QUANTUM AND SPECTRAL PROPERTIES OF THE LABYRINTH MODEL

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Abstract. We consider the Labyrinth model, which is a two-dimensional quasicrystal model. We show that the spectrum of this model, which is known to be a product of two Cantor sets, is an interval for small values of the coupling constant. We also consider the density of states measure of the Labyrinth model, and show that it is absolutely continuous with respect to Lebesgue measure for almost all values of coupling constants in small coupling regime.

1. Introduction

1.1. Quasicrystal and the Labyrinth model. Fibonacci Hamiltonian is a central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded self-adjoint operator in $l^2(\mathbb{Z})$:

$$(H_{\lambda,\beta}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \beta \mod 1)\psi(n), \tag{1.1}$$

where $\alpha = \frac{\sqrt{5}-1}{2}$ is the *frequency*, $\beta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the *phase*, and $\lambda > 0$ is the *coupling constant*. By minimality of the circle rotation and strong operator convergence, the spectrum is easily seen to be independent of β. With this specific choice of α, when $\beta = 0$ the potential of (1.1) coincides with the *Fibonacci substitution sequence* (for the precise definition, see section 2). Papers on this model include [**6**], [**8**], [**9**], [**10**], [**13**]. In [**9**], the authors showed that for sufficiently small coupling constant, the spectrum is a dynamically defined Cantor set, and the *density of states measure* is exact dimensional. Later, this result was extended for all values of the coupling constant [**13**].

In physics papers, it is more traditional to consider off-diagonal model, but in fact they are known to be very similar. The operator of off-diagonal model has the following form:

$$(H_\omega\psi)(n) = \omega(n+1)\psi(n+1) + \omega(n)\psi(n-1), \tag{1.2}$$

where the sequence ω is in the *hull* of the Fibonacci substitution sequence. For more details, see Section 2. This sequence takes two positive real values, say 1 and a. Let

$$\lambda = \frac{|a^2-1|}{a}, \tag{1.3}$$

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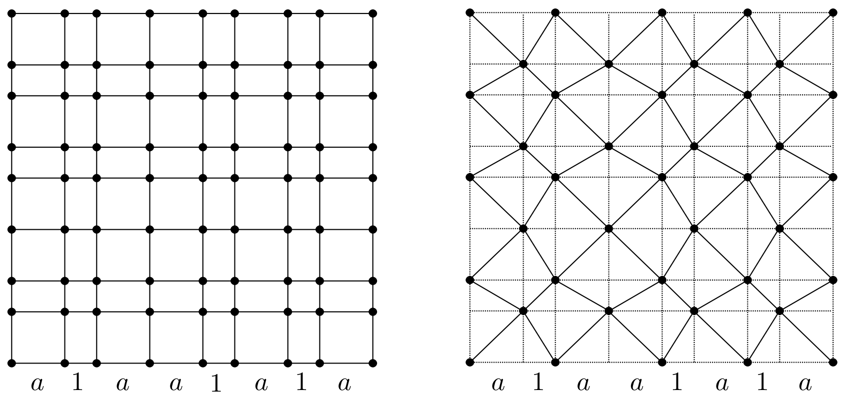

FIGURE 1. The square tiling (left) and the Labyrinth model (right).

and call this the *coupling constant*. The spectral properties of H_ω do not depend on the particular choice of ω, and depend only on the coupling constant λ. Recent mathematics papers of this operator include [**9**], [**26**], [**40**]. Recently, [**26**] considered tridiagonal substitution Hamiltonians, which include both (1.1) and (1.2) as special cases.

It is natural to consider higher dimensional models, but that is known to be extremely difficult. So, to get an idea of spectral properties of higher dimensional quasicrystals, simpler models have been considered. In two dimensional case, we have, for example, the square Fibonacci Hamiltonian [**12**], the square tiling, and the Labyrinth model, which is the main subject of this paper. These models are separable, so the exiting results of one-dimensional models can be applied in the study of their spectral properties.

The square Fibonacci Hamiltonian is constructed by two copies of Fibonacci Hamiltonian. Namely, this operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$\begin{aligned}\left[H_{\lambda_1,\lambda_2,\beta_1,\beta_2}\psi\right](m,n) &= \psi(m+1,n)+\psi(m-1,n)+\psi(m,n+1)+\psi(m,n-1)\\ &\quad+\left(\lambda_1\chi_{[1-\alpha,1)}(m\alpha+\beta_1 \mod 1)+\lambda_2\chi_{[1-\alpha,1)}(n\alpha+\beta_2 \mod 1)\right)\psi(m,n),\end{aligned}$$

where $\alpha = \frac{\sqrt{5}-1}{2}$, $\beta_1, \beta_2 \in \mathbb{T}$, and $\lambda_1, \lambda_2 > 0$. It is known that the spectrum of this operator is given by the sum of the spectra of the one-dimensional models, and the density of states measure of this operator is the convolution of the density of states measures of the one-dimensional models. See, for example, the appendix in [**12**]. Recently, it was shown that for small coupling constants the spectrum of the square Fibonacci Hamiltonian is an interval [**9**]. Furthermore, it was shown that for almost all pairs of the coupling constants, the density of states measure is absolutely continuous with respect to Lebesgue measure in weakly coupled regime [**12**].

The square tiling is constructed by two copies of off-diagonal models. The operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$\begin{aligned}\left[H_{\omega_1,\omega_2}\psi\right](m,n) &= \omega_1(m+1)\psi(m+1,n)+\omega_1(m)\psi(m-1,n)\\ &\quad+\omega_2(n+1)\psi(m,n+1)+\omega_2(n)\psi(m,n-1),\end{aligned}$$

where the sequences ω_1 and ω_2 are in the hull of the Fibonacci substitution sequence. All vertices are connected horizontally and vertically. See Figure 1. It has been studied mainly numerically by physicists (e.g., [**16**], [**18**], [**24**]). By repeating the argument from [**12**], one can show that the analogous results of the square Fibonacci Hamiltonian hold for the square tiling. Recently, [**17**] considered the square tridiagonal Fibonacci Hamiltonians, which include the square Fibonacci Hamiltonian and the square tiling as special cases.

The Labyrinth model is given by the following self-adjoint operator:

$$\begin{aligned}\left[\hat{H}_{\omega_1,\omega_2}\psi\right](m,n) = \omega_1(m+1)\omega_2(n+1)\psi(m+1,n+1)\\ + \omega_1(m+1)\omega_2(n)\psi(m+1,n-1)\\ + \omega_1(m)\omega_2(n+1)\psi(m-1,n+1)\\ + \omega_1(m)\omega_2(n)\psi(m-1,n-1),\end{aligned} \tag{1.4}$$

where the sequences ω_1 and ω_2 are in the hull of the Fibonacci substitution sequence. It is constructed by two copies of off-diagonal models. All vertices are connected diagonally, and the strength of the bond is equal to the product of the sides of the rectangle. See Figure 1. Without loss of generality, we can assume that ω_1 and ω_2 take values in $\{1, a_1\}$ and $\{1, a_2\}$, respectively. It can be shown that the spectral properties do not depend on the specific choice of ω_1 and ω_2, and only depend on their coupling constants. Therefore, we pick ω_1 and ω_2 arbitrarily and write this operator as $\hat{H}_{\lambda_1,\lambda_2}$, where λ_1 and λ_2 are the coupling constants of ω_1 and ω_2, respectively. Unlike the square Fibonacci Hamiltonian or the square tiling, the spectrum is the product (not the sum) of the spectra of the two one-dimensional models, and the density of states measure is not the convolution of the density of states measures of the one-dimensional models. This model was suggested in the late 1980s in [**32**], and so far this has been studied mostly by physicists, and their work is mainly relied on numerics [**3**], [**31**], [**32**], [**34**], [**35**], [**36**], [**37**], [**38**], [**41**]. Sire considered this model in [**32**], and the numerical experiments suggested that the density of states measure is absolutely continuous for small coupling constants, and is singular continuous for large coupling constants. By heuristic argument, he also estimated the critical value of which the transition from zero measure spectrum to positive measure spectrum occurs, and showed that it agrees with numerical experiment. In some papers, other substitution sequences, e.g., silver mean sequence or bronze mean sequence, are used to define the Labyrinth model. We consider more general case in this paper, and give rigorous proofs to some of the results predicted by physicists. In physicist's work, the coupling constants of two substitution sequences ω_1 and ω_2 are set as equal, but we consider the case that they may be different. We write the density of states measures of (1.2) and (1.4) as ν_λ and $\hat{\nu}_{\lambda_1,\lambda_2}$, respectively. The following theorems are the main results of this paper.

THEOREM 1.1. *The spectrum of* $\hat{H}_{\lambda_1,\lambda_2}$ *is a Cantor set of zero Lebesgue measure for sufficiently large coupling constants, and is an interval for sufficiently small coupling constants.*

THEOREM 1.2. *For any* $E \in \mathbb{R}$,

$$\hat{\nu}_{\lambda_1,\lambda_2}\left((-\infty, E]\right) = \iint_{\mathbb{R}^2} \chi_{(-\infty,E]}(xy)\, d\nu_{\lambda_1}(x) d\nu_{\lambda_2}(y). \tag{1.5}$$

The density of states measure $\hat{\nu}_{\lambda_1,\lambda_2}$ *is singular continuous for sufficiently large coupling constants. Furthermore, there exists* $\lambda^* > 0$ *such that for almost every pair* $(\lambda_1, \lambda_2) \in [0, \lambda^*) \times [0, \lambda^*)$*, the density of states measure* $\hat{\nu}_{\lambda_1,\lambda_2}$ *is absolutely continuous with respect to Lebesgue measure.*

2. Preliminaries

2.1. Linearly recurrent sequences. We recall some basic facts about subshifts over two symbols.

An *alphabet* is a finite set of symbols called *letters*. A *word* on $\mathcal{A}$ is a finite nonempty sequence of letters. Write $\mathcal{A}^+$ for the set of words. For $u = u_1u_2 \cdots u_n \in \mathcal{A}^+$, $|u| = n$ is the *length* of u. Define the *shift* T on $\mathcal{A}^{\mathbb{Z}}$ by

$$(Tx)_n = x_{n+1}$$

for $x \in \mathcal{A}^{\mathbb{Z}}$. Assume that $\mathcal{A}^{\mathbb{Z}}$ is equipped with the product topology. A *subshift* (X, T) on an alphabet $\mathcal{A}$ is a closed T-invariant subset X of $\mathcal{A}^{\mathbb{Z}}$, endowed with the restriction of T to X, which we denote again by T. Given $u = u_1u_2 \cdots u_n \in \mathcal{A}^+$ and an interval $J = \{i, \cdots, j\} \subset \{1, 2, \cdots, n\}$, we write u_J to denote the word $u_iu_{i+1} \cdots u_j$. A *factor* of u is a word v such that $v = u_J$ for some interval $J \subset \{1, 2, \cdots, n\}$. We extend this definition in obvious way to $u \in \mathcal{A}^{\mathbb{Z}}$. The *language* $\mathcal{L}(X)$ of a subshift (X, T) is the set of all words that are factors of at least one element of X.

Definition 2.1. Let (X, T) be a subshift. We say that $x \in X$ is *linearly recurrent* if there exists a constant $K > 0$ such that for every factor u, v of x, $K|u| < |v|$ implies that u is a factor of v.

We say that a subshift is *linearly recurrent* if it is minimal and contains a linearly recurrent sequence. Note that if a subshift is linearly recurrent, then by minimality all sequences belonging to X are linearly recurrent.

2.2. Metallic mean sequence. Let $\mathcal{A} = \{a, b\}$ be an alphabet, and consider the following substitution:

$$\mathcal{P}_s : \begin{cases} a \longrightarrow a^s b \\ b \longrightarrow a, \end{cases}$$

where s is a positive integer. Consider the iteration of $\mathcal{P}_s$ on a. For example, if $s = 1$,

$$a \longrightarrow ab \longrightarrow aba \longrightarrow abaab \longrightarrow abaababa \longrightarrow \cdots .$$

Let us write the nth iteration as $\mathcal{C}_s(n)$. It is easy to see that

$$\mathcal{C}_s(n+1) = (\mathcal{C}_s(n))^s \mathcal{C}_s(n-1).$$

Therefore, we can define a sequence $\{u_s(k)\}_{k=1}^{\infty}$ by $u_s = \lim_{n\to\infty} \mathcal{C}_s(n)$. They are called *metallic mean sequences*. In particular, when $s = 1$, it is called the *Fibonacci substitution sequence* or the *golden mean sequence*. When $s = 2, 3$, they are called the *silver mean sequence* and the *bronze mean sequence*, respectively.

Let us define the *hull* $\Omega^{(s)}_{a,b}$ of u_s by

$$\Omega^{(s)}_{a,b} = \left\{\omega \in \{a, b\}^{\mathbb{Z}} \mid \text{ every factor of } \omega \text{ is a factor of } u_s\right\}.$$

It is well known that $\Omega_{a,b}^{(s)}$ is compact and T-invariant, and $(\Omega_{a,b}^{(s)}, T)$ is linearly recurrent. See for example, [**29**] and references therein.

REMARK 2.1. Let us define a *rotation sequence* $v_{a,b,s,\beta}$ by

$$v_{a,b,s,\beta}(n) = \begin{cases} a & \text{if } n\alpha + \beta \mod 1 \in [1-\alpha, 1) \\ b & \text{o.w.,} \end{cases}$$

where α is given by

$$\alpha = \cfrac{1}{1 + s + \cfrac{1}{s + \cfrac{1}{s + \cfrac{1}{\ddots}}}} = \frac{s + 2 - \sqrt{s^2+4}}{2s}.$$

It is easy to see that the potential of the Fibonacci Hamiltonian (1.1) is $v_{\lambda,0,1,\beta}$. It is well known that $v_{a,b,s,0} = u_s$, so there is no need to distinguish rotation sequence and substitution sequence. But, it seems that it is more common to use rotation sequence in the definition of on-diagonal model and use substitution sequence in the definition of off-diagonal model. It is also known that

$$\Omega_{a,b}^{(s)} = \bigcup_{\beta \in \mathbb{T}} v_{a,b,s,\beta}.$$

See, for example [**25**].

REMARK 2.2. There seems to be some confusions about substitution sequences and rotation sequences in some papers. Let

$$\alpha^* = \cfrac{1}{s + \cfrac{1}{s + \cfrac{1}{s + \cfrac{1}{\ddots}}}}.$$

Using α^*, define $v_{a,b,s,\beta}^*$ analogously. In some papers it is stated that $v_{a,b,s,0}^* \in \Omega_{a,b}^{(s)}$, but this is obviously not true. What is true is $v_{a,b,1,0}^* = v_{b,a,1,0}$, so when $s = 1$ there is no actual harm. This seems to be (one of) the source of the confusion.

We simply write $\Omega_{a,b}^{(s)}$ as $\Omega^{(s)}$ below if there is no fear of confusion.

2.3. Necessary results. We will need the following definition and subsequent lemmas later.

DEFINITION 2.2. Let (X, T) be a linearly recurrent subshift, and let $x, y \in \mathcal{L}(X)$. If there exist disjoint intervals J_1 and J_2 such that

1) $J_i \subset \{1, 2, \cdots, |x|\}$ for $i = 1, 2$,
2) $J_1 = J_2 + k$ for some odd number k, and
3) $x_{J_1} = x_{J_2} = y$,

we say that y is *odd-twin* in x. Define *even-twin* analogously. For example, in the case of the subshift $\left(\Omega^{(1)}, T\right)$, ab is odd-twin in $abaab$, and even-twin in $abab$.

LEMMA 2.1. *For any $k \geqslant 1$, there exists $x \in \mathcal{L}(\Omega^{(s)})$ such that $|x| \leqslant 3|\mathcal{C}_s(k)|$ and $\mathcal{C}_s(k)$ is odd-twin in x.*

PROOF. In the proof below, we simply write $\mathcal{C}_s(n)$ as $\mathcal{C}(n)$. Recall that $\mathcal{C}(n)$ satisfies the concatenation rule

$$\mathcal{C}(n+1) = \mathcal{C}(n)^s \mathcal{C}(n-1).$$

Therefore, it is easy to see that $\mathcal{C}(k)\mathcal{C}(k)$ and $\mathcal{C}(k)\mathcal{C}(k-1)\mathcal{C}(k)$ are both factors of $\mathcal{C}(k+3)$. If $|\mathcal{C}(k)|$ is odd, $x = \mathcal{C}(k)\mathcal{C}(k)$ satisfies the desired properties. Suppose $|\mathcal{C}(k)|$ is even. Note that the sequence

$$\{|\mathcal{C}(n)| \mod 2\}$$

repeats $1, 1, 1, 1, \cdots$ if s is even, and $1, 0, 1, 1, 0, 1 \cdots$ if s is odd. Since $|\mathcal{C}(k)|$ is even, s has to be odd. Therefore $|\mathcal{C}(k-1)|$ is odd, so $x = \mathcal{C}(k)\mathcal{C}(k-1)\mathcal{C}(k)$ satisfies the desired properties. □

LEMMA 2.2. *For every $s \in \mathbb{N}$, there exists a constant $K_s > 0$ such that for any $x, y \in \mathcal{L}(\Omega^{(s)})$, $K_s|y| < |x|$ implies y is odd-twin in x. Analogous result holds for even-twin.*

PROOF. Let us show the statement for odd-twin. The latter statement is immediate. Let $K > 0$ be a number such that for any $x, y \in \mathcal{L}(\Omega^{(s)})$, y is a factor of x whenever $K|y| < |x|$. Let $y \in \mathcal{L}(\Omega^{(s)})$. Take $k > 0$ such that

$$|\mathcal{C}_s(k-1)| \leqslant K|y| < |\mathcal{C}_s(k)|.$$

Then y is a factor of $\mathcal{C}_s(k)$, and since $|\mathcal{C}_s(k)| < (s+1)|\mathcal{C}_s(k-1)|$, we have $|\mathcal{C}_s(k)| < (s+1)K|y|$. Therefore, by Lemma 2.1, there exists $v \in \mathcal{L}(\Omega^{(s)})$ such that $|v| < 3(s+1)K|y|$ and $\mathcal{C}_s(k)$ is odd-twin in v. Since y is a factor of $\mathcal{C}_s(k)$, y is odd-twin in v. Therefore,

$$K_s := K \cdot 3(s+1)K = 3(s+1)K^2$$

satisfies the desired properties. □

2.4. The off-diagonal model. Let $a, b > 0$ be real numbers, and let s be a positive integer. Let $\omega \in \Omega^{(s)}_{a,b}$. We define a Jacobi matrix H_ω acting on $l^2(\mathbb{Z})$ by

$$\left(H_\omega \psi\right)(n) = \omega(n+1)\psi(n+1) + \omega(n)\psi(n-1),$$

and set

$$\lambda = \left|\frac{a^2 - b^2}{ab}\right|.$$

We call this λ the *coupling constant*. We only consider the case that $a > b$ below. The argument is completely analogous in the case of $a < b$. With appropriate scaling, we can always assume $b = 1$. We assume this scaling all through this section. Note that this coincides with the definition (1.3). We call this family of self-adjoint operators $\{H_\omega\}$ the *off-diagonal model*. By the well known argument (minimality of the subshift and strong operator convergence), one can see that the spectrum of H_ω is independent of the specific choice of ω, and depends only on

λ and s. Therefore, when there is no fear of confusion, we pick ω arbitrarily from $\Omega_{a,b}^{(s)}$ and write H_ω as H_λ.

Definition 2.3. We define the *trace map* T_s by

$$T_s = U^s \circ P,$$

where

$$U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xz - y \\ x \\ z \end{pmatrix} \quad \text{and} \quad P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix}.$$

Let ℓ_λ be the line given by

$$\ell_\lambda = \left\{ \left(\frac{E^2 - a^2 - 1}{2a}, \frac{E}{2a}, \frac{E}{2} \right) : E \in \mathbb{R} \right\},$$

and call this the *line of initial condition*. Let us define the map $J_\lambda(\cdot)$ by

$$J_\lambda : E \mapsto \ell_\lambda(E). \tag{2.1}$$

The function

$$G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

is invariant under the action of T_s, and hence it preserves the family of surfaces

$$S_V = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz - 1 = \frac{V^2}{4} \right\}.$$

It is easy to see that $\ell_\lambda \subset S_\lambda$.

The following can be proven by repeating the argument of [**26**].

Theorem 2.1. *We have*

$$\sigma(H_\lambda) = \{E \in \mathbb{R} : \text{ the forward semi-orbit of } J_\lambda(E) \text{ is bounded}\}.$$

We immediately get the following:

Corollary 2.1. *The spectrum $\sigma(H_\lambda)$ contains* 0.

Proof. Note that

$$J_\lambda(0) = \left(-\frac{a^2 + 1}{2a}, 0, 0 \right). \tag{2.2}$$

It is easy to see that this point is periodic under the action of T_s. □

Remark 2.3. Analogous result of Theorem 2.1 holds for on-diagonal case (e.g., [**1**], [**7**], [**11**]), but it seems that the set of initial conditions is not quite right due to the confusion stated in Remark 2.2.

In what follows we are going to use some notations and results from the theory of hyperbolic dynamical systems, see [**19**] for some background on this subject.

Let us denote by Λ_λ the set of points whose orbits are bounded under T_s. The following theorem was first proven in [**33**] for the Fibonacci Hamiltonian.

Theorem 2.2 (Theorem 4.1 of [**25**]). *The set Λ_λ is a compact locally maximal T_s-invariant transitive hyperbolic subset of S_λ, and the periodic points of T_s form a dense subset of Λ_λ.*

We also have the following:

THEOREM 2.3 (Corollary 2.5 of [**15**]). *The forward semi-orbit of a point $p \in S_\lambda$ is bounded if and only if p lies on the stable lamination of Λ_λ.*

The following Theorem was proven in [**13**] for the Fibonacci Hamiltonian case, and recently it was extended to tridiagonal Fibonacci Hamiltonians in [**17**]. It follows by repeating the argument of [**13**].

THEOREM 2.4. *For all $\lambda > 0$, the intersections of the curve of initial condition ℓ_λ with the stable lamination is transverse.*

COROLLARY 2.2. *The spectrum of H_λ is a dynamically defined Cantor set.*

Now we define the density of states measure. The definition is analogous for higher dimensional models.

DEFINITION 2.4. Denote by $H_\lambda^{(N)}$ the restriction of H_λ to the interval $[0, N-1]$ with Dirichlet boundary conditions. The density of states measure ν_λ of H_λ is given by

$$\nu_\lambda\left((-\infty, E]\right) = \lim_{N\to\infty} \frac{1}{N} \# \left\{ \text{eigenvalues of } H_\lambda^{(N)} \text{ that are in } (-\infty, E] \right\},$$

where $E \in \mathbb{R}$.

The limit does not depend on the specific choice of ω, and in fact, the convergence is uniform in ω. This was shown in a more general setting [**22**].

It is well known that $\sigma(H_0) = [-2, 2]$, and

$$\nu_0\left((-\infty, E]\right) = \begin{cases} 0 & E \leqslant -2 \\ \frac{1}{\pi}\arccos(-\frac{E}{2}) & -2 < E < 2 \\ 1 & E \geqslant 2. \end{cases} \tag{2.3}$$

Let us write

$$\mathbb{S} = S_0 \cap \left\{ (x, y, z) \in \mathbb{R}^3 \mid |x| \leqslant 1, |y| \leqslant 1, |z| \leqslant 1 \right\}.$$

The trace map T_s restricted to $\mathbb{S}$ is a factor of the hyperbolic automorphism $\mathcal{A}$ of $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ given by

$$\mathcal{A} : \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \varphi \end{pmatrix}.$$

The semi-conjugacy is given by the map

$$F : (\theta, \varphi) \mapsto (\cos 2\pi(\theta + \varphi), \cos 2\pi\theta, \cos 2\pi\varphi).$$

A Markov partition of $\mathcal{A} : \mathbb{T}^2 \to \mathbb{T}^2$ when $s = 1$ is shown in Figure 2. For other values of $s \in \mathbb{N}$, the only difference is the slope of the stable and unstable manifolds. Its image under the map $F : \mathbb{T}^2 \to \mathbb{S}$ is a Markov partition for the pseudo-Anosov map $T_s : \mathbb{S} \to \mathbb{S}$. Write

$$I = \{(t, t) \mid 0 \leqslant t \leqslant 1/2\}.$$

Note that $F(I) \subset \ell_0$. The following lemma is immediate:

LEMMA 2.3. *The push-forward of the normalized Lebesgue measure on I under the semi-conjugacy F, which is a probability measure on $\ell_0 \cap \mathbb{S}$, corresponds to the free density of states measure (2.3) under the identification (2.1).*

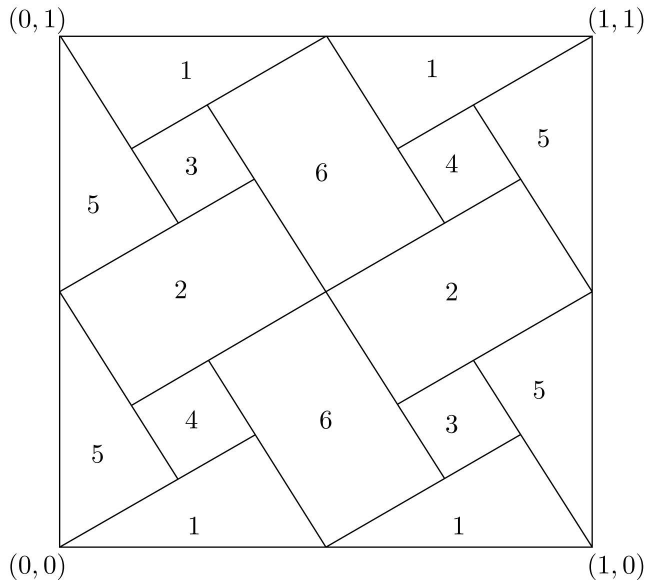

FIGURE 2. The Markov partition for the map $\mathcal{A}$.

Consider the union of elements of the Markov partition of $\mathbb{T}^2$, 1, 2, 4, 5, and 6, as in Figure 3. Let us denote the image of this union of elements under F by $\mathcal{R}_0$, and the continuation of $\mathcal{R}_0$ in $\lambda > 0$ by $\mathcal{R}_\lambda$. The following statement can be proven by repeating the proof of Claim 3.2 of [**10**].

PROPOSITION 2.1. *Consider the measure of maximal entropy of $T_s|_{\Lambda_\lambda}$, and restrict it to $\mathcal{R}_\lambda$. Normalize this measure, and project it to ℓ_λ along the stable manifolds. Then, the resulting probability measure on ℓ_λ corresponds to the density of states measure ν_λ under the identification (2.1).*

This immediately implies the following:

THEOREM 2.5. *For every $\lambda > 0$, the density of states measure ν_λ is exact-dimensional. That is, for ν_λ-almost every $E \in \mathbb{R}$, we have*

$$\lim_{\epsilon \downarrow 0} \frac{\log \nu_\lambda(E - \epsilon, E + \epsilon)}{\log \epsilon} = d_\lambda,$$

where d_λ satisfies

$$\lim_{\lambda \downarrow 0} d_\lambda = 1.$$

PROOF. The first claim is an immediate consequence of Proposition 2.1. The second claim follows from the verbatim repetition of Theorem 1.1 of [**10**]. □

We also have the following:

PROPOSITION 2.2 ([**28**]). *The stable and unstable Lyapunov exponents are analytic functions of $\lambda > 0$.*

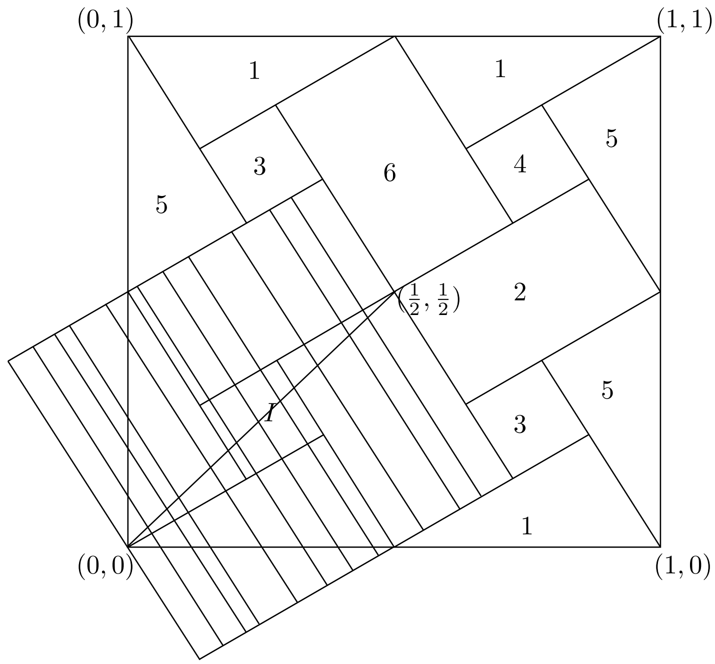

FIGURE 3. A Markov partition for the map $\mathcal{A}$, line segment I, and the stable manifolds.

For any Cantor set K, we denote the *thickness* of K by $\tau(K)$. For the definition of thickness, see for example, the chapter 4 of [**27**]. By Theorem 2 of [**6**], and by repeating the proof of Theorem 1.1 of [**9**], we get the following:

THEOREM 2.6. *We have*

$$\lim_{\lambda\to\infty} \dim_H \sigma(H_\lambda) = 0, \text{ and } \lim_{\lambda\downarrow 0} \tau(\sigma(H_\lambda)) = \infty.$$

PROPOSITION 2.3. *The density of states measure ν_λ of H_λ is symmetric with respect to the origin. In particular, the spectrum of H_λ is symmetric with respect to the origin.*

PROOF. Denote by $H_\lambda^{(N)}$ the restriction of H_λ to the interval $[0, N-1]$ with Dirichlet boundary conditions. Let ψ be an eigenvector of $H_\lambda^{(N)}$ and E be the corresponding eigenvalue. Let us define $\phi \in l^2([0, N-1])$ by

$$\phi(n) = (-1)^n \psi(n) \quad (n = 0, 1, \cdots, N-1).$$

Then, since

$$\begin{aligned}(H_\lambda^{(N)}\phi)(n) &= \omega(n+1)\phi(n+1) + \omega(n)\phi(n-1)\\ &= (-1)^{n+1}\omega(n+1)\psi(n+1) + (-1)^{n-1}\omega(n)\psi(n-1)\\ &= (-1)^{n+1}E\,\psi(n)\\ &= -E\,\phi(n),\end{aligned}$$

$-E$ is also an eigenvalue of $H_\lambda^{(N)}$. Therefore, the set of eigenvalues of $H_\lambda^{(N)}$ is symmetric with respect to the origin. Therefore, for any interval $A \subset (0, \infty)$,

$$\begin{aligned}\nu_\lambda(A) &= \lim_{N\to\infty} \frac{1}{N} \# \left\{\text{eigenvalues of } H_\lambda^{(N)} \text{ that are in } A\right\}\\ &= \lim_{N\to\infty} \frac{1}{N} \# \left\{\text{eigenvalues of } H_\lambda^{(N)} \text{ that are in } (-A)\right\}\\ &= \nu_\lambda(-A).\end{aligned}$$

This concludes the first claim. Since the spectrum is the topological support of the density of states measure, the second claim also follows. □

3. The Labyrinth Model

In this section, we define the Labyrinth model.

3.1. The Labyrinth model. Let $a_i, b_i > 0$ $(i = 1, 2)$ be real numbers, and let s be a positive integer. Let $\omega_i \in \Omega^{(s)}_{a_i,b_i}$ $(i = 1, 2)$. Write

$$A^e = \left\{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is even}\right\}, \text{ and } A^o = \left\{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is odd}\right\}.$$

Using ω_1, ω_2, we realign the lattices of A^e and A^o. See Figure 4. We denote this again by A^e and A^o (we use this identification freely). We define the operator $\hat{H}_{\omega_1,\omega_2}$, which acts on $l^2(A^e \cup A^o)$, by

$$\begin{aligned}\left[\hat{H}_{\omega_1,\omega_2}\psi\right](m, n) = \omega_1(m + 1)\omega_2(n + 1)\psi(m + 1, n + 1)\\ + \omega_1(m + 1)\omega_2(n)\psi(m + 1, n - 1)\\ + \omega_1(m)\omega_2(n + 1)\psi(m - 1, n + 1)\\ + \omega_1(m)\omega_2(n)\psi(m - 1, n - 1).\end{aligned}$$

Every lattice is connected diagonally, and the strength of the bond is equal to the product of the sides of the rectangle. With appropriate scaling, we can always assume that $b_i = 1$ $(i = 1, 2)$. In a similar way, we define the operators $\hat{H}^e_{\omega_1,\omega_2}$ and $\hat{H}^o_{\omega_1,\omega_2}$, which acts on $l^2(A^e)$ and $l^2(A^o)$, respectively. From here, we drop the subscripts ω_1, ω_2 if no confusion can arise. Note that

$$\hat{H} = \hat{H}^e \oplus \hat{H}^o.$$

It is natural to expect that the spectral properties of $\hat{H}^e$ and $\hat{H}^o$ are the same, and in fact, the spectra and the density of states measures do coincide for the three operators. For the proof, we need the notion of *Delone dynamical systems*, and *linear repetitivity* of Delone dynamical systems. See, for example, [**21**].

PROPOSITION 3.1. *Let us denote the density of states measures of $\hat{H}^e, \hat{H}^o$ and $\hat{H}$ by $\hat{\nu}^e, \hat{\nu}^o$ and $\hat{\nu}$, respectively. Then, $\hat{\nu}^e, \hat{\nu}^o$ and $\hat{\nu}$ define the same measure. In particular, the spectra of $\hat{H}^e, \hat{H}^o$ and $\hat{H}$ all coincide.*

PROOF. By Lemma 2.2, A^e and A^o are linearly repetitive. Therefore, by Theorem 6.1 of [**21**] and Theorem 3 of [**22**] and Lemma 2.2, we have $\hat{\nu}^e = \hat{\nu}^o$. Since $\hat{H} = \hat{H}^e \oplus \hat{H}^o$, we get $\hat{\nu}^e = \hat{\nu}^o = \hat{\nu}$. □

By this proposition, we will restrict our attention to $\hat{H}$ below.

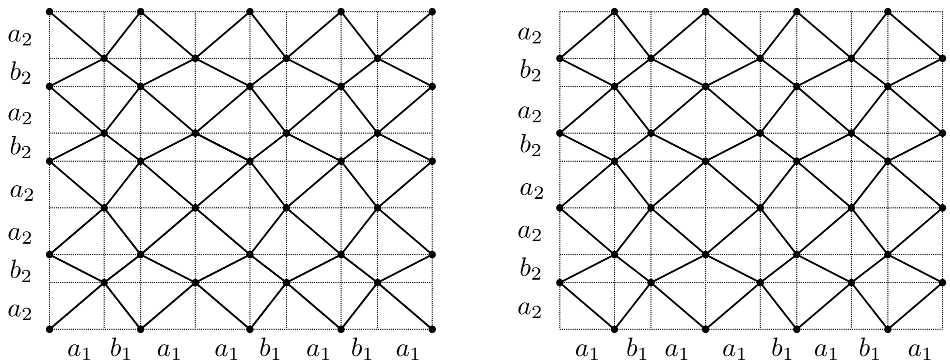

FIGURE 4. A^e (left) and A^o (right).

3.2. The spectrum of the Labyrinth model. Let $a_i, b_i > 0$ $(i = 1, 2)$ be real numbers, and let s be a positive integer. Let $\omega_i \in \Omega^{(s)}_{a_i,b_i}$ $(i = 1, 2)$.

PROPOSITION 3.2. *We have*

$$\sigma(\hat{H}_{\omega_1,\omega_2}) = \sigma(H_{\omega_1})\sigma(H_{\omega_2}).$$

In the proof below, we simply write $H_{\omega_1}, H_{\omega_2}$, and $\hat{H}_{\omega_1,\omega_2}$ as H_1, H_2, and $\hat{H}$, respectively.

PROOF. Let U be the unique unitary map from $l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z})$ to $l^2(\mathbb{Z}^2)$ so that for $\psi_1, \psi_2 \in l^2(\mathbb{Z})$, the elementary tensor $\psi_1 \otimes \psi_2$ is mapped to the element ψ of $l^2(\mathbb{Z}^2)$ given by $\psi(m,n) = \psi_1(m)\psi_2(n)$. We have

$$\begin{aligned}
\left[\hat{H}U(\psi_1 \otimes \psi_2)\right](m,n) &= \omega_1(m+1)\omega_2(n+1)\psi_1(m+1)\psi_2(n+1) \\
&\quad + \omega_1(m+1)\omega_2(n)\psi_1(m+1)\psi_2(n-1) \\
&\quad + \omega_1(m)\omega_2(n+1)\psi_1(m-1)\psi_2(n+1) \\
&\quad + \omega_1(m)\omega_2(n)\psi_1(m-1)\psi_2(n-1) \\
&= [H_1\psi_1](m)\,[H_2\psi_2](n) \\
&= [U(H_1 \otimes H_2)(\psi_1 \otimes \psi_2)](m,n),
\end{aligned}$$

for all $(m,n) \in \mathbb{Z}^2$. Therefore,

$$(U^*\hat{H}U)(\psi_1 \otimes \psi_2) = (H_1 \otimes H_2)(\psi_1 \otimes \psi_2).$$

Since the linear combinations of elementary tensors are dense in $l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z})$, we get $U^*\hat{H}U = H_1 \otimes H_2$. Therefore, the result follows from Theorem VIII 33 of [**30**]. $\square$

By this proposition, the spectrum of $\hat{H}_{\omega_1,\omega_2}$ does not depend on particular choice of ω_1 and ω_2, and only depends on the coupling constants.

PROOF OF THEOREM 1.1. The result for sufficiently small coupling constants follows from Theorem 2.6, Proposition 3.2, and Theorem 1.4 of [**39**]. Since

$$\dim_H \sigma(H_{\lambda_1}) + \dim_H \sigma(H_{\lambda_2}) < 1$$

for large λ_1, λ_2, the result for sufficiently large coupling constants also follows. See chapter 4 of [**27**]. □

3.3. Density of states measure of the Labyrinth model. In this section, we will prove Theorem 1.2. If there is no fear of confusion, we simply write the density of states measures of $\hat{H}$, H_1, and H_2 as $\hat{\nu}$, ν_1, and ν_2, respectively.

PROOF OF (1.5). The proof is essentially the repetition of the proof of Proposition A.3 of [**12**]. For the reader's convenience, we will repeat the argument.

Denote by $H_j^{(N)}$ $(j = 1, 2)$ the restriction of H_j to the interval $[0, N-1]$ with Dirichlet boundary conditions. Denote the corresponding eigenvalues and eigenvectors by $E_{j,k}^{(N)}$, $\phi_{j,k}^{(N)}$, where $j = 1, 2$ and $1 \leqslant k \leqslant N$. Recall that we have

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N \mid E_{j,k}^{(N)} \in (-\infty, E] \right\} = \nu_j\left((-\infty, E]\right) \tag{3.1}$$

for $E \in \mathbb{R}$.

Similarly, we denote by $\hat{H}^{(N)}$ the restriction of $\hat{H}$ to $[0, N-1]^2$ with Dirichlet boundary conditions. Denote the corresponding eigenvalues and eigenvectors by $E_k^{(N)}, \phi_k^{(N)}$ $(1 \leqslant k \leqslant N^2)$. Then, we have

$$\lim_{N\to\infty} \frac{1}{N^2} \# \left\{ 1 \leqslant k \leqslant N^2 \mid E_k^{(N)} \in (-\infty, E] \right\} = \hat{\nu}\left((-\infty, E]\right).$$

The eigenvectors $\phi_{j,k}^{(N)}$ of $H_j^{(N)}$ form an orthonormal basis of $l^2([0, N-1])$. Thus, the associated elementary tensors

$$\phi_{1,k_1}^{(N)} \otimes \phi_{2,k_2}^{(N)} \quad (1 \leqslant k_1, k_2 \leqslant N) \tag{3.2}$$

form an orthonormal basis of $l^2([0, N-1]) \otimes l^2([0, N-1])$, which is canonically isomorphic to $l^2([0, N-1]^2)$. Moreover, the vector in (3.2) is an eigenvector of $\hat{H}^{(N)}$, corresponding to the eigenvalue $E_{1,k_1}^{(N)} \cdot E_{2,k_2}^{(N)}$. By dimension count, these eigenvalues exhaust the entire set $\left\{ E_k^{(N)} \mid 1 \leqslant k \leqslant N^2 \right\}$. Therefore, for $E \in \mathbb{R}$,

$$\# \left\{ 1 \leqslant k \leqslant N^2 \mid E_k^{(N)} \in (-\infty, E] \right\} = \# \left\{ 1 \leqslant k_1, k_2 \leqslant N \mid E_{1,k_1}^{(N)} \cdot E_{2,k_2}^{(N)} \in (-\infty, E] \right\}.$$

Let $\nu_j^{(N)}$ $(j = 1, 2)$ be the probability measures on $\mathbb{R}$ with $\nu_j^{(N)}(E_{j,k}^{(N)}) = 1/N$ $(k = 1, 2, \cdots, N)$. Similarly, Let $\hat{\nu}^{(N)}$ be the probability measure on $\mathbb{R}$ with $\hat{\nu}^{(N)}(E_k^{(N)}) = 1/N^2$ $(k = 1, 2, \cdots, N^2)$. Then, by the above argument, we get

$$\hat{\nu}^{(N)}\left((-\infty, E]\right) = \iint_{\mathbb{R}^2} \chi_{(-\infty,E]}(xy)\, d\nu_1^{(N)}(x) d\nu_2^{(N)}(y).$$

By (3.1), $\nu_i^{(N)}$ converges weakly to ν_i (see, for example, chapter 13 of [**20**]). Therefore, $\nu_1^{(N)} \times \nu_2^{(N)}$ converges weakly to $\nu_1 \times \nu_2$. By Theorem 13.16 of [**20**], we have

$$\lim_{N\to\infty} \iint_{\mathbb{R}^2} \chi_{(-\infty,E]}(xy)\, d\nu_1^{(N)}(x) d\nu_2^{(N)}(y) = \iint_{\mathbb{R}^2} \chi_{(-\infty,E]}(xy)\, d\nu_1(x) d\nu_2(y).$$

The result follows from this. □

Let us define Borel measures $\bar{\nu}_i$ $(i = 1, 2)$ on $\mathbb{R}$ by

$$\bar{\nu}_i(A) = \nu_i(e^A),$$

where $A \subset \mathbb{R}$ is a Borel set. Then, the following holds.

Lemma 3.1. *The density of states measure of the Labyrinth model $\hat{\nu}$ is given by*

$$\hat{\nu}(A) = 2\left\{(\bar{\nu}_1 * \bar{\nu}_2)(\log A^+) + (\bar{\nu}_1 * \bar{\nu}_2)(\log A^-)\right\},$$

where A is a Borel set, and $A^+ = A \cap (0, \infty)$ and $A^- = (-A) \cap (0, \infty)$.

Proof. Let $A \subset (0, \infty)$ be a Borel set. Using Fubini's Theorem and change of coordinates, we get

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \chi_A(xy)\, d\nu_1(x) d\nu_2(y) &= \int_{\mathbb{R}^+}\left(\int_{\mathbb{R}^+} \chi_A(xy)\, d\nu_1(x)\right) d\nu_2(y) \\ &= \int_{\mathbb{R}^+}\left(\int_{\mathbb{R}} \chi_A\left(e^x y\right) d\bar{\nu}_1(x)\right) d\nu_2(y) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_A(e^x e^y)\, d\bar{\nu}_1(x) d\bar{\nu}_2(y) \\ &= \iint_{\mathbb{R}^2} \chi_{\log A}(x+y)\, d\bar{\nu}_1(x) d\bar{\nu}_2(y) \\ &= (\bar{\nu}_1 * \bar{\nu}_2)(\log A).\end{aligned}$$

Combining this with Proposition 2.3, the result follows. □

Therefore, the absolute continuity of $\hat{\nu}$ is equivalent to the absolute continuity of $\bar{\nu}_1 * \bar{\nu}_2$.

Theorem 3.2 from [**12**] implies the following:

Theorem 3.1. *Let $J \subset \mathbb{R}$ be an interval. Assume that for $\lambda \in J$, ν_λ is the density of states measure of H_λ. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism, and define a Borel measure μ_λ by*

$$\mu_\lambda(A) = \nu_\lambda(\gamma(A)),$$

where $A \subset \mathbb{R}$ is a Borel set. Then, for any compactly supported exact-dimensional measure η on $\mathbb{R}$ with

$$\dim_H \eta + \dim_H \mu_\lambda > 1$$

for all $\lambda \in J$, the convolution $\eta * \mu_\lambda$ is absolutely continuous with respect to Lebesgue measure for almost every $\lambda \in J$.

proof of theorem 1.2. The result for sufficiently large coupling constants immediately follows from Theorem 1.1.

By Theorem 2.5, there exists $\lambda^* > 0$ such that $\dim_H \nu_\lambda > \frac{1}{2}$ for all $\lambda \in [0, \lambda^*)$. Recall that $0 \in \sigma(H_{\lambda_i})$. Write

$$\sigma(H_{\lambda_i}) \cap (0, \infty) = \bigsqcup_{n=1}^{\infty} K_n^{(i)}(\lambda_i) \quad (i = 1, 2),$$

where $K_n^{(i)}(\lambda_i)$ are Cantor sets which depend naturally in λ_i. Let us define Borel measures $\bar{\nu}_{\lambda_i}^{(n)}$ $(i = 1, 2,\ n \in \mathbb{N})$ by

$$\bar{\nu}_{\lambda_i}^{(n)}(A) = \nu_{\lambda_i}|_{K_n^{(i)}(\lambda_i)}(e^A),$$

where $A \subset \mathbb{R}$ is a Borel set. Then, by Theorem 3.1, for each $(m, n) \in \mathbb{N} \times \mathbb{N}$ $\bar{\nu}_{\lambda_1}^{(m)} * \bar{\nu}_{\lambda_2}^{(n)}$ is absolutely continuous for almost all (λ_1, λ_2). This implies that $\bar{\nu}_{\lambda_1} * \bar{\nu}_{\lambda_2}$ is absolutely continuous for almost all (λ_1, λ_2). □

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