

Marginally Stable Circular Orbits in Schwarzschild Black Hole Surrounded by Quintessence Matter

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Abstract. Marginally stable circular orbits (MSCOs) of a massive test particle are investigated in the spacetime geometry of Schwarzschild black hole surrounded by quintessence. For that matter we consider three important scenarios where the equation of state parameter ω_q , has one of the following forms (i) $\omega_q = -1$ (ii) $\omega_q = -2/3$ and (iii) $\omega_q = -1/3$. The existence of such marginally stable circular orbits in these scenarios depend on the range of normalization factor α . Briefly, we show that in the first case such orbits exist only if $0 < \alpha < 4/16875$. Moreover in the second case which is a special Kiselev black hole it is found that MSCOs exist when the value of the normalization factor satisfy $0 < \alpha \leq 0.00536165238$. In the last case the MSCOs are also shown to exist.

Key words: Marginally stable circular orbits; Schwarzschild black hole; Quintessence matter; Normalization factor.

1. Introduction

Motion of particles in the spacetime geometry of black holes is an active topic of research for theoretical physicists. It may be helpful in understanding gravitational field around black holes. In this regard several black hole spacetimes have been investigated in the literature [1]-[26]. In the theory of general relativity the radius of circular orbits of particles in the vicinity of black holes has a lower bound which are known as inner most stable circular orbits (ISCOs). While the circular orbits with the upper bound on its radius are called outer most stable circular orbits (OSCOs). These two types of orbits form a boundary between the two regions, i.e., a stable region and an unstable region respectively. In the literature this boundary is known as the MSCO [26]. If in a spacetime geometry the number of MSCOs is only two then the smaller is called the ISCO and the larger is known as the OSCO. On the other hand if the number of MSCOs is greater than two then the smallest will be the ISCO and the largest will be the OSCO. A study of the ISCO for the Schwarzschild black hole can be found in the literature (see for example [27, 28]).

The existence of ISCOs may play an important role in the study of gravitational waves [29]. It is believed that a super massive black hole exists at the centre of each galaxy [30, 31, 32]. In the process of generation of gravitational waves, ISCOs are considered to be the location where an orbiting compact object e.g, another black hole orbiting around the super massive black hole goes from the inspiralling phase to the merging phase [29, 33]. ISCOs have their own importance in high energy astrophysics where the existence of these orbits is related to the inner edge of the accretion disks around black holes [34]. Therefore, the study of the ISCOs can provide useful information about the nonlinear spacetime geometry which is beyond the local tests of our solar system. The OSCO for a test particle in the vicinity of Kottler black hole spacetime has been investigated by Stuchlik and Hledik [1].

Cosmological observations like the Supernovae Ia, the Cosmic Microwave Background radiation anisotropies and X-ray experiments support the accelerated expansion of our Universe [35, 36, 37]. It is believed that dark energy is responsible for this accelerated expansion of our Universe. Several phenomenological models have been proposed to describe dark energy of which there is one model which examine the possibility of the presence of a scalar field known as quintessence (see e.g., [38]). This scalar field is defined by the equation of negative state parameter, which is the ratio of the pressure and density [39]. Kiselev derived a black hole solution of the Einstein field equations with quintessence matter [40]. This solution reduces to the Schwarzschild solution of the Einstein field equations when the quintessence term disappears. Null geodesics for the Schwarzschild black hole surrounded by quintessence have been investigated by Sharmanthie Fernando for particular values of the equation of state parameter $\omega_q = -2/3$ and the normalization factor $\alpha = 0.1, 0.01, 0.005$ [11] (the normalization factor is given in the metric coefficient of the time element of the quintessence black hole in the third section). For the same black hole time-like geodesics have been studied by Rashmi et. al [25]. They have considered three different values of the equation of state parameter $\omega_q = -1, -2/3, -1/3$ and four different values of the normalization factor $\alpha = 0.1, 0.08, 0.05, 0.005$. For unit mass of the black hole with $\omega_q = -1/3, -2/3$ and $\alpha = 0.1$ Rashmi et. al have shown that the radius of the ISCO has shifted to a larger distance from centre as compared to the pure Schwarzschild black hole. In the present study we are interested to analyze the MSCOs in the Schwarzschild back hole surrounded by quintessence scalar field for three different cases of the equation of state parameter $\omega_q = -1, -2/3, -1/3$. In particular we obtain upper and lower bounds on the value of the normalization factor α to find the radius of the MSCOs in the background of above geometry.

In the next Section we give the necessary and sufficient conditions for the existence of MSCOs in the case of a spherically symmetric static spacetime. In Section 3 we study MSCOs in the spacetime geometry of the Schwarzschild back hole surrounded by quintessence matter. A summary of the discussion is given in the last Section. In this paper we use $G = c = 1$. From here onwards we refer qSBH as a Schwarzschild black hole surrounded by quintessence.

2. Necessary and Sufficient Conditions for MSCOs

The line element for the general spherically symmetric static spacetime is given by [27]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + h(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $f(r) > 0$ and $h(r) > 0$, resulting in the signature $(-, +, +, +)$ of spacetime. In the above line element, $h(r)$ could be set equal to r^2 , without loss of generality. All other cases can be mapped to this case by an appropriate coordinate transformation which is not the main topic of discussion here. The necessary condition for the existences of MSCOs is given in terms of a second-order differential equation involving both $h(r)$ and $f(r)$ [22, 26]

$$\frac{d}{dr}\left(\frac{1}{f(r)}\right)\frac{d^2}{dr^2}\left(\frac{1}{h(r)}\right) - \frac{d}{dr}\left(\frac{1}{h(r)}\right)\frac{d^2}{dr^2}\left(\frac{1}{f(r)}\right) = 0. \quad (2)$$

For a given geometry of the black hole the functions $f(r)$ and $h(r)$ are specified and the above equation reduces to an algebraic equation in r , whose solutions provide us the required information about MSCOs. If in a spacetime, an MSCO exist then the solution of (2) gives us the radius of such an orbit. The constants of motion E and L are given by [26]

$$E^2 = -\frac{1}{D}\frac{d}{dr}\left(\frac{1}{h(r)}\right), \quad (3)$$

$$L^2 = -\frac{1}{D}\frac{d}{dr}\left(\frac{1}{f(r)}\right), \quad (4)$$

where

$$D = \frac{1}{h(r)}\frac{d}{dr}\left(\frac{1}{f(r)}\right) - \frac{1}{f(r)}\frac{d}{dr}\left(\frac{1}{h(r)}\right) \neq 0. \quad (5)$$

For any root of the equation (2) the sufficient condition for MSCO to exist, is that both constants of motion remains bounded, i.e.,

$$0 \leq E^2 < \infty, \quad 0 \leq L^2 < \infty. \quad (6)$$

If the condition given by (6) is satisfied for a root r of (2) then such a root is the radius of MSCO. Otherwise the root r is unphysical. The existence of such orbits for both Schwarzschild and Kottler black holes was proved in [26] with the application of Sturm's theorem. They also applied the analysis on spherically symmetric spacetimes in the Weyl conformal gravity. Here our purpose is to study the effect of quintessence on the stability of the circular orbits in a given black hole geometry. We now investigate the stability of such orbits in the Schwarzschild black hole geometry which is subjected to a quintessence field.

3. MSCOs in qSBH

For a Schwarzschild black hole surrounded by a quintessence matter field, the functions $f(r)$ and $h(r)$ in the equation (1) assumes the form [40]

$$f(r) = 1 - \frac{r_g}{r} - \frac{\alpha}{r^{3\omega_q+1}}, \quad h(r) = r^2. \quad (7)$$

Here $r_g = 2M$ and M is the mass of the black hole and the normalization factor satisfy $0 < \alpha < 1$ [25]. Note that for $\alpha = 0$, this black hole reduces to the case of pure (without quintessence) Schwarzschild black hole. The critical power value $\omega_q = -1/3$, reduces the quintessence term equal to a constant and we show that there exists an MSCO in this case. For $\omega_q = -1$, this black hole spacetime becomes the Schwarzschild black hole with the cosmological constant. For the horizon structure and other properties of this black hole one may study for example [11, 40]. Here we investigate MSCOs in the Schwarzschild black hole with quintessence by studying the necessary and sufficient conditions given by (2) and (6) for three different values of ω_q appearing in $f(r)$ given by (7).

3.1 qSBH with Equation of State ($\omega_q = -1/3$)

The condition given in the equation (2) becomes

$$r - \alpha r - 3r_g = 0, \quad (8)$$

where the case $\alpha = 0$, gives the equation for the Schwarzschild black hole. For unit mass of the black hole, $r_g = 2$ and we get

$$r = \frac{6}{1 - \alpha}. \quad (9)$$

In this case the constants of motion (3) and (4) simplifies into

$$E^2 = -\frac{(2 + r(\alpha - 1))^2}{r(3 + r(\alpha - 1))}, \quad (10)$$

$$L^2 = -\frac{r^2}{3 + r(\alpha - 1)}, \quad (11)$$

Note that as we have assumed $r_g = 2$, therefore in the above inequalities r carries a dimensionless form. Using the equation (9) we obtain

$$E^2 = \frac{8}{9}(1 - \alpha). \quad (12)$$

Since $\alpha < 1$, thus the above quantity satisfy the first sufficient condition (6). We now investigate the second constant of motion which becomes

$$L^2 = \frac{12}{(\alpha - 1)^2}. \quad (13)$$

This is again a positive quantity. Therefore an MSCO exist for the qSBH at $r = 6/(1-\alpha)$. This will give the radius of an ISCO if $\alpha \rightarrow 0$. Similarly it specifies the radius of an OSCO if $\alpha \rightarrow 1$, which will be a sufficiently large number.

3..2 qSBH with $\omega_q = -1$ (Cosmological Constant)

In this case the condition (2) takes the following form

$$8\alpha r^4 - 15\alpha r_g r^3 - r_g r + 3r_g^2 = 0. \quad (14)$$

It is convenient to write above equation in a dimensionless form, we introduce $x = r/r_g$ and $\lambda = \alpha r_g^2$. Then (14) becomes

$$8\lambda x^4 - 15\lambda x^3 - x + 3 = 0. \quad (15)$$

The above equation coincides with the equation obtained for the Schwarzschild-de Sitter spacetime [26]. From the applications of Sturm's theorem Toshika et. al [26], have shown that only for $0 < \lambda < 16/16875$, the necessary and sufficient conditions (2) and (6) are satisfied and two MSCOs exist. There is one ISCO and the other is the OSCO. From this we obtain bounds on the normalization factor as $0 < \alpha < 4/16875$. Thus we can say that when $\omega_q = -1$, then for the unit mass of the the Schwarzschild black hole with quintessence the MSCOs exist if $0 < \alpha < 4/16875$.

3..3 qSBH as a Kiselev Black Hole ($\omega_q = -2/3$)

This is a simplest nontrivial case of the Kiselev black hole [40], where the geometry is governed by the line element

$$ds^2 = -\left(1 - \frac{r_g}{r} - \alpha r\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_g}{r} - \alpha r\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (16)$$

which has the scalar curvature $R = 6\alpha/r$, where the inner and outer horizons exist at the coordinate singularities

$$r_{in} = \frac{1 - \sqrt{1 - 4\alpha r_g}}{2\alpha}, \quad r_{out} = \frac{1 + \sqrt{1 - 4\alpha r_g}}{2\alpha}, \quad (17)$$

provided $4r_g\alpha < 1$ and the two coincides if $4r_g\alpha = 1$, in which case we arrive at an extremal black hole. For a unit mass black hole, it results into $\alpha < 1/8$ and $\alpha = 1/8$, respectively.

The problem is to find the positive roots ($r > 0$) of the equation (2) with $f(r)$ and $h(r)$, defined in the above metric such that it also satisfies two additional constraints (6) in the form of inequalities given by

$$E^2 = -\frac{2(\alpha r^2 - r + r_g)^2}{r(\alpha r^2 - 2r + 3r_g)} > 0, \quad (18)$$

$$L^2 = -\frac{r^2(r_g - \alpha r^2)}{\alpha r^2 - 2r + 3r_g} > 0. \quad (19)$$

Before solving equation (2) for the above Kiselev black hole it is better to obtain the critical bounds on r on the positive r -axis for which there exists a solution. Let us determine the critical bounds on r using sufficient conditions. Note that in the first inequality (18) the numerator is a positive number so the inequality is true only if

$$\alpha r^2 - 2r + 3r_g < 0, \quad (20)$$

which can be factorized as follows

$$(r - r_-)(r - r_+) < 0, \quad (21)$$

where r_- and r_+ are given below

$$r_- = \frac{1 - \sqrt{1 - 3\alpha r_g}}{\alpha}, \quad r_+ = \frac{1 + \sqrt{1 - 3\alpha r_g}}{\alpha}. \quad (22)$$

This implies that $\alpha < 1/3r_g$, as otherwise we obtain complex constants of motion and both inequalities (18) and (19) are violated. In order to hold true the inequality (20) implies that both factors in (21) have opposite signs which is only possible when the value of r lies in the interval (r_-, r_+) . Since α is arbitrary therefore it can be used to identify the local bounds on r_- and r_+ , for which (20) is valid. We now employ Taylor expansion as $3r_g\alpha < 1$, to get

$$r_- = \frac{1}{\alpha} \left(1 - \left(1 - \frac{3\alpha r_g}{2} - \frac{(3\alpha r_g)^2}{8} - O(\epsilon^3) \right) \right), \quad (23)$$

$$= \frac{3r_g}{2} + \frac{9r_g^2\alpha}{8} + O(\epsilon^3), \quad \epsilon = 3\alpha r_g. \quad (24)$$

Therefore the least value of r_- is $3r_g/2$, as $r_- > 3r_g/2$. Similarly

$$r_+ = \frac{1}{\alpha} \left(1 + \left(1 - \frac{3\alpha r_g}{2} - \frac{(3\alpha r_g)^2}{8} - O(\epsilon^3) \right) \right), \quad (25)$$

$$= \frac{2}{\alpha} - \frac{3r_g}{2} - \frac{9r_g^2\alpha}{8} + O(\epsilon^3), \quad (26)$$

thus $r_+ < 2/\alpha$, so the largest value of r_+ is $2/\alpha$, hence we obtain

$$r \in \left(\frac{3r_g}{2} + \frac{9r_g^2\alpha}{8}, \frac{2}{\alpha} \right), \quad (27)$$

where the first constant of motion satisfy $E^2 > 0$, in (18). Since the denominator in the first inequality (18) is negative therefore the other inequality (19) holds true if we have

$$r_g - \alpha r^2 > 0, \quad (28)$$

which holds when

$$\sqrt{\frac{r_g}{\alpha}} > r. \quad (29)$$

Since $r_+ < 2/\alpha$, which follows from the condition (27) and as

$$\sqrt{\frac{r_g}{\alpha}} < \frac{2}{\alpha}, \quad \forall \quad \alpha < \frac{1}{4r_g} \quad (30)$$

therefore the upper bound r_+ is irrelevant for our purpose. Thus both inequalities (18) and (19) hold true provided the following lemma holds.

Lemma 1. *For a unit mass Schwarzschild black hole ($M = 1$) surrounded by a quintessence field with inner and outer horizons, the roots of equation (2) satisfy the inequalities (19), provided they satisfy the global bound*

$$r \in \left(\frac{3r_g}{2} + \frac{9r_g^2\alpha}{8}, \sqrt{\frac{r_g}{\alpha}} \right). \quad (31)$$

The above lemma provides a range on r for which both constants of motion satisfy the positivity criteria (18) and (19). Note that for an extremal Schwarzschild black hole surrounded by quintessence the above range reduces to

$$r \in \left(\frac{3r_g}{2} + \frac{9r_g^2}{64}, 4 \right), \quad (32)$$

which for a unit mass of the black hole gives $r \in (3.5625, 4)$.

Note that since $\alpha < 1/3r_g$, therefore the upper bound in (31) sharply moves away from origin as the value of α is chosen close to zero. However the lower bound contains α in the numerator which only contribute a small change in it. Therefore we expect to get more number of MSCOs in our analysis if the value of α lies in the close vicinity of zero. Indeed further analysis provide us sharper bounds on the values of α that yields a complete classification of MSCOs in Kieslev black holes. Thus, we only look for the

roots of algebraic equation which satisfy Lemma 1. In this case, we obtain the following constraint from equation (2)

$$\alpha^2 r^4 - 3\alpha r^3 + 6\alpha r_g r^2 + r_g r - 3r_g^2 = 0. \quad (33)$$

As required, for $\alpha = 0$, the above equation reduces to the equation for the Schwarzschild black hole. We write it in dimensionless form by defining $x = r/r_g$ and $\lambda = \alpha r_g$

$$\lambda^2 x^4 - 3\lambda x^3 + 6\lambda x^2 + x - 3 = 0. \quad (34)$$

We now proceed to find the roots of quartic equation (34) using Maple and it turns out that these can be converted into radicals

$$x = \frac{3\sigma_\omega^{1/6} \kappa_\sigma^{1/4} + \kappa_\sigma^{3/4} + \sqrt{(54 - 160\lambda)\sigma_\omega^{1/2} + ((18 - 32\lambda)\sigma_\omega^{1/3} - 4\lambda^{1/2})\kappa_\sigma^{1/2} - 4(\lambda\kappa_\sigma)^{1/2}\sigma_\omega^{2/3}}}{4\kappa_\sigma^{1/6} \sigma_\omega^{1/4}}, \quad (35)$$

where ω , κ_σ and σ_ω are defined below

$$\omega = 1024\lambda^3 - 640\lambda^2 + 100\lambda - 1, \quad (36)$$

$$\sigma_\omega = 32\lambda^{3/2} + \sqrt{\omega} - 10\sqrt{\lambda}, \quad (37)$$

$$\kappa_\sigma = 4\sqrt{\lambda}\sigma_\omega^{2/3} - 16\sigma_\omega^{1/3}\lambda + 9\sigma_\omega^{1/3} + 4\sqrt{\lambda}. \quad (38)$$

Note that the main term in the radical (35) is ω , while rest of the terms σ_ω and κ_σ are defined in terms of it. Furthermore, the main equation of ω , depends entirely on λ , which can be used to characterize the ranges of λ that yield feasible MSCOs. Therefore, we now consider ω , as a function of λ ,

$$\omega(\lambda) = 1024\lambda^3 - 640\lambda^2 + 100\lambda - 1, \quad (39)$$

which is a third degree equation in λ . We now examine the behavior of function $\omega(\lambda)$ in terms of λ , where we already have that $\lambda > 0$. Note that $\omega(\lambda)$ starts with a negative value $\omega(0) = -1$, and increases afterwards therefore the exact value of λ where after the graph of $\omega(\lambda)$ is positive can be obtained by finding the roots of above equation, i.e. $\omega(\lambda) = 0$ which gives

$$\lambda_1 = \frac{3 - 2\sqrt{2}}{16}, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_3 = \frac{3 + 2\sqrt{2}}{16}. \quad (40)$$

Note that the above values of λ satisfy $\lambda_1 < \lambda_2 < \lambda_3$, therefore the least value of λ is $\lambda = \lambda_1 = 0.0107233048$. The graph of the function is given in Figure 1. From the graph it is clear that $\omega(\lambda) > 0$ for all $\lambda \in (\lambda_1, \lambda_2)$, after which it again becomes negative in the interval $\lambda \in (\lambda_2, \lambda_3)$. Lastly for all $\lambda \in (\lambda_3, 1)$, the function ω is positive. We have found that in the intervals where ω , is positive gives rise to two real (in which one is positive and the other is negative) and two imaginary roots. However, there are three positive and one negative real root in the ranges where ω is negative. In short the feasible regions include a union of two disjoint intervals, i.e.

$$\omega(\lambda) > 0, \quad \forall \quad \lambda \in (0, 0.0107233048) \cup (0.25, 0.3642766952), \quad (41)$$

which in terms of the value α becomes (where $\lambda = 2\alpha$)

$$\omega(\alpha) > 0, \quad \forall \quad \alpha \in (0, 0.00536165238) \cup (0.125, 0.1821383476). \quad (42)$$

We now examine above regions in the light of Lemma 1. Note that $\alpha < 1/3r_g$, which for

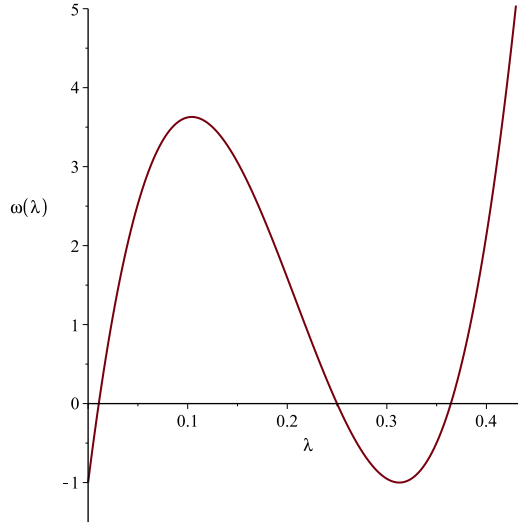


Figure 1: The graph of $\omega(\lambda)$, where it has three real roots.

a unit qSBH becomes $\alpha < 1/6$, i.e. $\alpha < 0.1666666667$. Since the value 0.1666666667 is smaller than 0.1821383476, therefore it provides a sharper upper bound on the range of α . It is easy to verify that in the interval $\alpha \in (0.125, 0.1666666667)$, we obtain four MSCOs (three positive and one negative) all of which fail to satisfy Lemma 1. Therefore, the only interval of interest is $(0, 0.00536165238)$, in which the condition of Lemma 1 is also fulfilled. In order to find the MSCOs we have to solve the quartic equation (34) and we find that in the regions where $\omega(\lambda)$ is positive it yields two real and two imaginary roots. On the other hand in the regions where $\omega(\lambda)$ is negative the equation results into three positive and one negative real root. We now consider a few particular cases to show that MSCOs exist. For example assume that $\lambda = 0.002$, i.e., $\alpha = 0.001$, then (34) becomes

$$0.000004x^4 - 0.006x^3 + 0.012x^2 + x - 3 = 0. \quad (43)$$

The real positive roots of this equation are obtained after using (36-38) are

$$x_1 = 3.059118379, \quad x_2 = 12.32978727, \quad x_3 = 1497.885976, \quad (44)$$

where we have discarded one negative root. Now for unit mass of the black hole this gives

$$r_1 = 6.118236758, \quad r_2 = 24.65957454, \quad r_3 = 2995.771952, \quad (45)$$

which means that both r_1 and r_2 lies in the required interval $(3, 44.72135)$ of Lemma 1. Since, r_3 does not lie in the interval therefore fail to satisfy Lemma 1. Therefore here

we have two MSCOs. One as ISCO with $r_{ISCO} = 6.119$ and the other as OSCO with $r_{OSCO} = 24.66$.

For another value of $\lambda = 0.01$, i.e. for $\alpha = 0.005$, we get two MSCOs. The ISCO with $r_{ISCO} = 7.2378$ and OSCO with $r_{OSCO} = 9.1628$. One can easily check that for other values $\alpha = 0.002$, $\alpha = 0.003$ and $\alpha = 0.004$ there exist MSCOs. Similarly if we consider $\alpha = 0.006$, then we get one positive real root of (2) for which $E^2 < 0$. Hence it is unphysical and no MSCO exist.

4. Summary

In this brief communication we have analyzed the MSCOs of a test particle in the vicinity of the Schwarzschild black hole with quintessence matter to obtain bounds on the value of the normalization factor α , for which such orbits exist. We have considered three different cases for the value of the equation of state parameter, i.e. $\omega_q = -1$, $\omega_q = -1/3$ and $\omega_q = -2/3$. In the case when $\omega_q = -1$, we have obtained $0 < \alpha < 4/16875$, for which there exist two MSCOs. For $\omega_q = -1/3$, we have seen that MSCO exists for all $\alpha \in (0, 1)$. While in the case of $\omega_q = -2/3$, we see that MSCOs exist if $0 < \alpha \leq 0.00536165238$.

Another observation is that in the presence of a quintessence field ($w_q = -1, -2/3$), the radius of the MSCOs gets larger as compared to the radius of MSCO of a pure unit mass Schwarzschild black hole for which it is 6. Recently the effect of a quintessence model on the energy content of the Riessner-Nordstrom black hole surrounded by the quintessence matter has been investigated in [41]. Both upper and lower bounds were obtained on the value of the normalization factor α , i.e. $0 < \alpha < 1$. The same bounds on the normalization factor α were also obtained for the Schwarzschild black hole with quintessence matter in a totally different scenario [25]. Therefore, it is worth exploring to check whether the same bounds on the normalization factor α obtained here, also exist in the case of the Riessner-Nordstrom black hole with quintessence matter.

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