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REALIZING ALGEBRAIC INVARIANTS OF HYPERBOLIC SURFACES

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Abstract. Let S_g ($g \geq 2$) be a closed surface of genus g. Let K be any real number field and A be any quaternion algebra over K. We show that there exists a hyperbolic structure on S_g such that K and A arise as its invariant trace field and invariant quaternion algebra.

1. Introduction

The invariant trace field and the quaternion algebra are basic algebraic invariants of a Kleinian group. Questions around these invariants have inspired extensive research in the study of hyperbolic 3-manifolds. The following Realization Conjecture due to W. Neumann [5] is one of the fundamental questions along these lines:

Conjecture 1. *Let K be any non-real complex number field and A be any quaternion algebra over K. Then there exists a hyperbolic 3-manifold M such that K and A arise as the invariant trace field and quaternion algebra of M.*

For the 2-dimensional case, since Mostow's rigidity theorem does not hold, trace fields and quaternion algebras are no longer topological invariants. But, in this case, we can ask the following analogous question instead, as suggested in [5]:

Question 1. *Let S_g ($g \geq 2$) be a closed surface of genus g, K be any real number field and A be any quaternion algebra over K.*[1] *Is there a hyperbolic structure on S_g whose invariant trace field and quaternion algebra are equal to K and A?*

This natural question had been discussed for some time, but the complete answer was unknown. Recently J. Kahn and V. Markovic announced a partial answer as follows:

Theorem 1.1. [3] *Let K be any real number field and A be any quaternion algebra over K. Then there exists a closed surface S_g and a hyperbolic structure on it such that K and A are its invariant trace field and quaternion algebra.*

Their proof uses the techniques developed in their recent proofs of two deep conjectures, the Surface Subgroup Conjecture and the Ehrenpreis Conjecture. Although Theorem 1.1 answers Question 1 partially, an important feature of this theorem is that they realize surfaces via integral traces. That is, all the traces in their construction are algebraic integers.

The aim of this paper is to provide a complete answer to Question 1. In other words, we prove the following theorem:

Theorem 1.2. *Let S_g ($g \geq 2$) be any closed surface of genus g. Let K be any real number field and A be any quaternion algebra over K. Then there exists a hyperbolic structure on S_g such that K and A arise as its invariant trace field and invariant quaternion algebra.*

Note that since invariant trace fields and quaternion algebras are invariants of commensurability classes, once we prove the above theorem for $g = 2$, the rest of the cases easily follow by looking at covering surfaces.

We can also ask a similar realization question about the (usual) trace field and quaternion algebra. In fact, we prove the following:

[1] We say K is a real number field if $K \subset \mathbb{R}$ and $[K : \mathbb{Q}] < \infty$.

Theorem 1.3. *Let S_2 be a genus 2 closed surface. Let K be any real number field and A be any quaternion algebra over K. Then there exists a hyperbolic structure on S_2 such that K and A arise as its trace field and quaternion algebra. Moreover, the invariant trace field and invariant quaternion algebra of it are equal to the trace field and quaternion algebra of it respectively.*

We split the proof of this theorem into two parts. We prove the first statement in Section 4 and, using the proof, we show the second statement in Section 5.

Our proofs are based on explicit computations as well as some elementary facts in number theory. The basic idea is to create a genus two Riemann surface by attaching two identical copies of a once-punctured torus. Then using the work of T. Gauglhofer [1], we prove that its trace field is fairly simple, generated only by the traces of three elements. We next convert the problem into a system of Diophantine equations and solve these equations. The whole process is completely elementary and natural.

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2. Preliminaries

In Sections 2.1 and 2.2, we quickly review basic definitions and facts which will be used below. For more details, see [4].

2.1. **Trace Field** Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ ($\cong \mathrm{Isom}^+(\mathbb{H}^2)$) be a Fuchsian group such that $\mathbb{H}^2/\Gamma$ is a closed hyperbolic surface. Let $\bar{\Gamma} \subset \mathrm{SL}(2,\mathbb{R})$ be the inverse image of Γ under the projection $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$. Then the *trace field* of Γ is defined by

$$\mathbb{Q}(\{\mathrm{tr}\ \gamma \mid \gamma \in \bar{\Gamma}\}).$$

This set is known to be a finite extension number field. For instance, if $\bar{\Gamma} = <\gamma_1, \cdots, \gamma_n \mid - >$, then its trace field is generated by the traces of the following elements:

$$\{\gamma_i,\ \gamma_{j_1}\gamma_{j_2},\ \gamma_{k_1}\gamma_{k_2}\gamma_{k_3} \mid 1 \le i \le n, 1 \le j_1 < j_2 \le n, 1 \le k_1 < k_2 < k_3 \le n\}. \tag{2.1}$$

The *invariant trace field* of Γ is defined by

$$\mathbb{Q}(\{\mathrm{tr}\ \gamma^2 \mid \gamma \in \bar{\Gamma}\}),$$

and it is an invariant of the commensurability class of Γ.

2.2. **Quaternion Algebra** Let K be a number field. A *quaternion algebra* over K is a four dimensional algebra with basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that

$$\mathbf{i}^2 = a,\ \mathbf{j}^2 = b,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$

for some $a, b \in K^*$ and this is often denoted by $\left(\frac{a,\ b}{K}\right)$. The following equivalence relations are well known:

$$\left(\frac{a,b}{K}\right) \cong \left(\frac{au^2, bv^2}{K}\right) \cong \left(\frac{au^2, bv^2 - abw^2}{K}\right) \tag{2.2}$$

where $u, v, w \in K$ such that $u, v, bv^2 - abw^2 \neq 0$.[2]

Let Γ be as in Section 2.1. The *quaternion algebra* of Γ is defined by

$$\Big\{\sum_{i=1}^{n} a_i\gamma_i \mid a_i \in \text{Trace field of } \Gamma,\ \ \gamma_i \in \bar{\Gamma}\Big\}.$$

---

[2]To show the equivalence between $A_1 = \left(\frac{a,b}{K}\right)$ and $A_2 = \left(\frac{au^2, bv^2 - abw^2}{K}\right)$, let $\varphi : A_1 \to A_2$ be a map such that $\varphi(\mathbf{1}) = \mathbf{1}, \varphi(\mathbf{i}) = u\mathbf{i}$ and $\varphi(\mathbf{j}) = u\mathbf{j} + w\mathbf{k}$. Then $\varphi(\mathbf{j})\varphi(\mathbf{k}) = -\varphi(\mathbf{k})\varphi(\mathbf{j})$ and so it can be naturally extended to an isomorphism between A_1 and A_2.

This set is a quaternion algebra over the trace field of Γ and equivalent to

$$\Big(\frac{\mathrm{tr}^2\,\gamma_1 - 4,\ \ \mathrm{tr}\,[\gamma_1,\gamma_2]-2}{\text{Trace field of }\Gamma}\Big) \tag{2.3}$$

where $\gamma_1, \gamma_2 \in \bar{\Gamma}$ are two hyperbolic elements such that $<\gamma_1,\gamma_2>$ is irreducible.

The *invariant quaternion algebra* of Γ is the algebra generated over the invariant trace field of Γ by the squares of the elements of $\bar{\Gamma}$. This set is also an invariant of the commensurability class of Γ.

2.3. **Trace Field Coordinates of Riemann Surfaces** Each $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ has two inverse images in $\mathrm{SL}(2,\mathbb{R})$. We denote the one having the positive trace by γ_+ and the one having the negative trace by γ_-.

Following the same notation given in [1], we use the matrices below as a basis of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathbf{1}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \mathbf{I}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \mathbf{J}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \mathbf{K}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

For example, in terms of the above basis, $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is represented by

$$\Big(\frac{a+d}{2}\Big)\mathbf{1}+\Big(\frac{a-d}{2}\Big)\mathbf{I}+\Big(\frac{b+c}{2}\Big)\mathbf{J}+\Big(\frac{b-c}{2}\Big)\mathbf{K}.$$

Let $\mathcal{T}$ be a hyperbolic once punctured torus whose fundamental domain and picture are shown in Figure 2 and Figure 1 respectively. Let ρ and σ be the hyperbolic elements given in Figure 2. Then ρ_+ and σ_+ can be represented in terms of the traces of $\rho_+, \sigma_+, (\rho\sigma)_+, [\rho,\sigma]_+$ plus the attracting fixed point of σ as follows:[3]

Theorem 2.1. [1] *Let tr $\rho_+ = 2r$, tr $\sigma_+ = 2s$, tr $(\rho\sigma)_+ = 2t$, tr $([\rho,\sigma]_+) = 2c$, and $M > 0$ be the attracting fixed point of σ (as shown in the picture). Then*

$$\begin{aligned}\rho_+ &= r\mathbf{1}+\frac{r(c+1)}{\sqrt{c^2-1}}\mathbf{I}-\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K})+\frac{2rs-t+\big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),\\ \sigma_+ &= s\mathbf{1}-\frac{s(c+1)}{\sqrt{c^2-1}}\mathbf{I}+\frac{\tilde{M}}{2}(\mathbf{J}+\mathbf{K})-\frac{c-1+2s^2}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),\end{aligned} \tag{2.4}$$

where

$$\tilde{M} = M\frac{s\sqrt{c^2-1}+(c-1)\sqrt{s^2-1}}{c-1},\qquad c = 4rst-2r^2-2s^2-2t^2+1 \quad \textit{and} \quad c,r,s,t>1, M>0. \tag{2.5}$$

Remark. In [1], it is proved that, for any $c, r, s, t, M, \tilde{M}$ satisfying (2.5), the group generated by ρ_+ and σ_+ act discretely on $\mathbb{H}$ and generate a fundamental domain as given in Figure 2.

Now we create a genus 2 surface $\mathcal{S}$ by attaching $\mathcal{T}$ to identical symmetric image $\mathcal{T}'$ along their common boundary as shown in Figure 3. (See Figure 4 for the corresponding fundamental domain.) Let ρ' and σ' be the elements corresponding to ρ and σ (Figure 4). Then $(\rho')_+$ and $(\sigma')_+$ are as

[3]Note that $[\rho_+,\sigma_+] = -[\rho,\sigma]_+ = [\rho,\sigma]_-$.

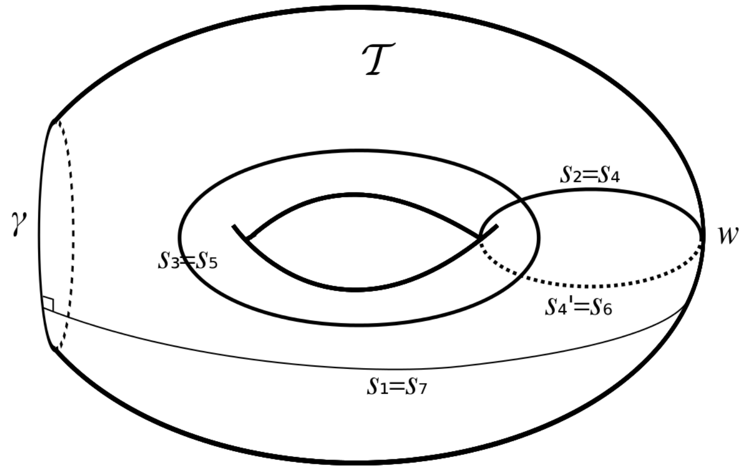

FIGURE 1.

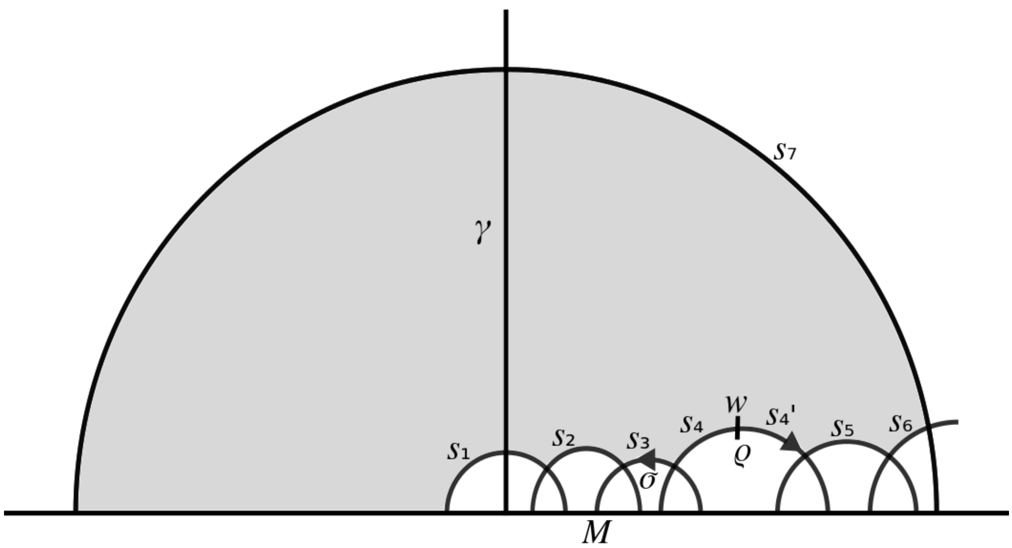

FIGURE 2. ρ and σ fix the geodesics containing $s_4 \cup s_4'$ and s_3 (respectively), and $[\rho, \sigma]$ fixes the y-axis. The side pairings are given as follows: $\rho(s_3) = s_5, \sigma(s_4) = s_2, \rho\sigma\rho^{-1}(s_4') = s_6$ and $[\rho, \sigma](s_1) = s_7$.

follows:[4]

$$
\begin{aligned}
(\rho')_+ &= r\mathbf{1} + \frac{r(c+1)}{\sqrt{c^2-1}}\mathbf{I} + \frac{\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) - \frac{2rs - t + \big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\big(\mathbf{J}-\mathbf{K}\big),\\
(\sigma')_+ &= s\mathbf{1} - \frac{s(c+1)}{\sqrt{c^2-1}}\mathbf{I} - \frac{\tilde{M}}{2}(\mathbf{J}+\mathbf{K}) + \frac{c-1+2s^2}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),
\end{aligned}
\tag{2.6}
$$

[4]If $\dfrac{az+b}{cz+d}$ is a Möbius transformation representing ρ, then it is easy to check that ρ' is of the form $\dfrac{az-b}{-cz+d}$. In terms of the basis we introduced, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to $\big(\frac{a+d}{2}\big)\mathbf{1} + \big(\frac{a-d}{2}\big)\mathbf{I} + \big(\frac{b+c}{2}\big)\mathbf{J} + \big(\frac{b-c}{2}\big)\mathbf{K}$ and $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ is equal to $\big(\frac{a+d}{2}\big)\mathbf{1} + \big(\frac{a-d}{2}\big)\mathbf{I} - \big(\frac{b+c}{2}\big)\mathbf{J} - \big(\frac{b-c}{2}\big)\mathbf{K}$. Thus the formula follows.

where

$$\tilde{M} = M\frac{s\sqrt{c^2-1}+(c-1)\sqrt{s^2-1}}{c-1}, \quad c = 4rst - 2r^2 - 2s^2 - 2t^2 + 1 \quad \text{and} \quad c, r, s, t > 1, M > 0.$$

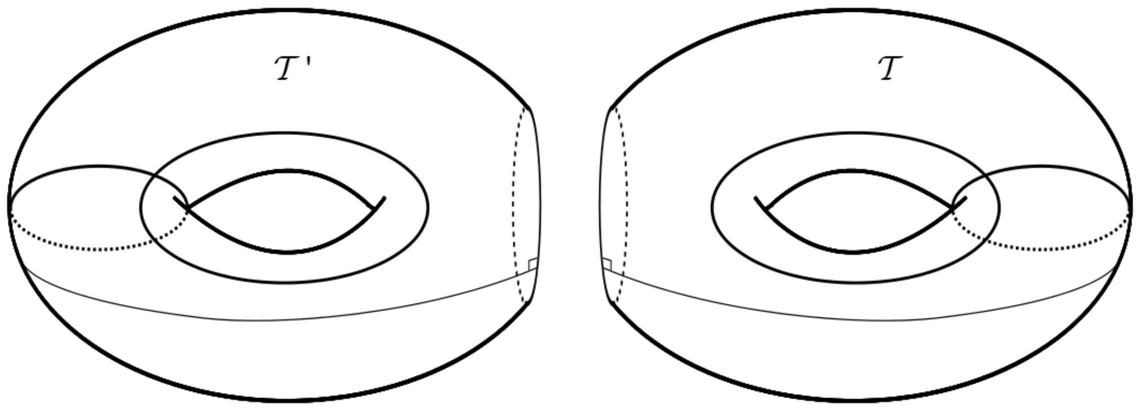

Figure 3.

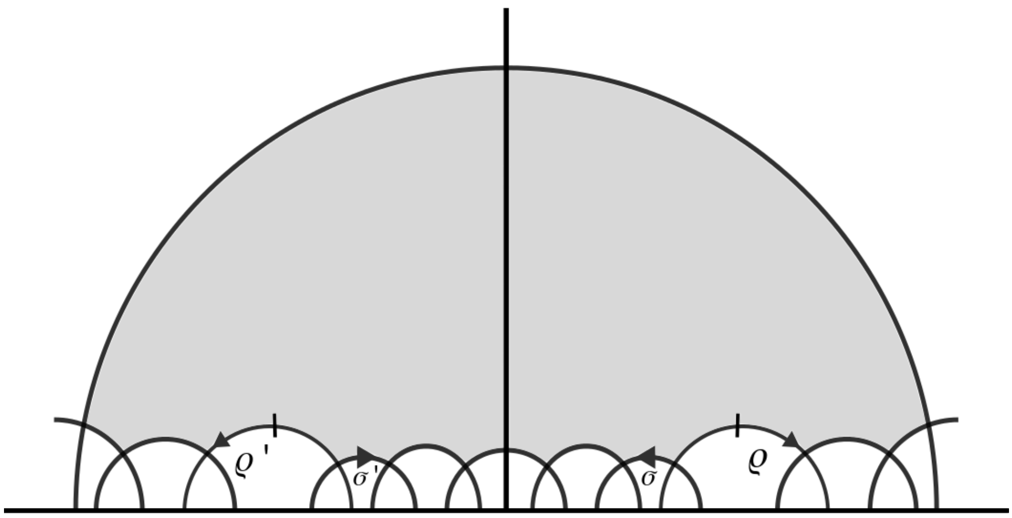

Figure 4.

3. **Trace Field of $\mathcal{S}$**

By slightly abusing notation, from now on, we denote $\rho_+, \sigma_+, (\rho')_+$ and $(\sigma')_+$ by ρ, σ, ρ' and σ' respectively.

By (2.1), the trace field of $\mathcal{S}$ is generated by the traces of the following elements:

$$\rho, \sigma, \rho', \sigma', \rho\sigma, \rho'\sigma', \rho\rho', \rho\sigma', \sigma\rho', \sigma\sigma', \rho\sigma\rho', \rho\sigma\sigma', \rho\rho'\sigma', \sigma\rho'\sigma'. \tag{3.1}$$

By the expressions given in (2.4) and (2.6), the trace field of $\mathcal{S}$ is clearly contained in

$$\mathbb{Q}(r,s,t,c,M,\sqrt{s^2-1},\sqrt{c^2-1}).$$

Now we further claim the following:

Theorem 3.1. *The trace field of $\mathcal{S}$ is $K=\mathbb{Q}(r,s,t)$.*

The proof is based on explicit computations. Note that we already know the traces of the first six elements in (3.1).[5] So, to prove the theorem, it is enough to show the traces of the following elements are contained in $\mathbb{Q}(r,s,t)$:

$$\rho\rho',\rho\sigma',\sigma\rho',\sigma\sigma',\rho\sigma\rho',\rho\sigma\sigma',\rho\rho'\sigma',\sigma\rho'\sigma'. \tag{3.2}$$

Also, by the symmetry between (2.4) and (2.6), we can further reduce (3.2) to the following five elements:

$$\rho\rho',\rho\sigma',\sigma\sigma',\rho\sigma\rho',\rho\sigma\sigma'. \tag{3.3}$$

Before computing the traces of (3.3), we first prove the following lemma which will make our computations more efficient:

Lemma 3.2. *Let*

$$A=a_0\mathbf{1}+a_1\mathbf{I}+a_2(\mathbf{J}+\mathbf{K})+a_3(\mathbf{J}-\mathbf{K}),$$
$$B=b_0\mathbf{1}+b_1\mathbf{I}+b_2(\mathbf{J}+\mathbf{K})+b_3(\mathbf{J}-\mathbf{K}).$$

Then their product AB is of the form

$$\begin{aligned}(a_0b_0+a_1b_1+2a_2b_3+2a_3b_2)\mathbf{1}+b_0\big(a_1\mathbf{I}+a_2(\mathbf{J}+\mathbf{K})+a_3(\mathbf{J}-\mathbf{K})\big)+a_0\big(b_1\mathbf{I}+b_2(\mathbf{J}+\mathbf{K})+b_3(\mathbf{J}-\mathbf{K})\big)\\ +(2a_2b_3-2a_3b_2)\mathbf{I}+(a_1b_2-a_2b_1)(\mathbf{J}+\mathbf{K})+(a_3b_1-a_1b_3)(\mathbf{J}-\mathbf{K}),\end{aligned} \tag{3.4}$$

and (thus) the trace of AB is

$$2(a_0b_0+a_1b_1+2a_2b_3+2a_3b_2). \tag{3.5}$$

Proof. Since

$$A=\begin{pmatrix}a_0+a_1 & 2a_2\\ 2a_3 & a_0-a_1\end{pmatrix} \text{ and } B=\begin{pmatrix}b_0+b_1 & 2b_2\\ 2b_3 & b_0-b_1\end{pmatrix},$$

AB is equal to

$$\begin{pmatrix}(a_0+a_1)(b_0+b_1)+4a_2b_3 & (a_0+a_1)2b_2+2a_2(b_0-b_1)\\ 2a_3(b_0+b_1)+(a_0-a_1)2b_3 & 4a_3b_2+(a_0-a_1)(b_0-b_1)\end{pmatrix}$$

$$=\begin{pmatrix}a_0b_0+a_1b_0+a_0b_1+a_1b_1+4a_2b_3 & 2a_0b_2+2a_1b_2+2a_2b_0-2a_2b_1\\ 2a_3b_0+2a_3b_1+2a_0b_3-2a_1b_3 & 4a_3b_2+a_0b_0-a_1b_0-a_0b_1+a_1b_1\end{pmatrix}$$

$$\begin{aligned}=&\begin{pmatrix}a_0b_0+a_1b_0+a_0b_1+a_1b_1+4a_2b_3 & 0\\ 0 & 4a_3b_2+a_0b_0-a_1b_0-a_0b_1+a_1b_1\end{pmatrix}\\ &+\begin{pmatrix}0 & 2a_0b_2+2a_1b_2+2a_2b_0-2a_2b_1\\ 2a_3b_0+2a_3b_1+2a_0b_3-2a_1b_3 & 0\end{pmatrix}\end{aligned}$$

$$\begin{aligned}=&\begin{pmatrix}a_0b_0+a_1b_1+2a_2b_3 & 0\\ 0 & 2a_3b_2+a_0b_0+a_1b_1\end{pmatrix}+\begin{pmatrix}a_1b_0+a_0b_1+2a_2b_3 & 0\\ 0 & 2a_3b_2-a_1b_0-a_0b_1\end{pmatrix}\\ &+\begin{pmatrix}0 & 2a_0b_2+2a_1b_2+2a_2b_0-2a_2b_1\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 2a_3b_0+2a_3b_1+2a_0b_3-2a_1b_3 & 0\end{pmatrix}\end{aligned}$$

[5] That is, $2r=\operatorname{tr}\rho=\operatorname{tr}\rho'$, $2s=\operatorname{tr}\sigma=\operatorname{tr}\sigma'$, and $2t=\operatorname{tr}\rho\sigma=\operatorname{tr}\rho'\sigma'$.

$$
\begin{aligned}
=&\begin{pmatrix} a_0b_0+a_1b_1+2a_2b_3+2a_3b_2 & 0 \\ 0 & 2a_3b_2+a_0b_0+a_1b_1+2a_2b_3 \end{pmatrix} \\
&+\begin{pmatrix} a_1b_0+a_0b_1+2a_2b_3-2a_3b_2 & 0 \\ 0 & 2a_3b_2-a_1b_0-a_0b_1-2a_2b_3 \end{pmatrix} \\
&+\begin{pmatrix} 0 & 2a_0b_2+2a_1b_2+2a_2b_0-2a_2b_1 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 2a_3b_0+2a_3b_1+2a_0b_3-2a_1b_3 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
=&\begin{pmatrix} a_0b_0+a_1b_1+2a_2b_3+2a_3b_2 & 0 \\ 0 & a_0b_0+a_1b_1+2a_2b_3+2a_3b_2 \end{pmatrix} \\
&+\begin{pmatrix} a_0b_1 & 0 \\ 0 & -a_0b_1 \end{pmatrix}+\begin{pmatrix} a_1b_0 & 0 \\ 0 & -a_1b_0 \end{pmatrix}+\begin{pmatrix} 2a_2b_3-2a_3b_2 & 0 \\ 0 & 2a_3b_2-2a_2b_3 \end{pmatrix} \\
&+\begin{pmatrix} 0 & 2a_0b_2 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 2a_2b_0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 2a_1b_2-2a_2b_1 \\ 0 & 0 \end{pmatrix} \\
&+\begin{pmatrix} 0 & 0 \\ 2a_0b_3 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 2a_3b_0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 2a_3b_1-2a_1b_3 & 0 \end{pmatrix}.
\end{aligned}
$$

Since

$$
\mathbf{1}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{J}+\mathbf{K}=\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \text{ and } \mathbf{J}-\mathbf{K}=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},
$$

the last given formula above is equal to

$$
\begin{aligned}
&(a_0b_0+a_1b_1+2a_2b_3+2a_3b_2)\mathbf{1}+(a_0b_1)\mathbf{I}+(a_1b_0)\mathbf{I}+(2a_2b_3-2a_3b_2)\mathbf{I} \\
&+a_0b_2(\mathbf{J}+\mathbf{K})+a_2b_0(\mathbf{J}+\mathbf{K})+(a_1b_2-a_2b_1)(\mathbf{J}+\mathbf{K})+a_0b_3(\mathbf{J}-\mathbf{K})+a_3b_0(\mathbf{J}-\mathbf{K})+(a_3b_1-a_1b_3)(\mathbf{J}-\mathbf{K}),
\end{aligned}
$$

which is the same as (3.4). □

Now we compute the traces of the elements in (3.3). We compute $\rho\rho', \rho\sigma', \sigma\sigma'$ first.

Lemma 3.3. *The traces of $\rho\rho'$, $\rho\sigma'$ and $\sigma\sigma'$ are contained in $\mathbb{Q}(r,s,t)$.*

Proof. Recall the formulas of ρ and ρ' in (2.4) and (2.6). By Lemma 3.2, the trace of $\rho\rho'$ is equal to

$$
\begin{aligned}
2\Bigg\{ & r^2+\Big(\frac{r(c+1)}{\sqrt{c^2-1}}\Big)\Big(\frac{r(c+1)}{\sqrt{c^2-1}}\Big)+2\Big(-\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\Big)\Big(-\frac{2rs-t+\big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\Big) \\
&+2\Big(\frac{2rs-t+\big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\Big)\Big(\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\Big)\Bigg\}
\end{aligned}
$$

which can be simplified as

$$
\begin{aligned}
&2r^2+\frac{2r^2(c+1)^2}{c^2-1}+\frac{\big(2rs-t+(c+\sqrt{c^2-1})t\big)\big(2rs-t+\big(c-\sqrt{c^2-1}\big)t\big)}{(c-1+2s^2)(c-1)} \\
&+\frac{\big(2rs-t+\big(c-\sqrt{c^2-1}\big)t\big)\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{(c-1)(c-1+2s^2)} \\
=&2r^2+\frac{2r^2(c+1)^2}{c^2-1}+\frac{\big((2rs-t+tc)+t\sqrt{c^2-1}\big)\big((2rs-t+tc)-t\sqrt{c^2-1}\big)}{(c-1+2s^2)(c-1)} \\
&+\frac{\big((2rs-t+tc)-t\sqrt{c^2-1}\big)\big((2rs-t+tc)+t\sqrt{c^2-1}\big)}{(c-1)(c-1+2s^2)}
\end{aligned}
$$

$$=2r^2+\frac{2r^2(c+1)^2}{c^2-1}+\frac{2\big((2rs-t+ct)^2-(c^2-1)t^2\big)}{(c-1+2s^2)(c-1)}.$$

Clearly this is contained in $\mathbb{Q}(r,s,t)$. (Recall $c\in\mathbb{Q}(r,s,t)$.)

Similarly, the trace of $\rho\sigma'$ is equal to

$$2\Bigg\{rs+\Big(\frac{r(c+1)}{\sqrt{c^2-1}}\Big)\Big(-\frac{s(c+1)}{\sqrt{c^2-1}}\Big)+2\Big(-\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\Big)\Big(\frac{c-1+2s^2}{2\tilde{M}(c-1)}\Big)$$
$$+2\Big(\frac{2rs-t+\big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\Big)\Big(-\frac{\tilde{M}}{2}\Big)\Bigg\},$$

which can be simplified as

$$\begin{aligned}&2rs-\frac{2rs(c+1)^2}{c^2-1}-\frac{2rs-t+(c+\sqrt{c^2-1})t}{c-1}-\frac{2rs-t+(c-\sqrt{c^2-1})t}{c-1}\\ =\ &2rs-\frac{2rs(c+1)}{c-1}-\frac{2(2rs-t+ct)}{c-1}.\end{aligned}$$

This is also an element of $\mathbb{Q}(r,s,t)$.

Lastly the trace of $\sigma\sigma'$ is equal to

$$\begin{aligned}&2\Bigg\{s^2+\Big(-\frac{s(c+1)}{\sqrt{c^2-1}}\Big)\Big(-\frac{s(c+1)}{\sqrt{c^2-1}}\Big)+2\Big(\frac{\tilde{M}}{2}\Big)\Big(\frac{c-1+2s^2}{2\tilde{M}(c-1)}\Big)+2\Big(-\frac{c-1+2s^2}{2\tilde{M}(c-1)}\Big)\Big(-\frac{\tilde{M}}{2}\Big)\Bigg\}\\ =\ &2s^2+\frac{2s^2(c+1)^2}{c^2-1}+\frac{c-1+2s^2}{(c-1)}+\frac{c-1+2s^2}{(c-1)}\\ =\ &2s^2+\frac{2s^2(c+1)}{c-1}+\frac{2(c-1+2s^2)}{(c-1)},\end{aligned}$$

which is contained in $\mathbb{Q}(r,s,t)$ as well. This completes the proof. □

Now we compute the traces of $\rho\sigma\rho'$ and $\rho\sigma\sigma'$. To do so, we need the explicit formula of $\rho\sigma$ first.

Lemma 3.4. *$\rho\sigma$ is*

$$t\mathbf{1}+\frac{t\sqrt{c^2-1}}{c-1}\mathbf{I}+\frac{(c+1+\sqrt{c^2-1})\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}(\mathbf{J}+\mathbf{K})$$
$$+\frac{(c+1-\sqrt{c^2-1})\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}\big(\mathbf{J}-\mathbf{K}\big).$$

Proof. We restate the formulas of ρ and σ given in (2.4):

$$\rho=r\mathbf{1}+\frac{r(c+1)}{\sqrt{c^2-1}}\mathbf{I}-\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K})+\frac{2rs-t+\big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\big(\mathbf{J}-\mathbf{K}\big),$$
$$\sigma=s\mathbf{1}-\frac{s(c+1)}{\sqrt{c^2-1}}\mathbf{I}+\frac{\tilde{M}}{2}(\mathbf{J}+\mathbf{K})-\frac{c-1+2s^2}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}).$$

To simply the notation, we abbreviate the above ρ and σ as

$$\rho=a_0\mathbf{1}+a_1\mathbf{I}+a_2(\mathbf{J}+\mathbf{K})+a_3(\mathbf{J}-\mathbf{K}),$$
$$\sigma=b_0\mathbf{1}+b_1\mathbf{I}+b_2(\mathbf{J}+\mathbf{K})+b_3(\mathbf{J}-\mathbf{K}).$$

Note that $a_0b_0 + a_1b_1 + 2a_2b_3 + 2a_3b_2 = t$ (since we got the matrices in (2.4) by initially assuming that the trace of $\rho\sigma$ is $2t$). We now compute each term appearing in (3.4) explicitly as follows:

(i) $b_0\big(a_1\mathbf{I} + a_2(\mathbf{J}+\mathbf{K}) + a_3(\mathbf{J}-\mathbf{K})\big) + a_0\big(b_1\mathbf{I} + b_2(\mathbf{J}+\mathbf{K}) + b_3(\mathbf{J}-\mathbf{K})\big)$

$$
\begin{aligned}
&= s\bigg(\frac{r(c+1)}{\sqrt{c^2-1}}\mathbf{I} - \frac{\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) + \frac{2rs - t + \big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\bigg)\\
&\quad + r\bigg(- \frac{s(c+1)}{\sqrt{c^2-1}}\mathbf{I} + \frac{\tilde{M}}{2}(\mathbf{J}+\mathbf{K}) - \frac{c-1+2s^2}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\bigg)\\
&= \bigg(\frac{sr(c+1)}{\sqrt{c^2-1}}\mathbf{I} - \frac{s\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) + \frac{s\big(2rs - t + (c-\sqrt{c^2-1})t\big)}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\bigg)\\
&\quad + \bigg(- \frac{rs(c+1)}{\sqrt{c^2-1}}\mathbf{I} + \frac{r\tilde{M}}{2}(\mathbf{J}+\mathbf{K}) - \frac{r(c-1+2s^2)}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\bigg)\\
&= \bigg(- \frac{s\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) + \frac{s\big(2rs - t + (c-\sqrt{c^2-1})t\big)}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\bigg)\\
&\quad + \bigg(\frac{r\tilde{M}}{2}(\mathbf{J}+\mathbf{K}) - \frac{r(c-1+2s^2)}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\bigg)\\
&= \bigg(\frac{r\tilde{M}}{2} - \frac{s\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\bigg)(\mathbf{J}+\mathbf{K})\\
&\quad - \bigg(\frac{r(c-1+2s^2)}{2\tilde{M}(c-1)} - \frac{s\big(2rs - t + (c-\sqrt{c^2-1})t\big)}{2\tilde{M}(c-1)}\bigg)(\mathbf{J}-\mathbf{K})\\
&= \bigg(\frac{r\tilde{M}(c-1+2s^2)}{2(c-1+2s^2)} - \frac{s\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\bigg)(\mathbf{J}+\mathbf{K})\\
&\quad - \bigg(\frac{r(c-1+2s^2)}{2\tilde{M}(c-1)} - \frac{s\big(2rs - t + (c-\sqrt{c^2-1})t\big)}{2\tilde{M}(c-1)}\bigg)(\mathbf{J}-\mathbf{K})\\
&= \frac{r\tilde{M}(c-1+2s^2) - s\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K})\\
&\quad - \frac{r(c-1+2s^2) - s\big(2rs - t + (c-\sqrt{c^2-1})t\big)}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\\
&= \frac{\tilde{M}(rc - r + 2s^2r) - \tilde{M}\big(2s^2r - st + st(c+\sqrt{c^2-1})\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K})\\
&\quad - \frac{rc - r + 2s^2r - \big(2s^2r - st + st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K})\\
&= \frac{\tilde{M}\big(r(c-1) + st - st(c+\sqrt{c^2-1})\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) - \frac{r(c-1) + st - st(c-\sqrt{c^2-1})}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K});
\end{aligned} \tag{3.6}
$$

(ii) $2(a_2b_3 - a_3b_2)\mathbf{I}$

$$
\begin{aligned}
&= 2\bigg\{\bigg(- \frac{\tilde{M}\big(2rs - t + (c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\bigg)\bigg(- \frac{c-1+2s^2}{2\tilde{M}(c-1)}\bigg) - \bigg(\frac{2rs - t + \big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\bigg)\bigg(\frac{\tilde{M}}{2}\bigg)\bigg\}\mathbf{I}\\
&= \bigg(\frac{2rs - t + (c+\sqrt{c^2-1})t}{2(c-1)} - \frac{2rs - t + (c-\sqrt{c^2-1})t}{2(c-1)}\bigg)\mathbf{I}
\end{aligned}
$$

$$= \frac{t\sqrt{c^2-1}}{(c-1)}\mathbf{I}; \tag{3.7}$$

(iii) $(a_1b_2 - a_2b_1)(\mathbf{J}+\mathbf{K})$

$$\begin{aligned}
&= \left\{\left(\frac{r(c+1)}{\sqrt{c^2-1}}\right)\left(\frac{\tilde{M}}{2}\right) - \left(-\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\right)\left(-\frac{s(c+1)}{\sqrt{c^2-1}}\right)\right\}(\mathbf{J}+\mathbf{K})\\
&= \left(\frac{\tilde{M}r(c+1)}{2\sqrt{c^2-1}} - \frac{s(c+1)\tilde{M}(2rs-t+(c+\sqrt{c^2-1})t)}{2\sqrt{c^2-1}(c-1+2s^2)}\right)(\mathbf{J}+\mathbf{K})\\
&= \left(\frac{\tilde{M}r(c+1)(c-1+2s^2)}{2\sqrt{c^2-1}(c-1+2s^2)} - \frac{s(c+1)\tilde{M}(2rs-t+(c+\sqrt{c^2-1})t)}{2\sqrt{c^2-1}(c-1+2s^2)}\right)(\mathbf{J}+\mathbf{K})\\
&= \frac{(c+1)\tilde{M}\big(r(c-1+2s^2)-2s^2r+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}(\mathbf{J}+\mathbf{K})\\
&= \frac{(c+1)\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}(\mathbf{J}+\mathbf{K});
\end{aligned} \tag{3.8}$$

(iv) $(a_3b_1 - a_1b_3)(\mathbf{J}-\mathbf{K})$

$$\begin{aligned}
&= \left\{\left(\frac{2rs-t+\big(c-\sqrt{c^2-1}\big)t}{2\tilde{M}(c-1)}\right)\left(-\frac{s(c+1)}{\sqrt{c^2-1}}\right) - \left(\frac{r(c+1)}{\sqrt{c^2-1}}\right)\left(-\frac{c-1+2s^2}{2\tilde{M}(c-1)}\right)\right\}(\mathbf{J}-\mathbf{K})\\
&= \left(-\frac{s(c+1)\big(2rs-t+(c-\sqrt{c^2-1})t\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}} + \frac{r(c+1)(c-1+2s^2)}{2\tilde{M}(c-1)\sqrt{c^2-1}}\right)(\mathbf{J}-\mathbf{K})\\
&= \frac{-s(c+1)\big(2rs-t+(c-\sqrt{c^2-1})t\big)+r(c+1)(c-1+2s^2)}{2\tilde{M}(c-1)\sqrt{c^2-1}}(\mathbf{J}-\mathbf{K})\\
&= \frac{(c+1)\big(-2s^2r+st-st(c-\sqrt{c^2-1})\big)+(c+1)\big(r(c-1)+2s^2r\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}(\mathbf{J}-\mathbf{K})\\
&= \frac{(c+1)\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}(\mathbf{J}-\mathbf{K}).
\end{aligned} \tag{3.9}$$

Combining (3.6) - (3.9) together with $a_0b_0 + a_1b_1 + 2a_2b_3 + 2a_3b_2 = t$, we get $\rho\sigma$ is equal to

$$\begin{aligned}
&t\mathbf{1} + \frac{t\sqrt{c^2-1}}{(c-1)}\mathbf{I}\\
&+ \frac{\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) + \frac{(c+1)\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}(\mathbf{J}+\mathbf{K})\\
&- \frac{\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)}\big(\mathbf{J}-\mathbf{K}\big) + \frac{(c+1)\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}(\mathbf{J}-\mathbf{K})\\
=\ &t\mathbf{1} + \frac{t\sqrt{c^2-1}}{(c-1)}\mathbf{I}\\
&+ \left(\frac{\sqrt{c^2-1}\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)} + \frac{(c+1)\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}\right)(\mathbf{J}+\mathbf{K})
\end{aligned}$$

$$+\left(-\frac{\sqrt{c^2-1}\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}+\frac{(c+1)\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}\right)(\mathbf{J}-\mathbf{K})$$

$$=t\mathbf{1}+\frac{t\sqrt{c^2-1}}{c-1}\mathbf{I}+\frac{(c+1+\sqrt{c^2-1})\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}(\mathbf{J}+\mathbf{K})$$

$$+\frac{(c+1-\sqrt{c^2-1})\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}\big(\mathbf{J}-\mathbf{K}\big).$$

This completes the proof of Lemma 3.4. □

Using the above lemma, we compute the traces of $\rho\sigma\rho'$ and $\rho\sigma\sigma'$.

Lemma 3.5. *The trace of $\rho\sigma\rho'$ is equal to*

$$2tr+\frac{2tr(c+1)}{c-1}+\frac{2(c+1)(r-2st)t}{c-1+2s^2}+\frac{2\big(2stc-r(c-1)\big)\big(2rs-t+tc\big)}{(c-1)(c-1+2s^2)}.$$

Proof. Recall the formulas of $\rho\sigma$ and ρ' given in Lemma 3.4 and (2.6). By Lemma 3.2, the trace of $\sigma\rho\rho'$ is equal to

$$2\Bigg\{tr+\left(\frac{t\sqrt{c^2-1}}{c-1}\right)\left(\frac{r(c+1)}{\sqrt{c^2-1}}\right)$$

$$+2\left(\frac{(c+1+\sqrt{c^2-1})\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}\right)\left(-\frac{2rs-t+(c-\sqrt{c^2-1})t}{2\tilde{M}(c-1)}\right)$$

$$+2\left(\frac{(c+1-\sqrt{c^2-1})\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}\right)\left(\frac{\tilde{M}\big(2rs-t+(c+\sqrt{c^2-1})t\big)}{2(c-1+2s^2)}\right)\Bigg\}$$

$$=2\Bigg\{tr+\frac{tr(c+1)}{c-1}$$

$$-\frac{\Big(c+1+\sqrt{c^2-1}\Big)\Big(r(c-1)+st-st(c+\sqrt{c^2-1})\Big)\Big(2rs-t+(c-\sqrt{c^2-1})t\Big)}{2(c-1)\sqrt{c^2-1}(c-1+2s^2)}$$

$$+\frac{\Big(c+1-\sqrt{c^2-1}\Big)\Big(r(c-1)+st-st(c-\sqrt{c^2-1})\Big)\Big(2rs-t+(c+\sqrt{c^2-1})t\Big)}{2(c-1)\sqrt{c^2-1}(c-1+2s^2)}\Bigg\}$$

$$=2tr+\frac{2tr(c+1)}{c-1}$$

$$-\frac{\Big(c+1+\sqrt{c^2-1}\Big)\Big(r(c-1)+st-stc-st\sqrt{c^2-1}\Big)\Big(2rs-t+tc-t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}$$

$$+\frac{\Big(c+1-\sqrt{c^2-1}\Big)\Big(r(c-1)+st-stc+st\sqrt{c^2-1}\Big)\Big(2rs-t+tc+t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}$$

$$=2tr+\frac{2tr(c+1)}{c-1}$$

$$-\frac{\Big(c+1+\sqrt{c^2-1}\Big)\Big((r-st)(c-1)-st\sqrt{c^2-1}\Big)\Big(2rs-t+tc-t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}$$

$$+\frac{\Big(c+1-\sqrt{c^2-1}\Big)\Big((r-st)(c-1)+st\sqrt{c^2-1}\Big)\Big(2rs-t+tc+t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}$$

$$
\begin{aligned}
=&2tr+\frac{2tr(c+1)}{c-1}\\
&-\frac{\Big((c^2-1)(r-st)+\big(-(c+1)st+(r-st)(c-1)\big)\sqrt{c^2-1}-st(c^2-1)\Big)\Big(2rs-t+tc-t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
&+\frac{\Big((c^2-1)(r-st)+\big((c+1)st-(r-st)(c-1)\big)\sqrt{c^2-1}-st(c^2-1)\Big)\Big(2rs-t+tc+t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
=&2tr+\frac{2tr(c+1)}{c-1}\\
&-\frac{\Big((c^2-1)(r-st)+\big(-2stc+r(c-1)\big)\sqrt{c^2-1}-st(c^2-1)\Big)\Big(2rs-t+tc-t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
&+\frac{\Big((c^2-1)(r-st)+\big(2stc-r(c-1)\big)\sqrt{c^2-1}-st(c^2-1)\Big)\Big(2rs-t+tc+t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
=&2tr+\frac{2tr(c+1)}{c-1}\\
&-\frac{\Big((c^2-1)(r-2st)-\big(2stc-r(c-1)\big)\sqrt{c^2-1}\Big)\Big((2rs-t+tc)-t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
&+\frac{\Big((c^2-1)(r-2st)+\big(2stc-r(c-1)\big)\sqrt{c^2-1}\Big)\Big((2rs-t+tc)+t\sqrt{c^2-1}\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
=&2tr+\frac{2tr(c+1)}{c-1}+\frac{2\big((c^2-1)(r-2st)\big)\big(t\sqrt{c^2-1}\big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}+\frac{2\Big(\big(2stc-r(c-1)\big)\sqrt{c^2-1}\Big)\Big(2rs-t+tc\Big)}{\sqrt{c^2-1}(c-1)(c-1+2s^2)}\\
=&2tr+\frac{2tr(c+1)}{c-1}+\frac{2(c+1)(r-2st)t}{c-1+2s^2}+\frac{2\big(2stc-r(c-1)\big)\big(2rs-t+tc\big)}{(c-1)(c-1+2s^2)}.
\end{aligned}
$$

This completes the proof. □

Lemma 3.6. *The trace of $\rho\sigma\sigma'$ is equal to*

$$2ts-\frac{2ts(c+1)}{c-1}+\frac{2\big(r(c-1)+st-stc\big)}{c-1}-\frac{2(c+1)st}{c-1}.$$

Proof. Recall the formulas of $\rho\sigma$ and σ' given in Lemma 3.4 and (2.4). By Lemma 3.2, the trace of $\rho\sigma\sigma'$ is equal to

$$
\begin{aligned}
2\Bigg\{&ts+\left(\frac{t\sqrt{c^2-1}}{c-1}\right)\left(-\frac{s(c+1)}{\sqrt{c^2-1}}\right)\\
&+2\left(\frac{\big(c+1+\sqrt{c^2-1}\big)\tilde{M}\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1+2s^2)}\right)\left(\frac{c-1+2s^2}{2\tilde{M}(c-1)}\right)\\
&+2\left(\frac{\big(c+1-\sqrt{c^2-1}\big)\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\tilde{M}(c-1)\sqrt{c^2-1}}\right)\left(-\frac{\tilde{M}}{2}\right)\Bigg\}.
\end{aligned}
$$

This can be simplified as

$$2\Bigg\{ts-\frac{ts(c+1)}{c-1}+\frac{\big(c+1+\sqrt{c^2-1}\big)\big(r(c-1)+st-st(c+\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1)}$$

$$
\begin{aligned}
&- \frac{(c+1-\sqrt{c^2-1})\big(r(c-1)+st-st(c-\sqrt{c^2-1})\big)}{2\sqrt{c^2-1}(c-1)}\Bigg\} \\
=&2ts - \frac{2ts(c+1)}{c-1} + \frac{(c+1+\sqrt{c^2-1})\big(r(c-1)+st-stc-st\sqrt{c^2-1}\big)}{\sqrt{c^2-1}(c-1)} \\
&- \frac{(c+1-\sqrt{c^2-1})\big(r(c-1)+st-stc+st\sqrt{c^2-1}\big)}{\sqrt{c^2-1}(c-1)} \\
=&2ts - \frac{2ts(c+1)}{c-1} + \frac{(c+1+\sqrt{c^2-1})\big(r(c-1)+st-stc-st\sqrt{c^2-1}\big)}{\sqrt{c^2-1}(c-1)} \\
&- \frac{(c+1-\sqrt{c^2-1})\big(r(c-1)+st-stc+st\sqrt{c^2-1}\big)}{\sqrt{c^2-1}(c-1)} \\
=&2ts - \frac{2ts(c+1)}{c-1} + \frac{2\big(r(c-1)+st-stc\big)\sqrt{c^2-1}}{\sqrt{c^2-1}(c-1)} - \frac{2(c+1)st\sqrt{c^2-1}}{\sqrt{c^2-1}(c-1)} \\
=&2ts - \frac{2ts(c+1)}{c-1} + \frac{2\big(r(c-1)+st-stc\big)}{c-1} - \frac{2(c+1)st}{c-1}.
\end{aligned}
$$

□

By Lemma 3.3, Lemma 3.5 and Lemma 3.6, the traces of the elements in (3.3) are all contained in $\mathbb{Q}(r,s,t)$, which implies Theorem 3.1.

4. Proof of Theorem 1.3 (Part I)

In this section, we prove the first part of Theorem 1.3. (The second part will be shown in the next section.)

By Theorem 3.1, the trace field of $\mathcal{S}$ is equal to

$$\mathbb{Q}(r,s,t), \tag{4.1}$$

and, by (2.3), the quaternion algebra of $\mathcal{S}$ is equivalent to

$$\left(\frac{(2r)^2-4,\ -2c-2}{\mathbb{Q}(r,s,t)}\right) \tag{4.2}$$

where $c = 4rst - 2r^2 - 2s^2 - 2t^2 + 1$ and $c, r, s, t > 1$.[6] Thus the following theorem implies the first part of Theorem 1.3:

Theorem 4.1. *Let K be any real number field and A be any quaternion algebra over it. Then there exists $r, s, t \in K$ such that*

$$K = \mathbb{Q}(r,s,t) \ \text{ and } \ A \cong \left(\frac{(2r)^2-4,\ -2c-2}{\mathbb{Q}(r,s,t)}\right) \tag{4.3}$$

where $c = 4rst - 2r^2 - 2s^2 - 2t^2 + 1$ and $c, r, s, t > 1$.

Let A be of the form $\left(\frac{a,b}{K}\right)$ for some $a, b \in K$. Since both $\left(\frac{a,b}{K}\right)$ and $\left(\frac{au^2,\ bv^2-abw^2}{K}\right)$ are equivalent (see (2.2)), to prove Theorem 4.1, it is enough to solve the following equality:

$$\left(\frac{(2r)^2-4,\ -2c-2}{\mathbb{Q}(r,s,t)}\right) = \left(\frac{au^2,\ bv^2-abw^2}{K}\right).$$

[6] Following the same notation given in Section 2, recall that $\mathrm{tr}\,([\rho_+,\sigma_+]) = -\mathrm{tr}\,([\rho,\sigma]_+) = -2c$.

In other words, for any given number field K and $a, b \in K$ it is enough to find $r, s, t, u, v, w \in K$ such that

$$\begin{aligned} c &= 4rst - 2r^2 - 2s^2 - 2t^2 + 1, \\ -2c - 2 &= bv^2 - abw^2, \\ (2r)^2 - 4 &= au^2, \\ \mathbb{Q}(r, s, t) &= K, \\ c, r, s, t &> 1. \end{aligned} \tag{4.4}$$

To simplify the notation, we let $2r = x', 2s = y', 2t = z, 2c = -c'$ and rewrite (4.4) as follows:

$$(x')^2 + (y')^2 + z^2 - x'y'z = c' + 2, \tag{4.5}$$

$$c' - 2 = bv^2 - abw^2, \tag{4.6}$$

$$z^2 - 4 = au^2, \tag{4.7}$$

$$\mathbb{Q}(x', y', z) = K, \tag{4.8}$$

$$x', y', z > 2, c' < -2. \tag{4.9}$$

To further simplify, we combine (4.5) and (4.6), and transform the resulting equation as follows:

$$\begin{aligned} & (x')^2 + (y')^2 + z^2 - x'y'z = c' + 2 \\ \text{(by (4.6))} \Rightarrow\ & (x')^2 + (y')^2 + z^2 - x'y'z = bv^2 - abw^2 + 4 \\ \Rightarrow\ & (x' - \frac{y'z}{2})^2 - \frac{(y')^2 z^2}{4} + (y')^2 + z^2 = bv^2 - abw^2 + 4 \\ \Rightarrow\ & (x' - \frac{y'z}{2})^2 - \frac{(y')^2}{4}(z^2 - 4) + z^2 = bv^2 - abw^2 + 4 \\ \text{(by (4.7))} \Rightarrow\ & (x' - \frac{y'z}{2})^2 - \frac{(y')^2}{4}(au^2) + au^2 + 4 = bv^2 - abw^2 + 4 \\ \Rightarrow\ & (x' - \frac{y'z}{2})^2 - \frac{(y')^2}{4}(au^2) + au^2 = bv^2 - abw^2 \\ \Rightarrow\ & (x' - \frac{y'z}{2})^2 - \frac{(y')^2}{4}(au^2) + au^2 - bv^2 + abw^2 = 0. \end{aligned}$$

Let $x = x' - \frac{y'z}{2}$ and $y = \frac{y'u}{2}$. Then (4.5) - (4.8) are reduced to

$$\begin{aligned} x^2 - ay^2 + au^2 - bv^2 + abw^2 &= 0, \\ z^2 - 4 &= au^2, \\ \mathbb{Q}(x, \frac{y}{u}, z) &= K\ (u \neq 0), \end{aligned} \tag{4.10}$$

and (4.9) is equivalent to

$$x + \frac{yz}{2u} > 2,\ \frac{y}{u} > 1,\ z > 2,\ abw^2 - bv^2 > 4. \tag{4.11}$$

Now Theorem 4.1 follows from the following theorem:

Theorem 4.2. *For any real number field K and any $a, b \in K$, there exists $x, y, z, u, v, w \in K$ satisfying the following:*

$$x^2 - ay^2 + au^2 - bv^2 + abw^2 = 0, \tag{4.12}$$

$$z^2 - 4 = au^2, \tag{4.13}$$

$$\mathbb{Q}(x, \frac{y}{u}, z) = K, \tag{4.14}$$

$$u \neq 0,\ x + \frac{yz}{2u} > 2,\ \frac{y}{u} > 1,\ z > 2,\ abw^2 - bv^2 > 4. \tag{4.15}$$

We resolve each equation step by step. First we parametrize the solutions of (4.12) and then, using the parametrization, we will find a subset of the solutions satisfying (4.13) and (4.14). Finally, we will further show that there exists a subset of the subset of the solutions satisfying (4.15).

Proof. **Step 1.** Having $(x, y, u, v, w) = (0, 1, 1, 0, 0)$ is as an initial solution, we employ a well-known technique in number theory to find other solutions of (4.12). Let

$$x = m_1 t, \quad y = m_2 t + 1, \quad u = m_3 t + 1, \quad v = m_4 t, \quad w = m_5 t. \tag{4.16}$$

Plugging (4.16) into (4.12), we get

$$\begin{aligned}
&(m_1 t)^2 - a(m_2 t + 1)^2 + a(m_3 t + 1)^2 - b(m_4 t)^2 + ab(m_5 t)^2 = 0\\
\Rightarrow\ &(m_1 t)^2 - a\big((m_2 t)^2 + 2m_2 t + 1\big) + a\big((m_3 t)^2 + 2m_3 t + 1\big) - b(m_4 t)^2 + ab(m_5 t)^2 = 0\\
\Rightarrow\ &(m_1 t)^2 - a\big((m_2 t)^2 + 2m_2 t\big) + a\big((m_3 t)^2 + 2m_3 t\big) - b(m_4 t)^2 + ab(m_5 t)^2 = 0\\
\Rightarrow\ &\big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big)t^2 - 2a\big(m_2 - m_3\big)t = 0\\
\Rightarrow\ &\big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big)t = 2a\big(m_2 - m_3\big)\\
\Rightarrow\ &t = \frac{2a\big(m_2 - m_3\big)}{m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2}.
\end{aligned} \tag{4.17}$$

So, for any $m_1, m_2, m_3, m_4, m_5 \in K$ such that

$$m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2 \neq 0, \tag{4.18}$$

the following is a solution of (4.12):

$$x = m_1 t, \quad y = (m_2 t + 1), \quad u = (m_3 t + 1), \quad v = m_4 t, \quad w = m_5 t \tag{4.19}$$

where t is the one given in (4.17). Furthermore since (4.12) is homogeneous, for any $d \in K$, the following is also a solution of (4.12):

$$x = m_1 td, \quad y = (m_2 t + 1)\, d, \quad u = (m_3 t + 1)\, d, \quad v = m_4 td, \quad w = m_5 td, \tag{4.20}$$

where m_i $(1 \leq i \leq 5)$ and t are the same as above.

Step 2. Using the common factor d in (4.20), we now resolve the second equation (4.13). Since

$$z^2 - 4 = au^2 = a(m_3 t + 1)^2 d^2,$$

by letting

$$z + 2 = a(m_3 t + 1)d \quad \text{and} \quad z - 2 = (m_3 t + 1)d,$$

we get

$$(z + 2) - (z - 2) = 4 = d(m_3 t + 1)(a - 1)$$

and so

$$d = \frac{4}{(m_3 t + 1)(a - 1)}. \tag{4.21}$$

Thus, by fixing d as in (4.21), we can always make (4.13) true for any m_i $(1 \leq i \leq 5)$ satisfying (4.18) and $m_3 t + 1 \neq 0$.[7]

[7]By (2.2), we can assume $a > 1$. We will talk more about this in **Step 4**.

Step 3. Next we resolve the third condition (4.14). Among three elements $x, \dfrac{y}{u}, z$, we use $\dfrac{y}{u}$ as a generator of K. That is, we will show that there exists a generator g of K and $m_1, m_2, m_3, m_4, m_5 \in K$ such that

$$\frac{y}{u} = \frac{m_2 t + 1}{m_3 t + 1} = g, \tag{4.22}$$

where $t = \dfrac{2a(m_2 - m_3)}{m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2}$. We first transform (4.22) as follows:

$$\begin{aligned}
&\frac{m_2 t + 1}{m_3 t + 1} = g \\
&\Rightarrow m_2 t + 1 = g(m_3 t + 1) \\
&\Rightarrow \frac{2a(m_2 - m_3)m_2}{m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2} + 1 = g\Big(\frac{2a(m_2 - m_3)m_3}{m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2} + 1\Big) \\
&\Rightarrow \frac{2a(m_2 - m_3)m_2 + \big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big)}{m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2} \\
&\quad = g\Big(\frac{2a(m_2 - m_3)m_3 + \big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big)}{m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2}\Big) \\
&\Rightarrow 2a(m_2 - m_3)m_2 + \big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big) \\
&\quad = g\Big(2a(m_2 - m_3)m_3 + \big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big)\Big) \\
&\Rightarrow (g-1)\big(m_1^2 - am_2^2 + am_3^2 - bm_4^2 + abm_5^2\big) + 2ga(m_2 m_3 - m_3^2) - 2a(m_2^2 - m_2 m_3) = 0 \\
&\Rightarrow (g-1)m_1^2 - (g-1)bm_4^2 + (g-1)abm_5^2 - (g-1)am_2^2 + (g-1)am_3^2 + 2gam_2 m_3 - 2gam_3^2 \\
&\quad - 2am_2^2 + 2am_2 m_3 = 0 \\
&\Rightarrow (g-1)m_1^2 - (g-1)bm_4^2 + (g-1)abm_5^2 - (g+1)am_2^2 - (g+1)am_3^2 + 2gam_2 m_3 + 2am_2 m_3 = 0 \\
&\Rightarrow (g-1)m_1^2 - (g-1)bm_4^2 + (g-1)abm_5^2 - (g+1)a(m_2^2 - 2am_2 m_3 + m_3^2) = 0 \\
&\Rightarrow (g-1)m_1^2 - (g-1)bm_4^2 + (g-1)abm_5^2 - (g+1)a(m_2 - m_3)^2 = 0 \\
&\Rightarrow m_1^2 - bm_4^2 + abm_5^2 - \frac{a(g+1)}{g-1}(m_2 - m_3)^2 = 0 \\
&\Rightarrow m_1^2 - bm_4^2 + abm_5^2 - a\Big(1 + \frac{2}{g-1}\Big)(m_2 - m_3)^2 = 0 \\
&\Rightarrow \frac{1}{a}m_1^2 - \frac{b}{a}m_4^2 + bm_5^2 - \Big(1 + \frac{2}{g-1}\Big)(m_2 - m_3)^2 = 0 \\
&\Rightarrow \frac{1}{a}\Big(\frac{m_1}{m_2 - m_3}\Big)^2 - \frac{b}{a}\Big(\frac{m_4}{m_2 - m_3}\Big)^2 + b\Big(\frac{m_5}{m_2 - m_3}\Big)^2 - \Big(1 + \frac{2}{g-1}\Big) = 0. \\
&\Rightarrow \frac{1}{a}\Big(\frac{m_1}{m_2 - m_3}\Big)^2 - \frac{b}{a}\Big(\frac{m_4}{m_2 - m_3}\Big)^2 + b\Big(\frac{m_5}{m_2 - m_3}\Big)^2 = 1 + \frac{2}{g-1}. \qquad (4.23)
\end{aligned}$$

Since $a, b \in K$ are arbitrary, finding $m_i \in K$ $(1 \leq i \leq 5)$ and a generator g of K satisfying (4.22) (or (4.23)) is now equivalent to the following lemma:

Lemma 4.3. *Let K be any real number field and a', b' be nonzero elements in K. Then there exist a generator g' of K and $x, y, z \in K$ satisfying*

$$a'x^2 + b'y^2 - a'b'z^2 = g'. \tag{4.24}$$

Before proving the lemma, we first state two lemmas which will be used in the proof.

Lemma 4.4. *Let $\alpha \in \bar{\mathbb{Q}}$. If a minimal polynomial of α contains a nonzero term of odd degree, then $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$. Thus, for any real number field K, the set of generators of K satisfying this property is dense in $\mathbb{R}$.*

Proof. Let $f(x) = 0$ be a minimal polynomial of α. Since it contains a nonzero term of odd degree, we can express f as follows:

$$f(x) = x(a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_2x^2 + a_0) + (b_{2m}x^{2m} + b_{2m-1}x^{2m-1} + \cdots + b_2x^2 + b_0).$$

Since $f(x)$ is a minimal polynomial of α, $a_{2n}\alpha^{2n} + a_{2n-1}\alpha^{2n-1} + \cdots + a_2\alpha^2 + a_0 \neq 0$, and so

$$\alpha = -\frac{b_{2m}\alpha^{2m} + b_{2m-1}\alpha^{2m-1} + \cdots + b_2\alpha^2 + b_0}{a_{2n}\alpha^{2n} + a_{2n-1}\alpha^{2n-1} + \cdots + a_2\alpha^2 + a_0},$$

which implies $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

Let α be a generator of K. Then, for any nonzero $r \in \mathbb{Q}$, either a minimal polynomial of α or a minimal polynomial of $\alpha + r$ contains a nonzero term of odd degree. The second statement easily follows. □

Lemma 4.5. *Let α, β be algebraic numbers. Then, except for finitely many $r \in \mathbb{Q}$, it always satisfies $\mathbb{Q}(\alpha + r\beta) = \mathbb{Q}(\alpha, \beta)$.*

Proof. This is a well-known fact and used to prove the existence of a primitive element in a number field. See [2], for instance, for a proof. □

Proof of Lemma 4.3. By Lemma 4.5, there exists $(x_0, y_0) \in \mathbb{Q}^2$ such that $\mathbb{Q}(a'x_0^2 + b'y_0^2) = \mathbb{Q}(a', b')$. Let $F = \mathbb{Q}(a', b')$ and z_0 be an element of K such that $K = F(z_0^2)$. By Lemma 4.5, we can further assume that x_0 and y_0 satisfy $\mathbb{Q}(a'x_0^2 + b'y_0^2 - a'b'z_0^2) = \mathbb{Q}(a'x_0^2 + b'y_0^2, a'b'z_0^2)$. Now the following equalities hold:

$$\mathbb{Q}(a'x_0^2 + b'y_0^2 - a'b'z_0^2) = \mathbb{Q}(a'x_0^2 + b'y_0^2, a'b'z_0^2) = \mathbb{Q}(a', b')(a'b'z_0^2) = \mathbb{Q}(a', b')(z_0^2) = F(z_0^2) = K.$$

Taking $g' = a'x_0^2 + b'y_0^2 - a'b'z_0^2$, we get the desired result. □

Step 4. Now we find a solution satisfying (4.15). For sufficiently small number $\epsilon > 0$, using (2.2), we first assume that a and b satisfy

$$1 < a,\ b < 1 + \epsilon. \tag{4.25}$$

Since $a > 1$, we have $z^2 > 4$ by (4.13). Also if $\frac{y}{u} > 1$ and $z > 2$, then $\frac{yz}{2u} > 1$. Thus (4.15) can be replaced by

$$u \neq 0, \quad x > 1, \quad \frac{y}{u} > 1, \quad abw^2 - bv^2 > 4. \tag{4.26}$$

Following the formulas given in (4.20), (4.26) is equivalent to

$$(m_3t + 1)d \neq 0, \quad m_1td > 1, \quad \frac{m_2t + 1}{m_3t + 1} > 1, \quad ab(m_5td)^2 - b(m_4td)^2 > 4. \tag{4.27}$$

By the proof of Lemma 4.3, the set of

$$\Big(\frac{m_1}{m_2 - m_3}, \frac{m_4}{m_2 - m_3}, \frac{m_5}{m_2 - m_3}\Big)$$

such that

$$\frac{1}{a}\Big(\frac{m_1}{m_2 - m_3}\Big)^2 - \frac{b}{a}\Big(\frac{m_4}{m_2 - m_3}\Big)^2 + b\Big(\frac{m_5}{m_2 - m_3}\Big)^2$$

is a generator of K is dense in $\mathbb{R}^3$. Thus for sufficiently large $L > 0$, we can pick m_i satisfying

$$L < m_1,\ m_2,\ m_4 < L + \epsilon, \quad 2L < m_5 < 2L + \epsilon, \quad 0 < m_3 < \epsilon. \tag{4.28}$$

Then

$$m_1^2 - a(m_2^2 - m_3^3) - bm_4^2 + abm_5^2 < (L + \epsilon)^2 - (L^2 - \epsilon^2) - L^2 + (1 + \epsilon)^2(2L + \epsilon)^2 < 4L^2$$

and

$$m_1^2 - a(m_2^2 - m_3^3) - bm_4^2 + abm_5^2 > L^2 - (1 + \epsilon)(L + \epsilon)^2 - (1 + \epsilon)(L + \epsilon)^2 + (2L)^2 > 2L^2.$$

Thus

$$\frac{1}{3L} < \frac{1}{2L}\Big(1-\frac{\epsilon}{L}\Big) = \frac{2(L-\epsilon)}{4L^2} < t = \frac{2a(m_2-m_3)}{m_1^2 - a(m_2^2-m_3^2) - bm_4^2 + abm_5^2} < \frac{2(1+\epsilon)(L+\epsilon)}{2L^2} < \frac{2}{L} \tag{4.29}$$

and

$$1 < m_3t+1 < \epsilon\frac{2}{L}+1. \tag{4.30}$$

Now (4.29) and (4.30) imply

$$d = \frac{1}{(m_3t+1)(a-1)} > \frac{1}{(\frac{2\epsilon}{L}+1)\epsilon} > \frac{1}{2\epsilon}.$$

Clearly we have $(m_3t+1)d \neq 0$, and

$$m_1td > L\Big(\frac{1}{3L}\Big)\Big(\frac{1}{2\epsilon}\Big) = \frac{1}{6\epsilon} > 1$$

for sufficiently small ϵ. Since $m_2 > m_3$, $\dfrac{m_2t+1}{m_3t+1} > 1$. Lastly

$$ab(m_5td)^2 - b(m_4td)^2 = t^2d^2(abm_5^2 - bm_4^2) > \Big(\frac{1}{3L}\Big)^2\Big(\frac{1}{2\epsilon}\Big)^2\big(4L^2-(1+\epsilon)(L+\epsilon)^2\big) > \frac{1}{18\epsilon^2} > 4$$

since ϵ is sufficiently small.

Thus it satisfies (4.27) and this completes the proof of Theorem 4.2. □

Remark. For any real number field K and $a, b \in K$, let $m_i = m_{i_0} \in K$ $(1 \le i \le 5)$ be numbers obtained by following the above proof of Theorem 4.2. (That is, we get x, y, z, u, v, w satisfying (4.12) - (4.15) by letting $m_i = m_{i_0}$ in (4.20).) Then

$$\frac{1}{a}\Big(\frac{m_{1_0}}{m_{2_0}-m_{3_0}}\Big)^2 - \frac{b}{a}\Big(\frac{m_{4_0}}{m_{2_0}-m_{3_0}}\Big)^2 + b\Big(\frac{m_{5_0}}{m_{2_0}-m_{3_0}}\Big)^2$$

is a generator of K, and, by (4.23), we have

$$\frac{1}{a}\Big(\frac{m_{1_0}}{m_{2_0}-m_{3_0}}\Big)^2 - \frac{b}{a}\Big(\frac{m_{4_0}}{m_{2_0}-m_{3_0}}\Big)^2 + b\Big(\frac{m_{5_0}}{m_{2_0}-m_{3_0}}\Big)^2 = 1 + \frac{2}{\frac{y_0}{u_0}-1} \tag{4.31}$$

where y_0 and u_0 are the values of y and u when $m_i = m_{i_0}$. Since

$$\frac{1}{a}\Big(\frac{rm_{1_0}}{m_{2_0}-m_{3_0}}\Big)^2 - \frac{b}{a}\Big(\frac{rm_{4_0}}{m_{2_0}-m_{3_0}}\Big)^2 + b\Big(\frac{rm_{5_0}}{m_{2_0}-m_{3_0}}\Big)^2$$

is also a generator of K for any $r \in \mathbb{Q}\backslash\{0\}$, by letting[8]

$$m_1 = rm_{1_0},\ \ m_2 = m_{2_0},\ \ m_3 = m_{3_0},\ \ m_4 = rm_{4_0},\ \ m_5 = rm_{5_0}, \tag{4.32}$$

we can produce another x, y, z, u, v, w satisfying (4.12) - (4.15). Let y_r and u_r be the values of y and u when m_i $(1 \le i \le 5)$ are as given in (4.32). Then the following equality holds by (4.23):

$$\frac{1}{a}\Big(\frac{rm_{1_0}}{m_{2_0}-m_{3_0}}\Big)^2 - \frac{b}{a}\Big(\frac{rm_{4_0}}{m_{2_0}-m_{3_0}}\Big)^2 + b\Big(\frac{rm_{5_0}}{m_{2_0}-m_{3_0}}\Big)^2 = 1 + \frac{2}{\frac{y_r}{u_r}-1}. \tag{4.33}$$

Finally (4.31) and (4.33) imply

$$r^2\Big(1+\frac{2}{\frac{y_0}{u_0}-1}\Big) = 1 + \frac{2}{\frac{y_r}{u_r}-1}. \tag{4.34}$$

[8]We also assume that r is sufficiently close to 1.

5. **Proof of Theorem 1.3 (Part II)**

Now we prove the second statement of Theorem 1.3. Remark that once the trace field is equal to the invariant trace field, then the quaternion algebra is also equal to the invariant quaternion algebra.[9]

In the previous section, we used the trace of σ (recall tr $\sigma = 2y/u$) to realize an arbitrary given real number field K as the trace field of $\mathcal{S}$. Since the invariant trace field of $\mathcal{S}$ contains tr σ^2 and

$$\mathrm{tr}\ \sigma^2 = (\mathrm{tr}\ \sigma)^2 - 4,$$

the second part of Theorem 1.3 follows from the following strengthened version of Theorem 4.2:

Theorem 5.1. *Theorem 4.2 still holds after replacing* (4.14) *by*

$$\mathbb{Q}\Big(\frac{y}{u}\Big) = \mathbb{Q}\Big(\frac{y^2}{u^2}\Big) = K.$$

Intuitively this lemma is clear since we have a dense set of solutions satisfying (4.12) - (4.15) in Theorem 4.2. But, for the sake of completeness, we will give the complete proof below. Before proving the theorem, we prove the following lemma first:

Lemma 5.2. *Let α be an algebraic number and $f(x) = a_n x^n + \cdots + a_0$ be a minimal polynomial of α. Then $(x-1)^n f\Big(1 + \dfrac{2}{x-1}\Big)$ is a minimal polynomial of $1 + \dfrac{2}{\alpha - 1}$, and the coefficient of x^{n-1} in $(x-1)^n f\Big(1 + \dfrac{2}{x-1}\Big)$ is*

$$\sum_{i=0}^{n} \big(-n + 2i\big) a_i.$$

Proof. Let $\beta = 1 + \dfrac{2}{\alpha - 1}$. Then $\alpha = 1 + \dfrac{2}{\beta - 1}$, and so the first part follows easily.

We next expand $f\Big(1 + \dfrac{2}{x-1}\Big)$ as below:

$$\begin{aligned}
&a_n\Big(1 + \frac{2}{x-1}\Big)^n + a_{n-1}\Big(1 + \frac{2}{x-1}\Big)^{n-1} + \cdots + a_1\Big(1 + \frac{2}{x-1}\Big) + a_0 \\
=&a_n\left(1 + n\Big(\frac{2}{x-1}\Big) + \cdots + n\Big(\frac{2}{x-1}\Big)^{n-1} + \Big(\frac{2}{x-1}\Big)^n\right) \\
&+ a_{n-1}\left(1 + (n-1)\Big(\frac{2}{x-1}\Big) + \cdots + (n-1)\Big(\frac{2}{x-1}\Big)^{n-2} + \Big(\frac{2}{x-1}\Big)^{n-1}\right) \\
&\cdots \\
&+ a_1\left(1 + \Big(\frac{2}{x-1}\Big)\right) + a_0.
\end{aligned} \tag{5.1}$$

Multiplying (5.1) by $(x-1)^n$, $(x-1)^n f\Big(1 + \dfrac{2}{x-1}\Big)$ is equal to

$$\begin{aligned}
&a_n\Big((x-1)^n + n \cdot 2(x-1)^{n-1} + \cdots + n \cdot 2^{n-1}(x-1) + 2^n\Big) \\
&+ a_{n-1}\Big((x-1)^n + (n-1)2(x-1)^{n-1} + \cdots + (n-1)2^{n-2}(x-1)^2 + 2^{n-1}(x-1)\Big) \\
&\cdots \\
&+ a_1\Big((x-1)^n + 2(x-1)^{n-1}\Big) + a_0(x-1)^n.
\end{aligned} \tag{5.2}$$

[9] Since both the quaternion algebra and the invariant quaternion algebra are then defined over the same field and the invariant quaternion algebra is a subalgebra of the quaternion algebra, both are the same.

The coefficient of x^{n-1} in (5.2) is

$$a_n\big(-n+n\cdot 2\big)+a_{n-1}\big(-n+(n-1)2\big)+\cdots+a_1\big(-n+2\big)+a_0\big(-n\big),$$

which is equal to

$$\sum_{i=0}^{n}(-n+2i)a_i. \tag{5.3}$$

□

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $y=y_0$, $u=u_0$ be the pair given in the remark in the previous section (after the proof of Theorem 4.2). Let $f(x)$ be a minimal polynomial of $1+\frac{2}{\frac{y_0}{u_0}-1}$. Then, by Lemma 5.2,

$$(x-1)^{\deg f} f\Big(1+\frac{2}{x-1}\Big) \tag{5.4}$$

is a minimal polynomial of $\frac{y_0}{u_0}$. If (5.4) contains a nonzero term of odd degree, then $\mathbb{Q}\Big(\frac{y_0}{u_0}\Big)=\mathbb{Q}\Big(\frac{y_0^2}{u_0^2}\Big)$ by Lemma 4.4, and so we are done.

Otherwise, suppose the degrees of all the nonzero terms of (5.4) are even. Let y_r and u_r be the same ones given in the remark in Section 4, and $f(x)$ be of the form $a_{2n}x^{2n}+\cdots+a_0$. Then

$$\begin{aligned} f\Big(\frac{x}{r^2}\Big) =& a_{2n}\Big(\frac{x}{r^2}\Big)^{2n}+a_{2n-1}\Big(\frac{x}{r^2}\Big)^{2n-1}+\cdots+a_1\Big(\frac{x}{r^2}\Big)+a_0 \\ =& \Big(\frac{a_{2n}}{r^{4n}}\Big)x^{2n}+\Big(\frac{a_{2n-1}}{r^{4n-2}}\Big)x^{2n-1}+\cdots+\Big(\frac{a_1}{r^2}\Big)x+a_0 \end{aligned} \tag{5.5}$$

is a minimal polynomial of $r^2\Big(1+\frac{2}{\frac{y_0}{u_0}-1}\Big)$. To simplify the notation, we denote the polynomial in (5.5) by $g(x)$ and $r^2\Big(1+\frac{2}{\frac{y_0}{u_0}-1}\Big)$ by α. Then $\alpha=1+\frac{2}{\frac{y_r}{u_r}-1}$ by (4.34), and so $\frac{y_r}{u_r}=1+\frac{2}{\alpha-1}$. Since $g(x)$ is a minimal polynomial of α, $(x-1)^{2n}g\Big(1+\frac{2}{x-1}\Big)$ is a minimal polynomial of $\frac{y_r}{u_r}$ by Lemma 5.2. By the same lemma, the coefficient of x^{2n-1} in $(x-1)^{2n}g\Big(1+\frac{2}{x-1}\Big)$ is

$$\sum_{i=0}^{2n}\big(-2n+2i\big)\frac{a_i}{r^{2i}}. \tag{5.6}$$

Clearly there are infinitely many $r\in\mathbb{Q}$ making (5.6) nonzero. By Lemma 4.4, we have $\mathbb{Q}\Big(\frac{y_r}{u_r}\Big)=\mathbb{Q}\Big(\frac{y_r^2}{u_r^2}\Big)$ for those r. This completes the proof. □

References

[1] T. Gauglhofer. *Trace coordinates of Teichmüller spaces of Riemann surfaces.* PhD thesis, EPFL, 2005.
[2] P. Grillet. *Abstract algebra*, Springer, New York, 2007.
[3] J. Kahn. and V. Markovic. *Finding cocompact Fuchsian groups of given trace field and quaternion algebra*, Talk at *Geometric structures on* 3-*manifolds*, IAS, Oct 2015.
[4] C. Maclachlan and A. Reid. *The arithmetic of hyperbolic 3-manifolds*, Springer, New York, 2003.
[5] W. Neumann. *Realizing arithmetic invariants of hyperbolic 3-manifolds*, Contem. Math. **541** (2011), 233-246.

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