Identities for partial Bell polynomials derived from identities for weighted integer compositions

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Abstract. We discuss closed-form formulas for the (n, k)-th partial Bell polynomials derived in Cvijović [2]. We show that partial Bell polynomials are special cases of weighted integer compositions, and demonstrate how the identities for partial Bell polynomials easily follow from more general identities for weighted integer compositions. We also provide short and elegant probabilistic proofs of the latter, in terms of sums of discrete integer-valued random variables. Finally, we outline further identities for the partial Bell polynomials.

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1. Introduction

In a recent note, Cvijović [2] has derived three new identities — which we list in Equations (5), (6), and (7) below — for the (n, k)-th partial Bell polynomials in the variables $x_1, x_2, \ldots, x_{n-k+1}$, which are defined as¹

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{\ell_1! \cdots \ell_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{\ell_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{\ell_{n-k+1}},$$
(1)

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¹Another way to define the partial Bell polynomials is by the formal power series expansion $\frac{1}{k!} \left(\sum_{m \ge 1} \frac{x_m}{m!} t^m \right)^k = \sum_{n \ge k} \frac{B_{n,k}(x_1, x_2, \dots, x_{n-k+1})}{n!} t^n$. This would also immediately lead to Corollary 1 below if we defined the subsequent concept of weighted integer compositions by the formal power series expansion $\left(\sum_{m \ge 0} f(m) t^m \right)^k = \sum_{n \ge 0} {k \choose n}_f t^n$, with notation as explicated below.

where the summation is over all solutions in nonnegative integers $\ell_1, \ldots, \ell_{n-k+1}$ of $\ell_1 + 2\ell_2 + \cdots + (n-k+1)\ell_{n-k+1} = n$ and $\ell_1 + \ell_2 + \cdots + \ell_{n-k+1} = k$. Such identities may be important for the efficient evaluation of the partial Bell polynomials, as hinted at in Cvijović [2], who also indicates applications of the polynomials. Importantly, they generalize the Stirling numbers S(n, k)of the second kind, since $S(n, k) = B_{n,k}(1, \ldots, 1)$.

The purpose of the present note is to show that these identities are special cases of identities for *weighted integer compositions* and to outline very short proofs for the more general identities, based on sums of discrete random variables. Our main point is to compile (identical or very similar) results developed within heterogeneous communities of research, and to combine these results. Finally, in Section 4, we present three additional identities for the partial Bell polynomials. Throughout, we write $B_{n,k}$ as a shorthand for $B_{n,k}(x_1, \ldots, x_{n-k+1})$, unless explicit reference to the indeterminates is critical.

2. Preliminaries on weighted integer compositions

An integer composition of a nonnegative integer n is a k-tuple (π_1, \ldots, π_k) , for $k \ge 1$, of nonnegative integers such that $\pi_1 + \cdots + \pi_k = n$. We call k the number of parts. Note that order of part matters, and this distinguishes integer compositions from *integer partitions*. Now, to generalize, we may consider f-colored integer compositions, where each possible part size $s \in \mathbb{N} = \{0, 1, 2, \ldots\}$ may come in f(s) different colors, whereby $f : \mathbb{N} \to \mathbb{N}$. For example, the integer n = 4 has nine distinct f-colored integer compositions, for f(0) = 2, f(1) = f(2) = 1, $f(3) = f(4) = f(5) = \cdots = 0$, with k = 3 parts, namely,

 $(2, 2, 0), (2, 2, 0^*), (2, 1, 1), (1, 2, 1), (2, 0, 2), (2, 0^*, 2), (1, 1, 2), (0, 2, 2), (0^*, 2, 2), (0, 2), (0, 2), (0, 2), (0, 2), (0, 2), (0, 2),$

where we use a star superscript to distinguish the two different colors of part size 0. For weighted integer compositions we let, more generally, f(s) be an arbitrary real number (or even a value in a commutative ring), the weight of part size s. Colored integer compositions have been discussed in [6, 9, 10], and weighted integer compositions have been under review in Eger [4], but have been investigated as early as Hoggatt and Lind's [7] work.

Let $\binom{k}{n}_f$ denote the *number* of *f*-weighted integer compositions of *n* with *k* parts, when the range of *f* is \mathbb{N} , and let $\binom{k}{n}_f$ denote the *total weight* of all *f*-weighted integer compositions of *n* with *k* parts, when the range of *f* is the set of reals \mathbb{R} . We then have the following theorem.

Theorem 1. Let $k, n \ge 0$ be integers, and let $f : \mathbb{N} \to \mathbb{R}$ be arbitrary. Then the following identities hold.

$$\binom{k}{n}_{f} = \sum f(\pi_{1}) \cdots f(\pi_{k}), \qquad (2)$$

$$\binom{k}{n}_{f} = \sum \binom{k}{\ell_0, \ell_1, \dots, \ell_n} f(0)^{\ell_0} \cdots f(n)^{\ell_n}, \tag{3}$$

where the sum in (2) is over all solutions in nonnegative integers π_1, \ldots, π_k of $\pi_1 + \cdots + \pi_k = n$, and the sum in (3) is over all solutions in nonnegative integers ℓ_0, \ldots, ℓ_n of $\ell_0 + \cdots + \ell_n = k$ and $0\ell_0 + 1\ell_1 + \cdots + n\ell_n = n$. Finally, $\binom{k}{\ell_0, \ell_1, \ldots, \ell_n} = \frac{k!}{\ell_0! \ell_1! \cdots \ell_n!}$ denote the multinomial coefficients.

The proof of Theorem 1 is simple. Identity (2) is a direct application of the definition of $\binom{k}{n}_f$ and (3) follows, combinatorially, from (2) by rewriting the summation over integer compositions into a summation over integer partitions and then adjusting each term in the sum appropriately (in particular, the multinomial coefficients account for distributing the parts in partitions). Alternatively, when $\binom{k}{n}_f$ is defined as coefficient of a certain polynomial, then (3) can also be arrived at via application of the multinomial theorem (see [4]) or the formula of Faà di Bruno [8] for the higher order derivatives of composite functions (see [7]).

Interpreting f(s), for $s \in \mathbb{N}$, as indeterminates, Theorem 1 identity (3) immediately implies that f-weighted integer compositions 'generalize' partial Bell polynomials.

Corollary 1. Let $k, n \ge 0$ be integers. Then:

$$\frac{k!}{n!}B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \binom{k}{n}_f,$$
(4)

whereby

$$f(0) = f(n - k + 2) = f(n - k + 3) = \dots = 0,$$

$$f(s) = \frac{x_s}{s!}, \text{ for } s \in \{1, 2, \dots, n - k + 1\}.$$

Corollary 1 has been established in [1] in the more particular setting of restricted integer compositions, or, classical 'multinomial/extended binomial coefficients', where, in our notation, f(s) = 1 for all $s \in \{1, \ldots, q\}$, for some positive integer q. This yielded the conclusion that partial Bell polynomials 'generalize' restricted integer compositions insofar as $B_{n,k}(1!, 2!, \ldots, q!, 0, \ldots) = \frac{n!}{k!} {k \choose n}_f$, where f is the indicator function on $\{1, \ldots, q\}$.

Representation (2) in Theorem 1 is very useful on its own since it captures an equivalence between f-weighted integer compositions and distributions of sums of independent and identically distributed (i.i.d.) discrete random variables as explicated in the following lemma.

Lemma 1. Let $k, n \ge 0$ be integers and let $f : \mathbb{N} \to \mathbb{R}$ be arbitrary. Then there exist i.i.d. nonnegative integer-valued random variables X_1, \ldots, X_k such that

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 $\binom{k}{n}_{f}$ is given as the distribution of the sum of X_{1}, \ldots, X_{k} (times a suitable normalization factor).

Conversely, the distribution $P_f[X_1 + \cdots + X_k = n]$ of arbitrary i.i.d. nonnegative integer-valued random variables X_1, \ldots, X_k with common distribution function f is given by $\binom{k}{n}_f$.

Proof. Consider $\binom{k}{n}_f$. When f is zero almost everywhere, i.e., f(s) = 0 for all s > x, for some $x \in \mathbb{N}$, then let \overline{F} denote the sum $\sum_{s' \in \mathbb{N}} f(s')$ and let $g(s) = \frac{f(s)}{F}$ for all $s \in \mathbb{N}$. Consider the i.i.d. random variables X_1, \ldots, X_k with common distribution function g, i.e., $P[X_i = s] = g(s)$. By definition, the distribution of the sum $X_1 + \cdots + X_k$ is given as

$$P_{g}[X_{1} + \dots + X_{k} = n] = \sum_{\pi_{1} + \dots + \pi_{k} = n} P[X_{1} = \pi_{1}] \cdots P[X_{k} = \pi_{k}]$$
$$= \sum_{\pi_{1} + \dots + \pi_{k} = n} g(\pi_{1}) \cdots g(\pi_{k})$$
$$= \frac{1}{\bar{F}^{n}} \sum_{\pi_{1} + \dots + \pi_{k} = n} f(\pi_{1}) \cdots f(\pi_{k}) = \frac{1}{\bar{F}^{n}} {k \choose n}_{f}$$

When f is not zero almost everywhere, then note that

$$\binom{k}{n}_f = \binom{k}{n}_{\hat{f}} = \bar{\hat{F}}^n P_{\hat{g}}[Y_1 + \dots + Y_k = n],$$

whereby $\hat{f}(s) = f(s)$ for $s \leq n$ and $\hat{f}(s) = 0$ for s > n, and Y_1, \ldots, Y_k are i.i.d. random variables with common distribution function $\hat{g}(s) = \frac{\hat{f}(s)}{\hat{F}}$, where $\bar{F} = \sum_{s' \in \mathbb{N}} \hat{f}(s')$.

Conversely, the distribution of the sum of i.i.d. nonnegative integervalued random variables X_1, \ldots, X_k with common distribution function f is given by, as above,

$$P_f[X_1 + \dots + X_k = n] = \sum_{\pi_1 + \dots + \pi_k = n} P[X_1 = \pi_1] \cdots P[X_k = \pi_k]$$
$$= \sum_{\pi_1 + \dots + \pi_k = n} f(\pi_1) \cdots f(\pi_k) = \binom{k}{n}_f,$$

applying Theorem 1 in the last equality.

Lemma 1 has appeared in [4] and, in the special case when f is the discrete uniform measure, in [1]. Lemma 1 allows us to prove properties of the weighted integer compositions $\binom{k}{n}_f$ by referring to properties of the distribution of the sum of discrete random variables, which is oftentimes convenient, as we shall see below.

3. Identities and proofs

The three identities for partial Bell polynomials that Cvijović [2] introduces are the following:

$$B_{n,k} = \frac{1}{x_1} \frac{1}{n-k} \sum_{\alpha=1}^{n-k} \binom{n}{\alpha} \left[(k+1) - \frac{n+1}{\alpha+1} \right] x_{\alpha+1} B_{n-\alpha,k},$$
(5)

$$B_{n,k_1+k_2} = \frac{k_1!k_2!}{(k_1+k_2)!} \sum_{\alpha=0}^n \binom{n}{\alpha} B_{\alpha,k_1} B_{n-\alpha,k_2},\tag{6}$$

$$B_{n,k+1} = \frac{1}{(k+1)!} \sum_{\alpha_1=k}^{n-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1} \binom{n}{\alpha_1} \cdots \binom{\alpha_{k-1}}{\alpha_k} x_{n-\alpha_1} \cdots x_{\alpha_k-1-\alpha_k} x_{\alpha_k}.$$
(7)

As shown in Cvijović [2], (7) easily follows inductively from (6). Therefore, we concentrate on the identities (5) and (6). Throughout, we let k, n be nonnegative integers and f be a function $f : \mathbb{N} \to \mathbb{R}$. As we will see now, identity (6) may be seen as a special case of the convolution formula for the sum of discrete random variables.

Lemma 2. Let n, k_1 and k_2 be nonnegative integers. Then:

$$\binom{k_1+k_2}{n}_f = \sum_{x+y=n} \binom{k_1}{x}_f \binom{k_2}{y}_f.$$

Proof. By our previous discussion, it suffices to prove the lemma for sums of i.i.d. integer-valued random variables $X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}$. Now,

$$P_f[(X_1 + \dots + X_{k_1}) + (Y_1 + \dots + Y_{k_2}) = n]$$

= $\sum_{x+y=n} P_f[X_1 + \dots + X_{k_1} = x]P_f[Y_1 + \dots + Y_{k_2} = y]$

by the discrete convolution formula for discrete random variables.

To formally complete the proof,² we apply Lemma 1 (or, more precisely, its proof), leading to:

$$\binom{k_1+k_2}{n}_f = \binom{k_1+k_2}{n}_{\hat{f}} = \bar{\hat{F}}^n P_{\hat{g}}[X_1+\dots+X_{k_1+k_2}=n]$$

$$= \bar{\hat{F}}^n \sum_{x+y=n} P_{\hat{g}}[X_1+\dots+X_{k_1}=x]P_{\hat{g}}[Y_1+\dots+Y_{k_2}=y]$$

$$= \sum_{x+y=n} \bar{\hat{F}}^x P_{\hat{g}}[X_1+\dots+X_{k_1}=x]\bar{\hat{F}}^y P_{\hat{g}}[Y_1+\dots+Y_{k_2}=y]$$

$$= \sum_{x+y=n} \binom{k_1}{x}_{\hat{f}}\binom{k_2}{y}_{\hat{f}} = \sum_{x+y=n} \binom{k_1}{x}_{f}\binom{k_2}{y}_{f}.$$

 $^{^2\}mathrm{We}$ omit this straightforward step in all subsequent proofs.

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Proof of identity (6). Using Lemma 2 and Corollary 1, we obtain that

$$B_{n,k_1+k_2} = \frac{n!}{(k_1+k_2)!} \binom{k_1+k_2}{n}_f = \frac{n!}{(k_1+k_2)!} \sum_{x+y=n} \binom{k_1}{x}_f \binom{k_2}{y}_f$$
$$= \frac{n!}{(k_1+k_2)!} \sum_{x+y=n} \frac{k_1!}{x!} B_{x,k_1} \frac{k_2!}{y!} B_{y,k_2}$$
$$= \frac{k_1!k_2!}{(k_1+k_2)!} \sum_{x+y=n} \frac{n!}{x!y!} B_{x,k_1} B_{y,k_2}$$
$$= \frac{k_1!k_2!}{(k_1+k_2)!} \sum_{x+y=n} \binom{n}{x} B_{x,k_1} B_{y,k_2}.$$

The proof of the following identity for weighted integer compositions which directly entails identity (5) is a straightforward variation of the proof outlined in DePril [3], who considers sums of integer-valued random variables.

 \Box

Lemma 3. Let $k, n \ge 0$ be integers and let $f : \mathbb{N} \to \mathbb{R}$ be arbitrary with f(0) = 0 and $f(1) \ne 0$. Then:

$$\binom{k}{n}_{f} = \frac{1}{f(1)(n-k)} \sum_{s \ge 1} \left(k+1 - \frac{n+1}{s+1}\right) (s+1)f(s+1)\binom{k}{n-s}_{f}.$$

Proof. Let X_1, \ldots, X_{k+1} be i.i.d. nonnegative integer-valued random variables, with distribution function f. Consider the conditional expectation $E[X_1 | X_1 + \ldots + X_{k+1} = n + 1]$. Due to identical distribution of the variables and linearity of $E[\cdot | \cdot]$, it follows that $E[X_1 | X_1 + \ldots + X_{k+1} = n + 1] = \frac{E[X_1 + \ldots + X_{k+1} | X_1 + \ldots + X_{k+1} = n + 1]}{k+1} = \frac{n+1}{k+1}$. Thus, $E[\frac{k+1}{n+1}X_1 - 1 | X_1 + \ldots + X_{k+1} = n + 1] = n + 1] = 0$, which means that

$$0 = \sum_{s=1}^{n} \left(\frac{k+1}{n+1}s - 1\right) \frac{P_f[X_1 = s, X_1 + \dots + X_{k+1} = n+1]}{P_f[X_1 + \dots + X_{k+1} = n+1]}$$

= $\frac{1}{\binom{k+1}{n+1}_f} \sum_{s=1}^{n} \left(\frac{k+1}{n+1}s - 1\right) P[X_1 = s] \cdot P_f[X_2 + \dots + X_{k+1} = n+1-s]$
= $\frac{1}{\binom{k+1}{n+1}_f} \sum_{s=1}^{n} \left(\frac{k+1}{n+1}s - 1\right) f(s) \binom{k}{n+1-s}_f,$

so that rewriting and shifting indices lead to the required expression.

We omit the proof of identity (5) since it is a simple application of Lemma 3.

4. More identities for the partial Bell polynomials

We mention the following three additional identities for partial Bell polynomials,

$$B_{n,k} = \frac{1}{k} \sum_{\alpha \ge 0} \binom{n}{\alpha} x_{\alpha} B_{n-\alpha,k-1},$$
(8)

$$B_{n,k} = \sum_{\alpha \ge 1} \binom{n-1}{\alpha-1} x_{\alpha} B_{n-\alpha,k-1}, \qquad (9)$$

$$B_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\alpha \ge 0} \binom{n}{\alpha} x_1^{\alpha} B_{n-\alpha,k-\alpha}(0, x_2, \dots, x_{n-k+1}).$$
(10)

(note that (8) is a special case of (6)) which straightforwardly follow from the following corresponding identities for weighted integer compositions,

$$\binom{k}{n}_f = \sum_{s \ge 0} f(s) \binom{k-1}{n-s}_f,\tag{11}$$

$$\binom{k}{n}_{f} = \frac{k}{n} \sum_{s \ge 1} sf(s) \binom{k-1}{n-s}_{f},$$
(12)

$$\binom{k}{n}_{f} = \sum_{i \ge 0} f(r)^{i} \binom{k}{i} \binom{k-i}{n-ri}_{\tilde{f}}.$$
(13)

In Equation (13), $\tilde{f}(r) = 0$ and $\tilde{f}(s) = f(s)$ for all $s \neq r$ (note that (10) is a special case of (13) in which r = 1). Proofs of identities (11) to (13) can, e.g., be found in Fahssi [5] and Eger [4], who also give further identities for weighted integer compositions.

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