A COMPARISON OF L-GROUPS FOR COVERS OF SPLIT REDUCTIVE GROUPS

MARTIN H. WEISSMAN

Abstract. In one article, the author has defined an L-group associated to a cover of a quasisplit reductive group over a local or global field. In another article, Wee Teck Gan and Fan Gao define (following an unpublished letter of the author) an L-group associated to a cover of a pinned split reductive group over a local or global field. In this short note, we give an isomorphism between these L-groups. In this way, the results and conjectures discussed by Gan and Gao are compatible with those of the author. Both support the same Langlands-type conjectures for covering groups.

Contents

Summary of two constructions

Let $\mathbf{G}$ be a split reductive group over a local or global field F. Choose a Borel subgroup $\mathbf{B} = \mathbf{TU}$ containining a split maximal torus $\mathbf{T}$ in $\mathbf{G}$. Let $X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ be the character lattice, and $Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ be the cocharacter lattice of $\mathbf{T}$. Let $\Phi \subset X$ be the set of roots and Δ the subset of simple roots. For each root $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the associated root subgroup. Let $\Phi^\vee$ and $\Delta^\vee$ be the associated coroots and simple coroots. The root datum of $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ is

$$\Psi = (X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee).$$

Fix a pinning (épinglage) of $\mathbf{G}$ as well – a system of isomorphisms $x_\alpha \colon \mathbf{G}_a \to \mathbf{U}_\alpha$ for every root α.

The following notions of covering groups and their dual groups match those in [Wei15]. Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ over F; in particular, $\#\mu_n(F) = n$. Here $\mathbf{G}'$ is a central extension of $\mathbf{G}$ by $\mathbf{K}_2$ in the sense of [B-D], and write $(Q, \mathcal{D}, f)$ for the three Brylinski-Deligne invariants of $\mathbf{G}'$. Assume that if n is odd, then $Q \colon Y \to \mathbb{Z}$ takes only even values (this is [Wei15, Assumption 3.1]).

Let $\tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee$ be the dual group of $\tilde{\mathbf{G}}$, and let $\tilde{Z}^\vee$ be the center of $\tilde{G}^\vee$. The group $\tilde{G}^\vee$ is a pinned complex reductive group, associated to the root datum

$$(Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta}).$$

Date: January 8, 2016.
2010 *Mathematics Subject Classification.* 11F70; 22E50; 22E55.

Here $Y_{Q,n} \subset Y$ is a sublattice containing nY. For each coroot $\alpha^\vee \in \Phi^\vee$, there is an associated positive integer n_α dividing n and a "modified coroot" $\tilde{\alpha}^\vee = n_\alpha \alpha^\vee \in \tilde{\Phi}^\vee$. The set $\tilde{\Phi}^\vee$ consists of the modified coroots, and $\tilde{\Delta}^\vee$ the modified simple coroots. Define $Y_{Q,n}^{\mathrm{sc}}$ to be the sublattice of $Y_{Q,n}$ generated by the modified coroots. Then

$$\tilde{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times) \text{ and } \tilde{Z}^\vee = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times).$$

Let $\bar{F}/F$ be a separable algebraic closure, and $\mathrm{Gal}_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. Fix an injective character $\epsilon \colon \mu_n(F) \hookrightarrow \mathbb{C}^\times$. From this data, the constructions of [Wei15] and [GG14] both yield an L-group of $\tilde{\mathbf{G}}$ via a Baer sum of two extensions. In both papers, an extension

$$\text{(First twist)} \qquad \tilde{Z}^\vee \hookrightarrow E_1 \twoheadrightarrow \mathrm{Gal}_F$$

is described in essentially the same way When F is local, this "first twist" E_1 is defined via a $\tilde{Z}^\vee$-valued 2-cocycle on Gal_F. See [GG14, §5.2] and [Wei15, §5.4] (in the latter, E_1 is denoted $(\tau_Q)_* \widetilde{\mathrm{Gal}_F}$). Over global fields, the construction follows from the local construction and Hilbert reciprocity.

Both papers include a "second twist". Gan and Gao [GG14, §5.2] describe an extension

$$\text{(Second twist)} \qquad \tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}_F,$$

following an unpublished letter (June, 2012) from the author to Deligne. In [Wei15], the second twist is the fundamental group of a gerbe, denoted $\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{s})$. In this article $\bar{s} = \mathrm{Spec}(\bar{F})$, and so we write $\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F})$ instead.

Both papers proceed by taking the Baer sum of these two extensions, $E = E_1 \dotplus E_2$, to form an extension $\tilde{Z}^\vee \hookrightarrow E \twoheadrightarrow \mathrm{Gal}_F$. The extension E is denoted ${}^{\mathsf{L}}\tilde{Z}$ in [Wei15, §5.4]. Then, one pushes out the extension E via $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$, to define the L-group

$$\text{(L-group)} \qquad \tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_F\,.$$

The two constructions of the L-group, from [GG14] and [Wei15] are the same, except for insignificant linguistic differences, and a significant difference between the "second twists". In this short note, by giving an isomorphism,

$$\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F}) \text{ (described by the author) } \xrightarrow{\sim} E_2 \text{ (described by Gan and Gao)}$$

we will demonstrate that the second twists, and thus the L-groups, of both papers are isomorphic. Therefore, the work of Gan and Gao in [GG14] supports the broader conjectures of [Wei15].

Remark 0.1. Among the "insignificant linguistic differences," we note that Gan and Gao use extensions of $F^\times/F^{\times n}$ (for local fields) or the Weil group $\mathcal{W}_F$ rather than Gal_F. But pulling back via the reciprocity map of class field theory yields extensions of Gal_F by $\tilde{Z}^\vee$ as above.

1. Computations in the gerbe

1.1. **Convenient base points.** Let $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$ be the gerbe constructed in [Wei15, §3]. Rather than using the language of étale sheaves over F, we work with $\bar{F}$-points and trace through the Gal_F-action. Let $\hat{T} = \mathrm{Hom}(Y_{Q,n}, \bar{F}^\times)$ and $\hat{T}_{\mathrm{sc}} = \mathrm{Hom}(Y_{Q,n}^{\mathrm{sc}}, \bar{F}^\times)$.

Let $p\colon \hat{T} \to \hat{T}_{\mathrm{sc}}$ be the surjective Gal_F-equivariant homomorphism dual to the inclusion $Y_{Q,n}^{\mathrm{sc}} \hookrightarrow Y_{Q,n}$. Define

$$\hat{Z} = \mathrm{Ker}(p) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \bar{F}^\times).$$

The reader is warned not to confuse $\hat{T}, \hat{T}_{\mathrm{sc}}, \hat{Z}$ with $\tilde{T}^\vee, \tilde{T}_{\mathrm{sc}}^\vee, \tilde{Z}^\vee$; the former are nontrivial Gal_F-modules (Homs into $\bar{F}^\times$) and the latter are trivial Gal_F-modules (Homs into $\mathbb{C}^\times$ as a trivial Gal_F-module).

Write $\bar{D} = \mathcal{D}(\bar{F})$ and $D = \mathcal{D}(F)$, where we recall $\mathcal{D}$ is the second Brylinski-Deligne invariant of the cover $\tilde{\mathbf{G}}$. We have a Gal_F-equivariant short exact sequence,

$$\bar{F}^\times \hookrightarrow \bar{D} \twoheadrightarrow Y.$$

By Hilbert's Theorem 90, the Gal_F-fixed points give a short exact sequence,

$$F^\times \hookrightarrow D \twoheadrightarrow Y.$$

Let $\bar{D}_{Q,n}$ and $\bar{D}_{Q,n}^{\mathrm{sc}}$ denote the preimages of $Y_{Q,n}$ and $Y_{Q,n}^{\mathrm{sc}}$ in $\bar{D}$. These are *abelian* groups, fitting into a commutative diagram with exact rows.

$$\begin{array}{ccccc}
\bar{F}^\times & \hookrightarrow & \bar{D}_{Q,n}^{\mathrm{sc}} & \twoheadrightarrow & Y_{Q,n}^{\mathrm{sc}} \\
\downarrow{=} & & \downarrow & & \downarrow \\
\bar{F}^\times & \hookrightarrow & \bar{D}_{Q,n} & \twoheadrightarrow & Y_{Q,n}
\end{array}$$

Let $\mathrm{Spl}(\bar{D}_{Q,n})$ be the $\hat{T}$-torsor of splittings of $\bar{D}_{Q,n}$, and similarly let $\mathrm{Spl}(\bar{D}_{Q,n}^{\mathrm{sc}})$ be the $\hat{T}_{\mathrm{sc}}$-torsor of splittings of $\bar{D}_{Q,n}^{\mathrm{sc}}$.

Let $\overline{\mathrm{Whit}}$ denote the $\hat{T}_{\mathrm{sc}}$-torsor of nondegenerate characters of $\mathbf{U}(\bar{F})$. An element of $\overline{\mathrm{Whit}}$ is a homomorphism (defined over $\bar{F}$) from $\mathbf{U}$ to $\mathbf{G}_a$ which is nontrivial on every simple root subgroup $\mathbf{U}_\alpha$. Gal_F acts on $\overline{\mathrm{Whit}}$, and the fixed points $\mathrm{Whit} = \overline{\mathrm{Whit}}^{\mathrm{Gal}_F}$ are those homomorphisms from $\mathbf{U}$ to $\mathbf{G}_a$ which are defined over F. The $\hat{T}_{\mathrm{sc}}$-action on $\overline{\mathrm{Whit}}$ is described in [Wei15, §3.3].

The pinning $\{x_\alpha : \alpha \in \Phi\}$ of $\mathbf{G}$ gives an element $\psi \in \mathrm{Whit}$. Namely, let ψ be the unique nondegenerate character of $\mathbf{U}$ which satisfies

$$\psi(x_\alpha(1)) = 1 \text{ for all } \alpha \in \Delta.$$

In [Wei15, §3.3], we define an surjective homomorphism $\mu\colon \hat{T}_{\mathrm{sc}} \to \hat{T}_{\mathrm{sc}}$, and a Gal_F-equivariant isomorphism of $\hat{T}_{\mathrm{sc}}$-torsors,

$$\bar{\omega}\colon \mu_* \overline{\mathrm{Whit}} \to \mathrm{Spl}(D_{Q,n}^{\mathrm{sc}}).$$

The isomorphism $\bar{\omega}$ sends ψ to the unique splitting $s_\psi \in \mathrm{Spl}(D_{Q,n}^{\mathrm{sc}})$ which satisfies

$$s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha}, \text{ with } r_\alpha = (-1)^{Q(\alpha^\vee)\cdot \frac{n_\alpha(n_\alpha - 1)}{2}}.$$

We describe the element $[e_\alpha] \in D$ concisely here, based on [B-D, §11] and [GG14, §2.4]. Let $F(\!(v)\!)$ be the field of Laurent series with coefficients in F. The extension $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ splits over any unipotent subgroup, and so the pinning homomorphisms $x_\alpha\colon F(\!(v)\!) \to \mathbf{U}_\alpha(F(\!(v)\!))$ lift to homomorphisms

$$\tilde{x}_\alpha\colon F(\!(v)\!) \to \mathbf{U}'_\alpha(F(\!(v)\!)).$$

Define, for any $u \in F(\!(v)\!)^\times$,

$$\tilde{n}_\alpha(u) = \tilde{x}_\alpha(u)\tilde{x}_{-\alpha}(-u^{-1})\tilde{x}_\alpha(u).$$

This yields an element

$$\tilde{t}_\alpha = \tilde{n}_\alpha(\upsilon) \cdot \tilde{n}_\alpha(-1) \in \mathbf{T}'(F(\!(\upsilon)\!)).$$

Then t_α lies over $\alpha^\vee(\upsilon) \in \mathbf{T}(F(\!(\upsilon)\!))$. Its pushout via $\mathbf{K}_2(F(\!(\upsilon)\!)) \xrightarrow{\partial} F^\times$ is the element we call $[e_\alpha] \in D$.

Remark 1.1. The element $s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha}$ coincides with what Gan and Gao call $s_{Q^{\mathrm{sc}}}(\tilde{\alpha}^\vee)$ in [GG14, §5.2]; the sign r_α arises from the formulae of [B-D, §11.1.4, 11.1.5].

Let $j_0 \colon \hat{T}_{\mathrm{sc}} \to \mu_* \overline{\mathrm{Whit}}$ be the unique isomorphism of $\hat{T}_{\mathrm{sc}}$-torsors which sends 1 to ψ (or rather the image of ψ via $\overline{\mathrm{Whit}} \to \mu_* \overline{\mathrm{Whit}}$). Since $\psi \in \mathrm{Whit}$ is Gal_F-invariant, this isomorphism j_0 is also Gal_F-invariant.

Finally, let $s \in \mathrm{Spl}(\bar{D}_{Q,n})$ be a splitting which restricts to s_ψ on $Y_{Q,n}^{\mathrm{sc}}$. Such a splitting s exists, since the map $\mathrm{Spl}(\bar{D}_{Q,n}) \to \mathrm{Spl}(\bar{D}_{Q,n}^{\mathrm{sc}})$ is surjective (since the map $\hat{T} \to \hat{T}_{\mathrm{sc}}$ is surjective). Note that s is not necessarily Gal_F-invariant (and often cannot be).

Let $h \colon \hat{T} \to \mathrm{Spl}(\hat{D}_{Q,n})$ be the function given by

$$h(x) = x^n * s \text{ for all } x \in \hat{T}.$$

The triple $\bar{z} = (\hat{T}, h, j_0)$ is an $\bar{F}$-object (i.e., a geometric base point) of the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$. Note that the construction of $\bar{z}$ depends on two choices: a pinning of $\mathbf{G}$ (to obtain $\psi \in \mathrm{Whit}$) and a splitting s of $\bar{D}_{Q,n}$ extending s_ψ. We call such a triple $\bar{z}$ a <u>convenient base point</u> for the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$.

1.2. **The fundamental group.** For a convenient base point $\bar{z}$ associated to s, we consider the fundamental group

$$\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) = \bigsqcup_{\gamma \in \mathrm{Gal}_F} \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z}).$$

This fundamental group fits into a short exact sequence

$$\tilde{Z}^\vee \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) \twoheadrightarrow \mathrm{Gal}_F,$$

where the fibre over $\gamma \in \mathrm{Gal}_F$ is $\mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$. Thus to describe the fundamental group, it suffices to describe each fibre (as a $\tilde{Z}^\vee$-torsor), and the multiplication maps among fibres.

The base point ${}^\gamma \bar{z}$ is the triple $({}^\gamma \hat{T}, \gamma \circ h, \gamma \circ j_0)$, where ${}^\gamma \hat{T}$ is the $\hat{T}$-torsor with underlying set $\hat{T}$ and twisted action

$$u *_\gamma x = \gamma^{-1}(u) \cdot x.$$

To give an element $f \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ is the same as giving an element $\zeta \in \tilde{Z}^\vee$ and a map of $\hat{T}$-torsors $f_0 \colon \hat{T} \to {}^\gamma \hat{T}$ satisfying

$$(\gamma \circ h) \circ f_0 = h \text{ and } (\gamma \circ j_0) \circ p_* f_0 = j_0.$$

Any such map of $\hat{T}$-torsors is uniquely determined by the element $\tau \in \hat{T}$ satisfying $f_0(1) = \tau$. The two conditions above are equivalent to the two conditions

$$\tau^n = \gamma^{-1} s / s \text{ and } \tau \in \hat{Z}. \tag{1.1}$$

Thus, to give an element $f \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ is the same as giving a pair $(\tau, \zeta) \in \hat{T} \times \tilde{Z}^\vee$, where τ satisfies the two conditions above. Therefore, in what follows, we write $(\tau, \zeta) \in \mathrm{Hom}(\bar{z}, {}^{\gamma}\bar{z})$ to indicate that τ satisfies the two conditions above, and to refer to the corresponding morphism in the gerbe $\mathsf{E}_{\epsilon}(\tilde{\mathbf{G}})$ in concrete terms.

We use $\epsilon \colon \mu_n(F) \xrightarrow{\sim} \mu_n(\mathbb{C})$ to identify $\hat{Z}_{[n]}$ with $\tilde{Z}^{\vee}_{[n]}$. Two pairs (τ, ζ) and (τ', ζ') are identified in $\mathrm{Hom}(\bar{z}, {}^{\gamma}\bar{z})$ if and only if there exists $\xi \in \hat{Z}_{[n]}$ such that

$$\tau' = \xi \cdot \tau \text{ and } \zeta' = \epsilon(\xi)^{-1} \cdot \zeta.$$

The structure of $\mathrm{Hom}(\bar{z}, {}^{\gamma}\bar{z})$ as a $\tilde{Z}^{\vee}$-torsor is by scaling the second factor in $(\tau, \zeta) \in \hat{T} \times \tilde{Z}^{\vee}$. To describe the fundamental group completely, it remains to describe the multiplication maps among fibres. If $\gamma_1, \gamma_2 \in \mathrm{Gal}_F$, and

$$(\tau_1, \zeta_1) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_1}\bar{z}) \text{ and } (\tau_2, \zeta_2) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_2}\bar{z}),$$

then their composition in $\pi_1^{\text{ét}}(\mathsf{E}_{\epsilon}(\tilde{\mathbf{G}}), \bar{z})$ is given by

$$(\tau_1, \zeta_1) \circ (\tau_2, \zeta_2) = (\gamma_2^{-1}(\tau_1) \cdot \tau_2, \zeta_1 \zeta_2).$$

Observe that

$$(\gamma_2^{-1}(\tau_1)\tau_2)^n = \gamma_2^{-1}\left(\gamma_1^{-1} s/s\right) \cdot \left(\gamma_2^{-1} s/s\right) = (\gamma_1\gamma_2)^{-1} s/s.$$

Therefore $(\gamma_2^{-1}(\tau_1) \cdot \tau_2, \zeta_1\zeta_2) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_1\gamma_2}\bar{z})$ as required.

2. Comparison to the second twist

2.1. **The second twist.** The construction of the second twist in [GG14] does not rely on gerbes at all, at the expense of some generality; it seems difficult to extend the construction there to nonsplit groups. But for split groups, the construction of [GG14] offers significant simplifications over [Wei15]. The starting point in [GG14] is the same short exact sequence of abelian groups as in the previous section,

$$F^{\times} \hookrightarrow D_{Q,n} \twoheadrightarrow Y_{Q,n}.$$

And as before, we utilize the splitting $s_{\psi} \colon Y^{\mathrm{sc}}_{Q,n} \hookrightarrow D^{\mathrm{sc}}_{Q,n}$. Taking the quotient by $s_{\psi}(Y^{\mathrm{sc}}_{Q,n})$, we obtain a short exact sequence

$$F^{\times} \hookrightarrow \frac{D_{Q,n}}{s_{\psi}(Y^{\mathrm{sc}}_{Q,n})} \twoheadrightarrow \frac{Y_{Q,n}}{Y^{\mathrm{sc}}_{Q,n}}.$$

Apply $\mathrm{Hom}(\bullet, \mathbb{C}^{\times})$ (and note $\mathbb{C}^{\times}$ is divisible) to obtain a short exact sequence,

$$\tilde{Z}^{\vee} \hookrightarrow \mathrm{Hom}\left(\frac{D_{Q,n}}{s_{\psi}(Y^{\mathrm{sc}}_{Q,n})}, \mathbb{C}^{\times}\right) \twoheadrightarrow \mathrm{Hom}(F^{\times}, \mathbb{C}^{\times}).$$

Define a homomorphism $\mathrm{Gal}_F \to \mathrm{Hom}(F^{\times}, \mathbb{C}^{\times})$ by the Artin symbol,

$$\gamma \mapsto \left(u \mapsto \epsilon\left(\frac{\gamma^{-1}(\sqrt[n]{u})}{\sqrt[n]{u}}\right)\right).$$

Pulling back the previous short exact sequence by this homomorphism yields a short exact sequence

$$\tilde{Z}^{\vee} \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}_F.$$

This E_2 is the second twist described in [GG14].

Remark 2.1. There is an insignificant difference here – at the last step, over a local field F, Gan and Gao pull back to $F^{\times}/F^{\times n}$ via the Hilbert symbol whereas we pull further back to Gal_F via the Artin symbol.

Write $E_{2,\gamma}$ for the fibre of E_2 over any $\gamma \in \mathrm{Gal}_F$. Again, to understand the extension E_2, it suffices to understand these fibres (as $\tilde{Z}^\vee$-torsors), and to understand the multiplication maps among them. The steps above yield the following (somewhat) concise description of $E_{2,\gamma}$.

$E_{2,\gamma}$ is the set of homomorphisms $\chi \colon D_{Q,n} \to \mathbb{C}^\times$ such that

- χ is trivial on the image of $Y_{Q,n}^{\mathrm{sc}}$ via the splitting s_ψ.
- For every $u \in F^\times$, $\chi(u) = \epsilon(\gamma^{-1} \sqrt[n]{u} / \sqrt[n]{u})$.

Multiplication among fibres is given by usual multiplication, $\chi_1, \chi_2 \mapsto \chi_1 \chi_2$. The $\tilde{Z}^\vee$-torsor structure on the fibres is given as follows: if $\eta \in \tilde{Z}^\vee$, then

$$[\eta * \chi](d) = \eta(y) \cdot \chi(d) \text{ for all } d \in D_{Q,n} \text{ lying over } y \in Y_{Q,n}.$$

2.2. **Comparison.** Now we describe a map from $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z})$ to E_2, fibrewise over Gal_F. From the splitting s (used to define $\bar{z}$ and restricting to s_ψ on $Y_{Q,n}^{\mathrm{sc}}$), every element of $\bar{D}_{Q,n}$ can be written uniquely as $s(y) \cdot u$ for some $y \in Y_{Q,n}$ and some $u \in \bar{F}^\times$. Such an element $s(y) \cdot u$ is Gal_F-invariant if and only if

$$\gamma(s(y))\gamma(u) = s(y)u, \text{ or equivalently } \frac{{}^{\gamma^{-1}}u}{u} \cdot \frac{{}^{\gamma^{-1}}s}{s}(y) = 1, \text{ for al } \gamma \in \mathrm{Gal}_F.$$

Suppose that $\gamma \in \mathrm{Gal}_F$ and $(\tau, 1) \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$. Define $\chi \colon D_{Q,n} \to \mu_n(\mathbb{C})$ by

$$\chi(s(y) \cdot u) = \epsilon\left(\gamma^{-1} \sqrt[n]{u} / \sqrt[n]{u} \cdot \tau(y)\right).$$

This makes sense, because Gal_F-invariance of $s(y) \cdot u$ implies

$$\left(\frac{\gamma^{-1} \sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau(y)\right)^n = \frac{{}^{\gamma^{-1}}u}{u} \cdot \frac{{}^{\gamma^{-1}}s}{s}(y) = 1.$$

To see that $\chi \in E_{2,\gamma}$, observe that

- χ is a homomorphism (a straightforward computation).
- If $y \in Y_{Q,n}^{\mathrm{sc}}$ then $\chi(s(y)) = \tau(y) = 1$ since $\tau \in \hat{Z}$.
- If $u \in F^\times$ then $\chi(u) = \epsilon(\gamma^{-1} \sqrt[n]{u} / \sqrt[n]{u})$ by definition.

Lemma 2.2. *The map sending $(\tau, 1)$ to χ, described above, extends uniquely to an isomorphism of $\tilde{Z}^\vee$-torsors from $\mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ to $E_{2,\gamma}$.*

Proof. If this map extends to an isomorphism of $\tilde{Z}^\vee$-torsors as claimed, the map must send an element $(\tau, \zeta) \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ to the element $\zeta * \chi \in E_{2,\gamma}$. To demonstrate that the map extends to an isomorphism of $\tilde{Z}^\vee$-torsors, it must only be checked that

$$(\xi \cdot \tau, 1) \text{ and } (\tau, \epsilon(\xi))$$

map to the same element of $E_{2,\gamma}$, for all $\xi \in \hat{Z}_{[n]}$. For this, we observe that $(\xi \cdot \tau, 1)$ maps to the character χ' given by

$$\chi'(s(y) \cdot u) = \epsilon\left(\frac{\gamma^{-1} \sqrt[n]{u}}{\sqrt[n]{u}} \xi(y) \tau(y)\right) = \epsilon(\xi(y)) \cdot \epsilon\left(\frac{\gamma^{-1} \sqrt[n]{u}}{\sqrt[n]{u}} \tau(y)\right) = \epsilon(\xi(y)) \cdot \chi(s(y) \cdot u).$$

Thus $\chi' = \epsilon(\xi) * \chi$ and this demonstrates the lemma. ☐

From this lemma, we have a well-defined "comparison" isomorphism of $\tilde{Z}^\vee$-torsors,

$$C_\gamma \colon \mathrm{Hom}(\bar{z}, {}^\gamma z) \to E_{2,\gamma},$$

$$C_\gamma(\tau, \zeta)(s(y) \cdot u) = \epsilon\left(\frac{\gamma^{-1} \sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau(y)\right) \cdot \zeta(y). \qquad \text{(Comparison)}$$

Checking compatibility with multiplication yields the following.

Lemma 2.3. *The isomorphisms C_γ are compatible with the multiplication maps, yielding an isomorphism of extensions of* Gal_F *by* $\tilde{Z}^\vee$,

$$C = C_{\bar{z}} \colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) \to E_2.$$

Proof. Suppose that $(\tau_1, \zeta_1) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_1}\bar{z})$ and $(\tau_2, \zeta_2) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_2}\bar{z})$. Their product in $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z})$ is $(\gamma_2^{-1}(\tau_1)\tau_2, \zeta_1\zeta_2)$. We compute

$$\begin{aligned}
C_{\gamma_1\gamma_2}(\tau_1\gamma^{-1}(\tau_2), \zeta_1\zeta_2)(s(y)\cdot u) &= \epsilon\left(\frac{(\gamma_1\gamma_2)^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \gamma_2^{-1}(\tau_1(y))\tau_2(y)\right) \cdot \zeta_1(y)\zeta_2(y) \\
&= \epsilon\left(\frac{\gamma_2^{-1}\gamma_1^{-1}\sqrt[n]{u}}{\gamma_2^{-1}\sqrt[n]{u}} \cdot \frac{\gamma_2^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \gamma_2^{-1}(\tau_1(y))\tau_2(y)\right) \\
&\quad \cdot \zeta_1(y)\zeta_2(y) \\
&= \epsilon\left(\gamma_2^{-1}\left(\frac{\gamma_1^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau_1(y)\right) \cdot \frac{\gamma_2^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau_2(y)\right) \\
&\quad \cdot \zeta_1(y)\zeta_2(y) \\
&= \epsilon\left(\frac{\gamma_1^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau_1(y)\right)\zeta_1(y) \\
&\quad \cdot \epsilon\left(\frac{\gamma_2^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau_2(y)\right)\zeta_2(y) \\
&= C_{\gamma_1}(\tau_1, \zeta_1)(s(y)\cdot u) \cdot C_{\gamma_2}(\tau_2, \zeta_2)(s(y)\cdot u)
\end{aligned}$$

In the middle step, we use the fact that $\left(\frac{\gamma_1^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau_1(y)\right)$ is an element of $\mu_n(F)$, and hence is Gal_F-invariant. This computation demonstrates compatibility of the isomorphisms C_γ with multiplication maps, and hence the lemma is proven. □

2.3. **Independence of base point.** Lastly, we demonstrate that the comparison isomorphisms

$$C_{\bar{z}} \colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) \to E_2$$

depend naturally on the choice of convenient base point. With the pinned split group $\mathbf{G}$ fixed, choosing a convenient base point is the same as choosing a splitting of $\bar{D}_{Q,n}$ which restricts to s_ψ.

So consider two convenient base points $\bar{z}_1$ and $\bar{z}_2$, arising from splittings s_1, s_2 of $\bar{D}_{Q,n}$ which restrict to s_ψ on $Y_{Q,n}^{\text{sc}}$. Any isomorphism ι from $\bar{z}_1$ to $\bar{z}_2$ in the gerbe $\mathbf{E}_\epsilon(\tilde{\mathbf{G}})$ defines an isomorphism

$$\iota \colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_1) \to \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_2).$$

See [Wei15, Theorem 19.6] for details. In fact, the isomorphism of fundamental groups above does not depend on the choice of isomorphism from $\bar{z}_1$ to $\bar{z}_2$; thus one may define a "Platonic" fundamental group

$$\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F})$$

without reference to an object of the gerbe.

Theorem 2.4. *For any two convenient base points $\bar{z}_1, \bar{z}_2$, and any isomorhpism $\iota \colon \bar{z}_1 \to \bar{z}_2$, we have $C_{\bar{z}_2} \circ \iota = C_{\bar{z}_1}$. Thus E_2 is isomorphic to the fundamental group $\pi_1(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F})$, as defined in [Wei15, Theorem 19.7, Remark 19.8].*

Proof. Choose any isomorphism from $\bar{z}_1 = (\hat{T}, h_1, j_0)$ to $\bar{z}_2 = (\hat{T}, h_2, j_0)$ in the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$. Here $h_1(1) = s_1$ and $h_2(1) = s_2$, and $j_0(1) = s_\psi$. Such an isomorphism $\bar{z}_1 \xrightarrow{\sim} \bar{z}_2$ is given by an isomorphism $\iota \colon \hat{T} \to \hat{T}$ of $\hat{T}$-torsors satisfying the two conditions

$$h_2 \circ \iota = h_1 \text{ and } j_0 \circ p_* \iota = j_0.$$

Such an ι is determined by the element $b = \iota(1) \in \hat{T}$. The two conditions above are equivalent to the two conditions

$$b^n = s_1/s_2 \text{ and } b \in \hat{Z}.$$

The isomorphism $\bar{z}_1 \xrightarrow{\sim} \bar{z}_2$ determined by such a $b \in \hat{T}$ yields an isomorphism ${}^\gamma\iota \colon {}^\gamma\bar{z}_1 \to {}^\gamma\bar{z}_2$, for any $\gamma \in \mathrm{Gal}_F$. The isomorphim ${}^\gamma\iota$ is given by the isomorphism of $\hat{T}$-torsors from ${}^\gamma\hat{T}$ to ${}^\gamma\hat{T}$, which sends 1 to $\gamma(b)$.

This allows us to describe the isomorphism

$$\iota \colon \pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_1) \to \pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_2)$$

fibrewise over Gal_F. Namely, for any $\gamma \in \mathrm{Gal}_F$, and any $f \in \mathrm{Hom}(\bar{z}_1, {}^\gamma\bar{z}_1)$, we find a unique element $\iota(f) \in \mathrm{Hom}(\bar{z}_2, {}^\gamma\bar{z}_2)$ which makes the following diagram commute.

$$\begin{array}{ccc} \bar{z}_1 & \xrightarrow{f} & {}^\gamma\bar{z}_1 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle {}^\gamma\iota} \\ \bar{z}_2 & \xrightarrow{\iota(f)} & {}^\gamma\bar{z}_2 \end{array}$$

If $f = (\tau, 1)$, then $\iota(f) = (\tau b / \gamma^{-1} b, 1)$. Indeed, when $\tau^n = \gamma^{-1} s_1 / s_1$, we have

$$\left(\frac{\tau b}{\gamma^{-1} b} \right)^n = \frac{\gamma^{-1} s_1}{s_1} \frac{b^n}{\gamma^{-1} b^n} = \frac{\gamma^{-1} s_1}{s_1} \frac{s_1}{s_2} \frac{\gamma^{-1} s_2}{\gamma^{-1} s_1} = \frac{\gamma^{-1} s_2}{s_2}.$$

Thus $\iota(f) \in \mathrm{Hom}(\bar{z}_2, {}^\gamma\bar{z}_2)$ as required. In this way,

$$\iota \colon \pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_1) \to \pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_2),$$

is given concretely on each fibre over $\gamma \in \mathrm{Gal}_F$ by

$$\iota(\tau, \zeta) = \left(\tau \cdot \frac{b}{\gamma^{-1} b}, \zeta \right).$$

Note that the conditions $b^n = s_1/s_2$ and $b \in \hat{Z}$ uniquely determine b up to multiplication by $\hat{Z}_{[n]}$. Since $\hat{Z}_{[n]}$ is a trivial Gal_F-module, the isomorphism ι of fundamental groups is independent of b. Finally, we compute, for any $y \in Y_{Q,n}, u \in \bar{F}^\times$ such that $s_1(y) \cdot u \in D_{Q,n}$, and any $(\tau, \zeta) \in \mathrm{Hom}(\bar{z}_1, {}^\gamma\bar{z}_1)$,

$$\begin{aligned} [C_{\bar{z}_2} \circ \iota](\tau, \zeta)(s_1(y) \cdot u) &= C_{\bar{z}_2}(\tau b / \gamma^{-1} b, \zeta)(s_1(y) \cdot u) \\ &= C_{\bar{z}_2}(\tau \gamma(b)/b, \zeta)(s_2(y) \cdot b^n(y) u) \\ &= \epsilon \left(\frac{\gamma^{-1} \sqrt[n]{b^n(y) u}}{\sqrt[n]{b^n(y) u}} \cdot \tau(y) \cdot \frac{b(y)}{\gamma^{-1}(b(y))} \right) \cdot \zeta(y) \\ &= \epsilon \left(\frac{\gamma^{-1} \sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau(y) \right) \cdot \zeta(u) \\ &= C_{\bar{z}_1}(\tau, \zeta)(s_1(y) \cdot u). \end{aligned}$$

$\square$

As noted in the introduction, this demonstrates compatibility between two approaches to the L-group.

Corollary 2.5. *The L-group defined in [Wei15] is isomorphic to the L-group defined in [GG14], for all pinned split reductive groups over local or global fields.*

References

[B-D] Jean-Luc Brylinski and Pierre Deligne. "Central extensions of reductive groups by $\mathbf{K}_2$". In: *Publ. Math. Inst. Hautes Études Sci.* 94 (2001), pp. 5–85.

[GG14] Wee Teck Gan and Fan Gao. "The Langlands-Weissman Program for Brylinski-Deligne extensions". In: *ArXiv e-prints* (Sept. 2014). arXiv:1409.4039 [math.NT].

[Wei15] M. H. Weissman. "L-groups and parameters for covering groups". In: *ArXiv e-prints* (July 2015). arXiv:1507.01042 [math.NT].

Yale-NUS College, 6 College Ave East, #B1-01, Singapore 138614
E-mail address: marty.weissman@yale-nus.edu.sg