$$h^0(X_b,\mathcal{E}_{m,b})\geq Cm^2,$$

(b) *for a general point $b \in B$ we have*

$$h^0(X_b, \mathcal{E}_{m,b}) = \frac{1}{2}m^2L^2 \cdot D + O(m) \quad \textit{and} \quad h^1(X_b, \mathcal{E}_{m,b}) = O(m).$$

Proof. For (a), Proposition 2.2(v) gives

$$\chi(X_b, \mathcal{E}_{m,b}) = \frac{1}{2}m^2L^2 \cdot D + O(m),$$

where $O(m)$ does not depend on b. Fix $b_0 \in B$, and for each positive integer m, let

$$U_m = \big\{b \in B \mid h^2(X_b, \mathcal{E}_{m,b}) \leq h^2(X_{b_0}, \mathcal{E}_{m,b_0})\big\}.$$

These sets are Zariski open in B by upper-semicontinuity; denote

$$U = \bigcap_{m \geq 1} U_m.$$

Since

$$h^2(X_{b_0},\mathcal{E}_{m,b_0}) = O(m)$$

by Theorem 2.9, there exists a constant $C_1 > 0$ such that

$$h^2(X_b, \mathcal{E}_{m,b}) \leq C_1 m \quad \text{for all } m \text{ and for all } b \in U.$$

Therefore, there is a constant $C > 0$ and a positive integer m_0 such that for all $m \geq m_0$ and all $b \in U$ we have

$$h^0(X_b, \mathcal{E}_{m,b}) \geq \frac{1}{2}m^2L^2 \cdot D + O(m) - C_1 m \geq Cm^2.$$

We conclude by upper-semicontinuity, since U is dense in B.

For (b), we have $h^0\big(X_b, \Omega^1_{X_b} \otimes \mathcal{O}_{X_b}(mL_b)\big) = 0$ by Proposition 3.4 and $h^1\big(X_b, \Omega^1_{X_b} \otimes \mathcal{O}_{X_b}(mL_b)\big) = O(m)$ by Corollary 3.5. Tensoring the residue sequence (2) associated to D_b by $\mathcal{O}_{X_b}(mL_b)$ and taking the long cohomology sequence, since $L_b|_{D_b}$ is ample by Lemma 2.5, Serre vanishing gives the exact sequence

$$\begin{aligned} 0 &\to H^0(X_b, \mathcal{E}_{m,b}) \to H^0(D_b, \mathcal{O}_{D_b}(mL_b)) \\ &\to H^1(X_b, \Omega^1_{X_b} \otimes \mathcal{O}_{X_b}(mL_b)) \to H^1(X_b, \mathcal{E}_{m,b}) \to 0. \end{aligned}$$

This immediately implies $h^1(X_b, \mathcal{E}_{m,b}) = O(m)$ and

$$h^0(X_b, \mathcal{E}_{m,b}) = h^0(D_b, \mathcal{O}_{D_b}(mL)) + O(m).$$

Riemann-Roch and Serre vanishing give

$$h^0(D_b, \mathcal{O}_{D_b}(mL_b)) = \chi(D_b, \mathcal{O}_{D_b}(mL_b)) = \frac{1}{2}m^2(L_b|_{D_b})^2 + O(m),$$

and (b) follows from the last two equations. □

Proposition 5.4. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef Cartier divisor on X with $\nu(X, L) = 2$ which is not semiample. Let $\mathcal{A}$ be the set of all ample prime divisors on X, and let $A \in \mathcal{A}$ be an element such that $A \cdot c_2(X)$ is minimal. If M is an integral divisor such that $M = aA + bL$ with $a > 0$, then $a \geq 1$.*

Proof. Assume $0 < a < 1$. Replacing M by $M - \lfloor b \rfloor L$, we may assume that $0 \leq b < 1$, and hence that M is ample. By (5), we have

$$\chi(X, M) = \frac{1}{6} M^3 + \frac{1}{12} M \cdot c_2(X).$$

Since M is ample, by Proposition 2.1 we have $\chi(X, M) > 0$, and hence $h^0(X, M) > 0$ by Kodaira vanishing. Pick $E \in |M|$. Since $L \cdot c_2(X) = 0$ by Proposition 2.2(ii), we have

$$E \cdot c_2(X) = (aA + bL) \cdot c_2(X) < A \cdot c_2(X),$$

hence $E \notin \mathcal{A}$ by the choice of A. Write $E = \sum a_i E_i$, where a_i are positive integers and E_i are prime divisors. By Lemma 2.3 we have $E_i \cdot c_2(X) > 0$ for all i, hence

$$E_i \cdot c_2(X) \leq E \cdot c_2(X) < A \cdot c_2(X).$$

However, since $\rho(X) = 2$ and since L lies on the boundary of the pseudoeffective cone, at least one E_{i_0} must be ample. This contradicts the choice of A. □

Lemma 5.5. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef Cartier divisor on X with $\nu(X, L) = 2$ which is not semiample. Assume that*

$$H^0(X, \Omega_X^q \otimes \mathcal{O}_X(kL)) = 0 \quad \textit{for all } q \textit{ and all } k.$$

Assume additionally that

$$D \sim_{\mathbb{Q}} \alpha A + \beta L,$$

where A is as in Proposition 5.4 and $\alpha, \beta \in \mathbb{Q}_{>0}$. Let $S \sim aD + bL$ be an effective divisor on X such that

$$\mathcal{O}_X(S) \subseteq \Omega_X^r(\log D) \otimes \mathcal{O}_X(mL)$$

for some r and m. Then:

(i) *$\alpha \geq 1$ and $0 \leq a \leq 1 - \frac{1}{\alpha}$, and $a > 0$ if $b \neq 0$,*
(ii) *the divisor $(1 - a)D + (m - b)L$ is pseudoeffective.*

Proof. Notice that $a \geq 0$, since S is effective and L lies on the boundary of the pseudoeffective cone. If $b \neq 0$, then additionally $a \neq 0$ since L is not effective. The inclusion

$$\mathcal{O}_X(S) \subseteq \Omega_X^r(\log D) \otimes \mathcal{O}_X(mL)$$

implies that

$$H^0\big(X, \Omega_X^r(\log D)(-D) \otimes \mathcal{O}_X((1 - a)D + (m - b)L)\big) \neq 0,$$

so that, via the inclusion $\Omega_X^r(\log D)(-D) \subseteq \Omega_X^r$,

$$H^0\big(X, \Omega_X^r \otimes \mathcal{O}_X((1 - a)D + (m - b)L)\big) \neq 0. \tag{13}$$

Hence we obtain an inclusion

$$\mathcal{O}_X\big((a - 1)D + (b - m)L\big) \to \Omega_X^r.$$

Since X is a Calabi-Yau threefold, the divisor

$$(1-a)D + (m-b)L \sim_{\mathbb{Q}} (1-a)\alpha A + \big(m - b - (a-1)\beta\big)L$$

is pseudoeffective by Theorem 2.7. Thus

$$(1-a)\alpha \geq 0, \tag{14}$$

and hence $a \leq 1$. If $a = 1$, then (13) implies

$$H^0\big(X, \Omega_X^r \otimes \mathcal{O}_X((m-b)L)\big) \neq 0,$$

which contradicts our assumption. Then from (14) and from Proposition 5.4 we have $\alpha \geq 1$ and $(1-a)\alpha \geq 1$, and the lemma follows. □

Proposition 5.6. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef Cartier divisor on X with $\nu(X, L) = 2$ which is not semiample. Assume Notation 5.2, and assume additionally that*

$$D \sim_{\mathbb{Q}} \alpha A + \beta L,$$

where A is as in Proposition 5.4 and $\alpha, \beta \in \mathbb{Q}_{>0}$.

Then, possibly after shrinking B, there exists a positive number n_1 and an algebraic set $\mathcal{V} \subsetneq \mathcal{X}$ such that for all irreducible curves $C \not\subseteq \mathcal{V}$ which are contracted by π, the restricted bundle $\mathcal{E}_{n_1}|_C$ is nef.

Proof. We note first that by Proposition 3.4, by passing to a multiple of L we may assume that

$$H^0(X, \Omega_X^q \otimes \mathcal{O}_X(kL)) = 0 \quad \text{for all } q \text{ and all } k. \tag{15}$$

We prove the proposition in several steps.

Step 1. For each m, let $U_m \subseteq B$ be the locus of points where the sheaf $\pi_*\mathcal{E}_m$ is locally free and has the base change property, and denote

$$U = \bigcap_{m \geq 1} U_m.$$

Fix $b_0 \in U$. By Lemma 5.3(a), there exist a positive constant C and a positive integer n_1 such that

$$h^0(X_{b_0}, \mathcal{E}_{m,b_0}) \geq Cm^2 \geq 2 \quad \text{for } m \geq n_1. \tag{16}$$

Since B is affine, by the definition of U the map

$$H^0(\mathcal{X}, \mathcal{E}_m) \to H^0(X_{b_0}, \mathcal{E}_{m,b_0})$$

is surjective for all m, cf. [Har77, III.12]. In particular,

$$\operatorname{rk} \operatorname{Im}(\pi^*\pi_*(\mathcal{E}_m) \to \mathcal{E}_m) \geq 1 \quad \text{for } m \gg 0.$$

Let $\mathcal{S}_m$ be the saturation of $\operatorname{Im}(\pi^*\pi_*(\mathcal{E}_m) \to \mathcal{E}_m)$ in $\mathcal{E}_m$, and let $\mathcal{S}_{m,b}$ be the saturation, in $\mathcal{E}_{m,b}$, of the sheaf generated by the global sections of $\mathcal{E}_{m,b}$. Then

$$\mathcal{S}_{m,b} = (\mathcal{S}_m|_{X_b})^{**},$$

and in particular, for every $b \in B$ we have $\mathcal{S}_m|_{X_b} = \mathcal{S}_{m,b}$ generically. Let

$$r_m = \operatorname{rk} \mathcal{S}_m = \operatorname{rk} \mathcal{S}_{m,b}.$$

By possibly replacing b_0, we may assume that

$$\mathcal{S}_m|_{X_{b_0}} = \mathcal{S}_{m,b_0} \quad \text{for all } m, \tag{17}$$

see for instance [GKP13, Proposition 5.2]. The sheaf $\det \mathcal{S}_{m,b_0}$ is a line bundle by [Har80, Proposition 1.9], and notice that $\det \mathcal{S}_{m,b_0}$ is effective by (16); say $\det \mathcal{S}_{m,b_0} \simeq \mathcal{O}_{X_{m,b_0}}(M)$ with an effective divisor M. Write

$$M \sim_{\mathbb{Q}} a_m D_{b_0} + b_m L_{b_0} \tag{18}$$

with rational numbers a_m and b_m. Since

$$\det \mathcal{S}_{m,b_0} \subseteq \Omega^{r_m}_{X_{b_0}}(\log D_{b_0}) \otimes \mathcal{O}_{X_{b_0}}(r_m m L_{b_0})$$

and (15) holds, Lemma 5.5 gives $\alpha \geq 1$,

$$0 \leq a_m \leq 1 - \frac{1}{\alpha} \quad \text{for all } m, \tag{19}$$

and that

(20) the divisor $(1 - a_m)D_{b_0} + (r_m m - b_m)L_{b_0}$ is pseudoeffective.

Step 2. Suppose $r_m = 3$ for some m, and we set

$$\mathcal{V} = \operatorname{Supp}\operatorname{Coker}(\pi^*\pi_*\mathcal{E}_m \to \mathcal{E}_m).$$

Then for every curve $C \not\subseteq \mathcal{V}$ contracted by π, the restriction $\mathcal{E}_m|_C$ is generically generated, hence nef.

Step 3. Hence we may assume that $r_m \leq 2$ for all m. In this step we assume that $r_m = 1$ for infinitely many m, and in particular, $\det \mathcal{S}_{m,b_0} = \mathcal{S}_{m,b_0}$.

By (5) and (18), together with Proposition 2.2(ii) we have

$$\chi(X_{b_0}, \mathcal{S}_{m,b_0}) = \frac{1}{2}a_m b_m^2 L^2 \cdot D + \frac{1}{2}a_m^2 b_m L \cdot D^2 + \frac{1}{6}a_m^3 D^3 + \frac{1}{12}a_m D \cdot c_2(X). \tag{21}$$

If $b_m < 0$ for infinitely many m, then these b_m are bounded from below, since a_m are bounded from above and the divisors $a_m D_{b_0} + b_m L$ are pseudoeffective. But then all $\mathcal{S}_{m,b_0}$ lie in a compact subset of $\operatorname{Pic}(X)$, hence take only finitely many values, which contradicts the fact that $h^0(X_{b_0}, \mathcal{S}_{m,b_0})$ grows quadratically with m by (16).

Therefore,

$$b_m \geq 0 \quad \text{for } m \gg 0, \tag{22}$$

and similar argument shows that, by (20), there exists a positive constant c such that

$$b_m - m < c \quad \text{for all } m. \tag{23}$$

From (23), (16) and by Lemma 5.5 we have $a_m > 0$, which implies that $\mathcal{S}_{m,b_0}$ is ample for $m \gg 0$. Hence, by (21), by Kodaira vanishing, and by

Lemma 5.3(b) we have

$$\begin{aligned} h^0(X_{b_0}, \mathcal{S}_{m,b_0}) &= \frac{1}{2} m^2 L^2 \cdot D + O(m) \\ &= \frac{1}{2} a_m b_m^2 L^2 \cdot D + \frac{1}{2} a_m^2 b_m L \cdot D^2 + \frac{1}{6} a_m^3 D^3 + \frac{1}{12} a_m D \cdot c_2(X), \end{aligned}$$

which is a contradiction by (19), (22) and (23).

Step 4. We are now reduced to the case $r_m = 2$ for all large m.

Let $p_X : \mathcal{X} \to X$ be the projection to X and recall that $\mathcal{L} = p_X^* L$. Consider the subsheaves

$$\mathcal{S}_{m+1} \subseteq \mathcal{E}_{m+1} \quad \text{and} \quad \mathcal{S}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L}) \subseteq \mathcal{E}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L}) = \mathcal{E}_{m+1}.$$

Introduce the torsion free rank 1 sheaves $\mathcal{Q}_m = \mathcal{E}_m / \mathcal{S}_m$ and consider the induced maps

$$\alpha_m : \mathcal{S}_{m+1} \to \mathcal{E}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L}) \to \mathcal{Q}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L})$$

and

$$\beta_m : \mathcal{S}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L}) \to \mathcal{E}_{m+1} \to \mathcal{Q}_{m+1}.$$

Suppose first that β_m is not the zero map for some m, and in particular, β_m is generically surjective since $\mathcal{Q}_{m+1}$ is of rank 1. Then $\mathcal{Q}_{m+1}$ is π-nef outside of a finite union of subvarieties by Lemma 2.6(a), hence Lemma 2.6(b) and the exact sequence

$$0 \to \mathcal{S}_{m+1} \to \mathcal{E}_{m+1} \to \mathcal{Q}_{m+1} \to 0$$

give that $\mathcal{E}_{m+1}$ is π-nef outside of an algebraic set, confirming the claim of the proposition. We note here for later use, that if $\mathcal{E}_m$ is not π-nef outside of an algebraic set for all m, then

$$h^0(\mathcal{X}, \mathcal{Q}_m) = 0 \quad \text{for all } m. \tag{24}$$

Otherwise there would exist a nontrivial morphism $\mathcal{O}_{\mathcal{X}} \to \mathcal{Q}_m$, which would imply as above that $\mathcal{Q}_m$ and $\mathcal{E}_m$ are π-nef outside of an algebraic set.

Thus we may assume that β_m is the zero map for all large m, and thus

$$\mathcal{S}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L}) \subseteq \mathcal{S}_{m+1} \quad \text{for} \quad m \gg 0.$$

Therefore $\mathcal{S}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L})$ and $\mathcal{S}_{m+1}$ are generically equal, thus α_m is generically the zero map, which implies that α_m is the zero map since the image of α_m is a torsion free coherent subsheaf of $\mathcal{Q}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L})$. We conclude that there exists $m_1 \gg 0$ such that

$$\mathcal{S}_{m+1} = \mathcal{S}_m \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{L}) \quad \text{for} \quad m \geq m_1,$$

and thus

$$\mathcal{S}_m = \mathcal{S}_{m_1} \otimes \mathcal{O}_{\mathcal{X}}\big((m - m_1)\mathcal{L}\big).$$

Restricting this last equation to X_{b_0}, we have

$$\mathcal{S}_{m,b_0} = \mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_{X_{b_0}}\big((m - m_1)L_{b_0}\big).$$

We claim that

$$h^0\big(X_{b_0}, \mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_{X_{b_0}}(mL_{b_0})\big) = \frac{1}{2}m^2L^2 \cdot c_1(\mathcal{S}_{m_1,b_0}) + O(m). \tag{25}$$

This immediately implies the proposition: indeed, by (19) we have that $c_1(\mathcal{S}_{m_1,b_0}) = a_{m_1}D_{b_0} + b_{m_1}L_{b_0}$, where a_{m_1} is bounded away from 1, hence

$$h^0\big(X_{b_0}, \mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_{X_{b_0}}(mL_{b_0})\big) = \frac{1}{2}a_{m_1}m^2L^2 \cdot D + O(m).$$

Since $h^0(X_{b_0}, \mathcal{E}_{m_1+m,b_0}) = h^0\big(X, \mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_{X_{b_0}}(mL_{b_0})\big)$, this is a contradiction by Lemma 5.3(b).

It remains to prove (25). First note that the Chern classes $c_1\big(\mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_X(mL_{b_0})\big)$ and $c_2\big(\mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_X(mL_{b_0})\big)$ are computed as in the locally free case, since $\mathcal{S}_{m_1,b_0}$ is locally free outside a finite set by [Har80, Corollary 1.4], whereas

$$c_3\big(\mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_X(mL_{b_0})\big) = c_3(\mathcal{S}_{m_1,b_0}),$$

cf. [Har80, p. 130]. Thus Riemann-Roch for coherent sheaves [OTT81] gives

$$\chi\big(X_{b_0}, \mathcal{S}_{m_1,b_0} \otimes \mathcal{O}_{X_{b_0}}(mL_{b_0})\big) = \frac{1}{2}m^2L^2 \cdot c_1(\mathcal{S}_{m_1,b_0}) + O(m). \tag{26}$$

Define sheaves $\mathcal{Q}_{m,b_0}$ by the short exact sequences

$$0 \to \mathcal{S}_{m,b_0} \to \mathcal{E}_{m,b_0} \to \mathcal{Q}_{m,b_0} \to 0, \tag{27}$$

and observe that $\mathcal{Q}_{m,b_0} = \mathcal{Q}_m|_{X_{b_0}}$ by (17). Since B is affine, (24) shows that the rank of the sheaf $\pi_*\mathcal{Q}_m$ is zero for every m, hence by possibly changing b_0, we have $h^0(X_{b_0}, \mathcal{Q}_{m,b_0}) = 0$ for every m. Then (27) and Lemma 5.3(b) give

$$h^1(X_{b_0}, \mathcal{S}_{m,b_0}) \leq h^1(X_{b_0}, \mathcal{E}_{m,b_0}) = O(m),$$

and since $h^2(X_{b_0}, \mathcal{S}_{m,b_0}) = h^3(X_{b_0}, \mathcal{S}_{m,b_0}) = O(m)$ by Theorem 2.9, the claim follows from (26). □

Proposition 5.7. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef Cartier divisor on X with $\nu(X, L) = 2$ which is not semiample. Assume Notation 5.2, and assume additionally that*

$$D \sim_{\mathbb{Q}} \alpha A + \beta L,$$

where A is as in Proposition 5.4 and $\alpha, \beta \in \mathbb{Q}_{>0}$. Then, possibly after shrinking B, there exists a positive number m_2 such that

$$\mathcal{E}_{m,b} = \Omega^1_{X_b}(\log D_b) \otimes \mathcal{O}_{X_b}(mL_b) \quad \textit{is nef}$$

for all $m \geq m_2$ and for all $b \in B$.

Proof. Step 1. In this step, we show that after possibly shrinking B, there exist a positive integer n_2 and an algebraic set $\mathcal{C} \subseteq \mathcal{X}$ of codimension ≥ 2, such that $\mathcal{E}_m|_C$ is nef for every $m \geq n_2$ and for every curve $C \not\subseteq \mathcal{C}$ which is contracted by π.

Indeed, by Proposition 5.6 , we find a positive integer n_1 and an algebraic set $\mathcal{V} \subseteq \mathcal{X}$ such that $\mathcal{E}_{n_1}|_C$ is nef for all curves $C \not\subseteq \mathcal{V}$ contracted by π. Let

$\mathcal{V}_1, \dots, \mathcal{V}_k$ and $\mathcal{W}_1, \dots, \mathcal{W}_l$ be the codimension 1, respectively codimension 2 irreducible components of $\mathcal{V}$. Set $\pi_j = \pi|_{\mathcal{V}_j}$. By possibly shrinking B, we may assume that $\mathcal{V}$ does not contain any fibre of π, and that each π_j is flat.

For each j, the line bundle $\mathcal{L}|_{\mathcal{V}_j}$ is clearly π_j-nef. Moreover, it is also π_j-big: indeed, for each $b \in B$, the set $\mathcal{V}_j \cap X_b$ is a surface in X_b, and since $\nu(X, L) = 2$, we have $(\mathcal{L}|_{\mathcal{V}_j \cap X_b})^2 > 0$ by Lemma 2.4. Therefore, by Kodaira's trick, there exist a π_j-ample $\mathbb{Q}$-divisor $\mathcal{A}_j$ and an effective $\mathbb{Q}$-divisor $\mathcal{B}_j$ such that

$$\mathcal{L}|_{\mathcal{V}_j} \sim_{\mathbb{Q}} \mathcal{A}_j + \mathcal{B}_j.$$

For k sufficiently divisible, the sheaf $\mathcal{E}_{n_1}|_{\mathcal{V}_j} \otimes \mathcal{O}_{\mathcal{V}_j}(k\mathcal{A}_j)$ is π_j-globally generated, and we conclude that for every curve C contracted by π which is not contained in the locus

$$\mathcal{C} = \bigcup \operatorname{Supp} \mathcal{B}_j \cup \bigcup \mathcal{W}_i,$$

the sheaf $\mathcal{E}_{n_1+k}|_C$ is nef. We set $n_2 = n_1 + k$.

Step 2. In the second step we show that there exist a positive integer m_2 and finitely many curves $C_1, \dots, C_s$ on X, such that $\mathcal{E}_m|_C$ is nef for all $m \geq m_2$ and for all curves $C \notin \{C_1, \dots, C_s\}$. The proposition then follows immediately from Lemma 4.1.

In order to prove the claim, let $\mathcal{C}_1, \dots, \mathcal{C}_s$ be the irreducible components of $\mathcal{C}$. Fix j, and consider the normalisation $\nu_j \colon \mathcal{C}_j^\nu \to \mathcal{C}_j$ and the Stein factorisation $\alpha_j \colon \mathcal{C}_j^\nu \to B_j$ of $\pi|_{\mathcal{C}_j} \circ \nu_j$.

$$\begin{array}{ccccc} \mathcal{C}_j^\nu & \xrightarrow{\nu_j} & \mathcal{C}_j & \xrightarrow{\pi|_{\mathcal{C}_j}} & B \\ & {\scriptstyle \alpha_j}\searrow & & \nearrow & \\ & & B_j & & \end{array}$$

After possibly shrinking B, we may assume that α_j is a smooth morphism. Let B_{jk} be the connected components of B_j and let $\mathcal{C}_{jk}^\nu = \alpha_j^{-1}(B_{jk})$ and $\mathcal{C}_{jk} = \nu_j(\mathcal{C}_{jk}^\nu)$. Then $\mathcal{C}_{jk}^\nu$ is irreducible since α_j has connected fibres and therefore $\mathcal{C}_{jk}$ is an irreducible component of $\mathcal{C}_j$, which maps onto B. Now, for fixed k we have that

$$\nu_j^* \mathcal{L} \cdot \alpha_j^{-1}(b) = \mathcal{L} \cdot (\nu_j)_* \alpha_j^{-1}(b) \quad \text{is constant for } b \in B_{jk}.$$

So if this constant is positive, then $\mathcal{L}|_{\mathcal{C}_{jk}}$ is $\pi|_{\mathcal{C}_{jk}}$-ample. In particular, increasing m_2, we conclude that $\mathcal{E}_m|_{\mathcal{C}_{jk}}$ is $\pi|_{\mathcal{C}_{jk}}$-nef.

Therefore, if $\mathcal{E}_m|_C$ is not nef, then C belongs to a family $\mathcal{C}_{jk}$ on which L is numerically trivial. Since there are only countably many L-trivial curves on X by Lemma 2.5, each family $\mathcal{C}_{jk}$ must be constant. This finishes the proof. □

The following proposition was asserted in [Wil94, p. 697], but we could not follow the arguments on p. 683, first lines of the proof.

Proposition 5.8. *Let X be a Calabi-Yau threefold with $\rho(X)=2$ and let L be a nef Cartier divisor on X with $\nu(X,L)=2$ which is not semiample. Then*

$$H^2\big(X,\Omega^1_X\otimes\mathcal{O}_X(mL)\big)=0\quad\textit{for } m\gg 0.$$

Proof. We assume Notation 5.2, and we assume further that $D\sim_{\mathbb{Q}}\alpha A+\beta L$, where A is as in Proposition 5.4 and $\alpha,\beta\in\mathbb{Q}_{>0}$.

Step 1. By Proposition 5.7, possibly after shrinking B, there exists a positive number m_2 such that

$$\mathcal{E}_{m,b}=\Omega^1_{X_b}(\log D_b)\otimes\mathcal{O}_{X_b}(mL_b)\quad\text{is nef}\tag{28}$$

for all $m\geq m_2$ and for all $b\in B$. Possibly further shrinking B, by Lemma 2.5 we may assume that

$$L_b|_{D_b}\quad\text{is ample for every } b\in B.\tag{29}$$

Fix $b_0\in B$, and simplifying notation, set $X=X_{b_0}$ and $L=L_{b_0}$. We may assume that $D=D_{b_0}$, and we claim that

$$H^2\big(X,\Omega^1_X(\log D)\otimes\mathcal{O}_X(\mu L)\big)=0\quad\text{for }\mu\gg 0.\tag{30}$$

The claim immediately implies the proposition. Indeed, tensoring the residue sequence (2) with $\mathcal{O}_X(\mu L)$ and taking cohomology gives the exact sequence

$$\begin{aligned}H^1\big(D,\mathcal{O}_D(\mu L)\big)&\to H^2\big(X,\Omega^1_X\otimes\mathcal{O}_X(\mu L)\big)\\&\to H^2\big(X,\Omega^1_X(\log D)\otimes\mathcal{O}_X(\mu L)\big),\end{aligned}$$

hence it suffices to show $H^1\big(D,\mathcal{O}_D(\mu L)\big)=0$. But this follows by (29) and by Serre vanishing as soon as μ is sufficiently large.

Step 2. It remains to prove (30). Tensoring the standard exact sequence associated to D with $\Omega^1_X(\log D)\otimes\mathcal{O}_X(\mu L+D)$ and taking cohomology, we get the exact sequence

$$\begin{aligned}H^1\big(D,\Omega^1_X(\log D)\otimes\mathcal{O}_D(\mu L+D)\big)&\to H^2\big(X,\Omega^1_X(\log D)\otimes\mathcal{O}_X(\mu L)\big)\\&\to H^2\big(X,\Omega^1_X(\log D)\otimes\mathcal{O}_X(\mu L+D)\big).\end{aligned}$$

Hence, it suffices to show that for $\mu\gg 0$ we have

$$H^1\big(D,\Omega^1_X(\log D)\otimes\mathcal{O}_D(\mu L+D)\big)=0\tag{31}$$

and

$$H^2\big(X,\Omega^1_X(\log D)\otimes\mathcal{O}_X(\mu L+D)\big)=0.\tag{32}$$

The equation (31) follows from Serre vanishing. For (32), we may further assume that m_2 is so large, so that

$$10m_2^2D\cdot L^2-D\cdot c_2(X)-c_3(X)>0.\tag{33}$$

Denote $\mathcal{E}=\Omega^1_X(\log D)\otimes\mathcal{O}_X(m_2L)$. By [Ful98, Chapter 3], by Proposition 2.2(vi) and by (33), we have

$$c_1\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\big)^5=s_3(\mathcal{E})>0,$$

and therefore, by (28) the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef and big on $\mathbb{P}(\mathcal{E})$. Noticing that $\det \mathcal{E} = \mathcal{O}_X(D + 3m_2 L)$, we have

$$\Omega^1_X(\log D) \otimes \mathcal{O}_X(\mu L + D) = \mathcal{E} \otimes \det \mathcal{E} \otimes \mathcal{O}_X\big((\mu - 4m_2)L\big),$$

and (32) follows by Lemma 2.8. This finishes the proof. □

Proposition 5.9. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef line bundle on X with $\nu(X, L) = 2$ which is not semiample. Then*

$$H^1\big(X, \Omega^1_X \otimes \mathcal{O}_X(-mL)\big) = H^2\big(X, \Omega^2_X \otimes \mathcal{O}_X(mL)\big) = 0 \quad \textit{for } m \gg 0.$$

Proof. Choose a smooth very ample divisor D on X, which is general in the linear system $|D|$. Tensoring the residue sequence (3) associated to $\Omega^1_X(\log D)$ with $\mathcal{O}_X(-mL)$, and taking cohomology, we obtain the exact sequence

$$H^1\big(X, \Omega^1_X(\log D) \otimes \mathcal{O}_X(-D - mL)\big) \to H^1\big(X, \Omega^1_X \otimes \mathcal{O}_X(-mL)\big) \to H^1\big(D, \Omega^1_D \otimes \mathcal{O}_D(-mL)\big). \tag{34}$$

Now since $D + mL$ is ample, we have

$$H^1\big(X, \Omega^1_X(\log D) \otimes \mathcal{O}_X(-D - mL)\big) = 0$$

by [EV92, Corollary 6.4], and as $L|_D$ is ample by Lemma 2.5, by Serre duality and Serre vanishing we have

$$H^1\big(D, \Omega^1_D \otimes \mathcal{O}_D(-mL)\big) \simeq H^1\big(D, \Omega^1_D \otimes \mathcal{O}_D(mL)\big) = 0 \quad \text{for } m \gg 0.$$

Therefore, (34) gives $H^1\big(X, \Omega^1_X \otimes \mathcal{O}_X(-mL)\big) = 0$, which together with Serre duality proves our assertion. □

Corollary 5.10. *Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef line bundle on X with $\nu(X, L) = 2$ which is not semiample. Then*

(i) $c_3(X) = 0$,
(ii) *there exists a positive integer k such that for all integers m we have*

$$H^q\big(X, \Omega^p_X \otimes \mathcal{O}_X(mkL)\big) = 0 \quad \textit{for all } p, q.$$

Proof. By Proposition 3.4 and by Serre duality, there exists a positive integer k such that for all integers m we have

$$h^0\big(X, \Omega^1_X \otimes \mathcal{O}_X(mL)\big) = h^3\big(X, \Omega^1_X \otimes \mathcal{O}_X(mL)\big) = 0$$

Therefore, by Propositions 2.2(iv) and 5.8, for $m \gg 0$ we have

$$-\frac{1}{2}c_3(X) = \chi\big(X, \Omega^1_X \otimes \mathcal{O}_X(mkL)\big) = -h^1\big(X, \Omega^1_X \otimes \mathcal{O}_X(mkL)\big) \leq 0,$$

and by Propositions 2.2(iv) and 5.9, for $m \gg 0$ we have

$$-\frac{1}{2}c_3(X) = \chi\big(X, \Omega^1_X \otimes \mathcal{O}_X(-mkL)\big) = h^2\big(X, \Omega^1_X \otimes \mathcal{O}_X(-mkL)\big) \geq 0.$$

This shows (i) and

$$h^1\big(X, \Omega^1_X \otimes \mathcal{O}_X(mkL)\big) = h^2\big(X, \Omega^1_X \otimes \mathcal{O}_X(-mkL)\big) = 0 \quad \text{for } m \gg 0.$$

The other equalities follow from Serre duality. □

Remark 5.11. Let X be a Calabi-Yau threefold with $\rho(X) = 2$ and let L be a nef line bundle on X with $\nu(X, L) = 2$ which is not semiample. Let D be any smooth divisor. Then from Corollary 5.10 and the standard residue sequences associated to $\Omega^1_X(\log D)$ and $\Omega^2_X(\log D)$, we deduce that there is a positive integer k such that for every integer m the residue maps

$$H^0\big(X, \Omega^1_X(\log D) \otimes \mathcal{O}_X(mkL)\big) \to H^0\big(D, \mathcal{O}_D(mkL)\big)$$

and

$$H^0\big(X, \Omega^2_X(\log D) \otimes \mathcal{O}_X(mkL)\big) \to H^0\big(D, \Omega^1_D \otimes \mathcal{O}_D(mkL)\big)$$

are isomorphisms. We expect that on a simply connected manifold this can never happen.

6. Proof of Corollary 1.5

In this section we prove Corollary 1.5. We follow the arguments in [Ogu93]. Choose a rational boundary ray R of $\mathrm{Nef}(X)$ and a Cartier divisor L such that $R = \mathbb{R}_+ L$. If $L^2 \neq 0$, then L is semiample by Theorem 1.2, and hence produces an elliptic fibration $f\colon X \to S$. Then X contains a rational curve by [Pet91, Ogu93]. If $L^2 = 0$, then we claim that the second ray R' is also rational. Indeed, fix any divisor L' such that $R' = \mathbb{R}_+ L'$. If $(L')^3 > 0$, then the claim follows from [Wil89] or from the Cone theorem. If $(L')^3 = 0$, fix an ample divisor H and consider the cubic polynomial

$$p(t) = (L + tH)^3 \in \mathbb{Z}[t].$$

There is a unique real number $t_0 < 0$ such that $-(L + t_0 H) \in R'$. Thus $p(t_0) = 0$. Since p has integer coefficients and a double zero at 0, the number t_0 is rational and hence the ray R' is rational. Since t_0 is a simple zero of p, we have $(L')^2 \neq 0$, and we conclude as above.

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